Parcial 2 - Métodos Computacionales
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Integración

1. Regla de trapecio simple

Se realiza la aproximación $f(x) \approx p_1(x) = \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)$

al tomar los puntos $(a, f(a))$, $(b, f(b))$ e interpolarlos.

Seguido a esto tenemos:

$$I = \int_a^b f(x)\,dx \approx \int_a^b p_1(x)\,dx$$

$$\int_a^b p_1(x)\,dx = \int_a^b \frac{x-b}{a-b} f(a) + \frac{x-a}{b-a} f(b)\,dx$$

$$= \frac{1}{2} \cdot \frac{f(a)}{a-b} (x-b)^2 + \frac{1}{2} \cdot \frac{f(b)}{b-a} (x-a)^2 \Big|_a^b$$

$$= \frac{1}{2} \frac{f(b)}{b-a} (b-a)^2 - \frac{1}{2} \frac{f(a)}{a-b} (a-b)^2$$

$$= \frac{1}{2} f(b)(b-a) - \frac{1}{2} f(a)(a-b)$$

$$= \frac{1}{2}(b-a)(f(a)+f(b))$$

Así, $I \approx \frac{1}{2}(b-a)(f(a)+f(b))$

3. Regla de Simpson simple

Se realiza la aproximación $f(x) \approx P_2(x) = \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a)$ $+ \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) + \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)$, al tomar los puntos $(a, f(a))$, $(x_m, f(x_m))$, $(b, f(b))$, donde $x_m = \frac{a+b}{2}$

Seguido a esto teremos:

$$I = \int_a^b f(x)\,dx \approx \int_a^b P_2(x)\,dx$$

$$\int_a^b P_2(x)\,dx = \int_a^b \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a)\,dx + \int_a^b \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m)\,dx$$

$$+ \int_a^b \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)\,dx$$

$$= \frac{f(a)}{(a-b)(a-x_m)} \int_a^b (x-b)(x-x_m)\,dx + \frac{f(x_m)}{(x_m-a)(x_m-b)} \int_a^b (x-a)(x-b)\,dx$$

$$+ \frac{f(b)}{(b-a)(b-x_m)} \int_a^b (x-a)(x-x_m)\,dx$$

Considere $x_m - a = h$ y $b - x_m = h$, es decir $b-a = 2h$. Note que

$$\int_a^b P_2(x)\,dx = \frac{f(a)}{2h^2} \int_a^{a+2h} (x-a-2h)(x-a-h)\,dx - \frac{f(x_m)}{h^2} \int_a^{a+2h} (x-a)(x-a-2h)\,dx$$

$$+ \frac{f(b)}{2h^2} \int_a^{a+2h} (x-a)(x-a-h)\,dx.$$

Si realizamos la sustitución $u = x-a$, donde para $x=a, u=0$ y para $x = a+2h$, $u = 2h$. $(du = dx)$.

$$= \frac{f(a)}{2h^2} \int_0^{2h} (u-2h)(u-h)\,du - \frac{f(x_m)}{h^2} \int_0^{2h} u(u-2h)\,du$$

$$+ \frac{f(b)}{2h^2} \int_0^{2h} u(u-h)\,du$$

$$= \frac{f(a)}{2h^2} \int_0^{2h} (u^2 - 3uh + 2h^2)\,du - \frac{f(x_m)}{h^2} \int_0^{2h} u^2 - 2uh\,du$$

$$+ \frac{f(b)}{2h^2} \int_0^{2h} u^2 - uh\,du$$

$$= \frac{f(a)}{2h^2}\left(\frac{1}{3}u^3 - \frac{3}{2}u^2h + 2h^2u\right)\Big|_0^{2h} - \frac{f(x_m)}{h^2}\left(\frac{1}{3}u^3 - u^2h\right)\Big|_0^{2h}$$

$$+ \frac{f(b)}{2h^2}\left(\frac{1}{3}u^3 - \frac{1}{2}u^2h\right)\Big|_0^{2h}$$

$$= \frac{f(a)}{2h^2}\left(\frac{8h^3}{3} - 6h^3 + 4h^3\right) - \frac{f(x_m)}{h^2}\left(\frac{8h^3}{3} - 4h^3\right)$$

$$+ \frac{f(b)}{2h^2}\left(\frac{8h^3}{3} - 2h^3\right)$$

$$= \frac{f(a)}{2}h\left(\frac{2}{3}\right) - f(x_m)\cdot h\left(-\frac{4}{3}\right) + \frac{f(b)}{2}h\left(\frac{2}{3}\right)$$

$$= \frac{h}{3}\left(f(a) + 4f(x_m) + f(b)\right)$$

Así, $\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f(a) + 4f(x_m) + f(b)\right)$