Parcial 4 - Métodos Computacionales
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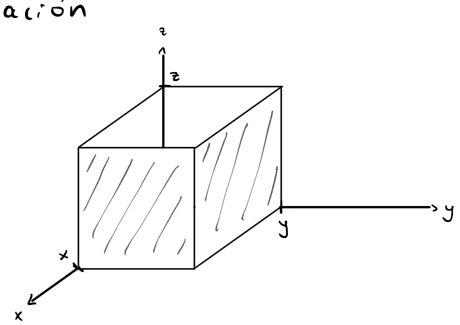
Optimización

3. a)



b) El área superficial corresponde a la suma de las áreas laterales y de la base. Como cada cara tiene área xy, xz o yz según corresponda, podemos ver a partir del dibujo que el área superficial es $A(x,y,z) = xy + 2yz + 2xz$. Como hacemos la restricción de que $A(x,y,z) = 12$, entonces

$$A(x,y,z) = xy + 2yz + 2xz = 12.$$

e)
$$V(x,y,z) = xyz$$

Restricción: $xy + 2yz + 2xz = 12$, $x,y,z \geq 0$.

Sea $A(x,y,z) = xy + 2yz + 2xz - 12$.

Lagrange:
$$\nabla V = \lambda \nabla A$$
$$(yz, xz, xy) = \lambda(y + 2z,\ x + 2z,\ 2y + 2x)$$

$$\begin{cases} yz = \lambda(y + 2z) & (1) \\ xz = \lambda(x + 2z) & (2) \\ xy = \lambda(2y + 2x) & (3) \end{cases}$$

$$\Rightarrow (1)\ y = \frac{2\lambda z}{z - \lambda},\ (2)\ x = \frac{2\lambda z}{z - \lambda} \Rightarrow y = x$$

$$(3)\ x^2 = \lambda(4x) \rightarrow x = 4\lambda$$

$$(2)\ 4\lambda z = 4\lambda^2 + 2z\lambda \rightarrow z = 2\lambda \Rightarrow x = 2z$$

Aplicando la restricción, vemos que $x = y = 2z = 2 \Rightarrow V(2,2,1) = 4.$

Distribuciones continuas de probabilidad:

1. Dada la función de probabilidad conjunta:

$$f(x,y) = \begin{cases} \frac{2}{3}(x+2y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otro caso} \end{cases}$$

a) Verifique que sea una función de densidad conjunta válida:

- Para $x \in \mathbb{R} \setminus [0,1]$ y $y \in \mathbb{R} \setminus [0,1]$, $f(x,y) = 0 \ge 0$.

  Para $x \in [0,1]$ y $y \in [0,1]$, $f(x,y) = \frac{2}{3}(x+2y) \ge 0$ pues $x+2y \ge 0$. $(x \ge 0$ y $y \ge 0)$.

- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dy dx = 0 + \int_0^1 \int_0^1 \frac{2}{3}(x+2y)\, dx dy = \int_0^1 \frac{1}{3}x^2 + \frac{4}{3}xy \Big|_0^1 dy = \int_0^1 \frac{1}{3} + \frac{4}{3}y\, dy$

  $= \frac{1}{3}y + \frac{2}{3}y^2 \Big|_0^1 = \frac{1}{3} + \frac{2}{3} = 1.$

b) Para $0 \le x \le 1$: $g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = 0 + \int_0^1 f(x,y)\, dy = \int_0^1 \frac{2}{3}(x+2y)\, dy$

$= \int_0^1 \frac{2}{3}x + \frac{4}{3}y\, dy = \frac{2}{3}xy + \frac{2}{3}y^2 \Big|_0^1 = \frac{2}{3}x + \frac{2}{3} = \frac{2}{3}(x+1)$

Para $x \in \mathbb{R} \setminus [0,1]$: $g(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = 0.$

$$g(x) = \begin{cases} \frac{2}{3}(x+1) & 0 \le x \le 1 \\ 0 & \text{otro caso.} \end{cases}$$

Para $0 \le y \le 1$: $h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = 0 + \int_0^1 \frac{2}{3}(x+2y)\, dx = \int_0^1 \frac{2}{3}x + \frac{4}{3}y\, dx$

$= \frac{1}{3}x^2 + \frac{4}{3}xy \Big|_0^1 = \frac{1}{3} + \frac{4}{3}y = \frac{1}{3}(1+4y).$

Para $y \in \mathbb{R} \setminus [0,1]$: $h(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = 0.$

$$h(y) = \begin{cases} \frac{1}{3}(1+4y) & 0 \le y \le 1 \\ 0 & \text{otro caso} \end{cases}$$

c) $E(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\, dx = 0 + \int_0^1\int_0^1 x \cdot \frac{2}{3}(x+2y)\, dy dx = \int_0^1\int_0^1 \frac{2}{3}x^2 + \frac{4}{3}xy\, dy dx$

$= \int_0^1 \frac{2}{3}x^2 y + \frac{2}{3}xy^2 \Big|_0^1 dx = \int_0^1 \frac{2}{3}x^2 + \frac{2}{3}x\, dx = \frac{2}{9}x^3 + \frac{1}{3}x^2 \Big|_0^1 = \frac{2}{9} + \frac{1}{3} = \frac{5}{9}$

d) $E(y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f(x,y)\,dy = 0 + \int_0^1\int_0^1 y\cdot\frac{2}{3}(x+2y)\,dy\,dx = \int_0^1\int_0^1 \frac{2}{3}xy + \frac{4}{3}y^2\,dy\,dx$

$= \int_0^1 \frac{1}{3}xy^2 + \frac{4}{9}y^3\Big|_0^1\,dx = \int_0^1 \frac{1}{3}x + \frac{4}{9}\,dx = \frac{1}{6}x^2 + \frac{4}{9}x\Big|_0^1 = \frac{1}{6} + \frac{4}{9} = \frac{11}{18}$

e) $Cov(x,y) = E(xy) - E(x)E(y)$. Solo falta calcular $E(xy)$:

$E(xy) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f(x,y)\,dy\,dx = \int_0^1\int_0^1 xy\cdot\frac{2}{3}(x+2y)\,dy\,dx = \int_0^1\int_0^1 \frac{2}{3}x^2y + \frac{4}{3}xy^2\,dy\,dx$

$= \int_0^1 \frac{1}{3}x^2y^2 + \frac{4}{9}xy^3\Big|_0^1\,dx = \int_0^1 \frac{1}{3}x^2 + \frac{4}{9}x\,dx = \frac{1}{9}x^3 + \frac{2}{9}x^2\Big|_0^1 = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

$Cov(x,y) = \frac{1}{3} - \frac{11}{18}\cdot\frac{5}{9} = -\frac{1}{162} \approx -0.00617.$

f) $cov(x,y) = E((x-\hat{\mu}_x)(y-\hat{\mu}_y)) = E((x-E(x))(y-E(y))) = E((x-\frac{5}{9})(y-\frac{11}{18}))$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\frac{5}{9})(y-\frac{11}{18}) f(x,y)\,dy\,dx = \int_0^1\int_0^1 \left(xy - \frac{5}{9}y - \frac{11}{18}x + \frac{55}{162}\right)\cdot\frac{2}{3}(x+2y)\,dy\,dx$

$= \int_0^1\int_0^1 xy\cdot\frac{2}{3}(x+2y)\,dy\,dx - \frac{5}{9}\int_0^1\int_0^1 y\cdot\frac{2}{3}(x+2y)\,dy\,dx - \frac{11}{18}\int_0^1\int_0^1 x\cdot\frac{2}{3}(x+2y)\,dy\,dx$

$+ \frac{55}{162}\int_0^1\int_0^1 \frac{2}{3}(x+2y)\,dy\,dx = E(xy) - \frac{5}{9}E(y) - \frac{11}{18}E(x) + \frac{55}{162}\cdot 1$

$= \frac{1}{3} - \frac{5}{9}\cdot\frac{11}{18} - \cancel{\frac{11}{18}\cdot\frac{5}{9}} + \cancel{\frac{55}{162}} = -\frac{1}{162} \approx -0.00617.$

g) Como $cov(x,y) \neq 0$, las variables x y y no son independientes.

Estimación de parámetros.

1. Sea $A(x_1, \dots, x_n)$ donde $A \sim N(\mu, \sigma)$ con parámetros μ y σ. Muestre que los estimadores máximo verosímiles son:

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

Para $N(\mu, \sigma)$ se tiene la función:

$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Así, para varias variables independientes que se distribuyen normalmente, se define la función de verosimilitud así:

$$\mathcal{L}(\mu, \sigma^2) = f(x_1, \dots, x_n \mid \mu, \sigma^2) = f(x_1 \mid \mu, \sigma^2) \cdot \ldots \cdot f(x_n \mid \mu, \sigma^2) = \prod_{i=1}^{n} f(x_i \mid \mu, \sigma^2)$$
$$= \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n e^{-\sum_{i=1}^{n}\frac{(x_i-\mu)^2}{2\sigma^2}} = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}$$

Para mayor comodidad, consideraremos $\tilde{\mathcal{L}}(\mu, \sigma^2) = \ln(\mathcal{L}(\mu, \sigma^2))$

$$\tilde{\mathcal{L}}(\mu, \sigma^2) = \ln\left(\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}\right) = \ln\left[\left(\frac{1}{2\pi\sigma^2}\right)^{n/2}\right] + \ln\left(e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}}\right)$$
$$= -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

Maximizaremos $\tilde{\mathcal{L}}$ (equivalente a maximizar $\mathcal{L}$ por la monotonía creciente de $\ln$).

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \mu} = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{\partial}{\partial \mu}(x_i-\mu)^2 = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\frac{\partial}{\partial \mu}(x_i^2 - 2x_i\mu + \mu^2) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}(-2x_i + 2\mu)$$
$$= -\frac{1}{2\sigma^2}\left(2n\mu - 2\sum_{i=1}^{n} x_i\right) = -\frac{n\mu}{\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \sigma} = -\frac{n}{2}\cdot\frac{4\pi\sigma}{2\pi\sigma^2} + \frac{1}{2\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2$$

Queremos $\frac{\partial \tilde{L}}{\partial \mu} = \frac{\partial \tilde{L}}{\partial \sigma} = 0$:

$$0 = -\frac{n\mu}{\sigma^2} + \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i \Rightarrow \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$0 = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n} (x_i-\mu)^2 \Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i-\hat{\mu})^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i-\bar{x})^2$$

Como buscábamos.