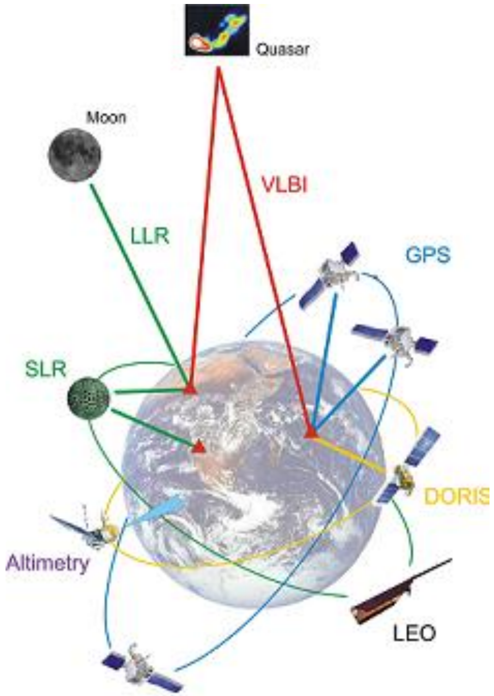
GGOS and Reference Systems

Geophysical models

2017-12-04



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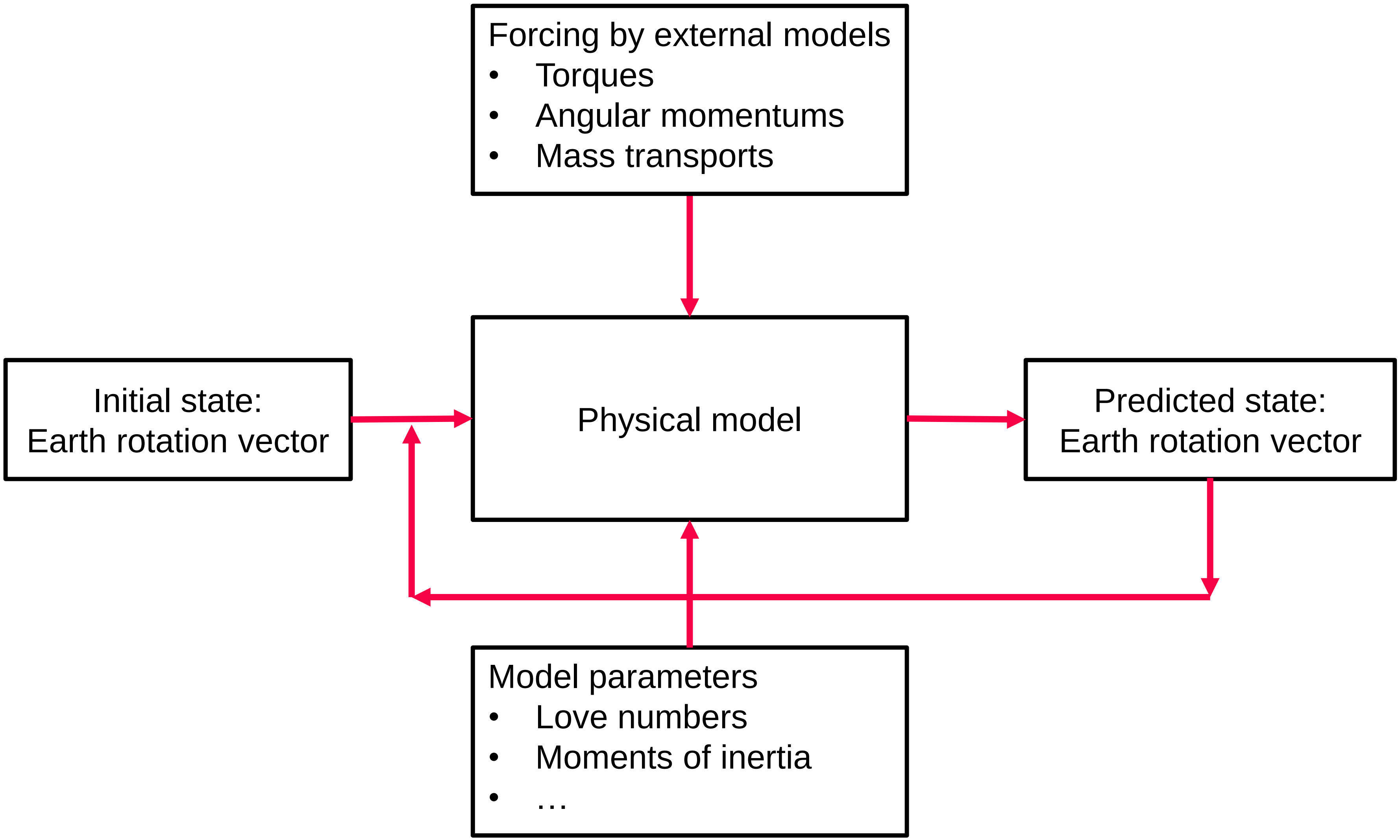
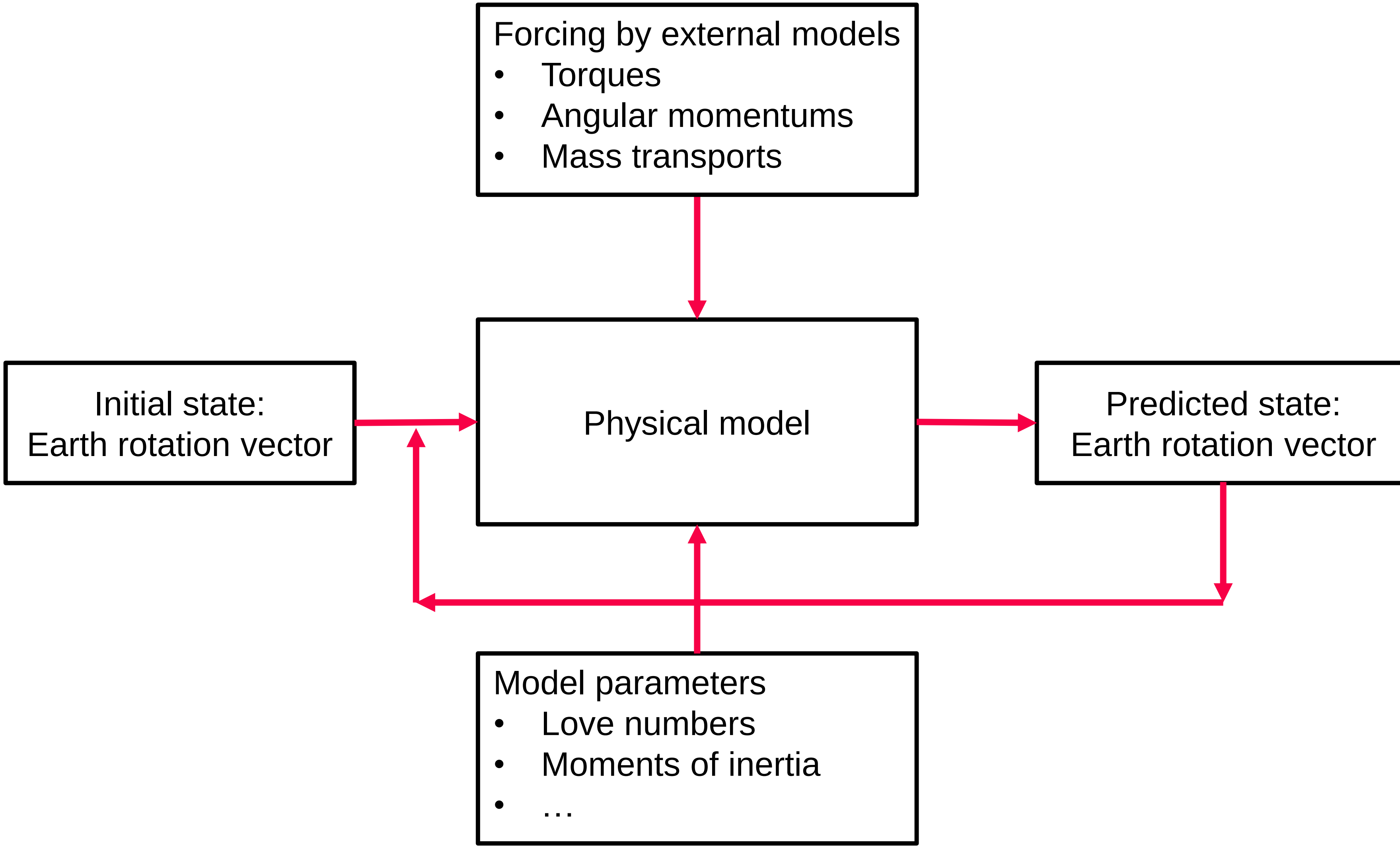
Geophysical model



Ocean currents

Geophysical model



Least squares adjustment

Gauß-Markoff Modell

Beobachtungsgleichungen

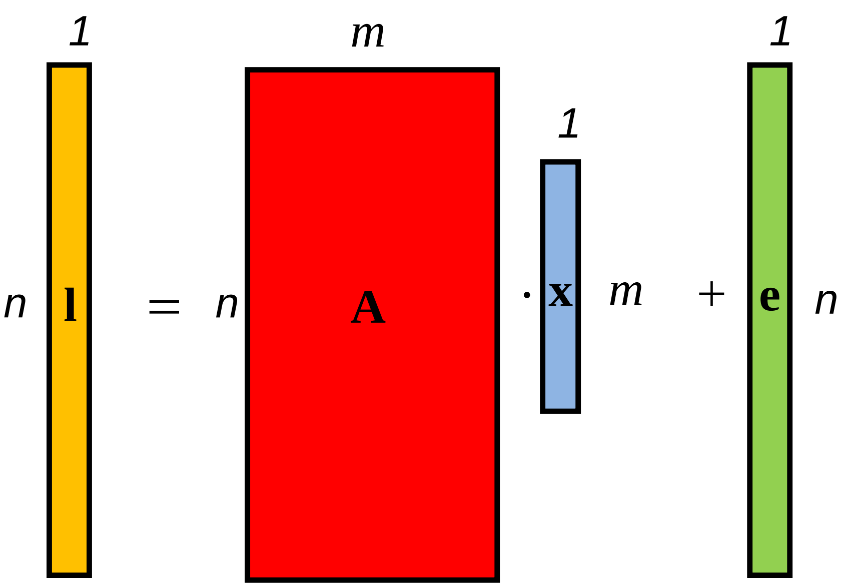
$$\mathbf{l} = \mathbf{A}\mathbf{x} + \mathbf{e}$$

(n x 1) Vektor
mit Residuen

(n x 1) Vektor
mit Beobachtungen

(m x 1) Vektor
mit unbekannten Parametern

(n x m) Designmatrix mit
bekannten Koeffizienten



Aufstellen und Berechung des Gauß-Markoff Modells

Gauß-Markoff Modell

Vektor mit allen Beobachtungen $\mathbf{l}_{(n\times 1)}$ Vektor mit unbekannten Parametern $\mathbf{x}_{(m\times 1)}$

1. Beobachtungsgleichungen

$$\mathbf{l} = \mathbf{f}(\mathbf{x}) + \mathbf{e}$$

n Funktionen

$$\begin{aligned} l_1 &= f_1(\mathbf{x}) + e_1 \\ l_2 &= f_2(\mathbf{x}) + e_2 \\ &\vdots \\ l_n &= f_n(\mathbf{x}) + e_n \end{aligned}$$

Gauß-Markoff Modell

Vektor mit allen Beobachtungen $\mathbf{l}_{(n\times 1)}$ Vektor mit unbekannten Parametern $\mathbf{x}_{(m\times 1)}$

1. Beobachtungsgleichungen

$$\mathbf{l} = \mathbf{f}(\mathbf{x}) + \mathbf{e}$$

2. Näherungswerte für die Unbekannten

$$\mathbf{x}_0$$

Modellkonstanten

Gauß-Markoff Modell

Vektor mit allen Beobachtungen $\mathbf{l}_{(n\times 1)}$ Vektor mit unbekannten Parametern $\mathbf{x}_{(m\times 1)}$

1. Beobachtungsgleichungen

$$\mathbf{l} = \mathbf{f}(\mathbf{x}) + \mathbf{e}$$

2. Näherungswerte für die Unbekannten

$$\mathbf{x}_0$$

Bei nicht-linearen Funktionen werden hierfür die Näherungswerte benötigt.

3. Gerechnete Beobachtungen

$$\mathbf{l}_0 = \mathbf{f}(\mathbf{x}_0)$$

4. Reduzierte Beobachtungen

$$\Delta\mathbf{l} = \mathbf{l} - \mathbf{l}_0$$

5. Designmatrix

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}_0}$$

$$\mathbf{A} = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & & \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}\right|_{\mathbf{x}_0}$$

Gauß-Markoff Modell

Vektor mit allen Beobachtungen $\mathbf{l}_{(n\times1)}$ Vektor mit unbekannten Parametern $\mathbf{x}_{(m\times1)}$

1. Beobachtungsgleichungen

$$\mathbf{l} = \mathbf{f}(\mathbf{x}) + \mathbf{e}$$

2. Näherungswerte für die Unbekannten

$$\mathbf{x}_0$$

3. Gerechnete Beobachtungen

$$\mathbf{l}_0 = \mathbf{f}(\mathbf{x}_0)$$

4. Reduzierte Beobachtungen

$$\Delta\mathbf{l} = \mathbf{l} - \mathbf{l}_0$$

5. Designmatrix

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}_0}$$

Linearisiertes Gauß-Markoff Modell

$$\Delta\mathbf{l} = \mathbf{A}\Delta\mathbf{x} + \mathbf{e}$$

6. Schätzung der Lösung

$$\Delta\hat{\mathbf{x}} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\Delta\mathbf{l} \quad \text{und} \quad \hat{\mathbf{x}} = \mathbf{x}_0 + \Delta\hat{\mathbf{x}}$$

7. Probe für die Güte der Linearisierung

$$\mathbf{f}(\hat{\mathbf{x}}) \;\; != \;\; \mathbf{l}_0 + \mathbf{A}\Delta\hat{\mathbf{x}} \qquad \text{sonst weiter bei 2.}$$

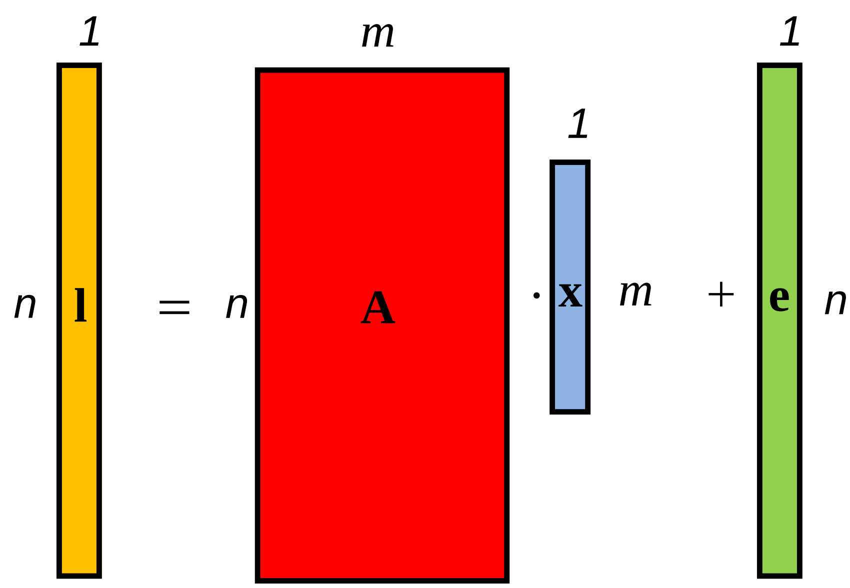
$$\boxed{\mathbf{f}(\hat{\mathbf{x}}) \approx \mathbf{f}(\mathbf{x}_0) + \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}_0}(\hat{\mathbf{x}} - \mathbf{x}_0) + \cdots}$$

Earth rotation model

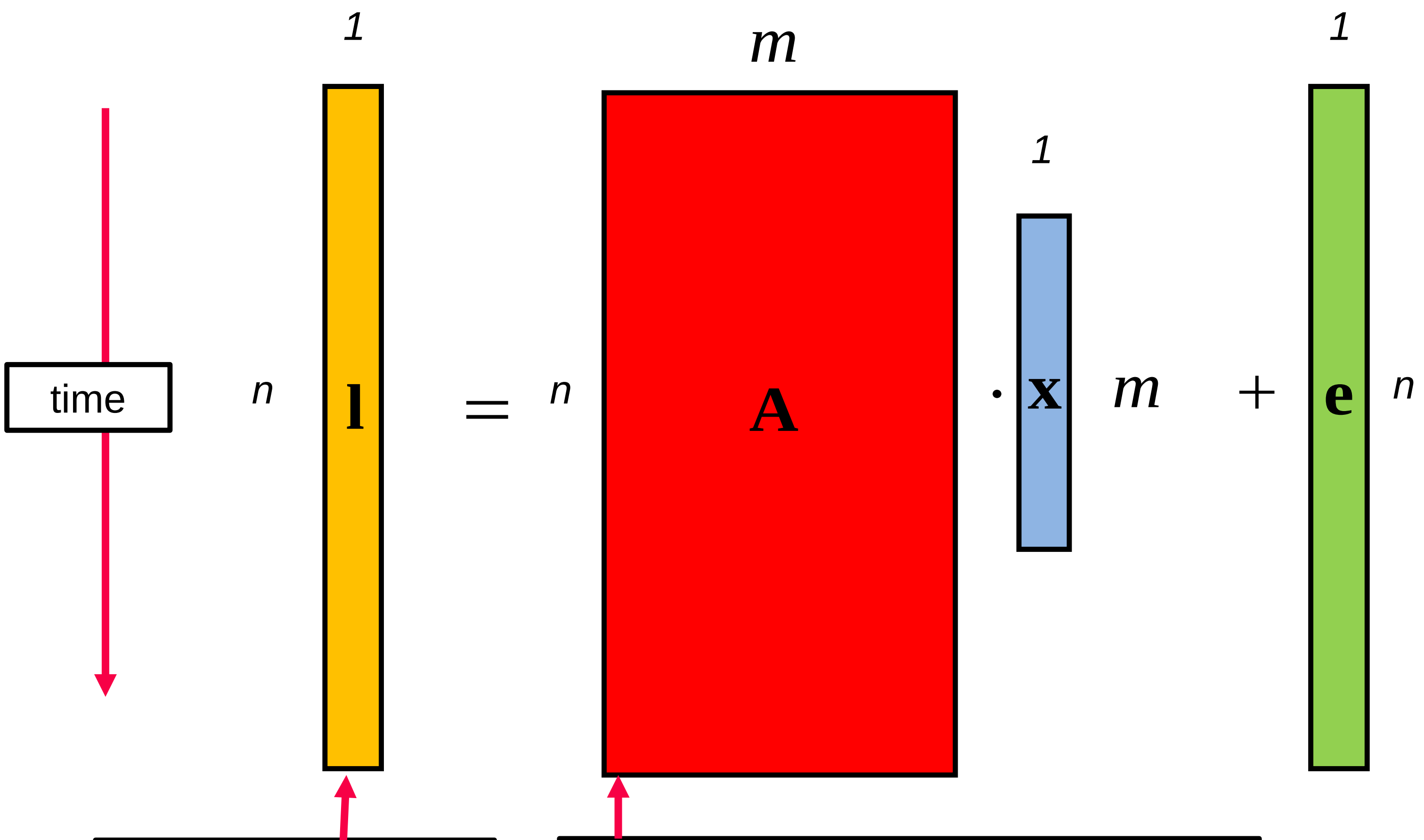
Unknown parameters
- Initial state
- Love numbers
- Factors of relative angular momentums
- …

Design matrix

$$\mathbf{A} = \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}_0} \approx \frac{\mathbf{f}(\mathbf{x}_0 + \Delta\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)}{\Delta\mathbf{x}}$$



Earth rotation model



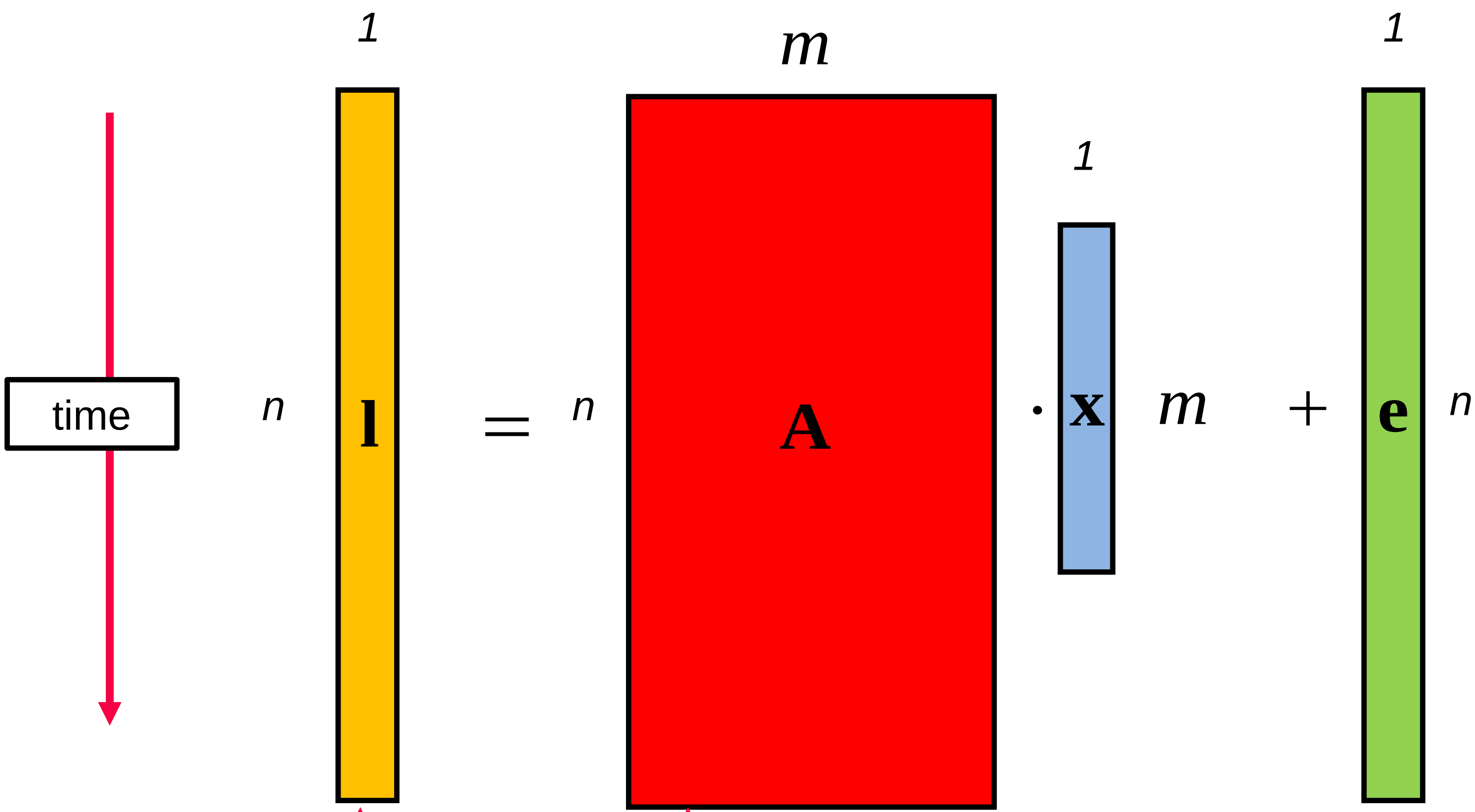
Standard model run

$$\mathbf{l}_0 = \mathbf{f}(\mathbf{x}_0)$$

First column:
Model with changed state omega0_x

$$\mathbf{a}_i = \frac{\mathbf{f}(\mathbf{x}_0 + \Delta\mathbf{x}_i) - \mathbf{f}(\mathbf{x}_0)}{\Delta\mathbf{x}_i}$$

Earth rotation model



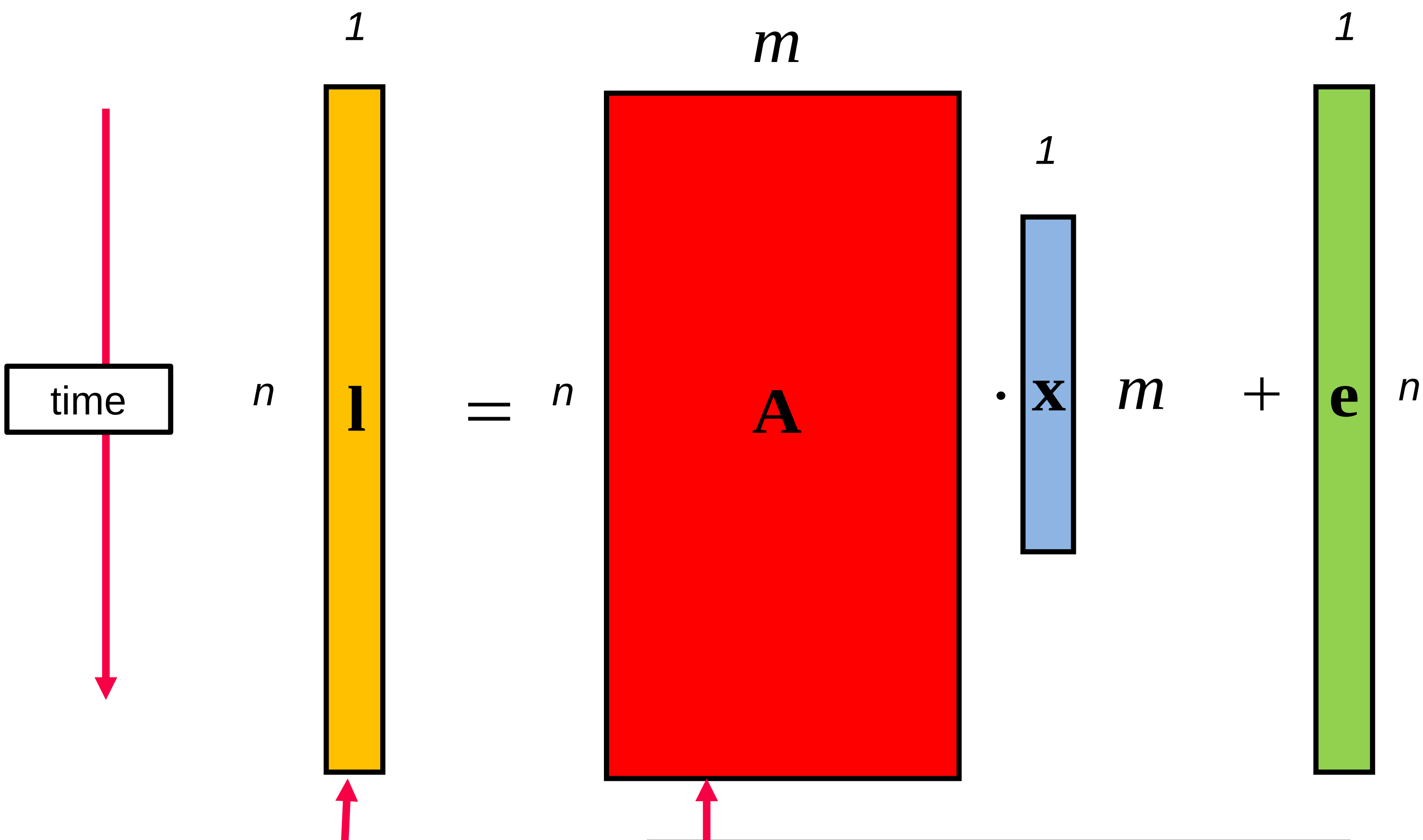
Standard model run

$$\mathbf{l}_0 = \mathbf{f}(\mathbf{x}_0)$$

Second column:
Model with changed state omega0_y

$$\mathbf{a}_i = \frac{\mathbf{f}(\mathbf{x}_0 + \Delta\mathbf{x}_i) - \mathbf{f}(\mathbf{x}_0)}{\Delta\mathbf{x}_i}$$

Earth rotation model



Standard model run

$$\mathbf{l}_0 = \mathbf{f}(\mathbf{x}_0)$$

Third column:
Model with changed state omega0_z

$$\mathbf{a}_i = \frac{\mathbf{f}(\mathbf{x}_0 + \Delta\mathbf{x}_i) - \mathbf{f}(\mathbf{x}_0)}{\Delta\mathbf{x}_i}$$

Data assimilation

Data assimilation

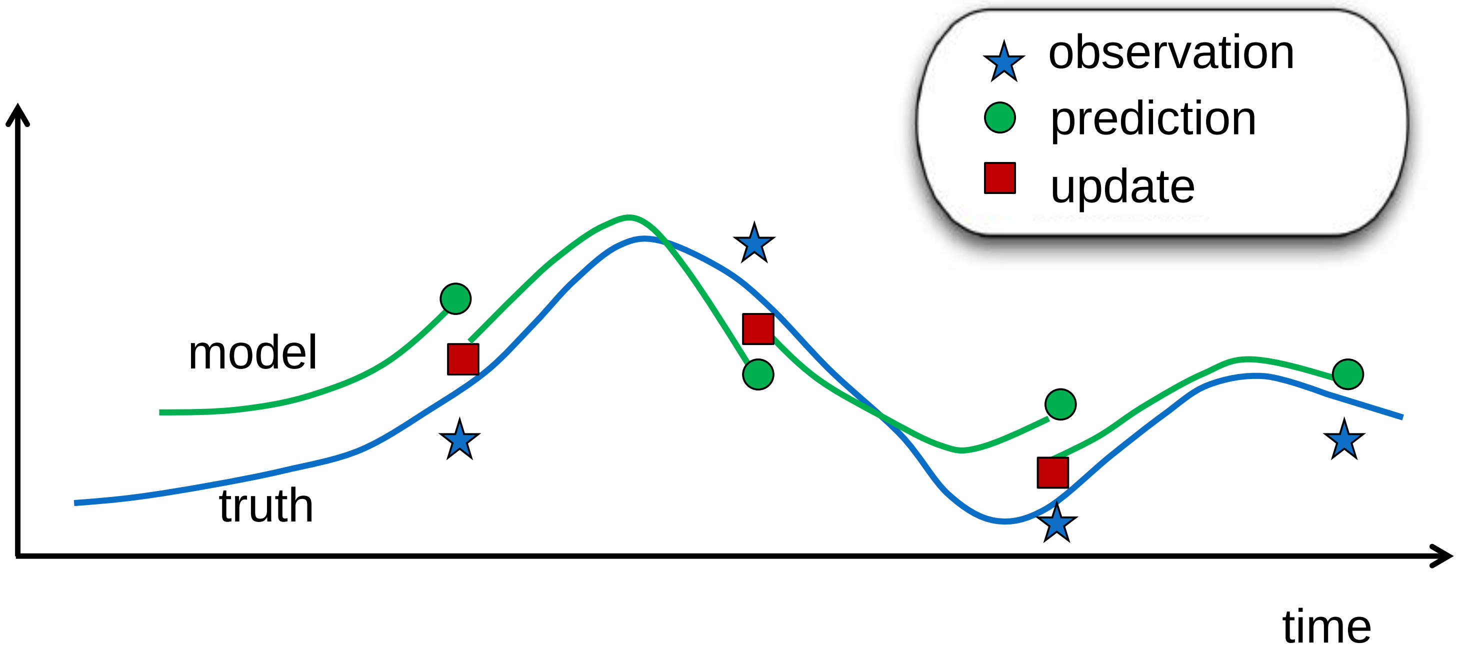
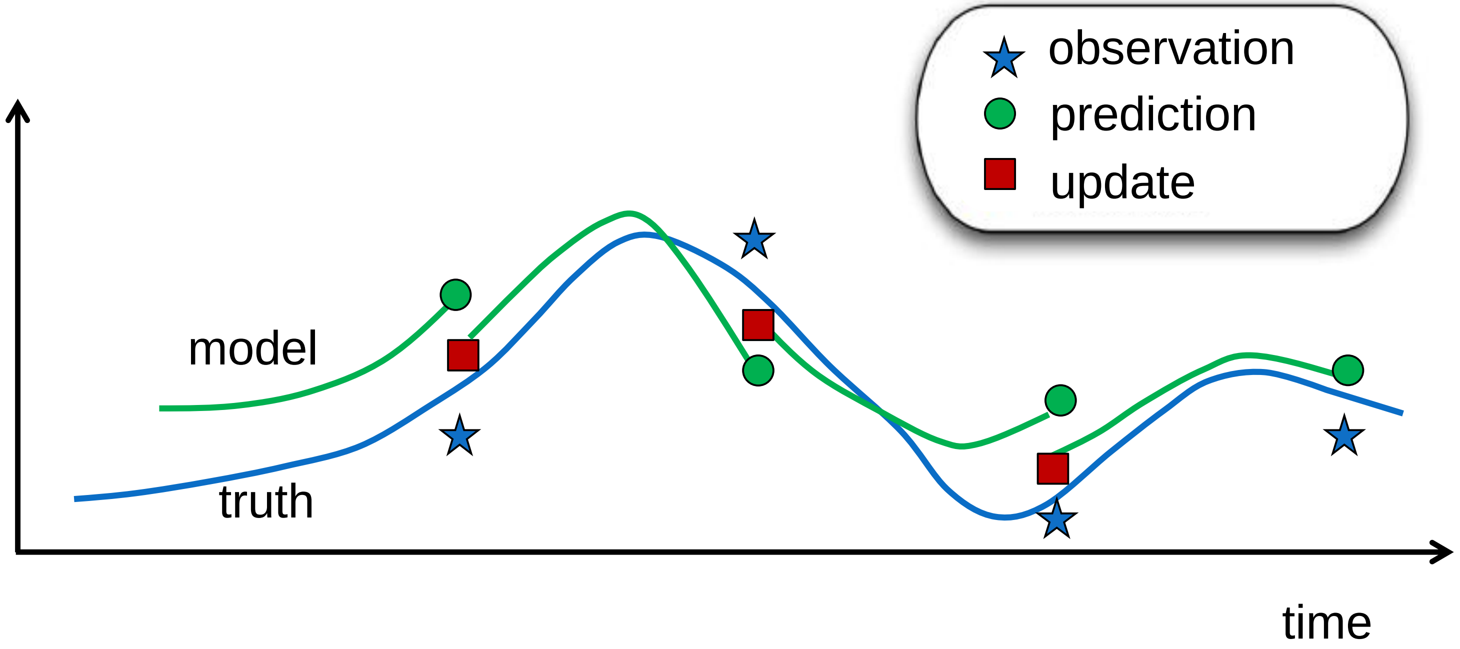


Figure modified from:
Introduction to Data Assimilation for Scientists and Engineers
O. Thual, *Open Learn. Res. Ed. INPT **0202*** (2013) 6h

Data assimilation



The update minimizes a combination of distances => cost function

$$J(\blacksquare) = \frac{1}{2}\left\| \blacksquare - \bullet \right\|_C^2 + \frac{1}{2}\left\| \bigstar - G(\blacksquare) \right\|_\Sigma^2$$

Data assimilation

The analysis minimizes a combination of distances => cost function

$$J(\blacksquare) = \frac{1}{2}\left\| \blacksquare - \bullet \right\|_C^2 + \frac{1}{2}\left\| \star - G(\blacksquare) \right\|_\Sigma^2$$

- $\star$ observation
- $\bullet$ prediction
- $\blacksquare$ update

We take the accuracies of the observations Σ and of the model prediction C into account

In general, the model output cannot be observed directly, but a function of the model results

Example:

$\blacksquare$ Earth rotation vector

$G(\blacksquare)$ Length of day

Data assimilation

The analysis minimizes a combination of distances => cost function

$$J(\blacksquare) = \frac{1}{2}\left\| \blacksquare - \bullet \right\|_C^2 + \frac{1}{2}\left\| \star - G(\blacksquare) \right\|_\Sigma^2$$

$\star$ observation
$\bullet$ prediction
$\blacksquare$ update

Different approaches:

- Nudging
  - simple weighting between model and observations
- 3D / 4D Var
  - mostly atmospheric community
  - optimization problem requires computation of gradients
    => adjoint model, computationally intensive!
  - no covariance matrix for assimilated model
- (Ensemble) Kalman filter
  - no gradient computation required
  - covariance matrix of model and observations required
  - not optimal in statistical sense

(Ensemble) Kalman filter

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

with $\mathbf{C}_{\mathbf{xx}}$

Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\boldsymbol{\Sigma}_{\mathbf{ll}}$

previous time step

model errors

prediction for this time stamp => „model state“

observations

State: Rotation vector

observation error

relation between model state and observations. (E.g. Length of day)

Kalman (1960)

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

with $\mathbf{C}_{\mathbf{xx}}$

Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\boldsymbol{\Sigma}_{\mathbf{ll}}$

Deriving the Kalman filter equation

Least squares adjustment with two sets of „observations“

$$\begin{pmatrix} \mathbf{x}_k^- \\ \mathbf{l}_k \end{pmatrix} + \underbrace{\begin{pmatrix} \boldsymbol{\delta} \\ \mathbf{e} \end{pmatrix}}_{\boldsymbol{\varepsilon}} = \begin{pmatrix} \mathbf{I} \\ \mathbf{H}_k \end{pmatrix} \mathbf{x}_k^+ \quad \text{with} \quad C(\boldsymbol{\varepsilon}) = \sigma^2 \begin{pmatrix} \mathbf{P}_{xx}^{-1} & 0 \\ 0 & \mathbf{P}_{ll}^{-1} \end{pmatrix}$$

$\mathbf{C}_{\mathbf{xx}}$

$\boldsymbol{\Sigma}_{\mathbf{ll}}$

Estimation of unknown parameters by accumulation of normal equations:

$$\hat{\mathbf{x}}^+ = \left(\mathbf{P}_{xx} + \mathbf{H}_k^T \mathbf{P}_{ll} \mathbf{H}_k\right)^{-1} \left(\mathbf{P}_{xx} \mathbf{x}_k^- + \mathbf{H}_k^T \mathbf{P}_{ll} \mathbf{l}_k\right)$$

matrix identities

$$\hat{\mathbf{x}}_k^+ = \mathbf{x}_k^- + \mathbf{C}_{xx} \mathbf{H}_k^T \left(\mathbf{H}_k \mathbf{C}_{xx} \mathbf{H}_k^T + \boldsymbol{\Sigma}_{ll}\right)^{-1} \left(\mathbf{l}_k - \mathbf{H}_k \mathbf{x}_k^-\right)$$

e.g.: Koch (1999)

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

with $\mathbf{C}_{\mathbf{xx}}$

$\Rightarrow$ Kalman Filter $\Leftarrow$

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\boldsymbol{\Sigma}_{\mathbf{ll}}$

Deriving the Kalman filter equation

Least squares adjustment with two sets of „observations“

$$\begin{pmatrix} \mathbf{x}_k^- \\ \mathbf{l}_k \end{pmatrix} + \underbrace{\begin{pmatrix} \boldsymbol{\delta} \\ \mathbf{e} \end{pmatrix}}_{\boldsymbol{\varepsilon}} = \begin{pmatrix} \mathbf{I} \\ \mathbf{H}_k \end{pmatrix} \mathbf{x}_k^+ \quad \text{with} \quad C(\boldsymbol{\varepsilon}) = \sigma^2 \begin{pmatrix} \mathbf{P}_{xx}^{-1} & 0 \\ 0 & \mathbf{P}_{ll}^{-1} \end{pmatrix}$$

$\mathbf{C}_{\mathbf{xx}}$ $\qquad$ $\boldsymbol{\Sigma}_{\mathbf{ll}}$

Estimation of unknown parameters by accumulation of normal equations:

$$\hat{\mathbf{x}}^+ = \left(\mathbf{P}_{xx} + \mathbf{H}_k^T \mathbf{P}_{ll} \mathbf{H}_k\right)^{-1} \left(\mathbf{P}_{xx} \mathbf{x}_k^- + \mathbf{H}_k^T \mathbf{P}_{ll} \mathbf{l}_k\right)$$

matrix identities

$$\hat{\mathbf{x}}_k^+ = \mathbf{x}_k^- + \boxed{\text{Kalman gain}} \left(\mathbf{l}_k - \mathbf{H}_k \mathbf{x}_k^-\right)$$

e.g.: Koch (1999)

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

with $\mathbf{C}_{\mathbf{xx}}$

Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\mathbf{\Sigma}_{\mathbf{ll}}$

Kalman filter update

$$\mathbf{x}^+ = \mathbf{x}^- + \boxed{\text{Kalman gain}} \cdot \left(\mathbf{l} - \mathbf{H}\mathbf{x}^-\right)$$

- The Kalman gain determines the influence of the observations
- It depends on the accuracies of
  - the observations $\mathbf{\Sigma}_{\mathbf{ll}}$
  - the model $\mathbf{C}_{\mathbf{xx}}$

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

with $\mathbf{C}_{\mathbf{xx}}$

Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\mathbf{\Sigma}_{\mathbf{ll}}$

Kalman filter update

$$\mathbf{x}^+ = \mathbf{x}^- + \underbrace{\mathbf{C}_{xx}\mathbf{H}^T\left(\mathbf{H}\mathbf{C}_{xx}\mathbf{H}^T + \mathbf{\Sigma}_{ll}\right)^{-1}}_{\text{gain matrix}}\left(\mathbf{l} - \mathbf{H}\mathbf{x}^-\right)$$

Problem:

In general, we do not know the model uncertainties C_{xx} !

=> Solution: Ensemble approach

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

sample 2

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

⋮

sample N

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\boldsymbol{\Sigma}_{\mathbf{ll}}$

Kalman filter update

$$\mathbf{x}^+ = \mathbf{x}^- + \underbrace{\mathbf{C}_{xx}\mathbf{H}^T\left(\mathbf{H}\mathbf{C}_{xx}\mathbf{H}^T + \boldsymbol{\Sigma}_{ll}\right)^{-1}}_{\text{gain matrix}}\left(\mathbf{l} - \mathbf{H}\mathbf{x}^-\right)$$

ensemble mean

$$\mathbf{C}_{xx} = \frac{1}{N-1}\sum_i (x_i^- - \bar{x}_i^-)\cdot(x_i^- - \bar{x}_i^-)^T$$

empirical ensemble covariance matrix

See also:
Evensen (2003)
Evensen (2009)

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

sample 2

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

⋮

sample N

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\boldsymbol{\Sigma}_{\mathbf{ll}}$

ensemble mean

$$\mathbf{C}_{xx} = \frac{1}{N-1}\sum_i (x_i^- - \bar{x}_i^-)\cdot(x_i^- - \bar{x}_i^-)^T$$

empirical ensemble covariance matrix

What causes uncertainties??

- model set-up (including calibration parameters
- start values of model run (initial state)
- climate forcing data

- calibration parameters: easy when given probability density function
- model uncertainty: use different versions of model equations, covariance inflation
- initial state: model spin-up runs
- climate forcing: e.g. use different data sets

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

sample 2

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

⋮

sample N

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

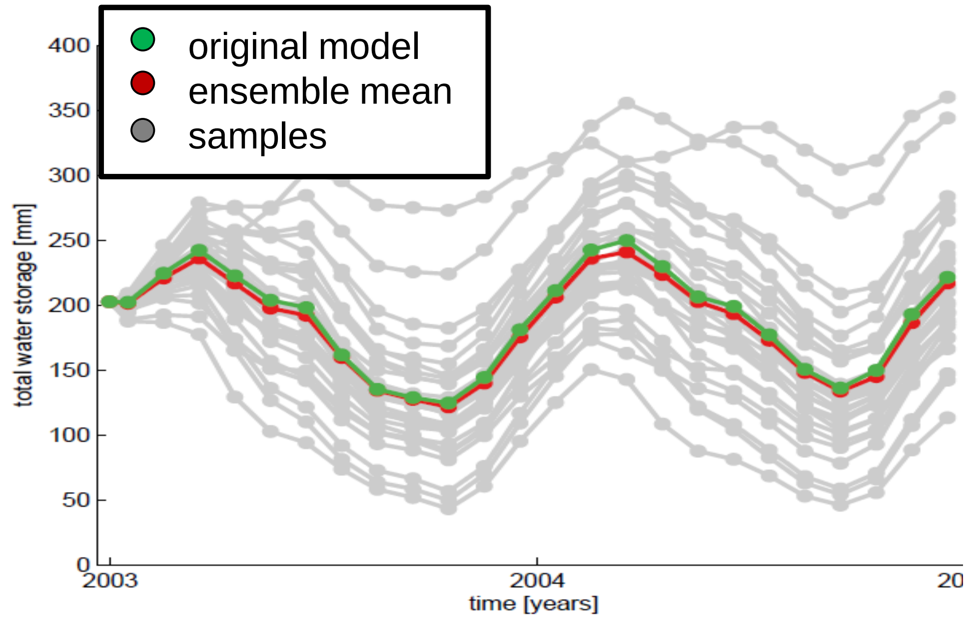
Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\mathbf{\Sigma}_{\mathbf{ll}}$

What causes uncertainties??



ensemble mean

$$\mathbf{C}_{xx} = \frac{1}{N-1}\sum_i (x_i^- - \overline{x}_i^-)\cdot(x_i^- - \overline{x}_i^-)^T$$

empirical ensemble covariance matrix

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

sample 2

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

$\vdots$

sample N

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\mathbf{\Sigma}_{\mathbf{ll}}$

Kalman filter update

$$\mathbf{X}^+ = \mathbf{X}^- + \underbrace{\mathbf{C}_{xx}\mathbf{H}^T\left(\mathbf{H}\mathbf{C}_{xx}\mathbf{H}^T + \mathbf{\Sigma}_{ll}\right)^{-1}}_{\text{gain matrix}}\left(\mathbf{L} - \mathbf{H}\mathbf{X}^-\right)$$

ensemble mean

$$\mathbf{C}_{xx} = \frac{1}{N-1}\sum_i (x_i^- - \bar{x}_i^-)\cdot(x_i^- - \bar{x}_i^-)^T$$

empirical ensemble covariance matrix

See also:
Evensen (2003)
Evensen (2009)

Kalman filter

prediction (model)

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

sample 2

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

⋮

sample N

$$\mathbf{x}_k^- = f(\mathbf{x}_{k-1})$$

Kalman Filter

observations

$$\mathbf{l}_k = \mathbf{H}_k \mathbf{x}_k$$

with $\boldsymbol{\Sigma}_{\mathbf{ll}}$

Kalman filter update

$$\mathbf{X}^+ = \mathbf{X}^- + \underbrace{\mathbf{C}_{xx}\mathbf{H}^T\left(\mathbf{H}\mathbf{C}_{xx}\mathbf{H}^T + \boldsymbol{\Sigma}_{ll}\right)^{-1}}_{\text{gain matrix}}\left(\mathbf{L} - \mathbf{H}\mathbf{X}^-\right)$$

ensemble mean

$$\mathbf{C}_{xx} = \frac{1}{N-1}\sum_i (x_i^- - \bar{x}_i^-)\cdot(x_i^- - \bar{x}_i^-)^T$$

empirical ensemble covariance matrix

See also:
Evensen (2003)
Evensen (2009)

Earth rotation model

Ensemble generation (ca. 100 runs ??)

- Add noise to initial state
- Add noise at every time step
  - Relative angular momentums
  - Potential coefficients
  - …