Stochastic Calculus, 2023–2024

Assignment 1: European and American put option pricing

This assignment has the following requirements:

(a) Write and submit a report (in pdf), which contains your answers to each question and explains how you have obtained them.

(b) Write and submit code (in R, Python or Matlab) to do the calculations and show the results. Use comments to explain the code.

Both should be submitted via Canvas. The deadline is **Wednesday March 13, 23:59**.
This is a group assignment; you may work in teams of one, two or three students and submit a common report. Please put the names of the group members in the file names that are submitted, as in `Name1Name2.pdf` and `Name1Name2.ipynb`.
It is important that you do your own programming; copies of other students' programs (or of programs found on the internet) with only different data are not acceptable. 100 points can be earned in total; points for each subquestion are indicated.

1. [**20 points**] Obtain, from the internet (or other sources),

   - the value of the closing price of a US stock on a given day (S_0);
   - the closing price, on the same day, of a **put** option (P_0) on the same stock that is at or slightly in the money, $K \approx S_0$ or $K > S_0$; choose an option that expires approximately 6 months from now, but also one which has sufficient open interest, and which has been last traded just before market closing (16:00 EST).
   - a risk-free US interest rate R (e.g., a Treasury bill rate) on the same day, and with approximately the same time-to-maturity as the option; make sure that the interest rate is positive.
   - a sample of daily historical prices $\{S_t\}_{t \leq 0}$ of the same stock (to be used for estimating the historical volatility).

   A good starting points for obtaining these data is `http://finance.yahoo.com`.

   Calculate and include in your report the values of the parameters that are needed to price the option:

   (a) The time to expiration T, measured in years, assuming that a year consists of 252 trading days. Standard options expire on the 3rd Friday of the expiration month; take proper account of the fact that not all days are trading days.

   (b) The continuously compounded interest rate $r = \log(1 + R)$; recall that log always denotes the *natural* logarithm.

   (c) From the historical returns, an estimate of σ, the per annum stock return volatility (see the slides of Lecture 1).

   In what follows, we will assume (or pretend) that the stock pays no dividend.

2. [**20 points**] Write a function that calculates, for an increasing number of steps $N = 2, \ldots, 100$, the binomial tree value of the put option (assumed for now to be a European option) for the values of (S_0, K, T, r, σ) obtained in Question 1. Also calculate the Black-Scholes value of the put option, and illustrate graphically that the binomial tree values converges to the Black-Scholes value at N increases (include the figure in your report).

3. **[40 points]** Individual stock options (such as those considered here) are American-style, meaning that they can be exercised early, before expiration time T. Write a function to compute the value of an *American* put option based on a binomial tree with arbitrary N, and another function that calculates this value for $N = 3$ and prints the entire option tree to the screen (this tree should be included in your report). Using these functions, and the function from Question 2, discuss in your report:

   (a) In the 3-step tree, is the American option worth more at time 0 than its (hypothetical) European counterpart, using the same parameters (S_0, K, T, r, σ) as before? If this is the case, at which point in the tree is it optimal to exercise the option?

   (b) Does the American put option price seem to converge to a limit as N increases, like the European put option value? Calculate the value of early exercise by subtracting the (binomial) European put option price from the American put option price with the same large N.

   (c) Investigate the effect of the interest rate r on the value of early exercise as found under (b); what happens if we set $r = 0$, and how does the value change if we increase r?

4. **[20 points]** The model price may differ substantially from the market value P_0, because the volatility over the period $[0, T]$ might not equal the historical volatility. Therefore, make a function to calculate the *American put option tree-implied volatility*, i.e., that value of σ for which the binomial tree price (with N chosen sufficiently large, and the other parameters (S_0, K, T, r) as before, with $r > 0$) is equal to the market price obtained in Question 1. Discuss the result in your report.