Lecture 2: Decisions

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Introduction

Decision theory as cornerstone of formal economics: models how individuals make decisions.

We generalise and formalise treatment of coin tossing game.

Axiom: maximise time-average growth rate of wealth.

Simple and powerful idea, which removes need for established but epistemologically troublesome techniques.

References

Some of this lecture is based on recent papers (open access):

- O. Peters and M. Gell-Mann. *Evaluating gambles using dynamics.* arXiv:1405.0585, 2014.
- O. Peters. *The time resolution of the St Petersburg paradox.* Phil. Trans. R. Soc. A 369, 4913–4931, 2011.
- O. Peters and A. Adamou. *Rational insurance with linear utility and perfect information.* arXiv:1507.04655, 2015.

Models and reality

Build our decision theory from mathematical model of the gamble problem. Define:

- the gamble, a mathematical object; and
- the decision criterion, a set of instructions containing mathematical manipulations of the gamble.

Gambles will resemble real-world situations.

Criteria may/may not resemble real decision-making.

Ignore realism for now: main task is to explore imaginary world of behaviours created by model (like science fiction).

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Gamble

Gamble as fundamental building block of decision theory.

Mathematical object resembling real situation whose outcome:

- is purely monetary;
- is uncertain; and
- occurs over a specified period of time.

Example: buying a lottery ticket.

Need formal mathematical definition...

Definition

DEFINITION: Gamble

A gamble is a pair of random variable, D, and duration, δt.

D is called the payout and takes one of N possible monetary values, $\{D_1, \ldots, D_N\}$.

Payouts are associated with probabilities, $\{p_1, \ldots, p_N\}$, where $\sum_{i=1}^{N} p_i = 1$.

Payouts can be positive, associated with monetary gain, or negative, associated with loss.

Order them such that $D_1 < \ldots < D_N$.

Examples

Betting on a fair coin

Imagine betting \$10 on the toss of a fair coin.

Model with payouts and probabilities:

$$\begin{aligned} D_1 &= -\$10, & p_1 &= 1/2; \\ D_2 &= +\$10, & p_2 &= 1/2. \end{aligned}$$

Duration: $\delta t = 2$ seconds.

Examples

Playing the lottery

Imagine a lottery, where individual pays F for a ticket which will win jackpot, J, with probability, p.

Payouts and probabilities are:

$$\begin{aligned} D_1 &= -F, \qquad & p_1 &= 1 - p; \\ D_2 &= J - F, \qquad & p_2 &= p. \end{aligned}$$

Duration: $\delta t = 1$ week.

Examples

Betting at fixed odds

Imagine a bet placed at fixed odds, *e.g.* on a horse race.

Bet F on *Ito* to win the Prix de l'Arc de Triomphe at Longchamp at odds of $50/1$.

Ito will win the race with unknown probability, p.

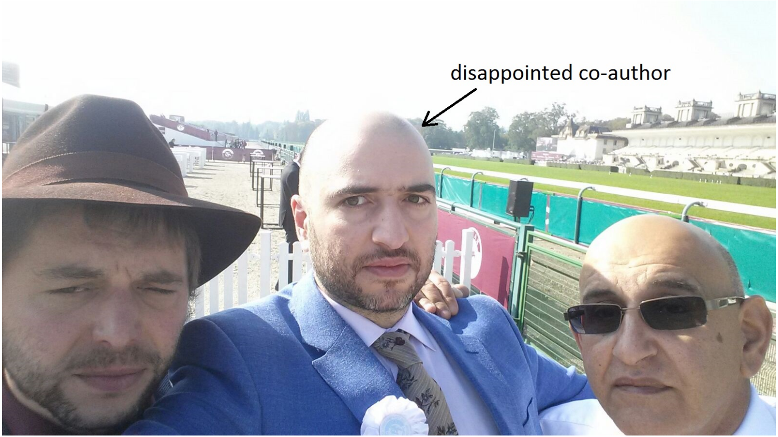
Payouts and probabilities are:

$$\begin{aligned} D_1 &= -F, & p_1 &= 1-p; \\ D_2 &= 50F, & p_2 &= p. \end{aligned}$$

Duration: $\delta t = 30$ minutes.

Examples

Ito did not win. (He didn't even run!)



The null gamble

Another example is the null gamble, in which a payout of zero is received with certainty:

$$D_1 = \$0, \quad p_1 = 1.$$

Useful for our theory as it represents the 'no bet' or 'do nothing' option.

Duration, δt, must be chosen appropriately (often by reference to an alternative gamble).

Discrete and continuous gambles

Gamble as simple but versatile model of uncertain future.

Used to model traditional wagers and other economic activities, *e.g.* stock market investments, insurance contracts, derivatives.

We presented discrete gamble, where payout has countable (usually small) number of possible outcomes.

Extension to continuous gambles is natural.

Used to model real-world scenarios with large number of possible outcomes, *e.g.* change in stock price.

The gamble problem

Suppose you must choose between two scenarios modelled as gambles (possibly including the null gamble).

Question

Which should you choose, and why?

This is the gamble problem.

Central question in decision theory and basis for most of mainstream economics.

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Single gamble

Gambles are mathematical abstractions, not yet connected to our model humans.

Relevant to people only in that they affect wealth process, $x(t)$.

Single round in isolation ("one-shot game") uninstructive: says only that $x(t + \delta t) = x(t) + D$, a random variable.

Tendency of $x(t)$ over time unclear as one time-step not enough for tendency to emerge.

Time has no significance in one-shot game. Value of δt irrelevant to analysis: could be heartbeat or life of universe!

Repeated gambles

Model wealth evolution by repeating gamble over many rounds.

We don't believe real-world situations repeat themselves, *e.g.* we don't have to keep betting on *Ito* (thankfully!)

Modelling device to extract otherwise invisible tendencies, reflects idea that consequences of decisions unfold over time.

Mode of repetition **not** specified in gamble: separate model component which must be specified.

Focus on two modes: **additive** and **multiplicative** repetition.

Additive repetition

Definition: Additive repetition

Define change in wealth over single round of gamble as

$$\delta x(t) \equiv x(t + \delta t) - x(t).$$

To repeat additively, simply add random payout to wealth at each round, *i.e.*

$$\delta x(t) = D.$$

Note that δx is a stationary random variable.

Wealth evolution

Starting at time t_0, wealth after T rounds is

$$x(t_0 + T\delta t) = x(t_0) + \sum_{\tau=1}^{T} D(\tau),$$

where $D(\tau)$ is realised payout in round τ.

Evolution equation for wealth following noisy additive dynamic.

Note that $x(t_0 + T\delta t)$ is itself a random variable.

Example

Additive coin toss

Return to first example gamble: \$10 fair coin toss.

Successive bets always \$10, regardless of wealth.

Suppose $x(t_0) = \$100$, then wealth after T rounds is

$$\begin{aligned} x(t_0 + T\delta t) &= \$100 + \$10k - \$10(T - k) \\ &= \$[100 + 10(2k - T)], \end{aligned}$$

where $0 \leq k \leq T$ is the number of winning tosses.

Note: wealth can go negative (no bankruptcy).

Multiplicative repetition

An alternative is multiplicative repetition.

In example, imagine first \$10 bet not as fixed size but instead as fixed fraction (10%) of starting wealth (\$100).

Successive bets are for same fraction of wealth, but different monetary amounts.

Formalise as follows...

Multiplicative repetition

Definition: Multiplicative repetition

Express payout, D, in first round as random wealth multiplier:

$$r \equiv \frac{x(t_0) + D}{x(t_0)}.$$

Repeat gamble by applying multiplier at subsequent rounds:

$$x(t + \delta t) = r x(t).$$

Wealth evolution

r is a stationary random variable but change in wealth,

$$\delta x(t) = (r-1)x(t),$$

depends on t through $x(t)$.

Wealth after T rounds of gamble is

$$x(t_0 + T\delta t) = x(t_0) \prod_{\tau=1}^{T} r(\tau),$$

where $r(\tau)$ is realised multiplier in round τ.

Example

Multiplicative coin toss

Repeat \$10 coin toss multiplicatively.

Random multiplier, r, has two possible possible outcomes:

$$r_1 = \frac{\$100 - \$10}{\$100} = 0.9, \qquad p_1 = 1/2;$$
$$r_2 = \frac{\$100 + \$10}{\$100} = 1.1, \qquad p_2 = 1/2.$$

Wealth after T rounds with k winning tosses:

$$x(t_0 + T\delta t) = \$100\,(1.1)^k\,(0.9)^{T-k}.$$

Note: no possibility of negative wealth or bankruptcy, since individual can lose at most 10% of wealth in a round.

Choose appropriate dynamic

Imagined mode of repetition not simply matter of taste.

Consequence of choice is huge: additive and multiplicative dynamics differ as starkly as linear and exponential functions.

Therefore must consider economic situation to be modelled and choose realistic dynamic.

e.g. Stock price fluctuations tend to be $\propto$ price $\rightarrow$ appropriate dynamic is multiplicative.

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Growth rates

We have linked $x(t)$ to the gamble *via* a dynamic.

Can now develop decision criterion.

Good advice: "Pick the gamble that will cause your wealth to grow fastest."

To formalise this we need to revisit growth rates.

Stationarity

Previously introduced growth rate, g, as rate of change of strictly increasing function, $v(x)$:

$$g(t, \Delta t) \equiv \frac{\Delta v(x(t))}{\Delta t}.$$

$v(x)$ chosen so that increment, $\Delta v(x(t))$, over general time period, Δt, is stationary random variable.

$\rightarrow g(t, \Delta t)$ also stationary: *distribution* independent of t.

Stationarity (*i.e.* statistical irrelevance of measurement time) is crucial $\rightarrow$ want distribution of g to convey robust information about wealth process, not merely when it was sampled.

Additive growth rate

Additive repetition: Δx already stationary.

$\rightarrow$ Appropriate mapping is $v(x) = x$.

$\rightarrow$ Growth rate under additive repetition is

$$g_{\mathrm{a}}(t, \Delta t) = \frac{\Delta x(t)}{\Delta t}.$$

Multiplicative growth rate

Multiplicative repetition: Δx not stationary $\rightarrow$ need mapping $v(x)$ with stationary increment. Note that over one round

$$\begin{aligned}\delta \ln x(t) &= \ln x(t+\delta t) - \ln x(t)\\ &= \ln r x(t) - \ln x(t)\\ &= \ln r,\end{aligned}$$

which is stationary because r is stationary.

$\rightarrow$ Appropriate mapping is $v(x) = \ln x$.

$\rightarrow$ Growth rate under multiplicative repetition is:

$$g_{\mathrm{m}}(t, \Delta t) = \frac{\Delta \ln x(t)}{\Delta t}.$$

Time-average growth rate

Distribution of $g(t, \Delta t)$ depends on Δt.

In general, distribution narrows as Δt increases, converging to a finite number as $\Delta t \to \infty$.

Repetition over long time exposes tendency of gamble.

DEFINITION: **Time-average growth rate**

Define time-average growth rate as

$$\bar{g} \equiv \lim_{\Delta t \to \infty} \{g(t, \Delta t)\}.$$

Ergodicity

$\bar{g}$ is growth rate over diverging rounds of gamble:

$$\bar{g} = \lim_{T\to\infty} \left\{ \frac{v(x(t+T\delta t)) - v(x(t))}{T\delta t} \right\}.$$

Expand numerator as sum of single-round increments:

$$\begin{aligned} \bar{g} &= \lim_{T\to\infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} \frac{\Delta v(x(t+\tau\delta t))}{\delta t} \right\} \\ &= \lim_{T\to\infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} g(t+\tau\delta t, \delta t) \right\} = \langle g(t,\delta t) \rangle \end{aligned}$$

by stationarity and independence of successive growth rates.

Restatement of ergodic property: time and ensemble averages of ergodic growth rate have same value.

Ergodic growth rates

Equivalences for additive and multiplicative dynamics:

$$\begin{aligned} \bar{g}_\mathrm{a} &= \lim_{\Delta t \to \infty} \left\{ \frac{\Delta x(t)}{\Delta t} \right\} = \left\langle \frac{\Delta x(t)}{\Delta t} \right\rangle ; \\ \bar{g}_\mathrm{m} &= \lim_{\Delta t \to \infty} \left\{ \frac{\Delta \ln x(t)}{\Delta t} \right\} = \left\langle \frac{\Delta \ln x(t)}{\Delta t} \right\rangle . \end{aligned}$$

General equivalence:

$$\bar{g} = \lim_{\Delta t \to \infty} \left\{ \frac{\Delta v(x(t))}{\Delta t} \right\} = \left\langle \frac{\Delta v(x(t))}{\Delta t} \right\rangle .$$

Value of Δt in $\langle\ \rangle$ immaterial. Gamble duration, δt, often used.

No special interpretation of $\langle g \rangle$. Just happens to coincide with quantity of interest, $\bar{g}$.

Scalars

General remarks on story so far.

Start with high-dimensional mathematical object: gamble.

Add dynamic: instructions for how wealth evolves when gamble repeated.

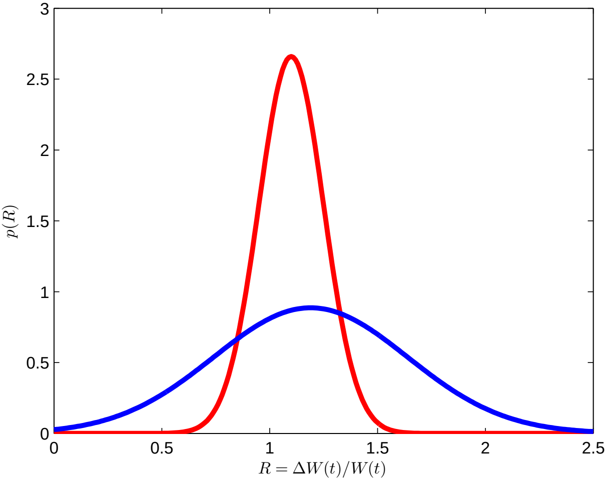
Collapse this information to a scalar (single number), $\bar{g}$, which summarises effect of gamble on wealth.

Scalars can be ranked unequivocally $\rightarrow$ clear decision criterion.

Hard to devise for high-dimensional objects, *e.g.* return distributions. . .

Scalars

Blue or red?



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Model rationale

Model rationale for deciding between gambles: maximise $\bar{g}$ for the specified mode of repetition.

In other words, model humans choose gamble which, if repeated indefinitely, causes wealth to grow fastest.

Indefinite repetition as conceptual device or thought experiment to elicit tendency of gamble.

In real world a particular choice between gambles may only occur once.

But also true that real decisions followed by many others.

Plausibility and parsimony

In model this decision rule outperforms all others in the long run $\rightarrow$ plausible logical basis.

Where model approximates real world, expect rationale to approximate real decisions.

Rationale is parsimonious, based on a single robust quantity derived from minimal model components (gamble, dynamic).

No arbitrary or unquantifiable human psychological factors.

Axiom

Nevertheless, decision criterion has status of *axiom*: it is asserted as premise of model.

Generates model world where certain behaviours observed.

Hope is for model world to resemble real world, but you may disagree.

Other decision axioms possible, *e.g.* classical decision theory based on axiom that humans maximise expected utility.

Algorithm

Decision criterion expressible as set of instructions.

Denote by $^{(m)}$ quantities relating to m^{th} available gamble.

Each gamble specified by:

- random payout, $D^{(m)}$; and
- duration, $\delta t^{(m)}$.

Common dynamic for all gambles.

Algorithm

Growth-optimal decision algorithm

1. Specify $D^{(m)}$ and $\delta t^{(m)}$ for gambles offered;
2. Specify wealth dynamic, *i.e.* relationship between $\delta x(t)$, $x(t)$, and D;
3. Determine function, $v(x)$, whose increments are stationary random variables under dynamic;
4. Determine time-average growth rates, $\bar{g}^{(m)}$, by taking time or ensemble averages of ergodic growth rates, $g^{(m)}(t, \Delta t)$;
5. Choose gamble, m, with the largest $\bar{g}^{(m)}$.

Example setup

Example decisions are between two gambles, *i.e.* $m \in \{1, 2\}$.

Decisions between three or more gambles straightforward extensions.

Often one option is the null gamble, which trivially has zero growth rate.

Then question posed is whether other gamble preferable to doing nothing, *i.e.* accept or decline?

Example game

Apply decision algorithm to original coin toss game.

Starting wealth $x(t_0) > \$0$, duration δt, random payout:

$$\begin{aligned} D_1^{(1)} &= -0.4x(t_0), &\qquad p_1^{(1)} &= 1/2; \\ D_2^{(1)} &= 0.5x(t_0), &\qquad p_2^{(1)} &= 1/2. \end{aligned}$$

Note that payouts $D_i^{(1)}$ are fixed monetary amounts, expressed here as fractions of $x(t_0)$.

Choice is between coin toss game and null gamble:

$$D_1^{(2)} = \$0, \quad p_1^{(2)} = 1.$$

Analyse under additive and multiplicative repetition.

Example

Additive repeated coin toss

Wealth evolution over T rounds (no bankruptcy):

$$
\begin{aligned}
x^{(1)}(t_0 + T\delta t) &= x(t_0) + \sum_{\tau=1}^{T} D^{(1)}(\tau); \\
x^{(2)}(t_0 + T\delta t) &= x(t_0).
\end{aligned}
$$

$v(x) = x$, so growth rates per unit time are:

$$
\begin{aligned}
g_{\mathrm{a}}^{(1)}(t) &= \frac{x^{(1)}(t + T\delta t) - x(t_0)}{T\delta t} = \frac{1}{T}\sum_{\tau=1}^{T} \frac{D^{(1)}(\tau)}{\delta t}; \\
g_{\mathrm{a}}^{(2)}(t) &= \frac{x^{(2)}(t + T\delta t) - x(t_0)}{T\delta t} = \$0.
\end{aligned}
$$

Example

Trivially $\bar{g}_\mathrm{a}^{(2)} = \$0$ per unit time. For coin toss,

$$\begin{aligned}\bar{g}_\mathrm{a}^{(1)} &= \left\langle \frac{D^{(1)}}{\delta t} \right\rangle \\ &= \frac{p_1^{(1)} D_1^{(1)} + p_2^{(1)} D_2^{(1)}}{\delta t} \\ &= \frac{x(t_0)}{20\delta t}\end{aligned}$$

per unit time. This is positive, so $\bar{g}_\mathrm{a}^{(1)} > \bar{g}_\mathrm{a}^{(2)}$.

Individual accepts coin toss under additive dynamics.

Realism

Decision rule under additive repetition is to maximise $\langle D/\delta t \rangle$, *i.e.* which is rate of change of expected wealth.

This coincides under additive dynamic with time-average growth rate.

See later that real people usually don't maximise $\langle D/\delta t \rangle$.

No great shock: additive repetition without bankruptcy is hardly most realistic model of wealth evolution.

Try multiplicative repetition instead.

Example

Multiplicative repeated coin toss

Payout, $D^{(1)}$, expressed as per-round multiplier,

$$r^{(1)} = \frac{x(t_0) + D^{(1)}}{x(t_0)},$$

with possible values:

$$r_1^{(1)} = \frac{x(t_0) + D_1^{(1)}}{x(t_0)} = 0.6, \qquad p_1^{(1)} = 1/2;$$

$$r_2^{(1)} = \frac{x(t_0) + D_2^{(1)}}{x(t_0)} = 1.5, \qquad p_2^{(1)} = 1/2.$$

Example

Wealth evolves according to:

$$
\begin{aligned}
x^{(1)}(t_0 + T\delta t) &= x(t_0) \prod_{\tau=1}^{T} r^{(1)}(\tau); \\
x^{(2)}(t_0 + T\delta t) &= x(t_0).
\end{aligned}
$$

Example

$v(x) = \ln x$ so growth rates per unit time are:

$$g_{\mathrm{m}}^{(1)}(t) = \frac{\ln x^{(1)}(t + T\delta t) - \ln x(t_0)}{T\delta t} \quad = \quad \frac{1}{T}\sum_{\tau=1}^{T} \frac{\ln r^{(1)}(\tau)}{\delta t};$$

$$g_{\mathrm{m}}^{(2)}(t) = \frac{\ln x^{(2)}(t + T\delta t) - \ln x(t_0)}{T\delta t} \quad = \quad 0$$

Trivially $\bar{g}_{\mathrm{m}}^{(2)} = 0$ per unit time. For the coin toss,

$$\bar{g}_{\mathrm{m}}^{(1)} = \left\langle \frac{\ln r^{(1)}}{\delta t} \right\rangle = \frac{\ln 0.9}{2\delta t},$$

per unit time. This is negative, so $\bar{g}_{\mathrm{m}}^{(1)} < \bar{g}_{\mathrm{m}}^{(2)}$.

Decline coin toss under multiplicative dynamics.

Warning

Decision rule under multiplicative repetition is to maximise

$$\left\langle \frac{\ln r}{\delta t} \right\rangle = \left\langle \frac{\ln(x(t_0)+D) - \ln x(t_0)}{\delta t} \right\rangle,$$

which coincides with the time-average growth rate.

For choice of gambles in example, get *opposite* of decision under additive repetition.

$\rightarrow$ Realistic dynamic crucial for modelling real decisions.

Inappropriate dynamic could lead to very bad advice!

Alternative presentation

Can also present multiplicative repetition without the logarithmic mapping.

Wealth after T rounds (with k winning tosses) is

$$x^{(1)}(t_0 + T\delta t) = x(t_0)\left(r_T^{(1)}\right)^T,$$

where

$$r_T^{(1)} = (0.6)^{(T-k)/T}(1.5)^{k/T}$$

is the equivalent per-round multiplier.

Alternative presentation

$r_T^{(1)}$ is a random variable but it converges to

$$r_\infty^{(1)} \equiv \lim_{T\to\infty} \left\{ r_T^{(1)} \right\} = (0.6)^{1/2}(1.5)^{1/2} = \sqrt{0.9},$$

since $k/T \to 1/2$ as $T \to \infty$ (fair coin).

$r_\infty^{(1)} < 1$ so wealth decays over time $\to$ coin toss declined.

Approaches are linked in that

$$\bar{g}_{\mathrm{m}}^{(1)} = \frac{\ln r_\infty^{(1)}}{\delta t}.$$

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Expected-wealth paradigm

Our decision rule under additive repetition is to maximise

$$\left\langle \frac{\Delta x}{\Delta t} \right\rangle = \left\langle \frac{\delta x}{\delta t} \right\rangle = \left\langle \frac{D}{\delta t} \right\rangle,$$

i.e. rate of change of expectation value of wealth.

"Expected-wealth paradigm" of 17^{th} century.

Axiom – not derived from dynamics or other considerations.

Simple rule, familiar average, incorporates gamble outcomes.

Has logical basis if game played many times in parallel to access multiple outcomes.

Risk neutrality

In language of economics, expected-wealth paradigm treats humans as "risk neutral".

i.e. No preference between gambles with equal changes in expected wealth over equal duration.

For example, fair coin toss at even odds same as doing nothing.

Recognised as unrealistic model since at least 1713, as predictions contradicted by observed human behaviour.

Predictive failure – a fix

Conventional reason for predictive failure is that value assigned by humans to possible changes in wealth depends on:

- existing wealth; and
- psychological risk preferences.

i.e. People don't treat equal amounts of money equally.

Makes intuitive sense:

- $\Delta x = \$10$ more significant to poor than to rich;
- some people gamble recreationally, others don't.

Expected-utility paradigm

D. Bernoulli 1738 built ideas into “expected-utility paradigm”.

Observation: usefulness of Δx not linear in Δx.

Model: individual has idiosyncratic utility function, $u(x)$, mapping wealth, x, to usefulness, u.

Claim: u is quantity whose rate of change of expected value,

$$\langle r_u \rangle \equiv \left\langle \frac{\Delta u(x)}{\Delta t} \right\rangle,$$

is maximised in a decision between gambles.

Axiom of utility theory $\rightarrow$ alternative decision algorithm...

Algorithm

Expected-utility decision algorithm

1. Specify $D^{(m)}$ and $\delta t^{(m)}$ for gambles offered;
2. Specify idiosyncratic utility function, $u(x)$, which maps wealth to utility;
3. Determine rate of change of expected utility, *e.g.* over single round of gamble,

$$\langle r_u \rangle^{(m)} = \left\langle \frac{u\left(x + D^{(m)}\right) - u(x)}{\delta t^{(m)}} \right\rangle ;$$

4. Choose gamble, m, with largest $\langle r_u \rangle^{(m)}$.

Equivalent approaches

Our decision theory and mainstream utility theory have different conceptual foundations but similar maximands:

$$\bar{g} = \left\langle \frac{\Delta v(x)}{\Delta t} \right\rangle; \quad \langle r_u \rangle = \left\langle \frac{\Delta u(x)}{\Delta t} \right\rangle.$$

Stationarity mapping, $v(x)$, depends on wealth dynamic.

Utility function, $u(x)$, depends on psychological attitude to risk.

Approaches mathematically equivalent if we identify v with u.

But suspect debates easier to settle empirically for dynamic than for psychology $\rightarrow$ better science.

Discussion

Utility theory consistent with $\bar{g}$ maximisation for matching pairs of dynamic and utility function.

But utility theory is a poorly constrained model, because infinitely many admissible u and, therefore, $\langle r_u \rangle$.

Our approach requires dynamic to be specified $\rightarrow$ unique v, $\bar{g}$.

$v = \ln x$ for multiplicative repetition.

$u = \ln x$ most widely used utility function.

Coincidence? Or humans evolved to make growth-optimal decisions in multiplicative environments?

Predictive failure – reason

Suggests different, deeper reason for predictive failure of expected-wealth paradigm.

Good model of decisions under additive repetition.

But additivity poor model of wealth evolution (*e.g.* bank interest payments independent of balance).

Fails to predict reality because implicitly contains unrealistic dynamic.

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Background

Devised by N. Bernoulli 1713 to test expected-wealth paradigm as model of human decision-making.

Hypothetical lottery with diverging rate of change of expected wealth for any ticket price.

Expected-wealth paradigm predicts people will pay any price.

Paradox: people observed to want to wager only small sums.

Motivation for development of expected-utility paradigm.

An unhelpful problem

Lottery deliberately provocative and unrealistic.

No need to invent gamble with $\langle \delta x \rangle \to \infty$ to expose flaws in expected-wealth paradigm.

Divergences in problem and proposed solutions confusing and allow non-fundamental objections of physical impossibility.

Such objections address only specific gamble, not paradigm.

But indelible part of history and current mainstream theory.

St Petersburg lottery

Traditional statement of lottery as follows:

Imagine starting prize of \$1 (originally ducats).

Toss a coin:

- if heads, prize is won and lottery ends;
- if tails, prize is doubled and the process is repeated.

Prize is \$2, \$4, \$8 if first head on 2nd, 3rd, 4th toss, *etc.*

Player pays F for lottery ticket.

Question: what is largest F they are willing to pay?

St Petersburg lottery

Lottery translates neatly into our gamble formalism:

$$D_k = \$2^{k-1} - F, \quad p_k = 2^{-k},$$

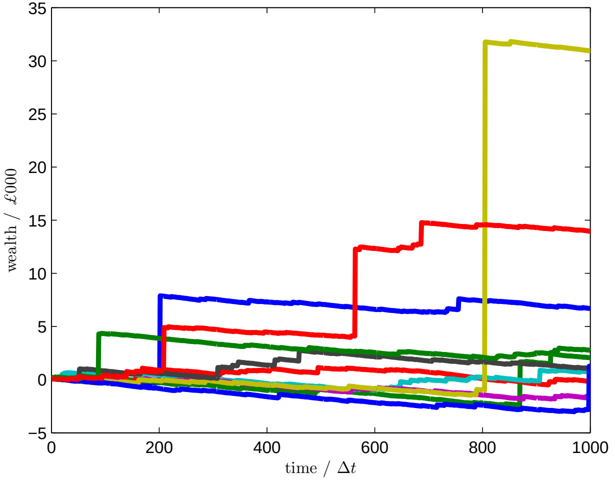
for $k \in \{1, 2, 3, \ldots\}$, *i.e.* set of positive integers.

Vast majority of *observed* payouts are small, but rarely extremely large payouts occur.

See simulated trajectories with lottery repeated additively.

Additive simulation

10 trajectories, 1000 rounds, $x(0) = \$100$, $F = \$10$.



Paradox

Coin tosses simply mechanism for selecting outcome $\rightarrow$ ignore.

Use compact definition of lottery and assume duration δt.

Rate of change of expected wealth:

$$
\begin{aligned}
\frac{\langle \delta x \rangle}{\delta t} &= \frac{1}{\delta t} \sum_{k=1}^{\infty} p_k D_k \\
&= \frac{1}{\delta t} \left(\$ \sum_{k=1}^{\infty} 2^{-k}\, 2^{k-1} - \sum_{k=1}^{\infty} 2^{-k} F \right) \\
&= \frac{1}{\delta t} \left(\$ \sum_{k=1}^{\infty} \frac{1}{2} - F \right) \rightarrow \infty \quad \underline{\text{for all } F}.
\end{aligned}
$$

Expected-wealth paradigm: lottery favourable at *any* price – including one's entire wealth!

Paradox

Reductio ad absurdum: real people do not behave this way.

D. Bernoulli resolved paradox using utility function, $u(x) = \ln x$.

(Equivalent to maximising $\bar{g}_{\mathrm{m}}$ under multiplicative repetition.)

Bernoulli made a mathematical error implementing his own paradigm.

We focus on what we presume he meant to write...

Resolution by logarithmic utility

Rate of change of expected logarithmic utility,

$$\frac{\langle \delta \ln x \rangle}{\delta t} = \frac{1}{\delta t} \sum_{k=1}^{\infty} p_k \left[\ln(x + D_k) - \ln x\right]$$
$$= \frac{1}{\delta t} \sum_{k=1}^{\infty} 2^{-k} \ln\left(\frac{x + \$2^{k-1} - F}{x}\right),$$

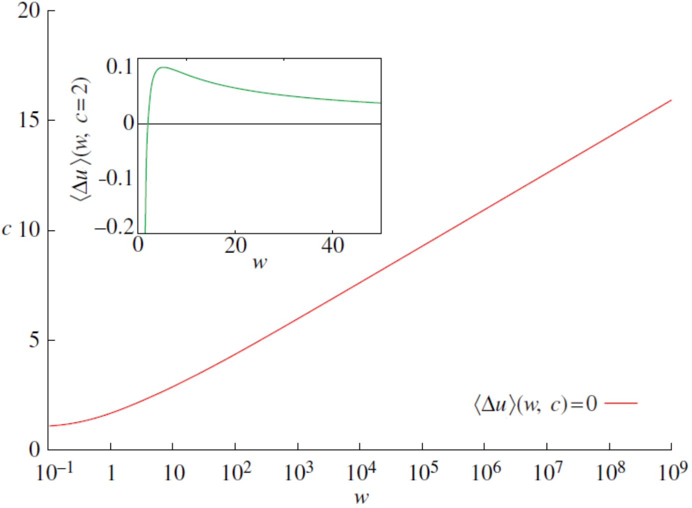
where x is the ticket buyer's wealth.

Finite for ticket price less than wealth plus booby prize: $F < x + \$1$ (*i.e.* price that prevents bankruptcy).

In particular, $\langle \delta \ln x / \delta t \rangle$ can be positive or *negative* $\rightarrow$ lottery can be declined in favour of null gamble.

Resolution by logarithmic utility

Locus in (x, F)-plane with $\langle \delta \ln x / \delta t \rangle = 0$.



Resolution by logarithmic utility

Bernoulli argued plausibly for this resolution: usefulness of monetary gain depends on how much money already have.

Also argued plausibly for logarithm: gain in usefulness proportional to fractional gain in wealth: $du = dx/x$.

Did not connect to repetition over time (or would have been less willing to accept Cramer's $u = \sqrt{x}$).

Time resolution

Resolve using our decision algorithm.

Assume lottery repeated multiplicatively: in effect, prizes and ticket price treated as fractions of player's wealth.

Get random wealth multiplier at each round:

$$r_k = \frac{x + \$2^{k-1} - F}{x}, \quad p_k = 2^{-k}.$$

Follows earlier treatment of multiplicative gamble $\rightarrow$ apply results directly.

Time resolution

Time-average growth rate is

$$\bar{g}_{\mathrm{m}} = \frac{1}{\delta t} \lim_{T\to\infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} \ln r(\tau) \right\} = \frac{1}{\delta t} \sum_{k=1}^{\infty} 2^{-k} \ln r_k,$$

identical to rate of change of expected log-utility.

(Because $\bar{g}_{\mathrm{m}}$ is the ergodic growth rate for multiplicative dynamics.)

Same result, different interpretation, fewer assumptions.

Time resolution

Plotted locus marks the decision threshold *vs.* null gamble under our decision rule.

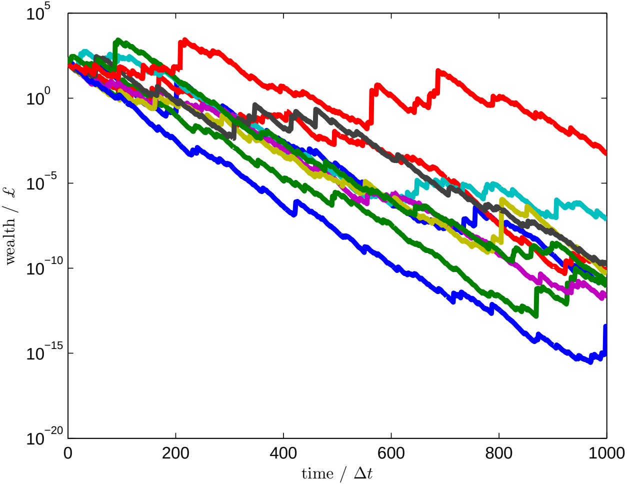
Player can sensibly decline the gamble $\rightarrow$ paradox resolved.

Can simulate trajectories for multiplicative repetition of same lotteries that we previously repeated additively.

Visualise decaying tendency in a realistic dynamic.

Multiplicative simulation

10 trajectories, 1000 rounds, $x(0) = \$100$, $F = \$10$.



Discussion

Treatments based on multiplicative repetition have appeared sporadically in the literature:

- explicitly by Whitworth 1870 (in appendix of textbook);
- related to Itô's lemma 1944;
- related to Kelly criterion 1956;
- explicitly by Cover & Thomas 1991 (again in textbook);
- rigorous treatment by Peters 2011.

Mainstream economics has ignored these.

Paradox remains a classic example and a topic of debate.

Agenda

Background

Insurance important and ubiquitous economic transaction, which can be modelled as gamble.

But in expected-wealth paradigm, buying insurance only rational at prices at which irrational to sell:

1. To be viable, insurer must charge at least expected value of possible claims ("net premium").
2. So buyer must be willing to pay more than net premium.
3. In expected-wealth paradigm, buyer should decline contract priced at more than net premium.
4. Therefore insurance contracts should not exist.

Puzzle

In this picture, insurance only ever beneficial to one party.

Anti-symmetry: expected value of one side's gain = expected value of other side's loss.

Puzzle: insurance contracts are observed to exist. Why?

Classical resolutions: utility theory (*i.e.* psychology) and asymmetric information (*i.e.* deception).

New resolution: contracts exist with range of prices that increase $\bar{g}_{\mathrm{m}}$ for both buyer and seller.

Illustrate with example from maritime trade...

Model shipping contract

Imagine shipowner sends cargo St Petersburg $\rightarrow$ Amsterdam:

- owner's wealth, $x_{\text{own}} = \$100,000$;
- gain on safe arrival of cargo, $G = \$4,000$;
- probability ship will be lost, $p = 0.05$;
- replacement cost of the ship, $C = \$30,000$;
- voyage time, $\delta t = 1$ month.

Insurer with wealth $x_{\text{ins}} = \$1,000,000$ offers to insure voyage for fee, $F = \$1,800$.

If ship lost, insurer pays $L = G + C$ for ship and lost profit.

Owner's gamble

Frame owner's decision as choice between two gambles.

Definition: Owner's decision

Sending the ship uninsured is gamble o1:

$$D_1^{(o1)} = G, \qquad p_1^{(o1)} = 1 - p;$$
$$D_2^{(o1)} = -C, \qquad p_2^{(o1)} = p.$$

Sending the ship fully insured is gamble o2:

$$D_1^{(o2)} = G - F \qquad p_1^{(o2)} = 1.$$

Trivial gamble as all risk has been transferred to the insurer.

Insurer's gamble

Also model insurer's decision as choice between gambles.

Definition: Insurer's decision

Not insuring the ship is gamble i1:

$$D_1^{(i1)} = 0 \qquad p_1^{(i1)} = 1.$$

This is the null gamble – the owner bears all the risk.

Insuring the ship is gamble i2:

$$D_1^{(i2)} = +F, \qquad p_1^{(i2)} = 1 - p;$$
$$D_2^{(i2)} = -L + F, \qquad p_2^{(i2)} = p.$$

Questions

Questions

1. Should the owner sign the insurance contract?
2. Should the insurer have offered it?

We will review the analysis in the expected-wealth paradigm.

Then present our approach based on maximising $\bar{g}_{\mathrm{m}}$.

Expected-wealth paradigm

Owner collapses gamble o1 to the scalar

$$\bar{g}_a^{(\text{o}1)} = \frac{1}{\delta t}\left\langle D^{(\text{o}1)}\right\rangle = \frac{(1-p)G + p(-C)}{\delta t} = \$2,300 \text{ pm}$$

and gamble o2 into the scalar

$$\bar{g}_a^{(\text{o}2)} = \frac{1}{\delta t}\left\langle D^{(\text{o}2)}\right\rangle = \frac{G-F}{\delta t} = \$2,200 \text{ pm}.$$

The difference between these scalars,

$$\delta\bar{g}_a^{\text{o}} = \bar{g}_a^{(\text{o}2)} - \bar{g}_a^{(\text{o}1)} = \frac{pL-F}{\delta t},$$

is the change in the rate of change of expected wealth when the owner takes insurance. He does so if $\delta\bar{g}_a^{\text{o}} > 0$.

Here $\delta\bar{g}_a^{\text{o}} = -\$100$ pm, so owner declines contract.

Expected-wealth paradigm

Insurer evaluates gambles i1 and i2 similarly:

$$\begin{aligned}\bar{g}_a^{(\mathrm{i1})} &= \$0 \text{ pm};\\ \bar{g}_a^{(\mathrm{i2})} &= \frac{F - pL}{\delta t} = \$100 \text{ pm}.\end{aligned}$$

Again we compute difference:

$$\delta \bar{g}_a^{\mathrm{i}} = \bar{g}_a^{(\mathrm{i2})} - \bar{g}_a^{(\mathrm{i1})} = \frac{F - pL}{\delta t}.$$

In the example this is $\delta \bar{g}_a^{\mathrm{i}} = \$100$ pm.

Insurer offers contract in world of expected-wealth paradigm.

Impossibility of insurance

No contract made as only one party willing to sign at that price.

Indeed, anti-symmetric property,

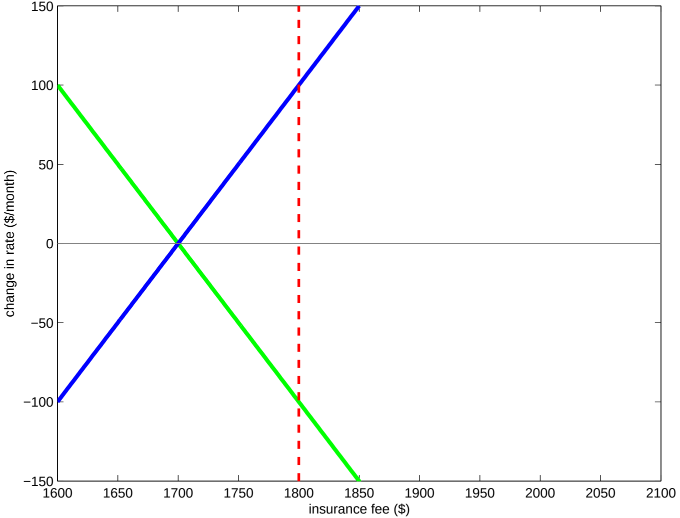
$$\delta\bar{g}^{\mathrm{o}}_{a} = -\delta\bar{g}^{\mathrm{i}}_{a},$$

means there is **no price** at which both parties will sign.

If one quantity is positive, the other must be negative $\rightarrow$ an unsavoury business.

Insurance contracts cannot exist in this world.

No price range



Fix – introduce asymmetry

In this picture, the observed existence of insurance contracts requires some asymmetry between the parties:

- different risk preferences;
- different information about voyage;
- different assessments of riskiness of voyage;
- one party to deceive, coerce, gull other into bad decision.

Hard to accept as basis for large and important global market.

Time paradigm

Resolve in time paradigm: assume multiplicative repetition and maximise $\bar{g}_{\mathrm{m}}$.

Here multiplicative repetition means that owner sends ship and cargo whose values $\propto$ his wealth.

Rich owner sends more cargo in larger ship or flotilla, while poor owner sends small boat.

Under additive repetition, sends same cargo on each voyage, regardless of wealth.

Large shipping companies, *e.g.* Maersk, inconceivable under additive repetition, where profits not reinvested.

Time paradigm

Both parties seek to maximise their

$$\bar{g}_m = \lim_{\Delta t \to \infty} \frac{\Delta v(x)}{\Delta t} = \frac{\langle \delta \ln x \rangle}{\delta t}.$$

Time paradigm

Owner without insurance has

$$\bar{g}_m^{(o1)} = \frac{(1-p)\ln(x_{\text{own}} + G) + p\ln(x_{\text{own}} - C) - \ln(x_{\text{own}})}{\delta t}$$

or 1.9% pm. Owner with insurance has

$$\bar{g}_m^{(o2)} = \frac{\ln(x_{\text{own}} + G - F) - \ln(x_{\text{own}})}{\delta t}$$

or 2.2% pm. Difference is

$$\delta \bar{g}_m^{o} = \bar{g}_m^{(o1)} - \bar{g}_m^{(o2)} \approx 0.24\% \text{ pm},$$

so owner signs contract in this world.

Time paradigm

An insurer who issues no insurance plays the null gamble:

$$\bar{g}_m^{(\mathrm{i1})} = \frac{0}{\delta t}$$

or 0% pm. With insurance he has

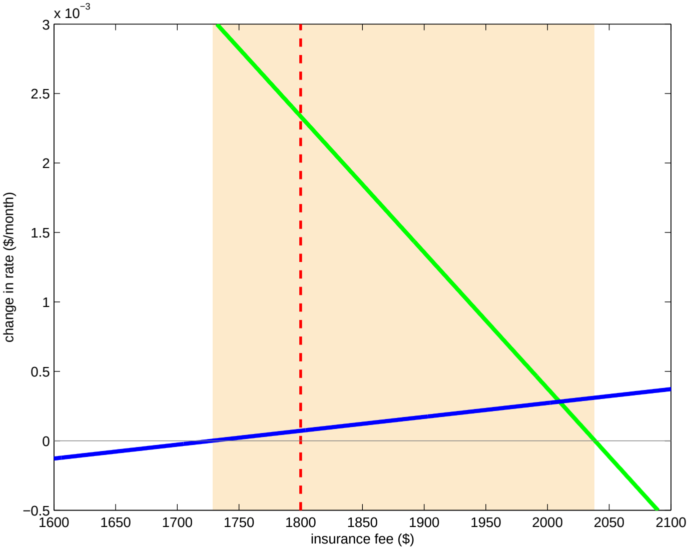
$$\bar{g}_m^{(\mathrm{i2})} = \frac{(1-p)\ln(x_{\mathrm{ins}} + F) + p\ln(x_{\mathrm{ins}} + F - L) - \ln(x_{\mathrm{ins}})}{\delta t}$$

or 0.0071% pm. Difference is

$$\delta\bar{g}_m^{\mathrm{i}} = \bar{g}_m^{(\mathrm{i2})} - \bar{g}_m^{(\mathrm{i1})} \approx 0.0071\%\ \mathrm{pm},$$

so he is also keen to do business.

Price range


x 10^−3
3
2.5
2
1.5
1
0.5
0
−0.5
change in rate ($/month)
1600
1650
1700
1750
1800
1850
1900
1950
2000
2050
2100
insurance fee ($)

Fundamental resolution

Time paradigm with multiplicative repetition creates a world where insurance contracts **can** exist.

We view this as the

Fundamental resolution of the insurance puzzle:

Buyer and seller of an insurance contract both sign when it increases the time-average growth rates of their wealths.

No appeal to asymmetries: arises naturally from our model of human decision-making.

Message: business happens when both parties gain.