Lecture 4: Markets

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Agenda

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Introduction

Apply our decision theory based on time-average growth rate maximisation to markets.

Based on two recent papers (open access):

- O. Peters. *Optimal leverage from non-ergodicity.* Quant. Fin. 11 (11), 1593–1602, 2011.
- O. Peters and A. Adamou. *Stochastic market efficiency.* Santa Fe Institute working paper 13-06-022, 2013.

Optimal leverage

1. Portfolio selection problem in simple market of two assets:
   - risky asset, like a stock;
   - riskless asset, like a bank deposit.

How should investor allocate money between these assets (the leverage problem)?

Classical approach $\rightarrow$ need to know investor risk preferences.

Our approach: maximise $\bar{g}_{\mathrm{m}}$ $\rightarrow$ unambiguous optimal leverage.

Stochastic market efficiency

2. Consider stability and efficiency of markets.

Does existence of objective optimal leverage constrain market prices and fluctuations?

Answer: yes, otherwise markets unstable and easy to beat.

Develop an accountable scientific theory:

- Quantify constraint on stochastic properties of market;
- Make prediction and account for imprecision;
- Test against real market data.

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Model market

Consider assets whose values follow multiplicative dynamics, which we model using GBM:

$$dx = x(\mu dt + \sigma dW).$$

μ is the drift and σ is the volatility.

Two assets

Simple model market of two assets.

Riskless, *e.g.* bank deposit: $\mu = \mu_r$; $\sigma = 0$.

Investment of x_0 grows deterministically:

$$dx_0 = x_0 \mu_r dt.$$

Future value of x_0 is known with certainty:

$$x_0(t_0 + \Delta t) = x_0(t_0) \exp(\mu_r \Delta t).$$

μ_r is the riskless drift.

Two assets

Simple model market of two assets.

Risky, *e.g.* stock: $\mu = \mu_s$; $\sigma = \sigma_s$.

Investment of x_1 grows noisily:

$$dx_1 = x_1(\mu_s dt + \sigma_s dW).$$

Future value of x_1 is a random variable:

$$x_1(t_0 + \Delta t) = x_1(t_0) \exp\left[\left(\mu_s - \frac{{\sigma_s}^2}{2}\right)\Delta t + \sigma_s W(\Delta t)\right].$$

$\mu_s > \mu_r$ is the risky drift, $\sigma_s > 0$ is the volatility.

Excess return

The risky asset has an excess drift over the riskless asset:

$$\mu_{\mathsf{e}} = \mu_{\mathsf{s}} - \mu_{\mathsf{r}}.$$

μ_{e} is known in finance as:

- excess return;
- risk premium;
- equity premium (in context of stock markets).

We can view μ_{e} as compensation for accepting an uncertain outcome rather than a guaranteed result.

Leverage

How to allocate investment between the two assets?

Simple portfolio setup:

- Total value x_ℓ;
- ℓx_ℓ invested in stock;
- $(1-\ell)x_\ell$ deposited in the bank.

ℓ is known as leverage.

Leverage range

$\ell = 0$ portfolio contains only bank deposits.

$\ell = 1$ portfolio contains only stock.

Not constrained to $0 \leq \ell \leq 1$.

Can make $\ell > 1$ by borrowing money from the bank (a negative deposit) to buy stock.

Can make $\ell < 0$ by borrowing stock, selling it, and putting the proceeds in the bank (known as short selling).

Leveraged SDE

Each component of the portfolio experiences same relative fluctuations as the asset in which it was made:

$$dx_\ell = \underbrace{(1-\ell)x_\ell}_{\text{in bank}} \frac{dx_0}{x_0} + \underbrace{\ell x_\ell}_{\text{in stock}} \frac{dx_1}{x_1}.$$

We already know $\frac{dx_0}{x_0}$ and $\frac{dx_1}{x_1}$ from the two asset SDEs.

Substitute in to get the SDE for leveraged portfolio:

$$dx_\ell = x_\ell[(\mu_\mathrm{r} + \ell\mu_\mathrm{e})dt + \ell\sigma_\mathrm{s} dW].$$

Leveraged SDE

Value of leveraged portfolio evolves according to:

$$x_\ell(t_0+\Delta t) = x_\ell(t_0)\exp\left[\left(\mu_\mathrm{r} + \ell\mu_\mathrm{e} - \frac{\ell^2{\sigma_\mathrm{s}}^2}{2}\right)\Delta t + \ell\sigma_\mathrm{s} W(\Delta t)\right].$$

We chose our notation for the two basic assets carefully:

- when $\ell = 0$, x_ℓ evolves as x_0;
- when $\ell = 1$, x_ℓ evolves as x_1.

Rebalancing

In our model ℓ is held constant over time.

Model portfolio must be rebalanced to ensure ratio of stock to total investment remains fixed.

Imagine stock $\downarrow$ and bank deposit $\uparrow$ over a time step.

Result: stock $< \ell x_\ell$; bank deposit $> (1-\ell)x_\ell$.

To restore leverage to ℓ, withdraw from bank and buy stock.

In the SDE we imagine this happens continuously.

Question

Simple model portfolio setup allows us to ask

Question:

What is the optimal value of ℓ?

Similar to the gamble problem, except we are now choosing from a continuum of gambles parametrised by ℓ.

Apply new decision theory $\rightarrow$ maximise time-average growth rate of investment under appropriate dynamic.

Classical portfolio theory

First let's review classical treatment of problem.

Intuition: there is a trade-off between risk and reward.

In GBM model, we might link risk to σ and reward to μ.

Ideal investment has large μ and small σ, but we acknowledge that larger μ tends to come with larger σ.

Hence excess return of risky over riskless asset: $\mu_\mathsf{e} > 0$.

Efficient portfolios

Markowitz first tried to treat this problem rigorously 1952.

Portfolio with (σ_i, μ_i) is "efficient" if no rival portfolio with (σ_j, μ_j) satisfies at least one of the statements:

- $\mu_j > \mu_i$ and $\sigma_j \leq \sigma_i$;
- $\sigma_j < \sigma_i$ and $\mu_j \geq \mu_i$.

Advice: invest only in efficient portfolios.

Efficient frontier

We compare portfolios with $(\ell\sigma_\mathsf{s}, \mu_\mathsf{r} + \ell\mu_\mathsf{e})$, which lie a on straight line in the (σ, μ)-plane:

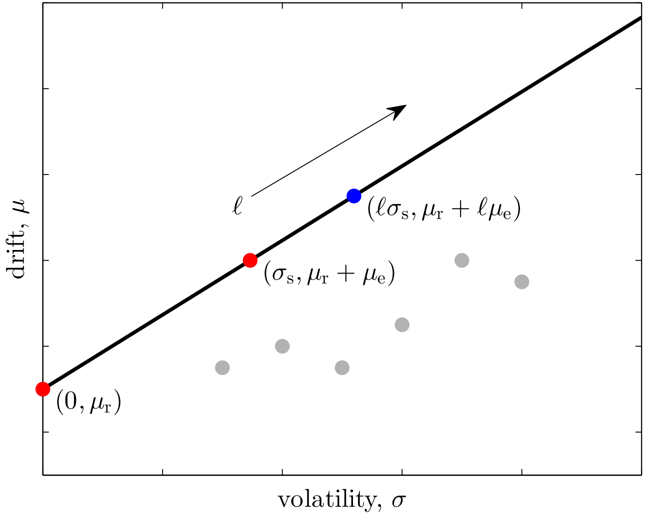
$$\mu = \mu_\mathsf{r} + \left(\frac{\mu_\mathsf{e}}{\sigma_\mathsf{s}}\right)\sigma.$$

In Markowitz's classification, all portfolios on this line are efficient wrt to each other (known as "efficient frontier").

All leverages equally investible $\rightarrow$ no answer to our question!

Classical approach requires additional information to distinguish between portfolios, *e.g.* investor risk preferences.

Efficient frontier



Sharpe ratio

Gradient of efficient frontier also known as the Sharpe ratio:

$$S = \frac{\mu_{\mathsf{e}}}{\sigma_{\mathsf{s}}}.$$

Shorthand for applying Markowitz's ideas: maximising S is equivalent to selecting an efficient portfolio.

S is insensitive to $\ell \rightarrow$ same non-advice as Markowitz approach.

Expected return

Expected value of stock grows as

$$\langle x_1(t_0 + \Delta t)\rangle = \langle x_1(t_0)\rangle \exp(\mu_s \Delta t),$$

with ensemble-average growth rate or "expected return":

$$g_m(\langle x_1\rangle) = \frac{\Delta \ln \langle x_1\rangle}{\Delta t} = \mu_s.$$

Follows that expected return of leveraged portfolio is:

$$g_m(\langle x_\ell\rangle) = \mu_r + \ell\mu_e.$$

Portfolio theory insensitive to ℓ is potentially dangerous.

If any ℓ is ok, then investor maximising expected return will seek $\ell \to \infty$, which will surely ruin him.

Time-average growth rate

We now understand the classical approach and its limitations.

What does our new decision theory have to say?

Time-average growth rate of leveraged portfolio is

$$\bar{g}_{\mathrm{m}}(\ell) \equiv \lim_{\Delta t \to \infty} \{g_{\mathrm{m}}(x_\ell, \Delta t)\} = \lim_{\Delta t \to \infty} \left\{ \frac{\Delta \ln x_\ell}{\Delta t} \right\}.$$

Depends on ℓ. Inserting solution for $x_\ell(t_0 + \Delta t)$ gives

$$\bar{g}_{\mathrm{m}}(\ell) = \lim_{\Delta t \to \infty} \left\{ \frac{1}{\Delta t} \left[\left(\mu_{\mathrm{r}} + \ell \mu_{\mathrm{e}} - \frac{\ell^2 {\sigma_{\mathrm{s}}}^2}{2} \right) \Delta t + \ell \sigma_{\mathrm{s}} W(\Delta t) \right] \right\}.$$

$$\boxed{\bar{g}_{\mathrm{m}}(\ell) = \mu_{\mathrm{r}} + \ell \mu_{\mathrm{e}} - \frac{\ell^2 {\sigma_{\mathrm{s}}}^2}{2}.}$$

Optimal leverage

$\bar{g}_{\mathrm{m}}(\ell)$ is a quadratic $\rightarrow$ unambiguous maximum at:

$$\boxed{\ell_{\mathrm{opt}} = \frac{\mu_{\mathrm{e}}}{{\sigma_{\mathrm{s}}}^2}.}$$

ℓ_{opt} is optimal leverage which defines portfolio with fastest long-run growth $\rightarrow$ single point on the efficient frontier.

Only need to know μ_{r}, μ_{s} and σ_{s} for the market assets.

No investor preferences (except that he maximises $\bar{g}_{\mathrm{m}}$).

Complete picture

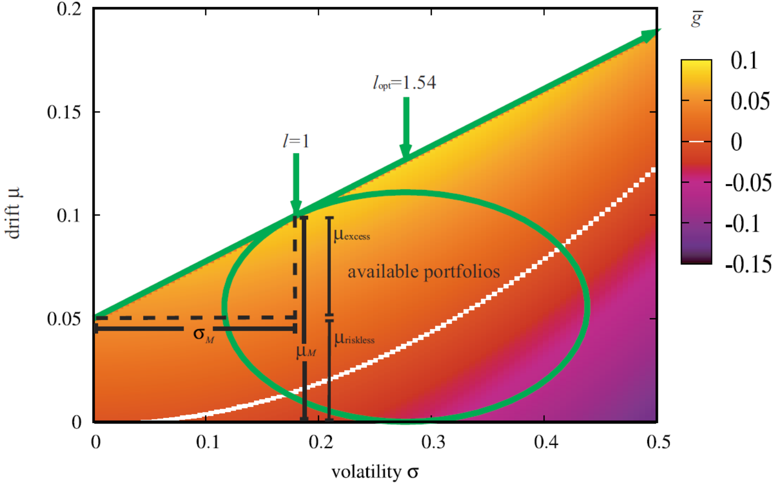
$\bar{g}_{\mathrm{m}}(\ell)$ is time-average growth rate along the efficient frontier.

Time-average growth rate is $\mu - \sigma^2/2$ for arbitrary point in (σ, μ)-plane $\rightarrow$ growth rate landscape.

Overlaying Markowitz picture on landscape gives missing information needed to distinguish portfolios.

Complete portfolio selection picture.

Complete picture



Excessive leverage

$\bar{g}_{\mathrm{m}}(\ell) < 0$ for $\ell < \ell^-$ and $\ell > \ell^+$, where

$$\ell^{\pm} = \ell_{\mathrm{opt}} \pm \sqrt{{\ell_{\mathrm{opt}}}^2 + \frac{2\mu_{\mathrm{r}}}{{\sigma_{\mathrm{s}}}^2}}$$

are the roots of $\bar{g}_{\mathrm{m}}(\ell) = 0$.

Investor maximising $|\ell|$ in either direction will cause his wealth to decay rapidly:

$$\bar{g}_{\mathrm{m}}(\ell) \to -\infty \quad \text{as} \quad |\ell| \to \infty.$$

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Fundamental measure

$S = \mu_{\mathrm{e}}/\sigma_{\mathrm{s}}$ is a dimensionful quantity: (time unit)$^{-1/2}$.

Value depends on time unit $\rightarrow$ arbitrary, *e.g.* $S = 5\ \mathrm{y}^{-1/2} \approx 0.26\ \mathrm{d}^{-1/2}$. Same portfolio, different numbers.

Tells us nothing fundamental about system under study.

$\ell_{\mathrm{opt}} = \mu_{\mathrm{e}}/{\sigma_{\mathrm{s}}}^2$ is a dimensionless quantity.

Value does not depend units and can carry fundamental information about system.

View ℓ_{opt} as fundamental measure of portfolio quality.

Question

However, significance of ℓ_{opt} runs deeper than this.

Optimal allocation of investment between risky and riskless assets in model market $\rightarrow$ tells us about market conditions.

High ℓ_{opt} indicates environment in risk-taking is incentivised. Low/negative ℓ_{opt} indicates the converse.

Question:

Are there any special values of ℓ_{opt} which describe different market regimes, or to which markets are attracted?

Relaxing the model

Assets in model market have static μ_{r}, μ_{s}, σ_{s} $\rightarrow$ static ℓ_{opt}.

Hard to explore question if ℓ_{opt} can't vary $\rightarrow$ need to relax model to make progress.

Real market assets don't follow GBM with static parameters.

In particular, real market conditions – represented in model by μ_{r}, μ_{s}, σ_{s} – change slowly (wrt to fluctuations).

Relax model constraint of static parameters and allow them to vary $\rightarrow$ allow ℓ_{opt} to change over time.

Classical efficiency

In general, an efficient system is one which is well optimised and can't be improved by simple actions.

Classical "efficient market hypothesis" treats markets as information processors (*e.g.* Hayek 1945, Fama 1965).

Claim: price of asset in efficient market accurately reflects all publicly available information.

Corollary: no market participant, without access to privileged information, can consistently beat the market simply by choosing prices at which he buys and sells.

Thought experiment

Consider a different efficiency involving leverage rather than price/information. Start with a thought experiment...

Imagine $\ell_{\text{opt}} > 1$ in our model market:

$\rightarrow$ Simple strategy of borrowing money to buy extra stock will achieve faster long-run growth than full investment in stock.

$\rightarrow$ Trivial matter for us to beat the market.

Similarly, imagine $\ell_{\text{opt}} < 1$: again easy to beat market by leaving some money in the bank (and, if $\ell_{\text{opt}} < 0$, short selling).

Stochastic efficiency

Hard to call a market efficient if it's so easy to beat.

Suggests different notion of efficiency $\rightarrow$ stochastic efficiency.

Claim: impossible for participant to beat a stochastically efficient market simply by applying leverage.

Hypothesis: stochastic market efficiency (strong form)

Real markets self-organise such that

$$\boxed{\ell_{\mathrm{opt}} = 1}$$

is an attractive point for their stochastic properties (represented by μ_{r}, μ_{s}, σ_{s} in the relaxed model).

Stability

Another approach: consider market stability and its dependence on ℓ_{opt}. Run another thought experiment...

Imagine $\ell_{\text{opt}} > 1$:

Objectively optimal $\rightarrow$ *everyone* in market should want to borrow money to buy stock.

But, if so, who will lend money and who will sell stock?

Imagine $\ell_{\text{opt}} < 0$:

Everyone should want to borrow stock and sell it for cash.

But who will lend stock and who will relinquish cash to buy it?

Stable range for ℓ_{opt}

Neither situation globally stable: $0 \leq \ell_{\text{opt}} \leq 1$ is stable range.

Hard to imagine deviations persisting for long before trading changes market parameters to return ℓ_{opt} to stable value.

Hypothesis: stochastic market efficiency (weak form)

Real markets self-organise such that

$$\boxed{0 \leq \ell_{\text{opt}} \leq 1}$$

is an attractive range for their stochastic properties.

Self-organisation

Self-organisation of parameters occurs *via* trading activity.

Feedback loops dynamically adjust prices and fluctuations to pull ℓ_{opt} back to stability when it strays.

Detailed mechanisms can be found in Peters & Adamou 2013.

Strong hypothesis favoured over weak as expect $\ell = 0$ a weaker attractor than $\ell = 1$ (to avoid economic paralysis).

Hypothesis: stochastic market efficiency (refined)

On sufficiently long time scales, $\ell_{\text{opt}} = 1$ is a strong attractor for the stochastic properties of real markets. Deviations from this attractor over shorter time scales are likely to be confined to the range $0 \leq \ell_{\text{opt}} \leq 1$.

Tests of the hypothesis

Test by simulating leveraged investments in a real stock market.

Treat U.S. S&P500 index as real-world equivalent of risky asset.

Treat bank deposits at historical interest rates as real-world equivalent of riskless asset.

Historical period: August 1955 to May 2013.

Data sets

Use FRED time series:

- `SP500` – daily closing prices of S&P500;
- `DFF` – effective federal funds rate;
- `DPRIME` – bank prime loan rate.

S&P500 $\rightarrow$ well diversified portfolio of large and successful companies $\rightarrow$ generous estimate of ℓ_{opt}.

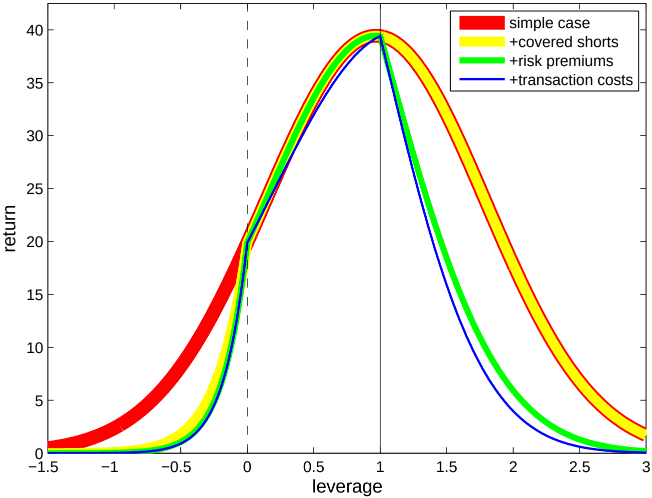
Simulation

Simulate investment of constant leverage over time window:

- Assume net value (equity) of \$1 at start of window: $\$\ell$ in S&P500; $\$(1-\ell)$ in bank;
- Update values of each component daily according to historical returns and rebalance portfolio;
- If equity falls below zero, investment is bankrupt and simulation stops;
- Proceed to last day of the window and record final equity.

Repeat simulation for different ℓ to find simulated optimal leverage, $\ell_{\mathrm{opt,s}}$, which maximises final equity.

Full time series



Full time series

Simulated return-leverage curves correspond to different sets of assumptions about interest rates and transaction costs.

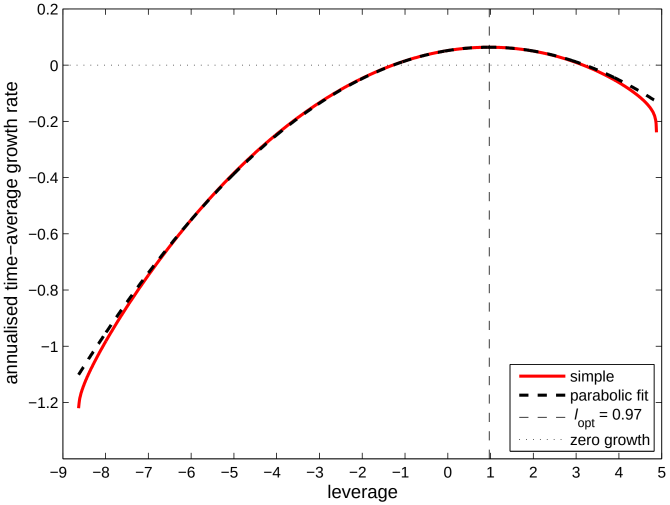
Labelled 1–4 in increasing complexity and resemblance to real market practices (call 1 "simple" and 4 "complex").

$\ell_{\text{opt}} \approx 0.97$ for simulations 1–3.

$\ell_{\text{opt}} = 1.00$ for simulation 4!

Results consistent with stochastic efficiency – discuss statistical significance shortly.

Parabolic fit



Parabolic fit

For $\Delta t = 58$ years, expect simulated growth rate $\approx \bar{g}_{\mathrm{m}}(\ell)$.

Parabolic form of GBM-based model clearly visible in plot.

Parameters of fitted parabola give effective values of μ_{r}, μ_{e}, σ_{s} for S&P500 over time series.

Least-squares fit:

- $\mu_{\mathrm{r}} = 5.2\%\ pa$;
- $\mu_{\mathrm{e}} = 2.4\%\ pa$;
- $\sigma_{\mathrm{s}} = 16\%\ p\sqrt{a}$.

Parabolic fit

Deviation from parabolic form for high and low leverages due to extreme fluctuations.

$\rightarrow$ Large losses and bankruptcy for high leverage portfolios.

Common criticism is that: real return distributions have fatter tails than in models.

Not relevant here: large deviations strengthen stochastic efficiency hypothesis because they penalise high leverage.

Equity premium puzzle

Results relevant to unexplained phenomenon in economics: the "equity premium puzzle".

Claim: observed equity premium (μ_{e} in model) incompatible with theory as it implies investors implausibly risk averse.

6% *pa* established in literature *vs.* 2.4% *pa* in our model.

Source of discrepancy unclear because classical estimates based on complicated behavioural models.

No puzzle in our model: μ_{e} consistent with stochastic efficiency.

Finite time scales

Can't observe $\bar{g}_{\mathrm{m}}(\ell)$ as this requires infinite observation time.

Can only observe finite-time growth rate $g_{\mathrm{m}}(x_\ell, \Delta t)$ over simulation window Δt.

Random variable whose distribution broadens as $\Delta t \to 0$.

Also can't observe ℓ_{opt} but only simulated optimal leverage $\ell_{\mathrm{opt,s}}(\Delta t)$ which maximises $g_{\mathrm{m}}(x_\ell, \Delta t)$.

$g_{\mathrm{m}}(x_\ell, \Delta t)$ worse estimator of $\bar{g}_{\mathrm{m}}(\ell)$ as $\Delta t \to 0$, so $\ell_{\mathrm{opt,s}}(\Delta t)$ worse estimator of ℓ_{opt}.

Significance

When is a single observation of $\ell_{\text{opt,s}}(\Delta t)$ consistent with our hypothesis significant?

Depends on width of distribution of $\ell_{\text{opt}}(\Delta t)$:

- if width much larger than hypothesised stable range of ℓ_{opt}, then can't read much into one observation;
- if width similar to or smaller than stable range, then observation outside is strong evidence that hypothesis is flawed, so observation inside significant.

Significance

Quantify these ideas in original model:

$$g_{\mathsf{m}}(x_\ell, \Delta t) = \mu_{\mathsf{r}} + \ell\mu_{\mathsf{e}} - \frac{\ell^2 {\sigma_{\mathsf{s}}}^2}{2} + \frac{\ell\sigma_{\mathsf{s}} W(\Delta t)}{\Delta t}.$$

$$\Rightarrow \quad \ell_{\mathsf{opt,s}}(\Delta t) = \ell_{\mathsf{opt}} + \frac{W(\Delta t)}{\sigma_{\mathsf{s}}\Delta t}.$$

Normally distributed with mean ℓ_{opt} and standard deviation:

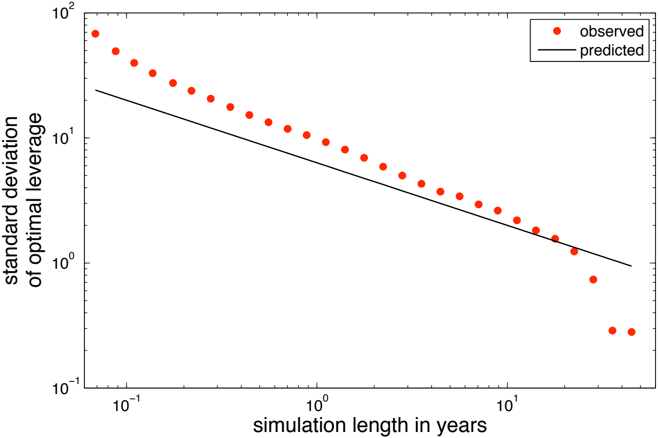
$$\mathrm{stdev}[\ell_{\mathsf{opt,s}}(\Delta t)] = \frac{1}{\sigma_{\mathsf{s}}\sqrt{\Delta t}}.$$

For $\Delta t \approx 58$ years and fitted σ_{s}, get $\mathrm{stdev}[\ell_{\mathsf{opt,s}}(\Delta t)] \approx 0.83$.

So single observation is significant corroboration of hypothesis.

Fluctuations

Test model prediction by running simulations with different window lengths and compiling histograms of $\ell_{\mathrm{opt,s}}(\Delta t)$:



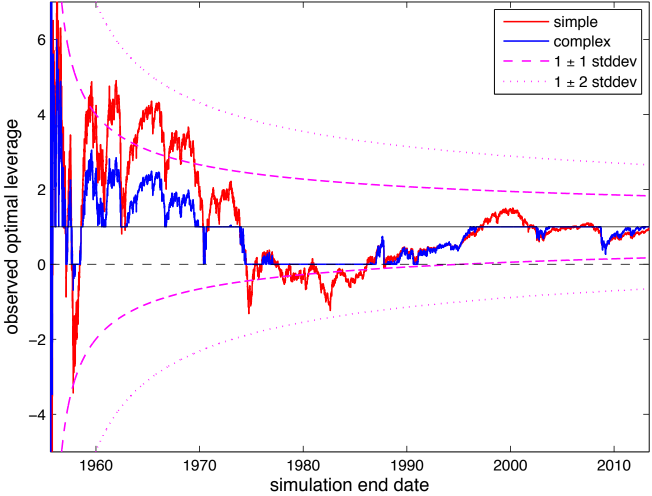
Expanding window

Run simulations starting on first day of time series with increasing window lengths.

Reduced fluctuations in $\ell_{\text{opt,s}}(\Delta t)$ with increasing window length consistent with model.

Evolution of $\ell_{\text{opt,s}}(\Delta t)$ supports hypothesis that ℓ_{opt} attracted to range $0 \leq \ell_{\text{opt}} \leq 1$ and, over long time scales, to $\ell_{\text{opt}} = 1$.

Expanding window



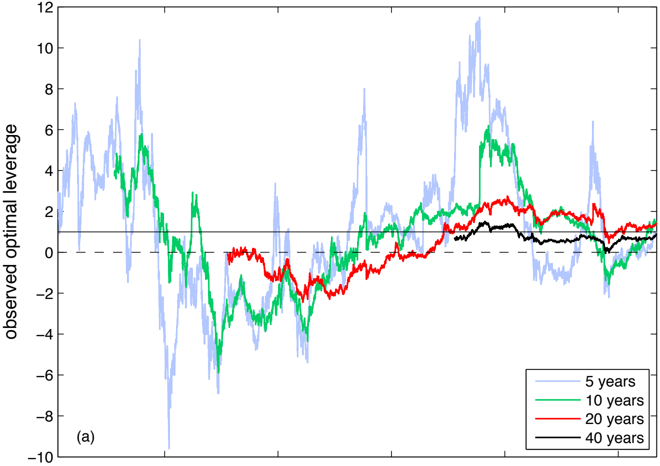
Moving windows

Can also plot $\ell_{\text{opt,s}}(\Delta t)$ for windows with fixed lengths and moving start date.

Run simple and complex simulations with window lengths ranging from 5 to 40 years...

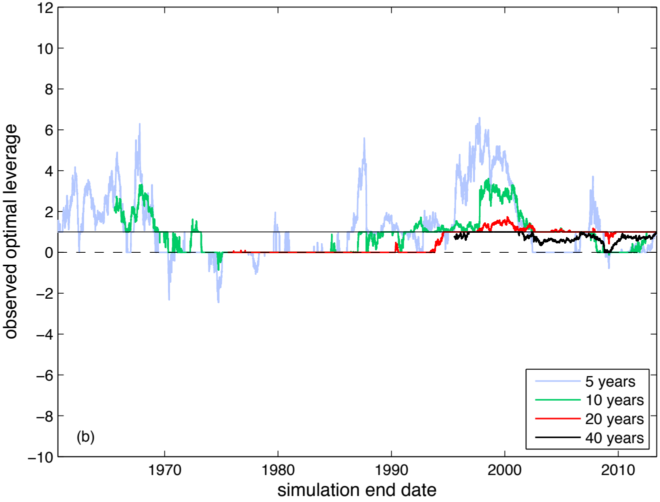
Moving windows

Simple simulation:



Moving windows

Complex simulation:



Moving windows

Can also plot $\ell_{\mathrm{opt,s}}(\Delta t)$ for windows with fixed lengths and moving start date.

Run simple and complex simulations with window lengths ranging from 5 to 40 years...

Observe:

- Strong fluctuations over short time scales;
- Weak fluctuations over long time scales;
- Attractive behaviour consistent with refined hypothesis.

Discussion

Mathematical formalism developed to conceptualise randomness in economics is very powerful.

Does more than just create a foundation for economic theory.

Framework is epistemologically sound $\rightarrow$ make testable predictions, *e.g.* stochastic efficiency.

Hard to guess from coin-tossing game that we could predict stochastic properties of U.S. stock market over 58 years!

What other surprising predictions can we make in this formalism?