Ergodicity Economics

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1.1 Reallocation

{section:reallocation}

1.1.1 Introduction

{section:RGBM_intro}

In Sec. **??** we created a model world of independent trajectories of GBM. We studied how the distribution of the resulting random variables evolved over time. We saw that this is a world of broadening distributions, increasing inequality, and wealth condensation. We introduced cooperation to it in Sec. **??** and saw how this increases the time-average growth rate for those who pool and share all of their resources. In this section we study what happens if a large number of individuals pool and share only a fraction of their resources. This is reminiscent of the taxation and redistribution – which we shall call "reallocation" – carried out by populations in the real world.

We will find that, while full cooperation between two individuals increases their growth rates, sufficiently fast reallocation from richer to poorer in a large population has two related effects. Firstly, everyone's wealth grows in the long run at a rate close to that of the expectation value. Secondly, the distribution of rescaled wealth converges over time to a stable form. This means that, while wealth can still be distributed quite unequally, wealth condensation and the divergence of inequality no longer occur in our model. Of course, for this to be an interesting finding, we will have to quantify what we mean by "sufficiently fast reallocation."

We will also find that when reallocation is too slow or, in particular, when it goes from poorer to richer – which we will quantify as negative reallocation – no stable wealth distribution exists. In the latter case, the population splits into groups with positive and negative wealths, whose magnitudes grow exponentially.

Finally, having understood how our model behaves in each of these reallocation regimes, we will fit the model parameters to historical wealth data from the real world, specifically the United States. This will tell us which type of model behaviour best describes the dynamics of the US wealth distribution in both the recent and more distant past. You might find the results surprising – we certainly did!

1.1.2 The ergodic hypothesis in economics

{section:RGBM_EH}

Of course, we are not the first to study resource distributions and inequality in economics. This topic has a long history, going back at least as far as Vilfredo Pareto's work in the late 19$^{\text{th}}$ century [20] (in which he introduced the power-law distribution we discussed in Sec. **??**). Economists studying such distributions usually assume that they converge in the long run to a unique and stable form, regardless of initial conditions. This allows them to study the stable distribution, for which many statistical techniques exist, and to ignore the transient phenomena preceding it, which are far harder to analyse. Paul Samuelson called this the "ergodic hypothesis" [26, pp. 11-12]. It's easy to see why: if this convergence happens, then the time average of the observable in question will equal its ensemble average over the stable distribution.[1]

Economics is often concerned with growth and a growing quantity cannot be ergodic in Samuelson's sense, because its distribution never stabilises. This

[1] Convergence to a unique and stable distribution is a sufficient but not necessary condition for an ergodic observable, as we have defined it.

suggests the simplifying ergodic hypothesis should never be made. Not so fast! Although rarely stated, a common strategy to salvage these techniques is to find a transformation of the non-ergodic process that produces a meaningful ergodic observable. If such an ergodic observable can be derived, then classical analytical techniques may still be used. We have already seen in the context of gambles that expected utility theory can be viewed as transformation of non-ergodic wealth increments into ergodic utility increments. Expectation values, which would otherwise be misleading, then quantify time-average growth of the decision-maker's wealth.

Studies of wealth distributions also employ this strategy. Individual wealth is modelled as a growing quantity. Dividing by the population average transforms this to a rescaled wealth, as in Sec. **??**, which is hypothesised to be ergodic. For example, [3, p. 130] "impose assumptions . . . that guarantee the existence and uniqueness of a limit stationary distribution." The idea is to take advantage of the simplicity with which the stable distribution can be analysed, *e.g.* to predict the effects of policies encoded in model parameters.

There is, however, an elephant in the room. To our knowledge, the validity of the ergodic hypothesis for rescaled wealth has never been tested empirically. It's certainly invalid for the GBM model world we studied previously because, as we saw in Sec. **??**, rescaled wealth has an ever-broadening lognormal distribution. That doesn't seem to say much, as most reasonable people would consider our model world – containing a population of individuals whose wealths multiply noisily and who never interact – somewhat unrealistic! The model we are about to present will not only extend our understanding from this simple model world to one containing interactions, but also will allows us to test the hypothesis. This is because it has regimes, *i.e.* combinations of parameters, for which rescaled wealth is and isn't ergodic. This contrasts with models typically used by economists, which have the ergodic hypothesis "baked in."

If it is reasonable to assume a stable distribution exists, we must also consider how long convergence to would take after a change of parameters. It's no use if convergence in the model takes millennia, if we are using it to estimate the effect of a new tax policy over the next election cycle. Therefore, treating a stable model distribution as representative of the empirical wealth distribution implies an assumption of fast convergence. As the late Tony Atkinson pointed out, that "the speed of convergence makes a great deal of difference to the way in which we think about the model" [2]. We will also use our model to discuss this point. Without further ado, let us introduce it.

1.1.3 Reallocating GBM

{section:RGBM_model}

Our model, called Reallocating Geometric Brownian Motion (RGBM), is a system of N individuals whose wealths, $x_i(t)$, evolve according to the stochastic differential equation,

$$dx_i = x_i\left[(\mu - \tau)dt + \sigma dW_i\left(t\right)\right] + tau\left\langle x\right\rangle_N \ dt, \qquad (1)$$ {eq:rgbm}

for all $i = 1 \ldots N$. In effect, we have added to the GBM model a simple reallocation mechanism. Over a time step, dt, each individual pays a fixed proportion of its wealth, $\tau x_i dt$, into a central pot ("contributes to society") and gets back an equal share of the pot, $\tau \left\langle x\right\rangle_N dt$, ("benefits from society"). We can think of this as applying a wealth tax, say of 1% per year, to everyone's wealth and then redistributing the tax revenues equally. Note that the reallocation parameter, τ, is, like μ, a rate with dimensions per unit time. Note also that when $\tau = 0$, we recover our old friend, GBM, in which individuals grow their wealths without interacting.

RGBM is our null model of an exponentially growing economy with social structure. It is intended to capture only the most general features of the dynamics of wealth. A more complex model would treat the economy as a system of agents that interact with each other through a network of relationships. These relationships include trade in goods and services, employment, taxation, welfare payments, using public infrastructure (roads, schools, a legal system, social security, scientific research, and so on), insurance, wealth transfers through inheritance and gifts, and everything else that constitutes an economic network. It would be a hopeless task to list exhaustively all these interactions, let alone model them explicitly. Instead we introduce a single parameter – the reallocation rate, τ – to represent their net effect. If τ is positive, the direction of net reallocation is from richer to poorer. If negative, it is from poorer to richer.

We will see shortly that RGBM has both ergodic and non-ergodic regimes, characterised to a good approximation by the sign of τ. $\tau > 0$ produces an ergodic regime, in which wealths are positive, distributed with a Pareto tail, and confined around their mean value. $\tau < 0$ produces a non-ergodic regime, in which the population splits into two classes, characterised by positive and negative wealths which diverge away from the mean.

We offer a couple of health warnings. In RGBM, like in GBM, there are no additive changes akin to labour income and consumption. This is unproblematic for large wealths, where additive changes are dwarfed by capital gains. For small wealths, however, wages and consumption are significant and empirical distributions look rather different for low and high wealths [11]. We modelled earnings explicitly in [5] and found this didn't generate insights different from RGBM when fit to real wealth data. We note also, as [18, p. 41] put it, that our agents "do not marry or have children or die or even grow old." Therefore, the individual in our setup is best imagined as a household or a family, *i.e.* some long-lasting unit into which personal events are subsumed.

Having specified the model, we will use insights from Sec. **??** to understand how rescaled wealth is distributed in the ergodic and non-ergodic regimes. Then we will show briefly our results from fitting the model to historical wealth data from the United States. The full technical details of this fitting exercise are beyond the scope of these notes – if you are interested, you can find "chapter and verse" in [5]. Fitting τ to data will allow us to answer the important questions:

- What is the net reallocating effect of socio-economic structure on the wealth distribution?
- Are observations consistent with the ergodic hypothesis that the rescaled wealth distribution converges to a stable distribution?
- If so, how long does it take, after a change in conditions, for the rescaled wealth distribution to reach the stable distribution?

1.1.4 Model behaviour

{section:RGBM_behaviour}

Equation (1) is our model for the evolution of wealth with social structure and the basis for the empirical study that follows. It is instructive to write it as

$$dx_i = \underbrace{x_i \left[\mu dt + \sigma dW_i\left(t\right)\right]}_{\text{Growth}} \underbrace{- \tau(x_i - \langle x\rangle_N)dt}_{\text{Reallocation}}. \tag{2}$$ {eq:rgbm_ou}

This can be thought of as GBM with a mean-reverting term like that of [28] in physics and [29] in finance. This representation exposes the importance of the sign of τ. We discuss the two regimes in turn.

Positive τ For $\tau > 0$, wealth, x_i, reverts to the population average, $\langle x\rangle_N$. The large-sample approximation, $\langle x\left(t\right)\rangle_N \propto e^{\mu t}$, is valid[2] and yields a simple differential equation for the rescaled wealth,

$$dy_i = y_i \sigma dW_i\left(t\right) - \tau(y_i - 1)dt, \tag{3}$$ {eq:rgbm_ou_re}

in which the common growth rate, μ, has been scaled out. The distribution of $y_i\left(t\right)$ can found by solving the corresponding Fokker-Planck equation (also known as the Kolmogorov forward equation). A stationary distribution exists with a Pareto tail, see Appendix 1.1.5. It is known as the inverse gamma distribution and has probability density function,

$$\mathcal{P}\left(y\right) = \frac{\left(\zeta - 1\right)^{\zeta}}{\Gamma\left(\zeta\right)} e^{-\frac{\zeta-1}{y}} y^{-(1+\zeta)}, \tag{4}$$ {eq:disti}

where $\zeta = 1 + \left(2\tau/\sigma^2\right)$ is the Pareto tail index, $\Gamma\left(\cdot\right)$ is the gamma function, and the index i has been dropped. Example forms of the stationary distribution are shown in Figure 1. The usual stylised facts are recovered: the larger σ (more randomness in the returns) and the smaller τ (less social cohesion), the smaller the tail index and the fatter the tail of the distribution. Moreover, the fitted τ values we obtain in Sec. **??** give typical ζ values between 1 and 2 for the different datasets analyzed, consistent with observed tail indices between 1.2 to 1.6 [13, 12, 9, 30]. Thus, not only does RGBM predict a realistic functional form for the distribution of rescaled wealth, but also it admits fitted parameters which match observed tail thicknesses. The inability to do the latter is a known shortcoming in models of earnings-based wealth accumulation, see Sec. **??**.

Equation (3) and extensions of it have received much attention in statistical mechanics and econophysics [7, 6]. As a combination of GBM and an Ornstein-Uhlenbeck process, it is a simple and analytically tractable stochastic process. [17] provide an overview of the literature and known results.

Negative τ For $\tau < 0$ the model exhibits mean repulsion rather than reversion. The ergodic hypothesis is invalid and no stationary wealth distribution exists. The population splits into those above the mean and those below the

[2] Strictly speaking, the large-sample approximation and resulting rescaled-wealth process, Equation (3), hold only for $\tau > \tau_c$. However, $\tau_c \approx 0$ for realistic model parameters and fits to data do not allow us to distinguish it from zero. Nonetheless, the derivation of τ_c is instructive, see Appendix **??**.

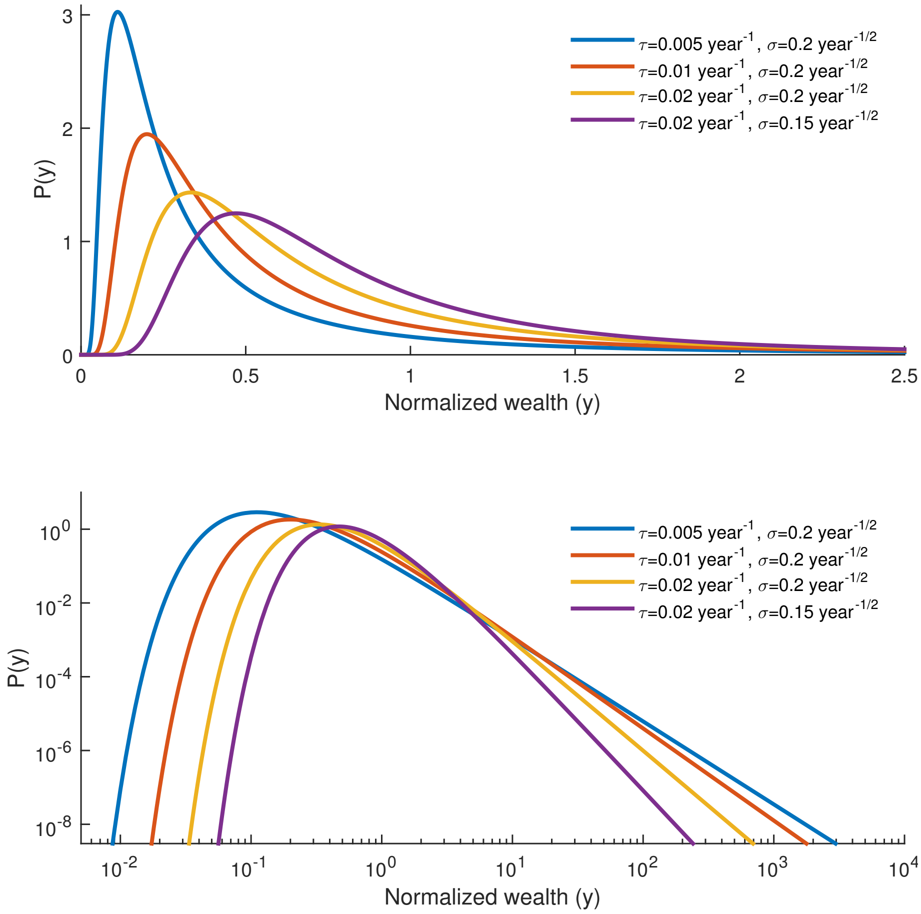


Figure 1: The stationary distribution for RGBM with positive τ. Top – linear scales; Bottom – logarithmic scales. {fig:dist}

mean. Whereas in RGBM with non-negative τ it is impossible for wealth to become negative, negative τ leads to negative wealth. No longer is total economic wealth a limit to the wealth of the richest individual because the poorest develop large negative wealth. The wealth of the rich in the population increases exponentially away from the mean, and the wealth of the poor becomes negative and exponentially large in magnitude, see Figure 2. Qualitatively, this echoes the findings that the rich are experiencing higher growth rates of their wealth than the poor [21, 31] and that the cumulative wealth of the poorest 50 percent of the American population was negative during 2008–2013 [24, 27].

Such splitting of the population is a common feature of non-ergodic processes. If rescaled wealth were an ergodic process, then individuals would, over long enough time, experience all parts of its distribution. People would spend 99 percent of their time as "the 99 percent" and 1 percent of their time as "the 1 percent". The social mobility that is, therefore, implicit in models that assume ergodicity might not exist in reality if that assumption is invalid. That inequality and immobility have been linked [10, 16, 4] may be unsurprising when both are viewed as consequences of non-ergodic wealth or income.

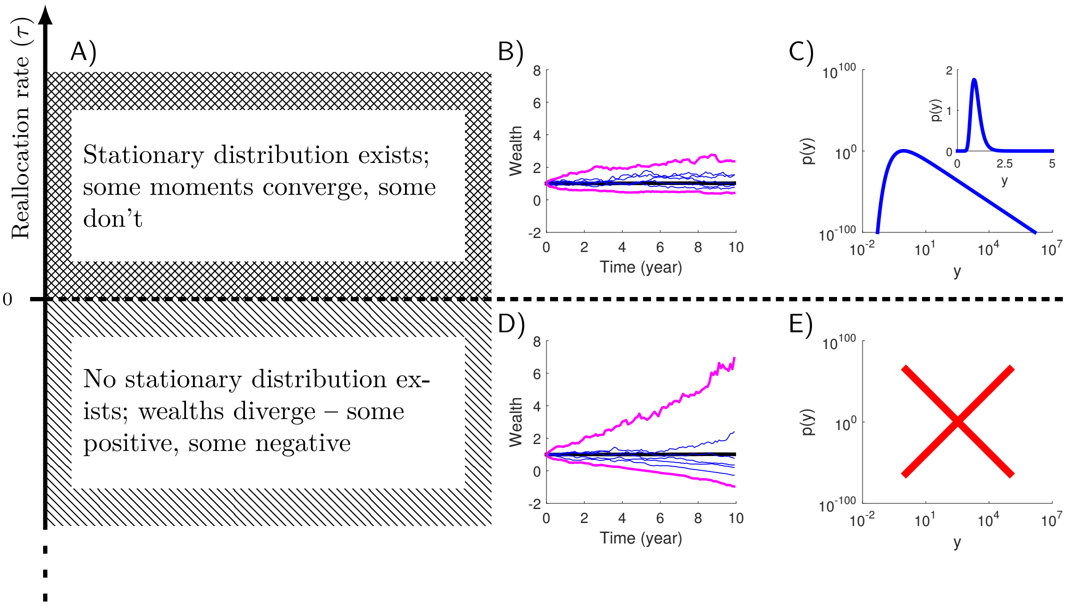


Figure 2: Regimes of RGBM. A) $\tau = 0$ separates the two regimes of RGBM. For $\tau > 0$, a stationary wealth distribution exists. For $\tau < 0$, no stationary wealth distribution exists and wealths diverge. B) Simulations of RGBM with $N = 1000$, $\mu = 0.021$ year^{-1} (presented after rescaling by $e^{\mu t}$), $\sigma = 0.14$ year$^{-1/2}$, $x_i(0) = 1$, $\tau = 0.15$ year^{-1}. Magenta lines: largest and smallest wealths, blue lines: five randomly chosen wealth trajectories, black line: sample mean. C) The stationary distribution to which the system in B) converges. Inset: same distribution on linear scales. D) Similar to B), with $\tau = -0.15$ year^{-1}. E) in the $\tau < 0$ regime, no stationary wealth distribution exists. {fig:regimes}

1.1.5 The derivation of the stationary distribution

{app:stat}

We start again with the SDE for the rescaled wealth,

$$dy = \sigma\, y\, dW - \tau\,(y-1)\,dt. \tag{5}$$

This is an Itô equation with drift term $A = \tau(y-1)$ and diffusion term $B = y\sigma$.

Such equations imply ordinary second-order differential equations that describe the evolution of the pdf, called Fokker-Planck equations. The Fokker-Planck equation describes the change in probability density, at any point in (relative-wealth) space, due to the action of the drift term (like advection in a fluid) and due to the diffusion term (like heat spreading). In this case, we have

$$\frac{dp\,(y,t)}{dt} = \frac{\partial}{\partial y}\left[Ap\,(y,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[B^2 p\,(y,t)\right]. \tag{6}$$

The steady-state Fokker-Planck equation for the pdf $p\,(y)$ is obtained by setting the time derivative to zero,

$$\frac{\sigma^2}{2}\left(y^2 p\right)_{yy} + \tau\left[(y-1)\,p\right]_y = 0. \tag{7}$$

{eq:fokker_planck}

Positive wealth subjected to continuous-time multiplicative dynamics with non-negative reallocation can never reach zero. Therefore, we solve Equation (7) with boundary condition $p(0) = 0$ to give

$$p(y) = C(\zeta)\, e^{-\frac{\zeta-1}{y}}\, y^{-(1+\zeta)}\,, \tag{8}$$

where

$$\zeta = 1 + \frac{2\tau}{\sigma^2} \tag{9}$$

and

$$C(\zeta) = \frac{(\zeta-1)^{\zeta}}{\Gamma(\zeta)}\,, \tag{10}$$

with the gamma function $\Gamma(\zeta) = \int_0^\infty x^{\zeta-1} e^{-x}\, \mathrm{d}x$. The distribution has a power-law tail as $y \to \infty$, resembling Pareto's often confirmed observation that the frequency of large wealths tends to decay as a power law. The exponent of the power law, ζ, is called the Pareto parameter and is one measure of economic inequality.

1.1.6 The derivation of the variance convergence time

{app:var_conv}

Our key finding is that under currently prevailing economic conditions it is not safe to assume the existence of stationary wealth distributions in models of wealth dynamics. Nevertheless, we present some results for the regime of our model where a stationary distribution exists. The full form of the distribution is derived in Appendix 1.1.5. Because it has a power-law tail for large wealths, only the lower moments of the distribution exist, while higher moments diverge. Below, we derive a condition for the convergence of the variance and calculate its convergence time.

The variance of y is a combination of the first moment, $\langle y \rangle$ (the average), and the second moment, $\langle y^2 \rangle$:

$$V(y) = \left\langle y^2 \right\rangle - \langle y \rangle^2 \tag{11}$$

We thus need to find $\langle y \rangle$ and $\langle y^2 \rangle$ in order to determine the variance. The first moment of the rescaled wealth is, by definition, $\langle y \rangle = 1$.

To find the second moment, we start with the SDE for the rescaled wealth:

$$dy = \sigma y\, dW - \tau(y-1)\, dt. \tag{12}$$ {eq:rescaledSDE}

This is an Itô process, which implies that an increment, df, in some (twice-differentiable) function $f(y,t)$ will also be an Itô process, and such increments can be found by Taylor-expanding to second order in dy as follows:

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial y} dy + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} dy^2\,. \tag{13}$$

We insert $f(y,t) = y^2$ and obtain

$$d\left(y^2\right) = 2y dy + (dy)^2\,. \tag{14}$$ {eq:diff2}

We substitute dy in Equation (14), which yields terms of order dW, dt, dW^2, dt^2, and $(dW\,dt)$. The scaling of Brownian motion allows us to replace dW^2 by dt, and we ignore $o\,(dt)$ terms. This yields

$$d\left(y^2\right) = 2\sigma y^2\,dW - \left(2\tau - \sigma^2\right) y^2\,dt + 2\tau y\,dt \tag{15}$$

Taking expectations on both sides, and using $\langle y \rangle = 1$, produces an ordinary differential equation for the second moment:

$$\frac{d\langle y^2 \rangle}{dt} = -\left(2\tau - \sigma^2\right)\langle y^2 \rangle + 2\tau \tag{16}$$ {eq:avediff2}

with solution

$$\langle y\,(t)^2 \rangle = \frac{2\tau}{2\tau - \sigma^2} + \left(\langle y\,(0)^2 \rangle - \frac{2\tau}{2\tau - \sigma^2} \right) e^{-\left(2\tau - \sigma^2\right)t}\,. \tag{17}$$ {eq:avediff3}

The variance $V\,(t) = \langle y\,(t)^2 \rangle - 1$ therefore follows

$$V\,(t) = V_\infty + \left(V_0 - V_\infty\right) e^{-\left(2\tau - \sigma^2\right)t}\,, \tag{18}$$ {eq:var1}

where V_0 is the initial variance and

$$V_\infty = \frac{2\tau}{2\tau - \sigma^2}\,. \tag{19}$$ {eq:varinf}

V converges in time to the asymptote, V_∞, provided the exponential in Equation (18) is decaying. This can be expressed as a condition on τ

$$\tau > \frac{\sigma^2}{2}\,. \tag{20}$$

Clearly, for negative values of τ the condition cannot be satisfied, and the variance (and inequality) of the wealth distribution will diverge. In the regime where the variance exists, $\tau > \sigma^2/2$, it also follows from Equation (18) that the convergence time of the variance is $1/\left(2\tau - \sigma^2\right)$.

As τ increases, increasingly high moments of the distribution become convergent to some finite value. The above procedure for finding the second moment (and thereby the variance) can be applied to the k^{th} moment, just by changing the second power y^2 to y^k, and any other cumulant can therefore be found as a combination of the relevant moments. For instance [17] also compute the third cumulant.

1.1.7 United States wealth data

{section:US_data}

Wealth share data We analyze the wealth shares of the top quantiles of the US population, as estimated by three sources using different methods:

- The income tax method ("capitalization method") that uses information on capital income from individual income tax returns to estimate the underlying stock of wealth [25, 27]. "If we can observe capital income $k = rW$, where W is the underlying value of an asset and r is the known rate of return, then we can estimate wealth based on capital income and capitalization factor $1/r$ defined using the appropriate choice of rate of return" [14, p. 54]. Data availability: the wealth shares of the top 5, 0.5, 0.1 and 0.01 percent for 1917–2012 and of the top 10 and 1 percent for 1913–2014 (annually).

- The estate multiplier method that uses data from estate tax returns to estimate wealth for the upper tail of the wealth distribution [15]. "The basic idea is to think of decedents as a sample from the living population. The individual-specific mortality rate m_i becomes the sampling rate. If m_i is known, the distribution for the living population can be simply estimated by reweighting the data for decedents by inverse sampling weights $1/m_i$, which are called 'estate multipliers' " [14, p. 53]. Data availability: the wealth shares of the top 1, 0.5, 0.25, 0.1, 0.05 and 0.01 percent for 1916–2000 (annually, with several missing years).

- The survey-based method that uses data from the Survey of Consumer Finances (SCF) conducted by the Federal Reserve, plus defined-benefit pension wealth, plus the wealth of the members of the Forbes 400 [8]. Data availability: the wealth shares of the top 1 and 0.1 percent for 1989–2013 (for every three years).

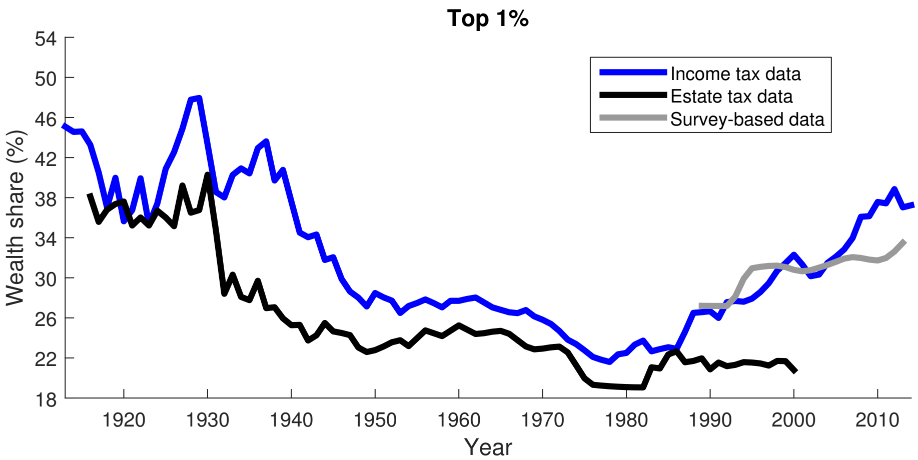
These sources are based on different datasets and for different time periods. In the overlapping periods, they sometimes report markedly different wealth share estimates (see Figure 3).

[14] reviewed the advantages and disadvantages of the different methods (see also the comment by Kopczuk on [8]). He observed that "the survey-based and estate tax methods suggest that the share of wealth held by the top 1 percent has not increased much in recent decades, while the capitalization method suggests that it has" [14, p. 48].

Which method best reflects the recent trends in wealth inequality is a matter of ongoing debate. Each method suffers from bias. For example, the survey-based method suffers from some underrepresentation of families who belong to the top end of the distribution. The income tax method suffers from some practical difficulties – "not all categories of assets generate capital income that appears on tax returns. [...] Owner-occupied housing does not generate annual taxable capital income" [14, p. 54]. The estate tax method suffers from the need to accurately estimate mortality rates for the wealthy, known to be lower than those for the rest of the population. We refer the reader to [14, 8] for a thorough discussion. We analyze each data source separately.

Wealth Growth Rate We find numerically that the results of our analysis do not depend on μ. This is because wealth shares depend only on the distribution of rescaled wealth and, for $\tau > 0$, it is possible to scale out μ completely from the wealth dynamic to obtain Equation (3) for rescaled wealth. The fitted $\tau < 0$ values we find are not large or persistent enough to make our simulations significantly μ-dependent. However, formally, since we allow negative τ, we must simulate Equation (1) and not Equation (3). This requires us to specify a value of μ, which we estimate as $\mu = 0.021 \pm 0.001$ year^{-1} by a least-squares fit of historical per-capita private wealth in the US [22] to an exponential growth curve.

Volatility We must also specify the volatility parameter, σ, in Equation (1). In principle, this can vary with time. We have no access to real individual wealth trajectories, so we resort to estimating $\sigma(t)$ from other data. We find numerically that our results are not very sensitive to the details, so we need only a good



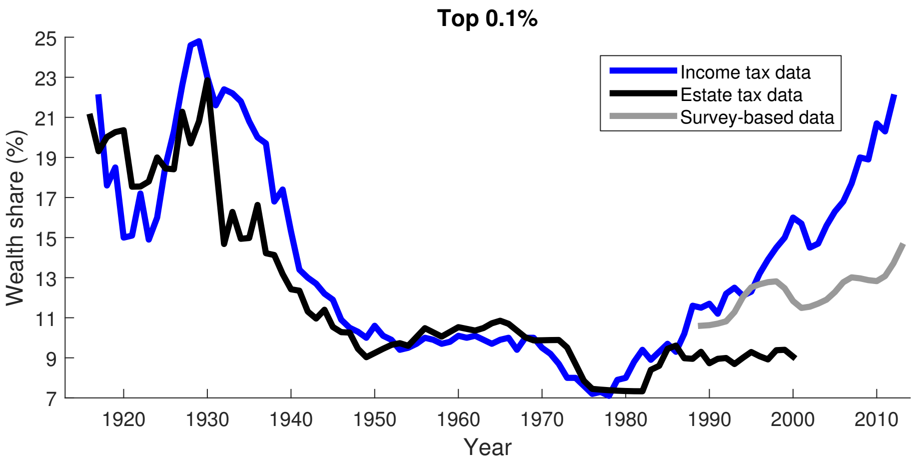


Figure 3: The top wealth shares in the US, 1913–2014. Sources – [25, 27] (blue); [15] (black); [8] (grey). {fig:data1}

"ballpark" estimate. We obtain that by assuming that the volatility in individual wealths tracks the volatility in the values of the companies that constitute the commercial and industrial base of the national economy. Therefore, for each year, we estimate $\sigma(t)$ as the standard deviation of daily logarithmic changes of the Dow Jones Industrial Average [23], which we annualise by multiplying by $(250/\text{year})^{1/2}$. The values usually lie between 0.1 and 0.2 $\text{year}^{-1/2}$, with an average of 0.16 $\text{year}^{-1/2}$. Running our empirical analysis with constant σ in this range had little effect on our results (see Appendix **??**) so, for simplicity, we present the analysis using $\sigma(t) = 0.16$ $\text{year}^{-1/2}$ for all t.

Fitting σ to stock market data means that we have only one model parameter – the effective reallocation rate, $\tau(t)$ – to fit to the historical wealth shares.

{sec:analysis}

Empirical Analysis The goal of the empirical analysis is to estimate $\tau(t)$ from the historical wealth data, using RGBM as our model. This estimation allows us to address two main questions:

1. Is it valid to assume ergodicity for the dynamics of relative wealth in the US? For the ergodic hypothesis to be valid, fitted values of $\tau(t)$ would have to be robustly positive.

2. If $\tau(t)$ is indeed positive, how long does it take for the distribution to converge to its asymptotic form?

We fit a time series, $\tau(t)$, that reproduces the annually observed wealth shares in the three datasets (see Sec. **??**): Income tax-based [25, 27], estate tax-based [15] and survey-based [8]. The wealth share, S_q, is defined as the proportion of total wealth, $\sum_i^N x_i$, owned by the richest fraction q of the population, *e.g.* $S_{10\%} = 80$ percent means that the richest 10 percent of the population own 80 percent of the total wealth.

For an empirical wealth share time series, $S_q^{\text{data}}(t)$, we proceed as follows.

– Step 1 Initialise N individual wealths, $\{x_i(t_0)\}$, as random variates of the inverse gamma distribution with parameters chosen to match $S_q^{\text{data}}(t_0)$.

– Step 2 Propagate $\{x_i(t)\}$ according to Equation (1) over Δt, using the value of τ that minimises the difference between the wealth share in the modelled population, $S_q^{\text{model}}(t + \Delta t, \tau)$, and $S_q^{\text{data}}(t + \Delta t)$. We use the Nelder-Mead algorithm [19].

– Step 3 Repeat Step 2 until the end of the time series.

We consider historical wealth shares of the richest $q = 10$, 5, 1, 0.5, 0.25, 0.1, 0.05 and 0.01 percent and obtain time series of fitted effective reallocation rates, $\tau_q(t)$, shown in Figure 4. For each value of q we perform a run of the simulation for $N = 10^8$. Since in practice dW is randomly chosen, each run of the simulation will result in slightly different $\tau_q(t)$ values. However, we found that the differences between such calculations are negligible.

Figure 4 (top) shows large annual fluctuations in $\tau_q(t)$. We are interested in longer-term changes in reallocation driven by structural economic and political changes. To elucidate these we smooth the data by taking a central 10-year moving average, $\widetilde{\tau}_q(t)$, where the window is truncated at the ends of the time series. To ensure the smoothing does not introduce artificial biases, we reverse the procedure and use $\widetilde{\tau}_q(t)$ to propagate the initially inverse gamma-distributed $\{x_i(t_0)\}$ and determine the wealth shares $S_q^{\text{model}}(t)$. The good agreement with $S_q^{\text{data}}(t)$ suggests that the smoothed $\widetilde{\tau}_q(t)$ is meaningful, see Figure 4 (bottom).

For the income tax method wealth shares [25], the effective reallocation rate, $\widetilde{\tau}(t)$, has been negative – *i.e.* from poorer to richer – since the mid-1980s. This holds for all of the inequality measures we derived from this dataset.

For the survey-based wealth shares [8], we observe briefer periods in which $\widetilde{\tau}(t) < 0$. The same is true for the estate tax data [15], see Figure 5. When $\tau(t)$ is positive, relevant convergence times are very long compared to the time scales of policy changes, namely at least several decades.

All three datasets indicate that making the ergodic hypothesis is an unwarranted restriction on models and analyses. The hypothesis makes it impossible to observe and reason about the most dramatic qualitative features of wealth dynamics, such as rising inequality, negative reallocation, negative wealth, and social immobility.

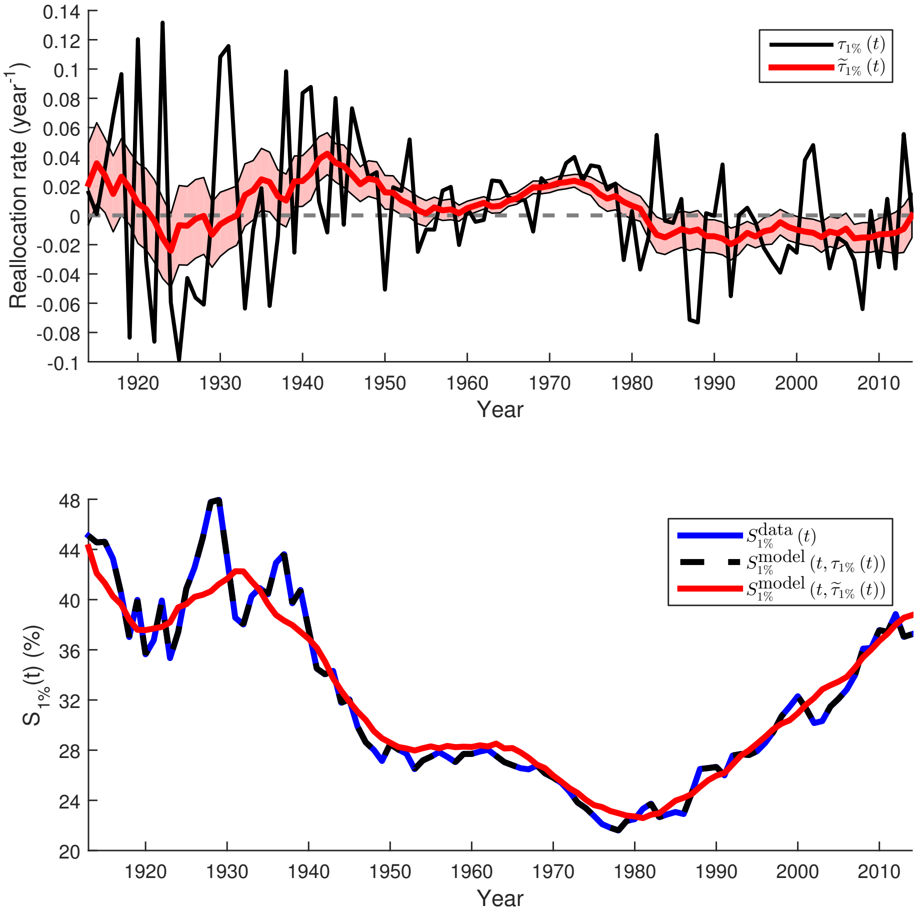


Figure 4: Fitted effective reallocation rates. Calculations done using $\mu = 0.021$ year^{-1} and $\sigma = 0.16$ year$^{-1/2}$. Top: $\tau_{1\%}(t)$ (black) and $\widetilde{\tau}_{1\%}(t)$ (red). Translucent envelopes indicate one standard error in the moving averages. Bottom: $S_{1\%}^{\mathrm{data}}$ (blue), $S_{1\%}^{\mathrm{model}}$ based on the annual $\tau_{1\%}(t)$ (dashed black), based on the 10-year moving average $\widetilde{\tau}_{1\%}(t)$ (red). {fig:tau}

Convergence times In the ergodic regime it is possible to calculate how fast the wealth shares of different quantiles converge to their asymptotic value. We do this numerically. Starting with a population of equal wealths and assuming $\mu = 0.021$ year^{-1}, $\sigma = 0.16$ year$^{-1/2}$, and $\tau = 0.04$ year^{-1}, we let the system equilibrate for 3000 years, long enough for the distribution to reach its asymptotic form to numerical precision. We then create a "shock", by changing τ to a different "shock value", and allow the system to equilibrate again for 3000 years, see top panel of Figure 6. Following the shock, the wealth shares converge to their asymptotic values. We fit this convergence numerically with an exponential function and interpret the inverse of the exponential convergence rate as the convergence time. The bottom panel of Figure 6 shows the convergence times versus the shock value of τ.

In addition, it is possible to calculate the convergence time of the variance of

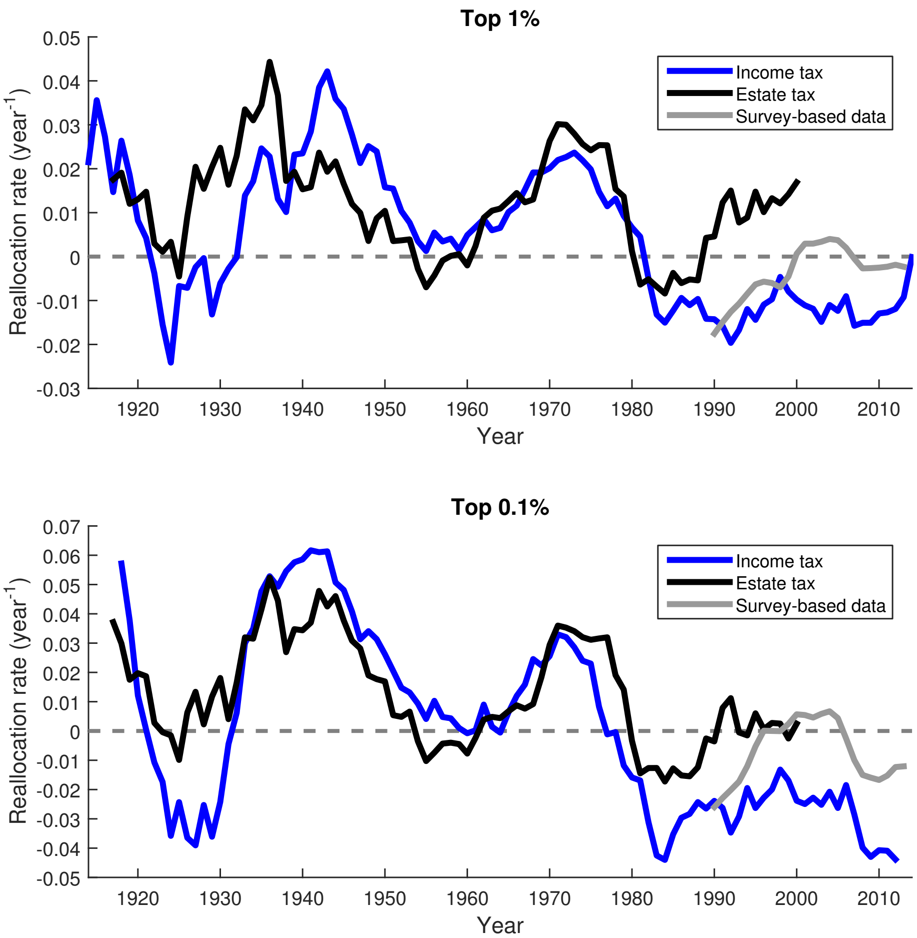


Figure 5: Effective reallocation rates for different datasets. {fig:shares_comp}

the stationary distribution (and other cumulants and moments of interest). In the ergodic regime the stationary distribution has a finite variance only if $\tau > \sigma^2/2$ [17]. Convergence of the actual variance to the stationary variance occurs exponentially over a timescale $1/(2\tau - \sigma^2)$. Figure 7 shows the convergence times for different values of σ. See Appendix 1.1.6 for more details.

Convergence times for wealth shares and variance are long, ranging from a few decades to several centuries. This implies that empirical studies which assume ergodicity and fast convergence will be inconsistent with the data. To test this, we simulate such a study by performing a different RGBM parameter fit. We find the reallocation rates, $\tau_q^{\text{eqm}}(t)$, that generate stationary distributions consistent with observed wealth shares. In other words, we assume instantaneous convergence.

Figure 8 contrasts $\tau_{1\%}^{\text{eqm}}(t)$ assuming ergodicity with $\widetilde{\tau}_{1\%}(t)$ without assuming ergodicity (using the income tax method dataset). If convergence were always possible and fast, then the two values would be identical within statistical uncertainties. They are not. In addition, the generally large discrepancies between the wealth inequality implied by $\tau_{1\%}^{\text{eqm}}(t)$ (bottom panel, Figure 8,

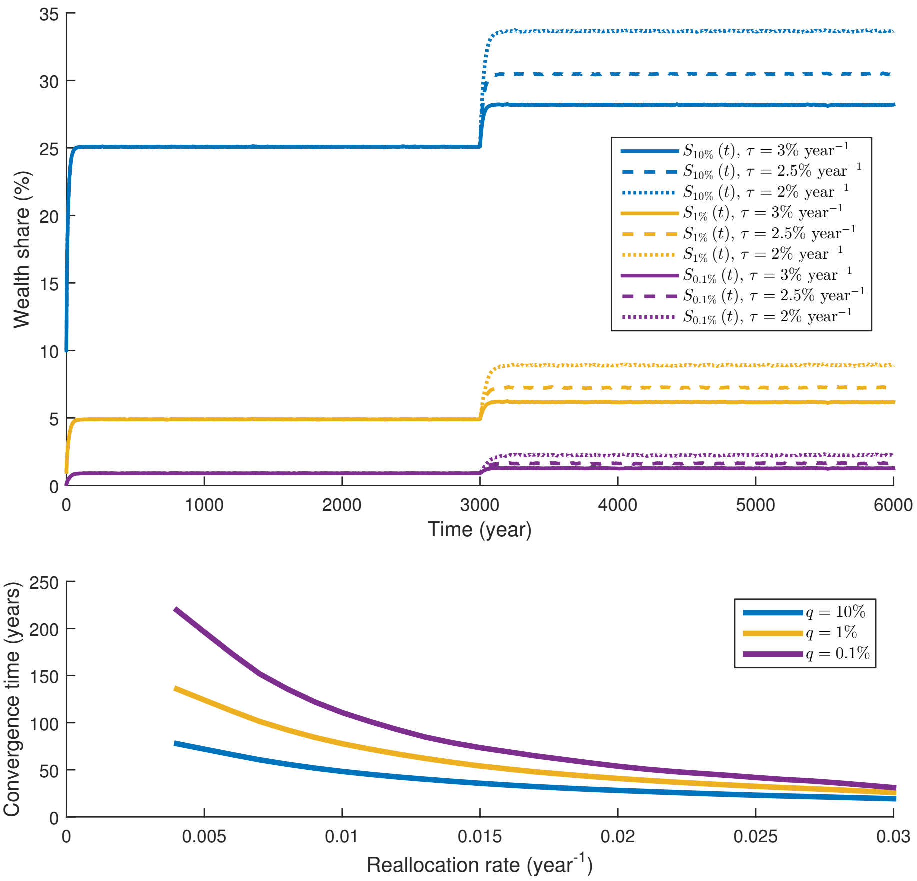


Figure 6: Wealth share convergence time. Top: The convergence of the wealth share for $q = 10$ percent (blue), $q = 1$ percent (yellow) and $q = 0.1$ percent (purple) following a change in the value of τ from 0.04 year^{-1} to 0.03 year^{-1} (solid), 0.025 year^{-1} (dashed) and 0.02 year^{-1} (dotted). Bottom: The wealth share exponential convergence time for $q = 10$ percent (blue), $q = 1$ percent (yellow) and $q = 0.1$ percent (purple) as a function of τ. {fig:conv}

green line) and as observed (bottom panel, Figure 8, blue line) indicate that the wealth distribution does not stay close to its asymptotic form. This means that the long convergence times we calculate are a practical methodological problem for conventional studies.

Conclusions Studies of economic inequality often assume ergodicity of relative wealth. This assumption also goes under the headings of equilibrium, stationarity, or stability [1]. Specifically, it is assumed that:

1. the system can equilibrate, *i.e.* a stationary distribution exists to which the observed distribution converges in the long-time limit; and
2. the system equilibrates quickly, *i.e.* the observed distribution gets close to the stationary distribution after a time shorter than other relevant

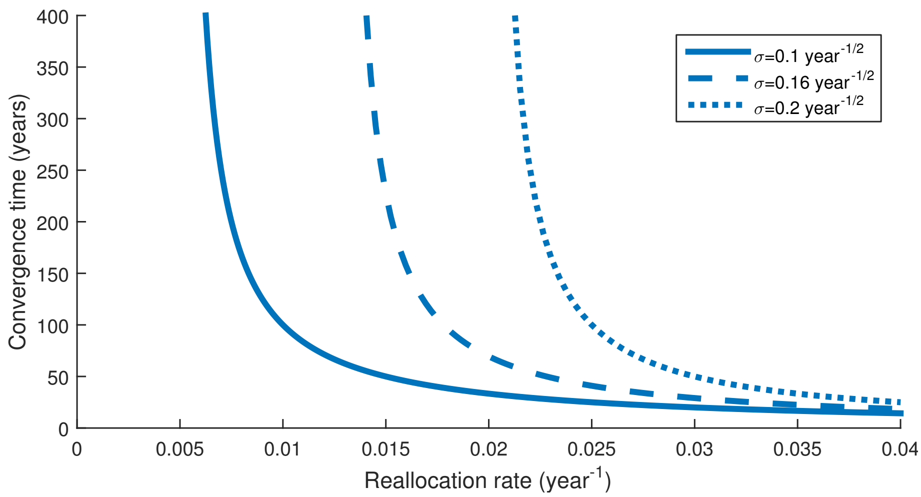


Figure 7: Variance convergence time {fig:convar}

timescales, such as the time between policy changes.

Assumption 2 is often left unstated, but it is necessary for the stationary (model) distribution to resemble the observed (real) distribution. This matters because the stationary distribution is often a key object of study – model parameters are found by fitting the stationary distribution to observed inequality, and effects of various model parameters on the stationary distribution are explored.

We do not assume ergodicity. Fitting τ in RGBM allows the data to speak without constraint as to whether the ergodic hypothesis is valid. We find it to be invalid because:

A. We observe negative τ values in all datasets analyzed, most notably using the income tax method, especially since about 1980. The wealth distribution is non-stationary and inequality increases for as long as these conditions prevail.

B. When we observe positive τ, the associated convergence times are mostly of the order of decades or centuries, see Figure 5 and Figure 6 (bottom). They are much longer than the periods over which economic conditions and policies change – they are the timescales of history rather than of politics.

The ergodic hypothesis precludes what we find. Item A above corresponds to reallocation that moves wealth from poorer to richer individuals, which is inconsistent with the ergodic hypothesis. In this sense the ergodic hypothesis is a set of blindfolds, hiding from view the most dramatic economic conditions. For the most recent data, the system is in a state best described by non-ergodic RGBM, $\tau < 0$ [25, 27] or $\tau \approx 0$ [8]. Therefore, each time we observe the wealth distribution, we see a snapshot of it either in the process of diverging or very far from its asymptotic form. It is much like a photograph of an explosion in space: it will show a fireball whose finite extent tells us nothing of the eventual distance between pieces of debris.

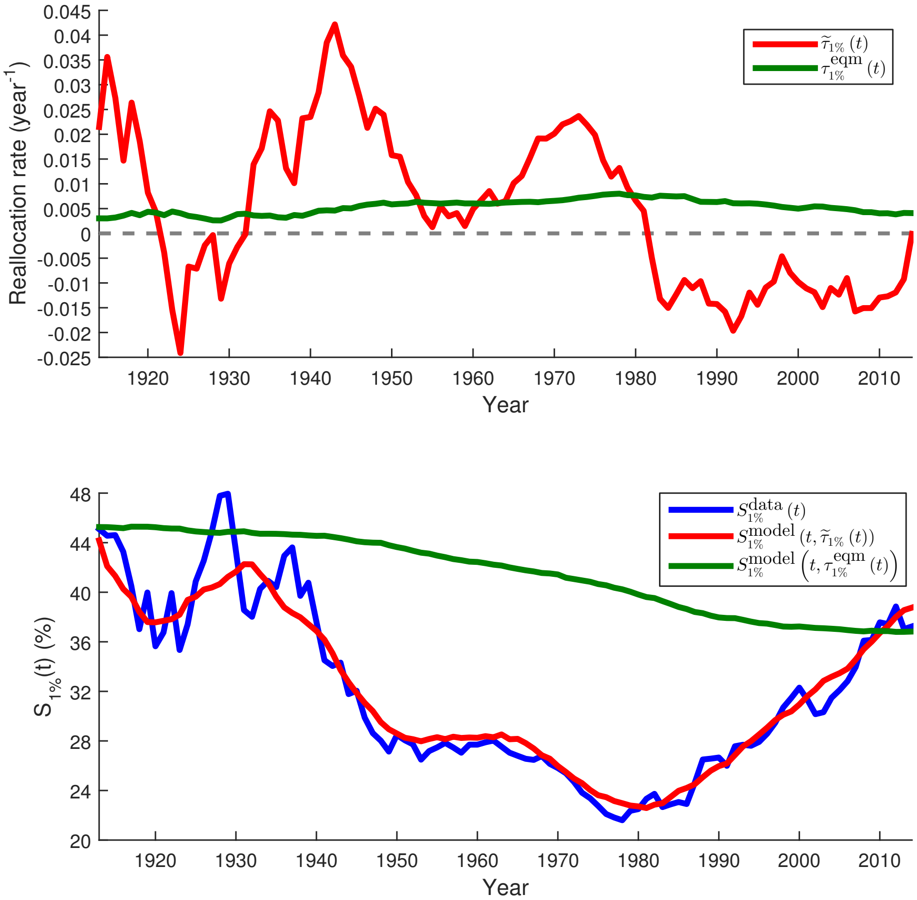


Figure 8: Comparison of dynamic and equilibrium reallocation rates. Top: $\widetilde{\tau}_{1\%}(t)$ (red, same as in the top of Figure 4). $\tau^{\text{eqm}}_{1\%}(t)$ (green), defined such that $\lim_{t' \to \infty} S^{\text{model}}_{1\%}\left(t', \tau^{\text{eqm}}_{1\%}(t)\right) = S^{\text{data}}_{1\%}(t)$. It is impossible by design for this value to be negative. The significant difference between the red and green lines demonstrates that the fast convergence assumption is invalid for the problem under consideration. Bottom: $S^{\text{data}}_{1\%}$ (blue), $S^{\text{model}}_{1\%}$ based on the 10-year moving average $\widetilde{\tau}_{1\%}(t)$ (red), based on $\tau^{\text{eqm}}_{1\%}(t)$ (green). The reallocation rates found under the fast convergence assumption generate model wealth shares which bear little relation to reality. {fig:asymptau}

We also find that changes in the earnings distribution do not provide an adequate alternative explanation of the described dynamics of the wealth distribution. Although earnings have become more unequal over the recent decades in which wealth inequality has increased, their effect on the wealth distribution has been small and generally stabilizing rather than destabilizing. Treating earnings explicitly in our model does not change fundamentally our conclusions.

The economic phenomena that trouble theorists most – such as diverging inequality, social immobility, and the emergence of negative wealth – are difficult to reproduce in a model that assumes ergodicity. In our simple model, this is easy to see: in the ergodic regime, $\tau > 0$, our model cannot reproduce these phenomena at all. One may be tempted to conclude that their existence is a sign of special conditions prevailing in the real world – collusion and conspiracies. But if we admit the possibility of non-ergodicity, $\tau \leq 0$, it becomes clear that these phenomena can easily emerge in an economy that does not actively guard against them.

List of Symbols

Δt A general time interval..

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