Ergodicity Economics

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Contents

1 Tossing a coin

In this introductory chapter we lay the conceptual foundation for the rest of what we have to say. We play a simple coin-toss game and analyze it by Monte Carlo simulation and with pen and paper. The game motivates the introduction of the expectation value and the time average, which in turn lead to a discussion of ergodic properties. We note the importance of rates of change and introduce Brownian motion and Geometric Brownian motion. Some historical perspective is provided to understand the prevalence or absence of key concepts in modern economic theory and other fields. The emphasis is on concepts, with more formal treatments and applications in later chapters.

1.1 The game

Imagine we offer you the following game: we toss a coin, and if it comes up heads we increase your monetary wealth by 50%; if it comes up tails we reduce your wealth by 40%. We're not only doing this once, we will do it many times, for example once per week for the rest of your life. Would you accept the rules of our game? Would you submit your wealth to the dynamic our game will impose on it?

Your answer to this question is up to you and will be influenced by many factors, such as the importance you attach to wealth that can be measured in monetary terms, whether you like the thrill of gambling, your religion and moral convictions and so on.

In these notes we will mostly ignore these. We will build an extremely simple model of your wealth, which will lead to an extremely simple powerful model of the way you make decisions that affect your wealth. We are interested in analyzing the game mathematically, which requires a translation of the game into mathematics. We choose the following translation: we introduce the key variable, $x(t)$, which we refer to as "wealth". We refer to t as "time". It should be kept in mind that "wealth" and "time" are just names that we've given to mathematical objects. We have chosen these names because we want to compare the behaviour of the mathematical objects to the behaviour of wealth over time, but we emphasize that we're building a model – whether we write $x(t)$ or wealth(time) makes no difference to the mathematics.

The usefulness of our model will be different in different circumstances, ranging from completely meaningless to very helpful. There is no substitute for careful consideration of any given situation, and labeling mathematical objects in one way or another is certainly none.

Having got these words of warning out of the way, we define our model of the dynamics of your wealth under the rules we specified. At regular intervals of duration δt we randomly generate a factor $r(t)$ with each possible value $r_i \in \{0.6, 1.5\}$ occurring with probability 1/2,

$$r(t) = \begin{cases} 0.6 & \text{with probability } 1/2 \\ 1.5 & \text{with probability } 1/2 \end{cases} \tag{1}$$

and multiply current wealth by that factor,

$$x(t + \delta t) = r(t)x(t). \tag{2}$$

Without discussing in depth how realistic a representation of your wealth this model is (for instance your non-gambling related income and spending are not represented in the model), and without discussing whether randomness truly exists and what the meaning of a probability is we simply switch on a computer and simulate what might happen. You may have many good ideas of how to analyze our game with pen and paper, but we will just generate possible trajectories of your wealth and pretend we know nothing about mathematics or economic theory. Figure 1 is a trajectory of your wealth, according to our computer model as it might evolve over the course of 52 time steps (corresponding to one year given our original setup).

A cursory glance at the trajectory does not reveal much structure. Of course there are regularities, for instance at each time step $x(t)$ changes, but no trend

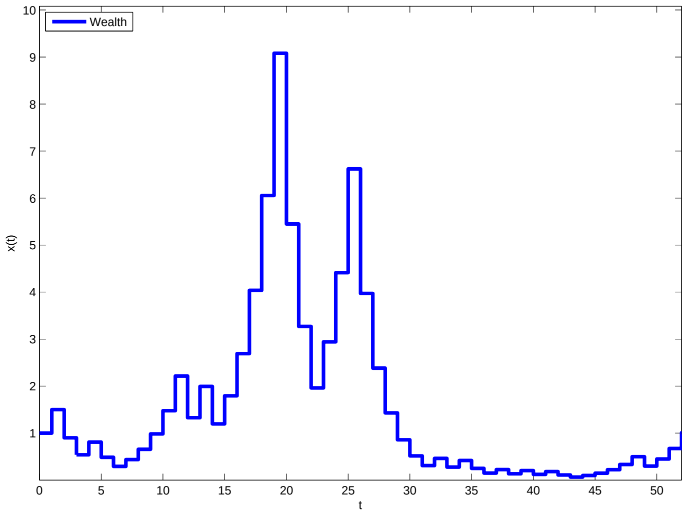


Figure 1: Wealth $x(t)$ resulting from a computer simulation of our game, repeated 52 times.

is discernible – does this trajectory have a tendency to go up, does it have a tendency to go down? Neither? What are we to learn from this simulation? Perhaps we conclude that playing the game for a year is quite risky, but is the risk worth taking?

1.1.1 Averaging over many trials

A single trajectory doesn't tell us much about overall tendencies. There is too much noise to discern a clear signal. A common strategy for getting rid of noise is to try again. And then try again and again, and look at what happens on average. For example, this is very successful in imaging – the 2014 Nobel Prize in chemistry was awarded for a technique that takes a noisy image again and again. By averaging over many images the noise is reduced and a resolution beyond the diffraction limit is achieved.

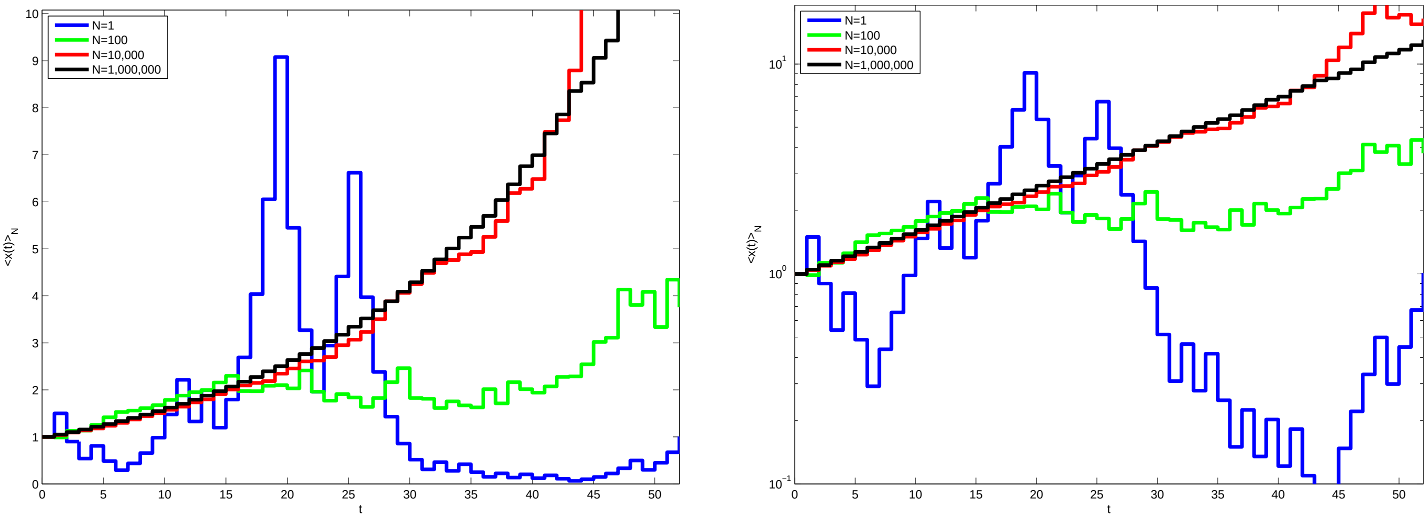
So let's try this in our case and see if we can make sense of the game. In Fig. 2 we average over finite ensembles of N trajectories, that is, we plot the finite-ensemble average.

DEFINITION: **Finite-ensemble average**

The finite-ensemble average of the observable A is

$$\langle A(t)\rangle_N = \frac{1}{N}\sum_i^N A_i(t), \tag{3}$$

where i indexes a particular realization of A and N is the number of realizations included in the average.



(A) (B)

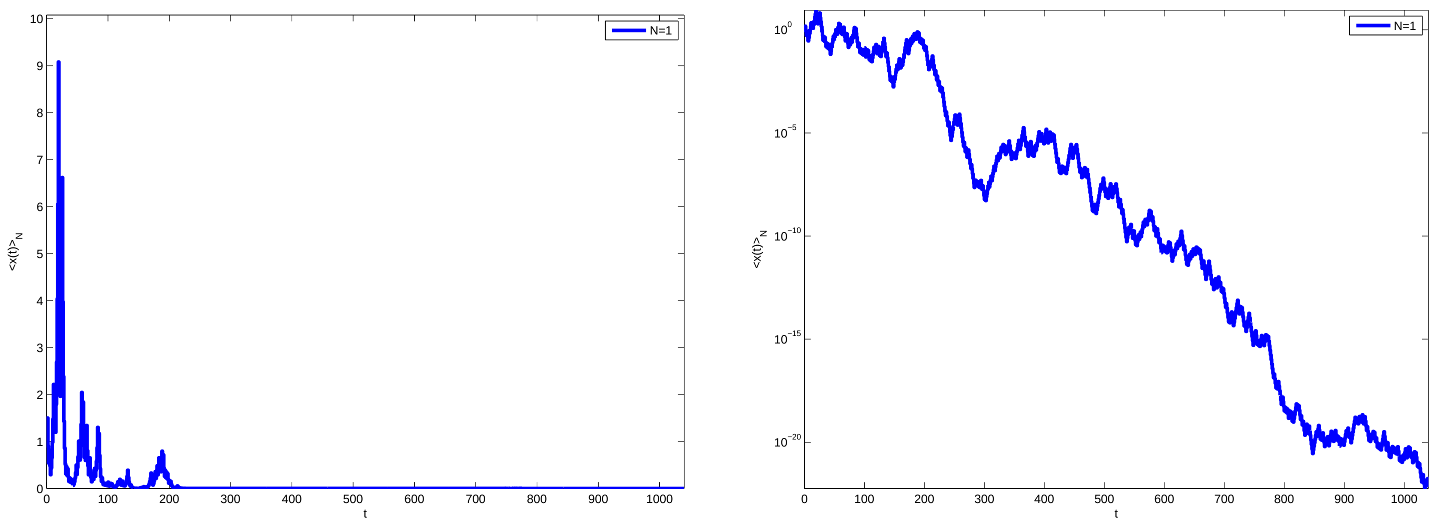
Figure 2: Partial ensemble averages $\langle x(t) \rangle_N$ for ensemble sizes $N = 1, 10^2, 10^4, 10^6$. (A) linear scales, (B) logarithmic scales.

As expected, the more trajectories are included in the average, the smaller the fluctuations of that average. For $N = 10^6$ hardly any fluctuations are visible. Since the noise-free trajectory points up it is tempting to conclude that the risk of the game is worth taking. This reasoning has dominated economic theory for about 350 years now. But it is flawed.

1.1.2 Averaging over time

Does our analysis necessitate the conclusion that the gamble is worth taking? Of course it doesn't, otherwise we wouldn't be belabouring this point. Our critique will focus on the type of averaging we have applied – we didn't play the game many times in a row as would correspond to the real-world situation of repeating the game once a week for the rest of your life. Instead we played the game many times in parallel, which corresponds to a different setup.

We therefore try a different analysis. Figure 3 shows another simulation of your wealth, but this time we don't show an average over many trajectories but a simulation of a single trajectory over a long time. Noise is removed also in this case but in a different way: to capture visually what happens over a long time we have to zoom out – more time has to be represented by the same amount of space on the page. In the process of this zooming-out, small short-time fluctuations will be diminished. Eventually the noise will be removed from the system just by the passage of time.



(A) (B)

Figure 3: Single trajectory over 1,040 time units, corresponding to 20 years in our setup. (A) linear scales, (B) logarithmic scales.

Of course the trajectory in Fig. 3 is random, but the apparent trend emerging from the randomness strongly suggests that our initial analysis does not reflect what happens over time in a single system. Jumping ahead a little, we reveal that this is indeed the case. If it seems counter-intuitive then this is because our intuition is built on so-called "ergodic processes", whereas x is non-ergodic. We will say more about this in Sec. 1.1.5. Several important messages can be derived from the observation that an individual trajectory grows more slowly over time than an average of a large ensemble.

1. An individual whose wealth follows (Eq. 2) will make poor decisions if he uses the finite-ensemble average of wealth as an indication of what is likely to happen to his own wealth.

2. The performance of the average (or aggregate) wealth of a large group of individuals differs systematically from the performance of an individual's wealth. In our case large-group wealth grows (think Gross domestic product (GDP)), whereas individual wealth decays.

3. For point 2 to be possible, *i.e.* for the average to outperform the typical individual, wealth must become increasingly concentrated in a few extremely rich individuals. The wealth of the richest individuals must be so large that the average becomes dominated by it, so that the average can grow although almost everyone's wealth decays. Inequality increases in our system.

The two methods we've used to eliminate the noise from our system are well known. The first method is closely related to the mathematical object called the expectation value, and the second is closely related to the object called the time average.

1.1.3 Expectation value

In this section we validate Fig. 2 by computing analytically the average of x over infinitely many trials, a quantity known as the expectation value. The expectation value is usually introduced as the sum of all possible values, weighted by their probabilities. We will define it as a limit instead, and then show that this limit is identical to the familiar expression.

Definition: Expectation value i

The expectation value of a quantity x is the large-ensemble limit of the finite-ensemble average (Eq. 3),

$$\langle x \rangle = \lim_{N\to\infty} \langle x \rangle_N \,. \tag{4}$$

This implies that in our first analysis of the problem – by averaging over N trajectories – we were approximately using the expectation value as a gauge of the desirability of the game. We will now prove that letting $N \to \infty$ is indeed the same as working with the more familiar definition of the expectation value.

Definition: Expectation value ii

The expectation value of a quantity x that can take discrete values x_j is the sum of all possible values weighted by their probabilities p_j

$$\langle x \rangle = \sum_j p_j x_j. \tag{5}$$

If x is continuous, the expectation value is the integral

$$\langle x \rangle = \int_{-\infty}^{+\infty} s\mathcal{P}_x(s) ds, \tag{6}$$

where $\mathcal{P}_x(s)$ is the Probability density function (PDF) of the random variable x at value s.

We now show that the two definitions of the expectation value are equivalent.

Proof. Consider the number of times the value x_j is observed at time t in an ensemble of N trajectories. Call this n_j. The finite-ensemble average can then be re-written as

$$\begin{aligned} \langle x(t) \rangle_N &= \frac{1}{N} \sum_i x_i(t) && (7) \\ &= \sum_j \frac{n_j}{N} x_j(t), && (8) \end{aligned}$$

where the subscript i indexes a particular realization of x, and the subscript j indexes a possible value of x. The fraction $\frac{n_j}{N}$ in the limit $N \to \infty$ is the probability p_j, and we find

$$\lim_{N\to\infty} \langle x(t) \rangle_N = \sum_j p_j x_j(t) \tag{9}$$

The Left-hand side (LHS) is the expectation value by the first definition as a limit, the Right-hand side (RHS) is the expectation value by the second definition as a weighted sum. This shows that the two definitions are indeed equivalent. □

We will use the terms "ensemble average" and "expectation value" as synonyms, carefully using the term "finite-ensemble average" for finite N.

We pretended to be mathematically clueless when carrying out the simulations, with the purpose to gain a deeper conceptual understanding of the expectation value. We now compute the expectation value exactly instead of approximating it numerically. Consider the expectation value of (Eq. 2)

$$\langle x(t+\delta t) \rangle = \langle x(t) r(t) \rangle \,. \tag{10}$$

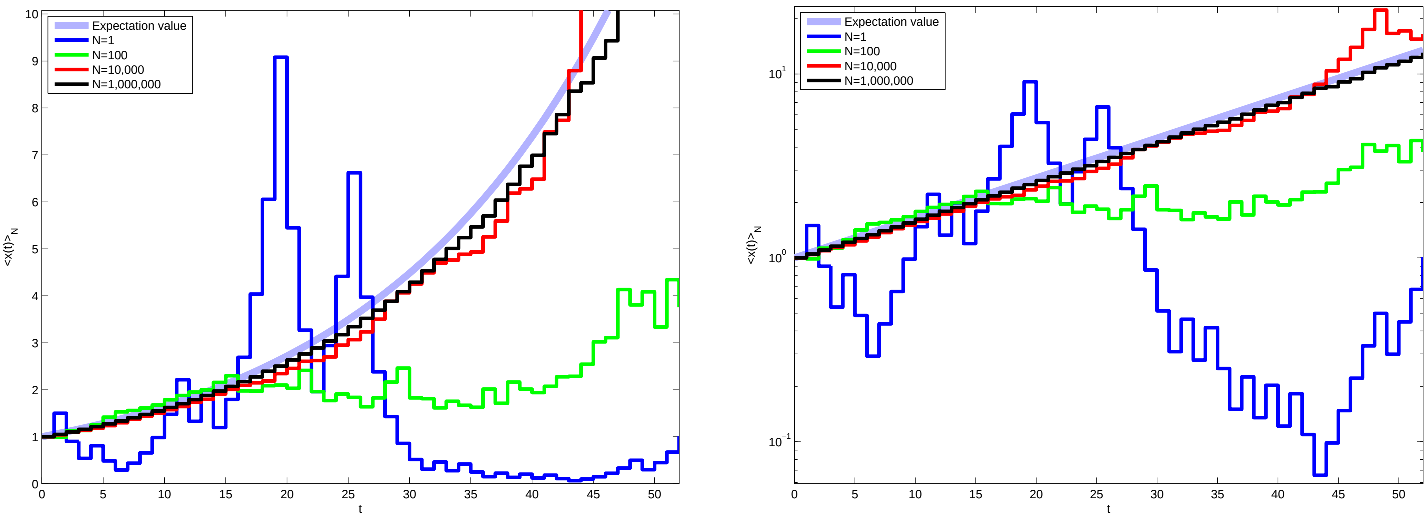
Since $r(t)$ is independent of $x(t)$ (we generate $r(t)$ independently of $x(t)$ in each time step), this can be re-written as

$$\langle x(t+\delta t) \rangle = \langle x(t) \rangle \langle r \rangle \,, \tag{11}$$

wherefore we can solve recursively for the wealth after T rounds, corresponding to a playing time of $\Delta t = T\delta t$,

$$\langle x(t+T\delta t) \rangle = x(t) \langle r \rangle^T \,. \tag{12}$$

The expectation value $\langle r \rangle$ is easily found from (Eq. 1) as $\langle r \rangle = \frac{1}{2} \times 0.6 + \frac{1}{2} \times 1.5 = 1.05$. Since this number is greater than one, $\langle x(t) \rangle$ grows exponentially in time at rate 1.05 per time unit, or expressed as a continuous growth rate, at $\frac{1}{\delta t} \ln \langle r \rangle \approx 4.9\%$ per time unit. This is what might have led us to conclude that the gamble is worth taking. Figure 4 compares the analytical result for the infinite ensemble to the numerical results of Fig. 2 for finite ensembles.



(A) (B)

Figure 4: Expectation value (thick light blue line) finite-ensemble averages. (A) linear scales, (B) logarithmic scales.

We stress that the expectation value is just some mathematical object – someone a long time ago gave it a suggestive name, but we certainly shouldn't give any credence to a statement like "we expect to see $\langle x \rangle$ because it's the expectation value." Mathematical objects are quite indifferent to the names we give them.

History: The invention of the expectation value

Expectation values were not invented in order to assess whether a gamble is worth taking. Instead, they were developed to settle a moral question that arises in the following somewhat contrived context: Imagine playing a game of dice with a group of gamblers. The rules of the game are simple: we roll the dice three times, and whoever rolls the most points gets the pot to which we've all contributed equal amounts. We've already rolled the dice twice when suddenly the police burst in because they've heard of our illegal gambling ring. We all avoid arrest, most of us escape through the backdoor, and to everyone's great relief you had the presence of mind to grab the pot before jumping out of a conveniently located ground-floor window. Later that day, under the cover of dusk, we meet behind the old oak tree just outside of town to split the pot in a fair way. But hold on – what does "fair" mean here? Some of us had acquired more points than others in the first two rolls of the dice. Shouldn't they get more? The game was not concluded, so wouldn't it be fair to return to everyone his wager and thank our lucky stars that we weren't arrested? Should we split the pot in proportion to each player's points? All of these solutions were proposed [8]. The question is fundamentally moral, and there is no mathematical answer. But Blaise Pascal, now famous for addressing theological questions using expectation values, put the problem to Fermat, and over the course of a few months' correspondence (the two never met in person) Fermat and Pascal agreed that fairness is achieved as follows: Imagine all (equally likely) possible outcomes of the third round of throws of the dice, call the number of all possibilities N. Now count those possibilities that result in player j winning, call this n_j. If D is the amount of money in the pot, then we split the pot fairly by giving each player $\frac{n_j}{N} \times D$. This is $\langle \Delta x \rangle$, according to (Eq. 5) because $\frac{n_j}{N} = p_j$ is the probability that player j wins the amount D. Later researchers called this amout the "mathematical expectation" or simply "expectation value". But this is really an unfortunate choice – no player "expected" to receive $\langle \Delta x \rangle$. Instead, each player expected to receive either nothing or D.

1.1.4 Time average

In this section we validate Fig. 3 and compute analytically what happens in the long-time limit. The blue line in Fig. 3 is not completely smooth, there's still some noise. It has some average slope, but that slope will vary from realisation to realisation. The longer we observe the system, *i.e.* the more time is represented in a figure like Fig. 3, the smoother the line will be. In the long-time limit, $\Delta t \to \infty$, the line will be completely smooth, and the average slope will be a deterministic number – in any realization of the process it will come out identical.

The dynamic is set up such that wealth at time t is

$$x(t + \Delta t) = x(t) \prod_{\tau=1}^{T} r(\tau), \tag{13}$$

which we can split up into two products, one for each possible value of $r(\tau)$, which we call r_1 and r_2. Let's denote the number of r_1s by n_1 and r_2s by n_2, so that

$$x(t + \Delta t) = x(t) r_1^{n_1} r_2^{n_2}. \tag{14}$$

We denote by r_{time} the factor by which $x(t)$ changes per round when the change is computed over a long time. This quantity is found by taking the t^{th} root of $\frac{x(t+\Delta t)}{x(t)}$ and considering the long-time limit

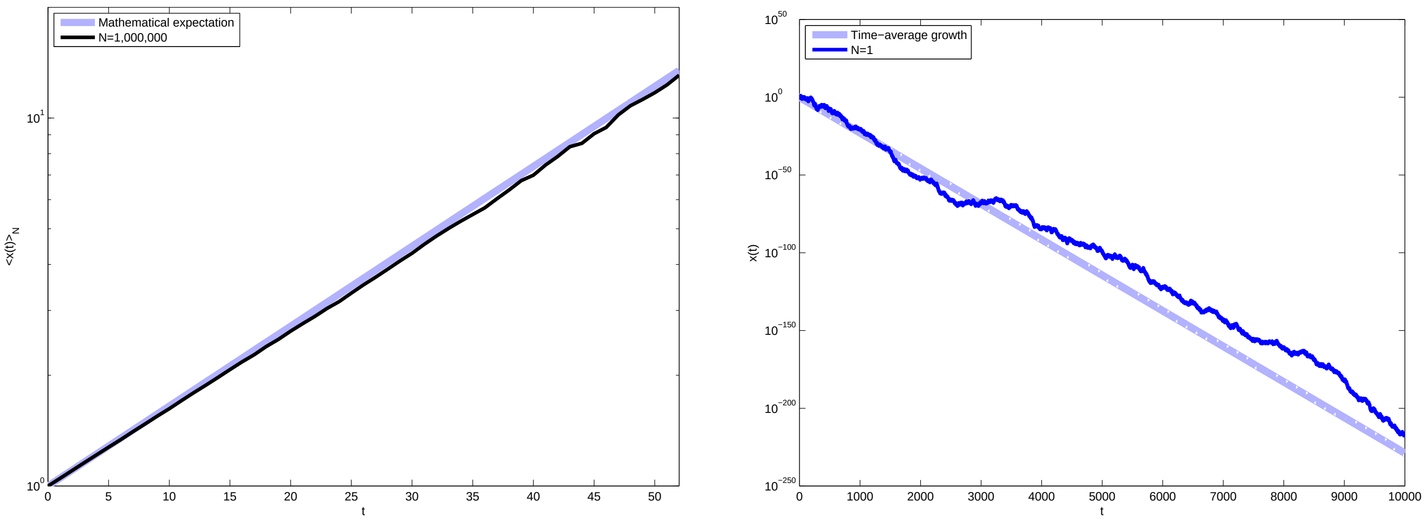
$$\begin{aligned} r_{\text{time}} &= \lim_{\Delta t \to \infty} \left(\frac{x(t + \Delta t)}{x(t)} \right)^{1/t} & (15) \\ &= \lim_{T \to \infty} r_1^{n_1/T} r_2^{n_2/T}. & (16) \end{aligned}$$

Identifying $\lim_{T \to \infty} n_1/T$ as the probability p_1 for r_1 to occur (and similarly $\lim_{T \to \infty} n_2/T = p_2$) this is

$$\lim_{T \to \infty} \left(\frac{x(t + T\delta t)}{x(t)} \right)^{1/T} = (r_1 r_2)^{1/2}, \tag{17}$$

or $\sqrt{0.9} \approx 0.95$, *i.e.* a number smaller than one, reflecting decay in the long-time limit for the individual trajectory. The trajectory in Fig. 3 was not a fluke: *every* trajectory will decay in the long run at a rate of $(r_1 r_2)^{1/2}$ per round.

Figure 5 (B) compares the trajectory generated in Fig. 3 to a trajectory decaying exactly at rate r_{time} and places it next to the average over a million systems.



(A) (B)

Figure 5: (A) Finite-ensemble average for $N = 10^6$ and 52 time steps, the light blue line is the expectation value.[2] (B) A single system simulated for 10,000 time steps, the light blue line decays exponentially with the time-average decay factor r_{time} in each time step.

Excursion: Scalars

$r(t)$ is a random variable, whereas both $r_{\langle\rangle}$ and r_{time} are scalars. Scalars have the so-called "transitive property" that is heavily relied upon in economic theory. Let a_i be a set of scalars. Transitivity means that if $a_1 > a_2$ and $a_2 > a_3$ we have $a_1 > a_3$. Notice that we cannot rank random variables in such a way. The "greater than" relation, $>$, is not defined for a pair of random variables, which is the mathematical way of saying that it is difficult to choose between two gambles, and it is why we went to the trouble

[2] In Fig. 5 (A) a slight discrepancy between the expectation value and the $N = 10^6$ finite-ensemble average is visible, especially for later times. This is not coincidence – in the long run, also the finite-ensemble average for 10^6 systems will decay at the time-average growth rate. A simple proof for this for the continuous process is in [29]. Moreover, much is known about the distribution for $\langle x(t)\rangle_N$ for any N and t (because it happens to be mathematically identical to the partition function of the random energy model, introduced and solved by Derrida [7]. We thank J.-P. Bouchaud for pointing this out to us).

of removing the randomness from the stochastic process $x(t)$. Removing randomness by averaging always involves a limiting process, and results are said to hold "with probability one". In the case of $r_{\langle\rangle}$ we considered the infinite-ensemble limit, $N \to \infty$, and in the case of r_{time} we considered the infinite-time limit, $\Delta t \to \infty$. If we use the scalars a_i to represent preferences, we can test for consistency among preferences. For instance, in such a model world where preferences are represented by scalars, the facts that "I prefer kangaroos to Beethoven" and "I prefer mango chutney to kangaroos" imply the fact "I prefer mango chutney to Beethoven". Translating back to reality, economists like to call individuals who make the first two statements but not the third "irrational."

Because transitivity makes for a nicely ordered world, it is useful to find scalars to represent preferences. We are skeptical about the attempt to map all preferences into scalars because the properties of mango chutney are too different, *qualitatively*, from the properties of Beethoven. We will restrict our analysis to money – the amount of money we will receive is random and this introduces a complication, but at least we know how to compare one amount to another in the limit of no randomness – there is no qualitative differences between \$1 and \$3, only a quantitative difference.

Both $r_{\langle\rangle}$ and r_{time} are scalars, and both are therefore potentially powerful representations of preferences. Your decision whether to accept our gamble could now be modelled as a choice between the value of the scalar r_{time} if you do not accept our game, namely $a_1 = 1$, and the value of the scalar r_{time} if you do accept, namely approximately $a_2 = 0.95$. In this model of your decision-making you would prefer not to play because $1 > 0.95$.

We have two averages, $r_{\langle\rangle}$ and r_{time} that we have determined numerically and analytically. Neither average is "wrong" in itself; instead each average corresponds to a different property of the system. Each average is the answer to a different question. Saying that "wealth goes up, on average" is clearly meaningless and should be countered with the question "on what type of average?"

History: William Allen Whitworth

$r_{\langle\rangle}$ and r_{time} are two different properties of the game. $r_{\langle\rangle}$ is the large-ensemble limit, r_{time} is the long-time limit, of wealth growth it induces. The Victorian mathematician William Allen Whitworth was aware that often r_{time} is the relevant property for an individual deciding whether to take part in a repeated gamble. He used this knowledge to write an appendix entitled "The disadvantage of gambling" to the 1870 edition of his book "Choice and Chance" [35]. He phrased his argument in terms of the difference of two squares. Imagine that you either win or lose, with equal probability, an amount $\epsilon x(t)$ in each round of a game. In the long run, positive and negative changes will occur equally frequently, and to determine the overall effect we just need to consider the effect of one positive and one negative change in a row. Over one up and one down-move wealth changes by the factor

$$(1+\epsilon)(1-\epsilon) = 1-\epsilon^2. \tag{18}$$

This factor is clearly less than one, meaning that what's often called a "fair gamble" – one that does not change the expectation value of the gambler's wealth – leads to an exponential decay of his wealth over time. Hence the title of the appendix "The disadvantage of gambling." We will see in Sec. 1.4 that Whitworth's work captured the essence of Itô's famous 1944 discovery [12] that was to form the basis of much of financial mathematics.

Whitworth was arguing against a dogma of expectation values of wealth, that had been established almost immediately following Fermat and Pascal's work. He hoped to show mathematically that gambling may not be a good idea even if the odds are favourable, and was a proponent of the notion that commerce should and does consist of mutually beneficial interactions rather than one winner and one loser. In the end his voice was not heard in the economics or mathematics communities. He quit mathematics to become a priest at All Saints Church in London's Margaret Street, only a 22 minute stroll away from the London Mathematical Laboratory, according to Google.

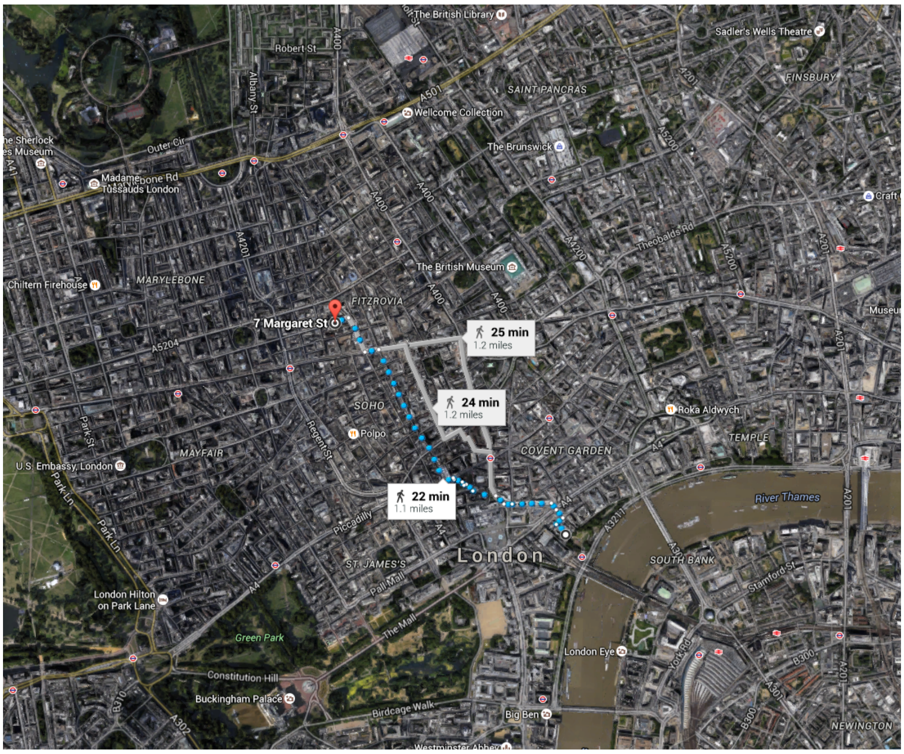


Figure 6: Location of All Saints Church and the London Mathematical Laboratory.

An observable that neatly summarises the two different aspects of multiplicative growth we have illustrated is the multiplicative (exponential) growth rate, observed over finite time Δt, in a finite ensemble of N realisations

$$g_{\mathrm{m}}(\langle x \rangle_N, \Delta t) = \frac{\delta \ln \langle x \rangle_N}{\Delta t}. \tag{19}$$

For $N, \Delta t$ finite this is a random variable. The relevant scalars arise as two different limits of the same stochastic object. The exponential growth rate of the expectation value (that's also $\frac{1}{\delta t} \ln r_{\langle\rangle}$) is

$$g_{\langle\rangle} = \lim_{N\to\infty} g_{\mathrm{m}}, \tag{20}$$

and the exponential growth rate followed by every trajectory when observed for a long time (that's also $\frac{1}{\delta t} \ln r_{\mathrm{time}}$) is

$$\bar{g} = \lim_{\Delta t\to\infty} g_{\mathrm{m}}. \tag{21}$$

We can also write (Eq. 19) as a sum of the logarithmic differences in the T individual rounds of the gamble that make up the time interval $\Delta t = T\delta t$

$$g_{\mathrm{m}}(\langle x\rangle_N, \Delta t) = \frac{1}{\delta T} \sum_{\tau=1}^{T} \Delta \ln \langle x\rangle_N (\tau). \tag{22}$$

This leads us to a technical definition of the time average.

Definition: **Finite-time average**

If the observable A only changes at $T = \Delta t/\delta t$ discrete times $\delta t, 2\delta t$ *etc.*, then the "finite-time average" is

$$\overline{A}_{\Delta t} = \frac{1}{T} \sum_{\tau=1}^{T} A(\tau\delta t). \tag{23}$$

If it changes continuously the finite-time average is

$$\overline{A}_{\Delta t} = \frac{1}{\Delta t} \int_0^{\Delta t} A(s) ds. \tag{24}$$

Definition: **Time average**

The "time average" is the long-time limit of the finite-time average

$$\overline{A} = \lim_{\Delta t\to\infty} \overline{A}_{\Delta t}. \tag{25}$$

According to this definition, $\bar{g}$ is the time average of the observable $\frac{\delta \ln x}{\delta t}$. It can be shown that the time-average growth rate of a single trajectory is the same as that of a finite-ensemble average of trajectories, $\lim_{\Delta t\to\infty} \frac{\Delta \ln x}{\Delta t} = \lim_{\Delta t\to\infty} \frac{\Delta \ln \langle x\rangle_N}{\Delta t}$, [29].

Excursion: Dimensional analysis

We will often and without qualm write the expression $\Delta \ln x$. Dimensional analysis suggests to think about this expression carefully, at least once. This may seem pedantic but the absence of this pedantry has caused sufficient confusion in economic theory for us to risk antagonizing you. "Dimension" in this context is closely related to the concept of "unit" – for instance, a dollar is a money unit, and the dimension function for money tells us how to convert from one currency into another. Similarly, length may have the unit "meter", and the dimension function for length tells us how to convert between different systems of units, such as meters and yards. We can only point to the subject here and recommend the book by BARENBLATT for a comprehensive treatment [1]. Dimensional analysis is a deeply fascinating and powerful tool that every physicist is drilled to use at all times. TAYLOR famously used it to compute the energy released by an early nuclear explosion at the Trinity site near Alamogordo, New Mexico, based on some grainy pictures published by Life magazine, at least that's the legend [**?**, **?**]. Fluid dynamicists in general use it to find meaningful quantities to distinguish different types of flow. In many problems involving random walks dimensional analysis immediately reveals scaling properties, supposed solutions to many problems can be seen at a glance to be wrong, and, conversely some complicated-looking problems can be solved as if by magic just by appealing to dimensional analysis.

BARENBLATT shows in his book that the dimension function must be a (scale-free) power-law monomial if there is to be no distinguished system of units. We can all agree that the unit of money is physically irrelevant – I can do exactly the same with the pennies in my bank account as I can do with the pounds those pennies correspond to. Since this is so, for functions of monetary amounts to be physically meaningful we want them to be power-law monomials. An amount of square-dollars, $\$^2$, may be meaningful, but an amount of logarithmic or exponential dollars cannot be meaningful. Hence $\ln(x)$ on its own is just some symbol spat on a page by a printer, but it has no physical meaning. The reason we're comfortable writing $\Delta \ln x$ is the unique property of the logarithmic function

$$\ln x_1 - \ln x_2 = \ln \left(\frac{x_1}{x_2} \right). \tag{26}$$

The quantity in brackets on the RHS is always dimensionless, it's a pure number because the dimension functions of two different values of x always cancel out. So do the units: $\$1/\$2 = 1/2$, which is a pure number without units. We will see that indeed only differences in logarithms of x will appear in these lecture notes or in any other reasonable lecture notes. Pedantically, we would refuse to write $\Delta \ln(x)$ and insist on writing $\ln \left(\frac{x_1}{x_2} \right)$. Since the first notation is shorter and one can make formal arguments for its validity, we are happy to use it here.

The issue is related to a result obtained by VON NEUMANN and MORGENSTERN in their famous but quite unhelpful book [34]: only differences in utility functions can have physical meaning.

1.1.5 Ergodic observables

We have encountered two types of averaging – the ensemble average and the time average. In our case – assessing whether it will be good for you to play our game, the time average is the interesting quantity because it tells you what happens to your wealth as time passes. The ensemble average is irrelevant because you do not live your life as an ensemble of many yous who can average over their wealths. Whether you like it or not, you will experience yourself owning your own wealth at future times; whether you like it not, you will never experience yourself owning the wealth of a different realization of yourself. The different realizations, and therefore the expectation value, are fiction, fantasy, imagined.

We are fully aware that it can be counter-intuitive that with probability one, a different rate is observed for the expectation value than for any trajectory over time. It sounds strange that the expectation value is completely irrelevant to the problem. A reason for the intuitive discomfort is history: since the 1650s we have been trained to compute expectation values, with the implicit belief that they will reflect what happens over time. It may be helpful to point out that all of this trouble has a name that's well-known to certain people, and that an entire field of mathematics is devoted to dealing with precisely this problem. The field of mathematics is call "ergodic theory." It emerged from the question under what circumstances the expectation value is informative of what happens over time, first raised in the development of statistical mechanics by Maxwell and Boltzmann starting in the 1850s.

History: Randomness and ergodicity in physics

The 1850s were about 200 years after FERMAT and PASCAL introduced expectation values into the study of random systems. Following the success of NEWTON's laws of motion, established around the same time as the expectation value, the notion of "proper science" had become synonymous with mechanics. Mechanics had no use for randomness and probability theory, and the success of mechanics was interpreted as a sign that the world was deterministic and that sooner or later we would understand what at the time still seemed random. At that point probability theory would become obsolete.

When BOLTZMANN hit upon the ingenious idea of introducing randomness into physics, to explain the laws of thermodynamics in terms of the underlying dynamics of large numbers of molecules, he was fighting an uphill battle. Neither molecules nor randomness were much liked in the physics community, especially in continental Europe, right up until the publication of EINSTEIN's 1905 paper on diffusion [9]. BOLTZMANN had to be more careful than FERMAT and PASCAL. He had to pre-empt predictable objections from his peers, and the question of ergodicity had to be answered – the usefulness of probability theory relies heavily on expectation values, but they are technically an average over imagined future states of the universe. BOLTZMANN's critics were aware of this and were not shy to voice their concerns. Under what circumstances are expectation values meaningful? BOLTZMANN gave two answers. Firstly, expectation values are meaningful when the quantity of interest really is an average (or a sum) over many approximately independent systems. An average over a finite ensemble will be close to the expectation value if the ensemble is large enough. Secondly, expectation values are meaningful, even if only a single system exists, if they reflect what happens over time.

BOLTZMANN called a system "ergodic[a]" if the possible states of the system could be assigned probabilities in such a way that the expectation value of any observable with respect to those probabilities would be the same as its time average with probability 1.

[a]The word "ergodic" was coined by BOLTZMANN. He initially proposed the word "monodic", from Greek $\mu o \nu o$ (unique)+$o\delta o\sigma$ (path) suggesting that a single path when followed for a sufficiently long time will explore all there is to explore and reflect what happens in an ensemble. The term "ergodic" refers to the specific system BOLTZMANN was considering, namely an energy ($\epsilon\rho\gamma o\nu$) shell across which a path is being traced out.

To convey concisely that we cannot use the expectation value and the time average interchangeably in our game, we would say "x is not ergodic."

DEFINITION: **Ergodic property**

In these notes, an observable A is called ergodic if its expectation value is constant in time, $\frac{d\langle A \rangle}{dt} = 0$, and its time average converges to this value with probability one

$$\lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_0^{\Delta t} A(s) ds = \lim_{N \to \infty} \frac{1}{N} \sum_i^N A_i. \tag{27}$$

We stress that in a given setup, some observables may have this property even if others do not. Language therefore must be used carefully. Saying our game is non-ergodic really means that some key observables of interest, most notably wealth x, are not ergodic. Wealth $x(t)$, defined by (Eq. 1), is clearly not ergodic – with $A = x$ the LHS of (Eq. 27) is zero, and the RHS is not constant in time but grows. The expectation value $\langle x \rangle (t)$ is simply not informative about the temporal behavior of $x(t)$.

This does not mean that no ergodic observables exist that are related to x. Such observables do exist, and we have already encountered two of them. In fact, we will encounter a particular type of them frequently – in our quest for an observable that tells us what happens over time in a stochastic system we will find them automatically. However, again, the issue is subtle: an ergodic observable may or may not tells us what we're interested in. It may be ergodic but not indicate what happens to x. For example, the multiplicative factor $r(t)$ is an ergodic observable that reflects what happens to the expectation value of x, whereas changes in the logarithm of wealth, $\Delta \ln x = \ln r$, are also ergodic and reflect what happens to x over time.

Proposition: $r(t)$ and $\Delta \ln x$ are ergodic.

Proof. According to (Eq. 4) and (Eq. 3), the expectation value of $r(t)$ is

$$\langle r \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_i^N r_i, \tag{28}$$

and according to (Eq. 23), the time average of $r(t)$ is

$$\overline{r} = \lim_{T \to \infty} \frac{1}{T} \sum_\tau^T r_\tau. \tag{29}$$

The only difference between the two expressions is the label we have chosen for the dummy variable (i in (Eq. 28) and τ in (Eq. 29)). Clearly, the expressions yield the same value. $r(t)$ is stationary and takes independent identically distributed values.

The same argument holds for $\Delta \ln x$. □

Whether we consider (Eq. 29) an average over time or over an ensemble is only a matter of our choice of words.

$\langle \delta \ln x \rangle$ is important, historically. DANIEL BERNOULLI noticed in 1738 [3] that people tend to optimize $\langle \delta \ln x \rangle$, whereas it had been assumed that they should optimize $\langle \delta x \rangle$. Unaware of the issue of ergodicity (200 years before the concept was discovered and the word was coined), BERNOULLI had no good explanation for this empirical fact and simply stated that people tend to behave as though they valued money non-linearly. We now know what is actually going on: δx is not ergodic, and $\langle \delta x \rangle$ is of no interest – it doesn't tell us what happens over time. However, $\delta \ln x$ is ergodic, $\langle \delta \ln x \rangle$ specifies what happens to x over time, and it is the right object to optimize.

When the foundations of economic theory were laid, specifically in BERNOULLI's seminal paper of 1738 [3], the distinction between ergodic and non-ergodic observables was unknown. Researchers thought that the expectation value of δx reflected what happens over time but observed that real people behaved according to what the expectation value of $\delta \ln x$ would suggest. The origin of this discrepancy remained mysterious and numerous puzzles and paradoxes in economic theory arose as a result. The paradigm we outline here resolves these puzzles.

1.2 Rates

The ergodic observable $\delta \ln x$, identified in the previous section, is basically a rate. If we divide it by the duration of the time step we obtain exactly the exponential growth rate of x, namely $\frac{1}{\delta t} \delta \ln x$. Finding good rates of change will be important, wherefore we now discuss the notion of a rate of change and the notion of stationarity. To do this properly let's think about the basic task of science. This may be described as the search for stable structure. Science attempts to build models of the world whose applicability does not vary over time. This does not mean that the world does not change, but the way in which the models describe change does not change. The model identifies something stable. This is implied by the fact that we can write equations (or English sentences) in ink on paper, with the equation (or sentence) remaining useful over time. The ink won't change over time, so if an article written in 1905 is useful today then it must describe something that hasn't changed in the meantime. These "somethings" are often somewhat grandiosely called laws of nature.

NEWTON's laws are a good illustration of this. They are part of mechanics, meaning that they are an idealized mathematical model of the behavior of positions, time, and masses. For instance, NEWTON's second law, $\mathcal{F} = \mathrm{m}\frac{d^2 \mathrm{x}}{dt^2}$, states that the mass multiplied by the rate of change of the rate of change of its position equals the force. The law is an unchanging law about positions, time, and masses, but it does not say that positions don't change, it doesn't even say that rates of change of positions don't change. It does say that the rate of change of the rate of change of a position remains unchanged so long as the force and the mass remain unchanged. NEWTON's deep insight was to transform an unstable thing – the position of a mass – until it became stable: he fixed the force and considered rates of changes of rates of changes, et voilá!, a useful equation could be written down in ink, remaining useful for 350 years so far.

Like NEWTON's laws (a mathematical model of the world), our game is a prescription of changes. Unlike NEWTON's laws it's stochastic, but it's a prescription of changes nonetheless. Our game is also a powerful mathematical model of the world, as we will see in subsequent lectures.

We're very much interested in changes of x – we want to know whether we're winning or losing – but changes in x are not stable. Under the rules of the game the rate of change of wealth, $\frac{\delta x(t)}{\delta t}$, is a different random variable for each t because it is proportional to $x(t)$. But not to worry, in NEWTON's case changes in the position are not stable either, even in a constant force field. Nonetheless NEWTON found a useful stable property. Maybe we can do something similar. We're looking for a function $v(x)$ that satisfies two conditions: it should indicate what happens to x itself, and its changes should be stationary.

The first condition is that $v(x)$ must tells us whether $x(t)$ is growing or shrinking – this just means that $v(x)$ has to be monotonic in x. We know that there is something stationary about x because we were able to write down in ink how x changes. So we only need to find the monotonic function of x that inherits the stationarity of the ink in (Eq. 1). The game is defined by a set of factors of increase in $x(t)$, (Eq. 2). Therefore, the fractional change in x, namely $\frac{x(t+\delta t)}{x(t)}$, has a stationary distribution. Which function responds additively to a multiplicative change in its argument? The answer is the logarithm, *i.e.* only the logarithm satisfies

$$v[x(t+\delta t)] - v[x(t)] = v\left[\frac{x(t+\delta t)}{x(t)}\right] \tag{30}$$

and we conclude that for our game $v(x) = \ln(x)$. For multiplicative dynamics, *i.e.* if $\frac{x(t+\delta t)}{x(t)}$ is stationary, the expectation value of the rate of change of the logarithm of $x(t)$ determines whether the game is long-term profitable for an individual.

More generally, when evaluating a gamble that is represented as a stochastic process

1. Find a monotonically increasing function $v[x(t)]$ such that $\frac{\Delta v[x(t)]}{\delta t}$ are independent instances of a stationary random variable.

2. Compute the expectation value of $\frac{\Delta v[x(t)]}{\delta t}$. If this is positive then $x(t)$ grows in the long run, if it is negative then $x(t)$ decays.

1.3 Brownian motion

In the previous section we established that the discrete increments of the logarithm of x, which we called v, are stationary and independent in our game. A quantity for which this is the case performs a random walk. Indeed, the blue line for a single system in Fig. 2 (B) shows 52 steps of a random walk trajectory.

Random walks come in many forms – in all of them v changes discontinuously by an amount δv drawn from a stationary distribution, at time intervals that are themselves drawn from a stationary distribution.

We are interested only in the simple case where v changes at regular intervals, $\delta t, 2\delta t \ldots$. For the distribution of increments we only insist on the existence of the variance, meaning we insist that $\mathrm{var}(\delta v) = \langle \delta v^2 \rangle - \langle \delta v \rangle^2$ be finite. The change in v after a long time is then the sum of many stationary independent increments,

$$v(t + T\delta t) - v(t) = \sum_i^T \delta v_i \tag{31}$$

The Gaussian central limit theorem tells us that such a sum, properly rescaled, will be Gaussian distributed

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}} \sum_i^T \delta v_i - T \langle \delta v \rangle \sim \mathcal{N}(0, \mathrm{var}(\delta v)), \tag{32}$$

where we call $\frac{\langle \delta v \rangle}{\delta t}$ the drift term. The logarithmic change in the long-time limit that was of interest to us in the analysis of the coin toss game is thus Gaussian distributed.

Let's also ask about the re-scaling that was applied in (Eq. 32). Scaling properties are very robust, and especially the scaling of random walks for long times will be useful to us. We work with the simplest setup: we start at zero, $v = 0$, and in each time step, we either increase or decrease v by 1, with probability 1/2. We are interested in the variance of the distribution of v as T increases, which we obtain by computing the first and second moments of the distribution. The expectation value (the first moment) of v is $\langle v \rangle (T) = 0$, by symmetry for all times. We obtain the second moment by induction[3]: Whatever the second moment, $\langle v(T)^2 \rangle$, is at time T, we can write down its value at time $T+1$ as

$$\begin{aligned} \langle v(T+1)^2 \rangle &= \frac{1}{2}\left[\langle (v(T)+1)^2 \rangle + \langle (v(T)-1)^2 \rangle \right] && (33)\\ &= \frac{1}{2}\left[\langle v(T)^2 + 1 + 2v(T) \rangle + \langle (v(T)^2 + 1 - 2v(T)) \rangle \right] && (34)\\ &= \langle v(T)^2 \rangle + 1 && (35) \end{aligned}$$

In addition, we know the initial value of $v(0) = 0$. By induction it follows that the second moment is

$$\langle v(T)^2 \rangle = T \tag{36}$$

and, since the first moment is zero, the variance is

$$\mathrm{var}(v(T)) = T. \tag{37}$$

The standard deviation – the width of the distribution – of changes in a quantity following a random walk thus scales as the square-root of the number of steps that have been taken, $\sqrt{T}$. This square-root behaviour leads to many interesting

[3]The argument is nicely illustrated in [10, Volume 1, Chapter 6-4], where we first came across it.

results. It can make averages stable (because $\sqrt{T}/T$ converges to zero for large T), and sums unstable (because $\sqrt{T}$ diverges for large T).

Imagine simulating a single long trajectory of v and plotting it on paper[4]. The amount of time that has to be represented by a fixed length of paper increases linearly with the simulated time because the paper has a finite width to accommodate the horizontal axis. If $\langle \delta v \rangle \neq 0$ then the amount of variation in v that has to be represented by a fixed amount of paper also increases linearly with the simulated time. However, the departures of Δv from its expectation value $T \langle \delta v \rangle$ only increase as the square-root of T. Thus, the amount of paper-space given to these departures scales as $T^{-1/2}$, and for very long simulated times the trajectory will look like a straight line on paper.

In an intermediate regime, fluctuations will still be visible but they will also be approximately Gaussian distributed. In this regime it is often easier to replace the random walk model with the corresponding continuous process. That process is called Brownian Motion (BM) and is intuitively thought of as the limit of a random walk where we shorten the duration of a step $\delta t \to 0$, and scale the width of an individual step so as to maintain the random-walk scaling of the variance, meaning $|\delta v| = \sqrt{\delta t}$. In the limit $\delta t \to 0$, this implies that $\frac{\delta v}{\delta t}$ diverges, so BM trajectories are infinitely jagged, or – in mathematical terms – they are not differentiable. However, the way in which they become non-differentiable, though the $\sqrt{\delta t}$ factor, leaves the trajectories just continuous (this isn't the case for $|\delta v| = \delta t^{\alpha}$, where α is less than 0.5. Continuity of v means that it is possible to make the difference $|v(t) - v(t+\epsilon)|$ arbitrarily small by choosing ϵ sufficiently small. This would not be possible if there was a jump in $v(t)$, so continuity roughly means that there are no jumps. These subtleties make BM a topic of great mathematical interest, and many books have been written about it. We will pick from these books only what is immediately useful to us. To convey the universality of BM we define it formally as follows:

Definition: **Brownian motion i**

If a stochastic process has continuous paths, stationary independent increments, and is distributed according to $\mathcal{N}(\mu t, \sigma^2 t)$ then it is a Brownian motion.

The process can be defined in different ways. Another illuminating definition is this:

Definition: **Brownian motion ii**

If a stochastic process is continuous, with stationary independent increments, then the process is a Brownian motion.

We quote from [**?**]: *"This beautiful theorem shows that Brownian motion can actually be defined by stationary independent increments and path continuity alone, with normality following as a consequence of these assumptions.*

[4]This argument is inspired by a colloquium presented by Wendelin Werner in the mathematics department of Imperial College London in January 2012. Werner started the colloquium with a slide that showed a straight horizontal line and asked: what is this? Then answered that it was the trajectory of a random walk, with the vertical and horizontal axes scaled equally.

This may do more than any other characterization to explain the significance of Brownian motion for probabilistic modeling."

Indeed, BM is not just a mathematically rich model but also – due to its emergence through the Gaussian central limit theorem – a model that represents a large universality class, *i.e.* it is a good description of what happens over long times in many other models. BM is a process with two parameters, μ and σ. It can be written as a stochastic differential equation

$$dv = \mu dt + \sigma dW \tag{38}$$

where dW is a Wiener increment. The Wiener increment can be defined by its distribution and correlator,

$$dW \sim \mathcal{N}(0, \sqrt{dt}) \tag{39}$$

$$\langle dW(t) dW(t') \rangle = dt\ \delta(t, t'), \tag{40}$$

where $\delta(t, t')$ is the Kronecker delta – zero if its two arguments differ ($t \neq t'$), and one if they are identical ($t = t'$).[5] In simulations BM paths can be constructed from a discretized version of (Eq. 38)

$$v(t + \delta t) = v(t) + \mu \delta t + \sigma \sqrt{\delta t} \xi_t, \tag{41}$$

where ξ_t are instances of a standard normal distribution.

BM itself is not stationary, which makes it non-ergodic according to our definition. This is easily seen by comparing time average and expectation value

$$\langle v \rangle_N \sim \mu t + \mathcal{N}(0, t/N) \tag{42}$$

$$\bar{v}_t \sim \mu t/2 + \sigma \mathcal{N}(0, t/3) \tag{43}$$

The expectation value, *i.e.* the limit $N \to \infty$ of (Eq. 4), converges to μt with probability one, so it's not stationary, it depends on time, and it's unclear what to compare it to. Its limit $t \to \infty$ does not exist.

The time average, the limit $t \to \infty$ of (Eq. 43) diverges unless $\mu = 0$, but even with $\mu = 0$ the limit is a random variable with diverging variance – something whose density is zero everywhere. In no meaningful sense do the two expressions converge to the same scalar in the relevant limits.

Clearly, BM, whose increments are independent and stationary, is not ergodic. That doesn't make it unmanageable or unpredictable – we know the distribution of BM at any moment in time. But the non-ergodicity has surprising consequences of which we mention one now. We already mentioned that if we plot a Brownian trajectory on a piece of paper it will turn into a straight line for long enough simulation times. This suggests that the randomness of a Brownian trajectory becomes irrelevant under a very natural rescaling. Inspired by this insight let's hazard a guess as to what the time-average of zero-drift BM might be. The simplest form of zero-drift BM starts at zero, $v(0) = 0$ and has

[5] Physicists often write $dW = \eta dt$, where $\langle \eta \rangle = 0$ and $\langle \eta(t) \eta(t') \rangle = \delta(t - t')$, in which case $\delta(t - t')$ is the Dirac delta function, defined by the integral $\int_{-\infty}^{\infty} f(t) \delta(t - t') dt = f(t')$. Because of its singular nature ($\eta(t)$ does not exist ("is infinite"), only its integral exists) it can be difficult to develop an intuition for this object, and we prefer the dW notation.

variance $\text{var}(v(t)) = t$ (this process is also known as the Wiener process). The process is known to be recurrent – it returns to zero, arbitrarily many times, with probability one in the long-time limit. We would not be mad to guess that the time average of zero-drift BM,

$$\bar{v} = \lim_{t\to\infty} \frac{1}{t} \int_0^t dt' v(t') \tag{44}$$

will converge to zero with probability one. But we would be wrong. Yes, the process has no drift, and yes it returns to zero infinitely many times, but its time average is not a delta function at zero. It is, instead normally distributed with infinite variance according to the following limit

$$\bar{v} \sim \lim_{t\to\infty} \mathcal{N}(0, t/3). \tag{45}$$

Averaging over time, in this case, does not remove the randomness. A sample trajectory of the time average is shown in Fig. 7. In the literature the process $\frac{1}{t}\int_0^t dt' v(t')$ is known as the "random acceleration process" [5].

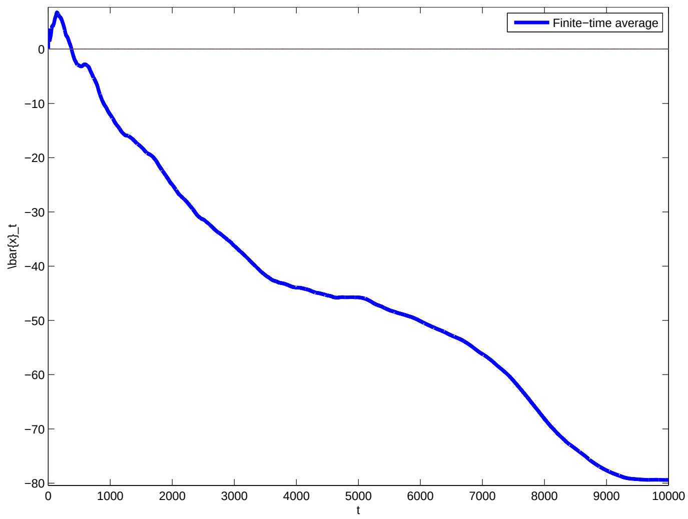


Figure 7: Trajectory of the finite-time average of a zero-drift BM. The time average does not converge to some value with probability one, but is instead distributed according to $\mathcal{N}(0, t/3)$ for all times. It is the result of integrating a BM; integration is a smoothing operation, and as a consequence the trajectories are smoother than BM (unlike a BM trajectory, they are differentiable).

1.4 Geometric Brownian motion

DEFINITION: **Geometric Brownian motion**

If the logarithm of a quantity performs Brownian motion, the quantity itself performs "geometric Brownian motion."

While in Sec. 1.3 $v(x) = \ln(x)$ performed BM, x itself performed Geometric

Brownian motion (GBM). The change of variable from x to $v(x) = \ln(x)$ is trivial in a sense but it has interesting consequences. It implies, for instance, that

- $x(t)$ is log-normally distributed
- increments in x are neither stationary nor independent
- $x(t)$ cannot become negative
- the most likely value of x (the mode) does not coincide with the expectation value of x. The median is the same as the mode.

The log-normal distribution is not symmetric, unlike the Gaussian distribution.

Again, it is informative to write GBM as a stochastic differential equation.

$$dx = x(\mu dt + \sigma dW). \tag{46}$$

Trajectories for GBM can be simulated using the discretized form

$$\Delta x = x(\mu\delta t + \sigma\sqrt{\delta t}\xi_t), \tag{47}$$

where $\xi_t \sim \mathcal{N}(0,1)$ are instances of a standard normal variable. In such simulations we must pay attention that the discretization does not lead to negative values of x. This happens if the expression in brackets is smaller than -1 (in which case x changes negatively by more than itself). To avoid negative values we must have $\mu\delta t + \sigma\sqrt{\delta t}\xi_t > -1$, or $\xi_t < \frac{1+\mu\delta t}{\sigma\sqrt{\delta t}}$. As δt becomes large it becomes more likely for ξ_t to exceed this value, in which case the simulation fails. But ξ_t is Gaussian distributed, meaning it has thin tails, and choosing a sufficiently small value of δt makes these failures essentially impossible.

On logarithmic vertical scales, GBM looks like BM, and we've already seen some examples. But it is useful to look at a trajectory of GBM on linear scales to develop an intuition for this important process.

The basic message of the game from Sec. 1.1 is that we may obtain different values for growth rates, depending on how we average – an expectation value is one average, a time average is quite a different thing. The game itself is sometimes called the multiplicative binomial process [30], we thank S. Redner for pointing this out to us. GBM is the continuous version of the multiplicative binomial process, and it shares the basic feature of a difference between the growth rate of the expectation value and time-average growth.

The expectation value is easily computed – the process is not ergodic, but that does not mean we cannot compute its expectation value. We simply take expectations of both sides of (Eq. 46),

$$\begin{aligned} \langle dx \rangle &= \langle x(\mu dt + \sigma dW) \rangle && (48) \\ = d\langle x \rangle &= \langle x \rangle \mu dt. && (49) \end{aligned}$$

This differential equation has the solution

$$\langle x(t) \rangle = x_0 \exp(\mu t), \tag{50}$$

which determines the growth rate of the expectation value as

$$g_{\langle\rangle} = \mu. \tag{51}$$

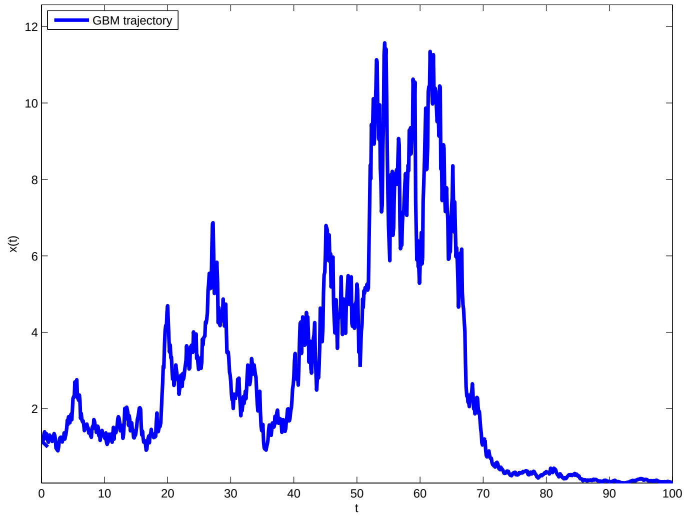


Figure 8: Trajectory of a GBM. The trajectory seems to know about its history – for instance, unlike for BM, it is difficult to recover from a low value of x, and trajectories are likely to get stuck near zero. Occasional excursions are characterised by large fluctuations. Parameters are $\mu = 0.05$ per time unit and $\sigma = \sqrt{2\mu}$, corresponding to zero growth rate in the long run. It would be easy to invent a story to go with this (completely random) trajectory – perhaps something like "things were going well in the beginning but then a massive crash occurred that destroyed morale."

As we know, this growth rate is different from the growth rate that materializes with probability 1 in the long run. Computing the time-average growth rate is only slightly more complicated. We will follow this plan: consider the discrete process (Eq. 47) and compute the changes in the logarithm of x, then we will let δt become infinitesimal and arrive at the result for the continuous process. We know $\Delta \ln(x(t))$ to be ergodic, wherefore we will proceed to take its expectation value to compute the time average of the exponential growth rate of the process.

The change in the logarithm of x in a time interval δt is

$$\begin{aligned}
\ln x(t+\delta t) - \ln x(t) &= \ln[x(1+\mu\delta t + \sigma\sqrt{\delta t}\xi_t)] - \ln x(t) && (52)\\
&= \ln x + \ln(1+\mu\delta t + \sigma\sqrt{\delta t}\xi_t) - \ln x(t) && (53)\\
&= \ln(1+\mu\delta t + \sigma\sqrt{\delta t}\xi_t), && (54)
\end{aligned}$$

which we Taylor-expand as ln(1+something small) because we will let δt become small. Expanding to second order,

$$\ln x(t+\delta t) - \ln x(t) = \mu\delta t + \sigma\sqrt{\delta t}\xi_t - \frac{1}{2}\left(\mu\sigma\delta t^{3/2}\xi_t + \sigma^2\delta t\xi_t^2\right) + o(\delta t^2), \quad (55)$$

using "little-o notation" to denote terms that are of order δt^2 or smaller. Finally, because $\Delta \ln x(t)$ is ergodic, by taking the expectation value of this equation we find the time average of $\Delta \ln x(t)$

$$\langle \ln x(t+\delta t) - \ln x(t) \rangle = \mu \delta t - \frac{1}{2}\left(\mu^2 \delta t^2 + \sigma^2 \delta t\right) + o(\delta t^2). \tag{56}$$

Letting δt become infinitesimal the higher-order terms in δt vanish, and we find

$$\langle \ln x(t+dt) - \ln x(t) \rangle = \mu dt - \frac{1}{2}\sigma^2 dt \tag{57}$$

so that the time-average growth rate is

$$\bar{g} = \frac{d\,\langle \ln x \rangle}{dt} = \mu - \frac{1}{2}\sigma^2. \tag{58}$$

We could have guessed the result by combining Whitworth's argument on the disadvantage of gambling with the scaling of BM. Let's re-write the factor $1-\epsilon$ in (Eq. 18) as $1-\sigma\sqrt{\delta t}$. According to the scaling of the variance in a random walk, (Eq. 37), this would be a good coarse-graining of some faster process (with shorter time step) underlying Whitworth's game. To find out what happens over one single time step we take the square root of (Eq. 18),

$$[(1+\sigma\sqrt{\delta t})(1-\sigma\sqrt{\delta t})]^{1/2} = [1-\sigma^2\delta t]^{1/2}. \tag{59}$$

Letting δt become infinitesimally small, we replace δt by dt, and the first-order term of a Taylor-expansion becomes exact,

$$[(1+\sigma\sqrt{\delta t})(1-\sigma\sqrt{\delta t})]^{1/2} \to 1 - \frac{\sigma^2}{2}dt, \tag{60}$$

in agreement with (Eq. 58) if the drift term $\mu = 0$, as assumed by Whitworth.

1.4.1 Itô calculus

We have chosen to work with the discrete process here and have arrived at a result that is more commonly shown using Itô's formula. We will not discuss Itô calculus in depth but we will use some of its results. The key insight of Itô was that the non-differentiability of so-called Itô processes leads to a new form of calculus, where in particular the chain rule of ordinary calculus is replaced. An Itô process is a Stochastic differential equation (SDE) of the following form

$$dx = a(x,t)dt + b(x,t)dW. \tag{61}$$

If we are interested in the behaviour of some other quantity that is a function of x, let's say $v(x)$, then Itô's formula tells us how to derive the relevant SDE as follows:

$$dv = \left(\frac{\partial v}{\partial t} + a(x,t)\frac{\partial v}{\partial x} + \frac{b(x,t)^2}{2}\frac{\partial^2 v}{\partial x^2}\right)dt + b(x,t)\frac{\partial v}{\partial x}dW. \tag{62}$$

Derivations of this formula can be found on Wikipedia. Intuitive derivations, such as [11], use the scaling of the variance, (Eq. 37), and more formal derivations, along the lines of [?], rely on integrals. We simply accept (Eq. 62) as given. It makes it very easy to re-derive (Eq. 58), which we leave as an exercise:

use (Eq. 62) to find the SDE for $\ln(x)$, take its expectation value and differentiate with respect to t. We will use (Eq. 62) in Sec. 3.3. The above computations are intended to give the reader intuitive confidence that Itô calculus can be trusted[6]. We find that, though phrased in different words, our key insight – that the growth rate of the expectation value is not the time-average growth rate – has appeared in the literature not only in 1870 but also in 1944. And in 1956 [14], and in 1966 [**?**], and in 1991 [6], and at many other times. Yet the depth of this insight remained unprobed.

Equation (58), which agrees with Itô calculus, may be surprising. Consider the case of no noise $dx = x\mu dt$. Here we can identify $\mu = \frac{1}{x}\frac{dx}{dt}$ as the infinitesimal increment in the logarithm, $\frac{d\ln(x)}{dt}$, using the chain rule of calculus. A naïve application of the chain rule to (Eq. 46) would therefore also yield $\frac{d\langle\ln(x)\rangle}{dx} = \mu$, but the fluctuations in GBM have a non-linear effect, and it turns out that the usual chain rule does not apply. Itô calculus is a modified chain rule, (Eq. 62) which leads to the difference $-\frac{\sigma^2}{2}$ between the expectation-value growth rate and the time-average growth rate.

This difference is sometimes called the "spurious drift", but at the London Mathematical Laboratory (LML) we call it the "Weltschmerz" because it is the difference between the many worlds of our dreams and fantasies, and the one cruel reality that the passage of time imposes on us.

[6] Itô calculus is one way of interpreting the non-differentiability of dW. Another interpretation is due to Stratonovich, which is not strictly equivalent. However, the key property of GBM that we make extensive use of is the difference between the growth rate of the expectation value, $g_{\langle\rangle}$, and the time-average growth rate, $\bar{g}$. This difference is the same in the Stratonovich and the Itô interpretation, and all our results hold in both cases.

2 Decision theory

Decision theory is a cornerstone of formal economics. As the name suggests, it models how people make decisions. In this chapter we will generalise and formalise the treatment of the coin tossing game to introduce our approach to decision theory. Our central axiom will be that people attempt to maximize the rate at which wealth grows when averaged over time. This is a surprisingly powerful idea. In many cases it eliminates the need for well established but epistemologically troublesome techniques.

2.1 Models and science fiction

We will do decision theory by using mathematical models, and since this can be done in many ways we will be explicit about how we choose to do it. We will define a gamble, which is a mathematical object, and we will define a decision criterion. The gamble will be reminiscent of real-world situations; and the decision criterion may or may not be reminiscent of how real people make decisions. We will not worry too much about the accuracy of these reminiscences. Instead we will "shut up and calculate" – we will let the mathematical model create its world. Writing down a mathematical model is like laying out the premise for a science-fiction novel. We may decide that people can download their consciousness onto a computer, that medicine has advanced to eliminate aging and death – these are premises we are at liberty to invent. Once we have written them down we begin to explore the world that results from those premises. A choice of decision criterion implies an endless list of behaviors that will be observed. For example, some criteria will lead to cooperation, others will not, some will lead to the existence of insurance contracts, others will not *etc.* We will explore the worlds created by the different models. Once we have done so we invite you to judge which model may be more useful for your understanding of the world. Of course, having spent many years thinking about these issues we have come to our own conclusions, and we will put them forward because we believe them to be helpful.

To keep the discussion to a manageable volume we will only consider the gamble problem. This is reminiscent of making decisions in an uncertain context – where we have to decide on a course of action now although we don't know with certainty what will happen to us under any of our choices. To limit the debate even further, we will only consider a setup that corresponds to making purely financial decisions. We may bet on a horse or take out personal liability insurance. This chapter will not tell you whom you should marry or even whose economics lectures you should attend.

2.2 Gambles

One fundamental building block of mathematical decision theory is the gamble. This is a mathematical object that resembles a number of situations in real life, namely situations where we face a decision whose consequences will be purely financial and are somewhat uncertain when we make the decision. A real-world example would be buying a lottery ticket. We define the gamble mathematically as follows.

> DEFINITION: **Gamble**
>
> A gamble is a pair of a random variable, D, and a duration, δt.
>
> D is called the payout and takes one of N (mutually exclusive) possible monetary values, $\{D_1, \ldots, D_N\}$, associated with probabilities, $\{p_1, \ldots, p_N\}$, where $\sum_{i=1}^{N} p_i = 1$. Payouts can be positive, associated with a monetary gain, or negative, associated with a loss. We order them such that $D_1 < \ldots < D_N$.

Everything we need to know about the gamble is contained in the payouts, probabilities, and duration. We relate it to reality through a few examples:

Example: Betting on a fair coin

Imagine betting \$10 on the toss of a fair coin. We would model this with the following payouts and probabilities:

$$D_1 = -\$10, \qquad p_1 = 1/2; \tag{63}$$
$$D_2 = +\$10, \qquad p_2 = 1/2. \tag{64}$$

The duration would be $\delta t = 2$ seconds.

Example: Playing the lottery

We can also imagine a gamble akin to a lottery, where our individual pays an amount, F, for a ticket which will win the jackpot, J, with probability, p. The corresponding payouts and probabilities are:

$$D_1 = -F, \qquad p_1 = 1 - p; \tag{65}$$
$$D_2 = J - F, \qquad p_2 = p. \tag{66}$$

Note that we deduct the ticket price, F, in the payout D_2. The duration would be $\delta t = 1$ week.

Example: Betting at fixed odds

A bet placed at fixed odds, for example on a horse race can also be modelled as a gamble. Suppose we bet on the horse *Ito* to win the 2015 Prix de l'Arc de Triomphe in Paris at odds of 50/1 (the best available odds on 20th September 2015). *Ito* will win the race with unknown probability, p. If we bet F, then this is modelled by payouts and probabilities:

$$D_1 = -F, \qquad p_1 = 1 - p; \tag{67}$$
$$D_2 = 50F, \qquad p_2 = p. \tag{68}$$

The duration would be $\delta t = 30$ minutes.

Example: The null gamble

It is useful to introduce the null gamble, in which a payout of zero is received with certainty: $D_1 = \$0$, $p_1 = 1$. This represents the 'no bet' or 'do nothing' option. The duration, δt, has to be chosen appropriately. The meaning of the duration will become clearer later on – often it is the time between two successive rounds of a gamble.

The gamble is a simple but versatile mathematical model of an uncertain future. It can be used to model not only traditional wagers, such as sports bets and lotteries, but also a wide range of economic activities, such as stock market investments, insurance contracts, derivatives, and so on. The gamble we have presented is discrete, in that the payout, D, is a random variable with a countable (and, we usually assume, small) number of possible outcomes.

The extension to continuous random variables is natural and used frequently to model real-world scenarios where the number of possible outcomes, *e.g.* the change in a stock price over one day, is large.

Suppose now that you have to choose between two options that you've modeled as two gambles (possibly including the null gamble). Which should you choose, and why? This is the gamble problem, the central question of decision theory, and the basis for almost everything in mainstream economics.

2.3 Repetition and wealth evolution

To solve the gamble problem we must propose a criterion to choose between two gambles. Different criteria will result in different decisions – by writing down a criterion we build a model world of model humans who behave in ways that may seem sensible to us or crazy – if the behavior seems crazy we have probably not chosen a good criterion and we should try a different one.

The wealth process $x(t)$ is connected to the gambles our model humans choose to play. Precisely *how* it is affected remains to be specified.

Considering a single round of a gamble in isolation – the so-called 'one-shot game' of game theory – is relatively uninstructive in this regard. All we know is that one of the possible payouts will be received, leading to the random variable $x(t + \delta t) = x(t) + D$. We don't know the tendency the gamble will give to $x(t)$ over time, since one time step is generally not enough for any tendencies to become apparent. The one-shot game takes one random variable, D, and turns it trivially into another, $x(t) + D$. Time has no significance in a one-shot game. An amount δt elapses, but this could be a single heartbeat or the lifetime of the universe, for all the difference it makes to the analysis.

To establish how your wealth evolves, we must imagine that the world does not come to a grinding halt after the gamble. Instead we imagine that the gamble is repeated over many rounds.[7] This does not mean that we actually believe that a real-world situation will repeat itself over and over again, *e.g.* we don't believe that we will bet on the horse *Ito* at odds 50/1 many times in a row. Instead, imagining repetition is a methodological device that allows us to extract tendencies where they would otherwise be invisible. It is the model analogue of the idea that individuals live in time and that their decisions have consequences which unfold over time.

Crucially, *the mode of repetition is not specified in the gamble itself.* It is a second component of the model, which must be specified separately. We shall focus on two modes: *additive* and *multiplicative* repetition.

[7] In fact, to make the problem tractable mathematically, it will be necessary to imagine the gamble will be repeated indefinitely. More on this later.

Definition: Additive repetition

If a gamble is repeated additively then the random payout, D, is simply added to $x(t)$ at each round. We define the change in wealth occurring over a single round as

$$\delta x(t) \equiv x(t+\delta t) - x(t). \tag{69}$$

In the additive case, we have

$$\delta x(t) = D. \tag{70}$$

In other words, under additive repetition, δx is a stationary random variable.[a] Starting at time, t_0, wealth after T rounds is

$$x(t_0 + T\delta t) = x(t_0) + \sum_{\tau=1}^{T} D(\tau), \tag{71}$$

where $D(\tau)$ is the realisation of the random variable in round τ. This is an evolution equation for wealth following a noisy additive dynamic. Note that $x(t_0 + T\delta t)$ is itself a random variable.

[a] This seemingly mundane corollary stems from our definitions of D as a monetary payout and δx as an additive change in wealth, expressed in monetary units. It would not hold if, for example, we had defined δx to be some other type of change, such as a relative change.

Example: Additive repetition

We return to our first example of a gamble: a \$10 bet on a coin toss. Under additive repetition, successive bets will always be \$10, regardless of how rich or poor you become. Suppose your starting wealth is $x(t_0) = \$100$. Then, following (Eq. 71), your wealth after T rounds will be

$$\begin{aligned} x(t_0 + T\delta t) &= \$100 + \$10k - \$10(T-k) & (72)\\ &= \$[100 + 10(2k - T)], & (73) \end{aligned}$$

where $0 \leq k \leq T$ is the number of tosses you've won. Note that we have assumed your wealth is allowed to go negative. If not, then the process would stop when $x < \$10$, since you would be unable to place the next \$10 bet.

An alternative is multiplicative repetition. In the example above, let us imagine that the first \$10 bet were viewed not as a bet of fixed monetary size, but as a fixed fraction of the starting wealth (\$100). Under multiplicative repetition, each successive bet is for the same fraction of wealth which, in general, will be a different monetary amount.

The formalism is as follows. The payout, D, in the first round is expressed instead as a random wealth multiplier,

$$r \equiv \frac{x(t_0) + D}{x(t_0)}. \tag{74}$$

The gamble is repeated by applying the same multiplier at all subsequent rounds:

$$x(t + \delta t) = rx(t). \tag{75}$$

From (Eq. 74) we see that r is a stationary random variable, since it depends only on D, which is stationary, and the starting wealth, $x(t_0)$, which is fixed. However, successive changes in wealth,

$$\delta x(t) = (r - 1)x(t), \tag{76}$$

are not stationary, as they depend on t through $x(t)$. The wealth after T rounds of the gamble is

$$x(t_0 + T\delta t) = x(t_0) \prod_{\tau=1}^{T} r(\tau), \tag{77}$$

where $r(\tau)$ is the realisation of the random multiplier in round τ.

Example: Multiplicative repetition

The \$10 bet on a coin toss is now re-expressed as a bet of a fixed fraction of wealth at the start of each round. Following (Eq. 74), the random multiplier, r, has two possible possible outcomes:

$$r_1 = \frac{\$100 - \$10}{\$100} = 0.9, \qquad p_1 = 1/2; \tag{78}$$

$$r_2 = \frac{\$100 + \$10}{\$100} = 1.1, \qquad p_2 = 1/2. \tag{79}$$

The wealth after T rounds is, therefore,

$$g(t_0 + T\delta t) = \$100\ (1.1)^k\ (0.9)^{T-k}, \tag{80}$$

where $0 \leq k \leq T$ is the number of winning tosses. In this example there is no need to invoke a 'no bankruptcy' condition, since our individual can lose no more than 10% of his wealth in each round.

The difference between the two modes of repetition might easily be mistaken for a matter of taste. When the \$10 bet was first offered, what difference does it make whether our individual imagined this to be a bet of a fixed size or of a fixed fraction of his wealth? However, the consequences of this choice between imagined situations are enormous. As we saw in the previous lecture, additive and multiplicative dynamics differ as starkly as the linear and exponential functions. It matters, therefore, that we consider carefully the economic situation we wish to model in order to choose the most realistic mode of repetition. For example, fluctuations in the price of a stock tend to be proportional to the price, *cf.* (Eq. 76), so multiplicativity is the appropriate paradigm here.

Now that we've established how the gamble is related to $x(t)$ we can begin to think about decision criteria. Not surprisingly, appropriate growth rates are useful decision criteria – "pick the gamble that will lead your wealth to grow the fastest" is generally good advice. To be able to follow this advice we will think again about growth rates.

2.4 Growth rates

In the previous lecture we introduced the concept of a growth rate, g, which is the rate of change of a monotonically increasing function of wealth, $v(x(t))$:

$$g(t, \Delta t) \equiv \frac{\Delta v(x(t))}{\Delta t}. \tag{81}$$

The function, v, is chosen such that the increment, $\Delta v(x(t))$, over the period Δt,[8] is a stationary random variable whose distribution does not depend on when the period starts. The growth rate is, therefore, also stationary. Indeed, we consider it a function of t only inasmuch as this labels a particular realisation of the randomness at a particular point in time.

Stationarity – the statistical irrelevance of the time of measurement – is important because we want the distribution of the random growth rate to convey robust information about the underlying process, rather than mere happenstance about when it was sampled.

Under additive repetition, we know from (Eq. 70) that Δx is already stationary, so we know immediately that the correct mapping is the identity: $v(x) = x$.[9] This gives the growth rate for an additive process (denoted by the subscript 'a'):

$$g_{\mathrm{a}}(t, \Delta t) = \frac{\Delta x(t)}{\Delta t}. \tag{82}$$

For a multiplicative dynamic, however, using δx in the numerator of the rate will not do, as we know from (Eq. 76) that changes in $x(t)$ depend on $x(t)$. Instead we must find the mapping $v(x)$ whose increment has a stationary distribution. The correct mapping is the logarithm, since the increment over a single round is

$$\begin{aligned} \delta \ln x(t) &= \ln x(t + \delta t) - \ln x(t) && (83)\\ &= \ln r x(t) - \ln x(t) && (84)\\ &= \ln r, && (85) \end{aligned}$$

where (Eq. 74) has been used in the second line. This inherits its stationarity from r. Thus the appropriate growth rate for a multiplicative process (denoted by the subscript 'm') over an arbitrary time period is

$$g_{\mathrm{m}}(t, \Delta t) = \frac{\Delta \ln x(t)}{\Delta t}. \tag{86}$$

The distribution of the stationary random variable $g(t, \Delta t)$ depends on Δt. Subject to certain conditions on $\Delta v(x(t))$, the distribution of $g(t, \Delta t)$ narrows as Δt increases, converging to a finite number in the limit $\Delta t \to \infty$. In other words, as the effect of the gamble manifests itself over an increasingly long time, the noise is eliminated to reveal a growth rate reflecting the gamble's underlying tendency.

We define this time-average growth rate, $\bar{g}$, as

$$\bar{g} \equiv \lim_{\Delta t \to \infty} \{g(t, \Delta t)\}. \tag{87}$$

[8]Note that we use a general time period, Δt, here and not the period of the gamble, δt.

[9]In fact, any linear function $v(x) = \alpha x + \beta$ has stationary increments and is monotonically increasing provided $\alpha > 0$. However, there is nothing gained by choosing anything other $v(x) = x$.

This is the growth rate that an individual will experience almost surely (*i.e.* with probability approaching one) as the number of rounds of the gamble diverges. Indeed, we can express $\bar{g}$ in precisely these terms,

$$\bar{g} = \lim_{T\to\infty} \left\{ \frac{v(x(t+T\delta t)) - v(x(t))}{T\delta t} \right\}, \tag{88}$$

where T is the number of rounds. Expanding the numerator as a sum of increments due to individual rounds of the gamble gives

$$\begin{aligned}
\bar{g} &= \lim_{T\to\infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} \frac{\Delta v(x(t+\tau\delta t))}{\delta t} \right\} && (89)\\
&= \lim_{T\to\infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} g(t+\tau\delta t, \delta t) \right\} && (90)\\
&= \langle g(t,\delta t) \rangle, && (91)
\end{aligned}$$

where the final line follows from the stationarity and independence of the successive per-round growth rates. This is a restatement of the ergodic property of the previous lecture, namely that the time-average growth rate can be expressed equivalently as the long-time limit and as the ensemble average of the properly chosen ergodic growth rate. For additive and multiplicative dynamics, we obtain the following equivalences:

$$\bar{g}_{\mathrm{a}} = \lim_{\Delta t\to\infty} \left\{ \frac{\Delta x(t)}{\Delta t} \right\} = \left\langle \frac{\Delta x(t)}{\Delta t} \right\rangle; \tag{92}$$

$$\bar{g}_{\mathrm{m}} = \lim_{\Delta t\to\infty} \left\{ \frac{\Delta \ln x(t)}{\Delta t} \right\} = \left\langle \frac{\Delta \ln x(t)}{\Delta t} \right\rangle. \tag{93}$$

These follow the form of the general expression,

$$\bar{g} = \lim_{\Delta t\to\infty} \left\{ \frac{\Delta v(x(t))}{\Delta t} \right\} = \left\langle \frac{\Delta v(x(t))}{\Delta t} \right\rangle. \tag{94}$$

The value of Δt in the ensemble averages is immaterial. In calculations, it is often set to the period, δt, of a single round of the gamble.

Where we are interested in the value of $\bar{g}$, knowing that it is equal to the value of $\langle g \rangle$ may provide a convenient method of calculating it. However, we will attach no special interpretation to the fact that $\langle g \rangle$ is an expectation value. It is simply a quantity whose value happens to coincide with that of the quantity we're interested in, *i.e.* the time-average growth rate.

Let's take a step back and remark more generally on what we have done so far. We started with a high-dimensional mathematical object, namely the probability distribution of the payout, D, of the gamble. To this we added two model components: the time period, δt, over which the gamble unfolds; and a dynamic, in essence a set of instructions, specifying how the repeated gamble causes your wealth to evolve. We then collapsed all of this information into a single number, $\bar{g}$, which characterises the effect of the gamble. The collapse from distribution to single number (or, equivalently, from uncertain to certain quantity) allows different gambles to be compared and, in particular, ranked. This permits an unequivocal decision criterion, which would be much harder to formulate for higher-dimensional objects, such as the two distributions shown in Fig. 9.

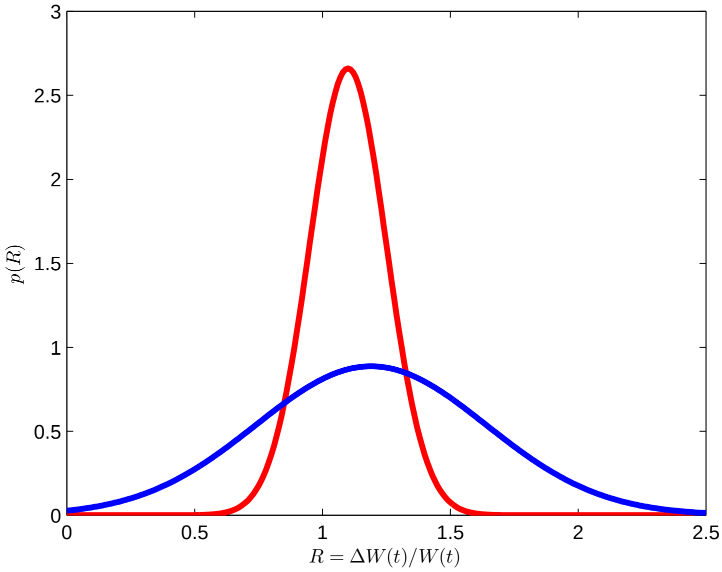


Figure 9: Two possible probability density functions for the per-round multiplier, r, defined in (Eq. 74). The distribution denoted by the blue line has a higher mean and a higher variance than the one in red. How are we to decide which represents the more favourable gamble?

2.5 The decision axiom

Our model rationale for deciding between two gambles is simple: given a model for the mode of repetition, choose the gamble with the largest time-average growth rate. In other words, choose the gamble which, if repeated indefinitely, causes your wealth to grow fastest.

We are not saying that real repetitions are necessary. We merely create model humans who base decisions on what would happen to their wealth if they were repeated (infinitely many times). It is a conceptual device – in effect a thought experiment – to elicit the underlying tendency of each gamble. A particular choice that can be represented by gambles may be offered only once and, indeed, in the real world this will often be the case. However, in the real world it is also the case that a decision is likely to be followed by many others, a scenario to which indefinite repetition is a plausible approximation.

In our thought experiment this decision rule outperforms any other decision rule almost surely: so, at least in that imagined world, the rationale has a logical basis. If our thought experiment is a good approximation to real-world decision scenarios, then our rationale should be a good model of real-world decisions. Certainly it is parsimonious, based on a single, robust quantity and requiring only the gamble and its mode of repetition to be specified. Unlike some treatments of human decision-making, it contains no arbitrary or hard-to-measure psychological factors.

Having said all this, while we think that the decision axiom is reasonable, we stress that it is an *axiom*: it defines a model world where certain types of behaviour will be observed. We feel reminded of reality by this model world, but you may disagree or you may prefer a different model that creates a different model world that also reminds you of reality. Other decision axioms are possible and, indeed, have been proposed. For instance, classical decision theory is defined by the axiom that decision makers maximize expected utility.

Our decision rationale can be expressed as a set of instructions. We denote quantities relating to the m^{th} available gamble with the superscript (m). Each gamble is specified by its random payout, $D^{(m)}$, and per-round period, $\delta t^{(m)}$.

Growth-optimal decision algorithm

1. Specify $D^{(m)}$ and $\delta t^{(m)}$ for the gambles offered;
2. Specify the wealth dynamic, *i.e.* the relationship between $\delta x(t)$, $x(t)$, and D;
3. Determine the function, $v(x)$, of the wealth whose increments are stationary random variables under this dynamic;
4. Determine the time-average growth rates, $\bar{g}^{(m)}$, either by taking the long-time limits of the growth rates, $g^{(m)}(t, \delta t)$, or by invoking the ergodic property to take their ensemble averages;
5. Choose the gamble, m, with the largest time-average growth rate.

In examples, we will focus on choices between two gambles, *i.e.* $m \in \{1, 2\}$. Decisions between three or more gambles are straightforward extensions. Often one of the gambles presented is the null gamble, which trivially has a growth rate of zero. In this case, the question then posed is whether the other gamble presented is preferable to doing nothing, *i.e.* bet or no bet?

We will illustrate the decision algorithm by applying it to the coin toss game of the previous lecture. $x(t_0) > \$0$ is the starting wealth and δt is the time between rounds for each of the gambles offered. We recall that the coin toss gamble is specified by the random payout:

$$D_1^{(1)} = -0.4x(t_0), \qquad p_1^{(1)} = 1/2; \tag{95}$$

$$D_2^{(1)} = 0.5x(t_0), \qquad p_2^{(1)} = 1/2. \tag{96}$$

Note that, in our setup, the payouts $D_i^{(m)}$ are always fixed monetary amounts. Here they are expressed as fractions of $x(t_0)$, which is itself a fixed monetary wealth.[10] We shall ask our individual to choose between the coin toss and the null gamble,

$$D_1^{(2)} = \$0, \quad p_1^{(2)} = 1. \tag{97}$$

Both the additive and multiplicative versions of the repeated gamble are analysed.

[10]Even though this formulation might appear to encode a multiplicative dynamic (largely because it comes from an imagined multiplicative game), we have arranged things so that formally it does not. Indeed, it encodes no dynamic at all: that is specified separately.

Example: Additive coin toss game

If the repetition is additive, the wealth evolves over T rounds according to:

$$x^{(1)}(t_0 + T\delta t) = x(t_0) + \sum_{\tau=1}^{T} D^{(1)}(\tau); \quad (98)$$

$$x^{(2)}(t_0 + T\delta t) = x(t_0). \quad (99)$$

Here we have assumed that the wealth is free to go negative, *i.e.* that there is no bankrupt state from which the individual can't recover.[11] The appropriate utility function is the identity, $v(x) = x$, so the growth rates are simply the rates of change of wealth itself. We can express these over the T rounds as:

$$g_{\mathrm{a}}^{(1)}(t) = \frac{x^{(1)}(t + T\delta t) - x(t_0)}{T\delta t} = \frac{1}{T}\sum_{\tau=1}^{T} \frac{D^{(1)}(\tau)}{\delta t}; \quad (100)$$

$$g_{\mathrm{a}}^{(2)}(t) = \frac{x^{(2)}(t + T\delta t) - x(t_0)}{T\delta t} = \$0 \quad (101)$$

per unit time. The long-time limit of the growth rate for the null gamble is trivially $\bar{g}_{\mathrm{a}}^{(2)} = \$0$ per unit time. For the coin toss, we calculate it as

$$\bar{g}_{\mathrm{a}}^{(1)} = \lim_{T\to\infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} \frac{D^{(1)}(\tau)}{\delta t} \right\} \quad (102)$$

$$= \left\langle \frac{D^{(1)}}{\delta t} \right\rangle \quad (103)$$

$$= \frac{p_1^{(1)} D_1^{(1)} + p_2^{(1)} D_2^{(1)}}{\delta t} \quad (104)$$

$$= \frac{x(t_0)}{20\delta t}, \quad (105)$$

which is positive (assuming $x(t_0) > \$0$). Therefore, $\bar{g}_{\mathrm{a}}^{(1)} > \bar{g}_{\mathrm{a}}^{(2)}$ and our individual should accept the coin toss gamble under additive dynamics.

We see from this example that the decision rule under additive repetition is to maximise $\langle D/\delta t \rangle$.[12] This is the rate of change of the expected wealth, which, as we know from (Eq. 92), happens to coincide under this dynamic with the time-average growth rate. We will see later that humans tend not to act to maximise $\langle D/\delta t \rangle$ in reality. This may not be a great shock: additive repetition without bankruptcy isn't going to win many prizes for the most realistic model

[11]We could model this realistic feature by an absorbing boundary on $x(t)$, if we were so minded.

[12]Too frequently in presentations of decision theory, it is assumed implicitly that δt is the same for all available gambles and the decision algorithm is presented as the maximisation of the expected payout, $\langle D \rangle$. While this is equivalent if all the $\delta t^{(m)}$ are identical, it is not if they aren't. Moreover, the assumption is usually left unstated. This is unhelpful in that it masks the important role of time in the analysis and, in particular, the fact that our individual is maximising a *rate*.

of wealth evolution.

Let's try multiplicative repetition instead.

Example: Multiplicative coin toss game

The payout, $D^{(1)}$, is re-expressed as a per-round multiplier,

$$r^{(1)} = \frac{x(t_0) + D^{(1)}}{x(t_0)}, \tag{106}$$

which takes the values:

$$r_1^{(1)} = \frac{x(t_0) + D_1^{(1)}}{x(t_0)} = 0.6, \qquad p_1^{(1)} = 1/2; \tag{107}$$

$$r_2^{(1)} = \frac{x(t_0) + D_2^{(1)}}{x(t_0)} = 1.5, \qquad p_2^{(1)} = 1/2. \tag{108}$$

Under multiplicative dynamics, the wealth evolves according to:

$$x^{(1)}(t_0 + T\delta t) = x(t_0) \prod_{\tau=1}^{T} r^{(1)}(\tau); \tag{109}$$

$$x^{(2)}(t_0 + T\delta t) = x(t_0). \tag{110}$$

The appropriate utility function is the logarithm, $v(x) = \ln x$. We have already discussed why, but one way of seeing this is to take logarithms of (Eq. 109). This converts the product into a sum of stationary and independent random variables:

$$\ln x^{(1)}(t_0 + T\delta t) = \ln x(t_0) + \sum_{\tau=1}^{T} \ln r^{(1)}(\tau); \tag{111}$$

$$\ln x^{(2)}(t_0 + T\delta t) = \ln x(t_0). \tag{112}$$

Therefore, $\ln x$ is the desired quantity whose increments are stationarily distributed. The growth rates, expressed over T rounds, are:

$$g_{\mathrm{m}}^{(1)}(t) = \frac{\ln x^{(1)}(t + T\delta t) - \ln x(t_0)}{T\delta t} = \frac{1}{T} \sum_{\tau=1}^{T} \frac{\ln r^{(1)}(\tau)}{\delta t}; \tag{113}$$

$$g_{\mathrm{m}}^{(2)}(t) = \frac{\ln x^{(2)}(t + T\delta t) - \ln x(t_0)}{T\delta t} = 0 \tag{114}$$

per unit time. As in the additive case, we take the $T \to \infty$ limits. For the null gamble this is trivial: $\bar{g}_{\mathrm{m}}^{(2)} = 0$ per unit time. For the coin toss gamble, we get

$$\bar{g}_{\mathrm{m}}^{(1)} = \left\langle \frac{\ln r^{(1)}}{\delta t} \right\rangle = \frac{\ln 0.9}{2\delta t}, \tag{115}$$

which is negative. Thus, $\bar{g}_{\mathrm{m}}^{(1)} < \bar{g}_{\mathrm{m}}^{(2)}$ under multiplicative dynamics and our individual should decline the coin toss. That this is the opposite of his decision under additive repetition highlights the importance of specifying a dynamic that corresponds well to what is happening in reality.

Another way of presenting the repeated coin toss is to express the wealth after T rounds as

$$x^{(1)}(t_0 + T\delta t) = x(t_0)\left(r_T^{(1)}\right)^T, \tag{116}$$

where

$$r_T^{(1)} = (0.6)^{(T-k)/T}(1.5)^{k/T} \tag{117}$$

is the equivalent per-round multiplier after T rounds and $0 \leq k \leq T$ is the number of winning rounds. r_T is a random variable but it converges to an almost sure quantity in the long time limit,

$$r_\infty^{(1)} \equiv \lim_{T\to\infty}\{r_T^{(1)}\} = (0.6)^{1/2}(1.5)^{1/2} = \sqrt{0.9}, \tag{118}$$

since $k/T \to 1/2$ as $T \to \infty$ (the coin is fair). $r_\infty^{(1)} < 1$ so the individual's wealth is sure to decay over time and he should decline the gamble. The two approaches are, of course, linked, in that

$$\bar{g}_{\mathrm{m}}^{(1)} = \frac{\ln r_\infty^{(1)}}{\delta t}. \tag{119}$$

Here we see that our decision rule boils down to maximising

$$\left\langle \frac{\ln r}{\delta t} \right\rangle = \left\langle \frac{\ln(x(t_0) + D) - \ln x(t_0)}{\delta t} \right\rangle. \tag{120}$$

This coincides with the time-average growth rate in (Eq. 93).

2.6 The expected-wealth and expected-utility paradigms

Our decision rule under additive repetition of the gamble is to maximise

$$\left\langle \frac{\delta x}{\delta t} \right\rangle = \left\langle \frac{D}{\delta t} \right\rangle, \tag{121}$$

i.e. the rate of change of the expectation value of wealth. This was, in fact, the first decision rule to be suggested when gamble problems were considered in the early days of probability theory in the 17$^{\text{th}}$ century. We will call this the 'expected-wealth paradigm'. It was not derived as we have derived it, from a criterion to maximise growth over repetition. Instead, it was essentially proposed as the decision axiom itself, with no reference to dynamics. It is easy to see why: it is a simple rule containing a familiar type of average, which incorporates all the possible outcomes of the game. Indeed, it would be logically sound if we could play the game many times in parallel, thereby accessing all the possible outcomes.

In the language of economics, the expected-wealth paradigm treats humans as 'risk neutral', *i.e.* they have no preference between gambles whose expected changes in wealth are identical (over a given time interval). This treatment has been known to be a flawed model of human decision-making since at least 1713 [22, p. 402], in that it does not accord well with observed behaviour.

The conventionally offered reason for this predictive failure is that the value to an individual of a possible change in wealth depends on how much wealth he already has and his psychological attitude to taking risks. In other words, people do not treat equal amounts of money equally. This makes intuitive sense: an extra \$10 is much less significant to a rich man than to a pauper for whom it represents a full belly; an inveterate gambler has a different attitude to risking \$100 on the spin of a roulette wheel than a prudent saver, their wealths being equal.

In 1738 Daniel Bernoulli [3], after correspondence with Cramer, devised the 'expected-utility paradigm' to model these considerations. He observed that money may not translate linearly into usefulness and assigned to an individual an idiosyncratic utility function, $u(x)$, that maps his wealth, x, into usefulness, u. He claimed that this was the true quantity whose rate of change of expected value,

$$\langle r_u \rangle \equiv \left\langle \frac{\delta u(x)}{\delta t} \right\rangle, \tag{122}$$

is maximised in a decision between gambles.

This is the axiom of utility theory. It leads to an alternative decision algorithm, which we summarise here:

Expected-utility decision algorithm

1. Specify $D^{(m)}$ and $\delta t^{(m)}$ for the gambles offered;
2. Specify the individual's idiosyncratic utility function, $u(x)$, which maps his wealth to his utility;
3. Determine the rate of change of his expected utility, *e.g.* over a single round of the gamble,

$$\langle r_u \rangle^{(m)} = \left\langle \frac{u\left(x + D^{(m)}\right) - u(x)}{\delta t^{(m)}} \right\rangle; \tag{123}$$

4. Choose the gamble, m, with the largest $\langle r_u \rangle^{(m)}$.

Despite their conceptually different foundations, we note the similarities between the maximands[13] of our growth-optimal decision theory, (Eq. 94), and the expected-utility paradigm, (Eq. 122). Our decision theory contains a mapping which transforms wealth into a variable, v, whose increments are stationary random variables. This mapping depends on the wealth dynamic which describes how the gamble is repeated. The second contains a function which transforms wealth into usefulness. This is determined by the idiosyncratic risk preferences of the individual. If we identify the stationarity transformation of the former with the utility transformation of the latter, then the expressions are the same.

[13] The quantities to be maximised.

In other words, the expected utility hypothesis of classical decision theory is consistent with time-average growth rate optimisation for a particular choice of utility function. While it can be made to replicate the results of our decision axiom, it is a poorly constrained model for doing so, since it admits any number of alternative utility functions which do not replicate those results. For multiplicative dynamics, the necessary choice is the logarithm. That this is the most widely used utility function in both theory and practice might be a psychological fluke; or it might indicate that our brains have evolved to produce growth-optimal decisions in a world governed by multiplicative dynamics, *i.e.* where resources can be deployed to make more of themselves.

Thus we can offer a different reason for the predictive failure of the expected-wealth paradigm. In our framework this corresponds to good decisions under additive repetition, which we claim is generally a poor model of how wealth evolves.[14] It fails, therefore, because it corresponds to an unrealistic dynamic.

2.7 The St Petersburg paradox

The problem known today as the St Petersburg paradox was suggested by Nicolaus Bernoulli[15] in 1713 in his correspondence with Montmort [22]. It involves a hypothetical lottery for which the rate of change of expected wealth diverges for any finite ticket price. The expected-wealth paradigm would predict, therefore, that people are prepared to pay any price to enter the lottery. However, when the question is put to them, they are seldom observed to want to wager more than a few dollars. This is the paradox. It is the first well-documented example of the inadequacy of the expected-wealth paradigm as a model of human rationality. It was the primary motivating example for Daniel Bernoulli's and Cramer's development of the expected-utility paradigm [3].

In some sense it is a pity that this deliberately provocative and unrealistic lottery has played such an important role in the development of classical decision theory. It is quite unnecessary to invent a gamble with a diverging change in expected wealth to expose the flaws in the expected-wealth paradigm. The presence of infinities in the problem and its variously proposed solutions has caused much confusion, and permits objections on the grounds of physical impossibility. Such objections are unhelpful because they are not fundamental: they address only the gamble and not the decision paradigm. Nevertheless, the paradox is an indelible part not only of history but also of the current debate [25], and so we recount it here. We'll start by defining the lottery.

Example: St Petersburg lottery

The classical statement of the lottery is to imagine a starting prize of \$1 (originally the prize was in ducats). A fair coin is tossed: if it lands heads, the player wins the prize and the lottery ends; if it lands tails, the prize is doubled and the process is repeated. Therefore, the player wins \$2, \$4, \$8 if the first head lands on the second, third, fourth toss, and so on. The

[14] It would correspond, for example, to the interest payments in your bank account being independent of your balance!

[15] Daniel's cousin. The Bernoulli family produced a remarkable number of famous mathematicians in the 17^{th} and 18^{th} centuries, who helped lay the foundations of applied mathematics and physics.

player must buy a ticket, at price F, to enter the lottery. The question usually posed is what is the largest F they are willing to pay.

The lottery can be translated neatly into our gamble formalism:

$$D_k = \$2^{k-1} - F, \quad p_k = 2^{-k}, \tag{124}$$

for $k \in \{1, 2, 3, \ldots\}$, *i.e.* the set of positive integers. The vast majority of observed payouts are small, but occasionally an extremely large payout (corresponding to a very long unbroken sequence of tails in the classical description) occurs. This is shown in the example trajectories in Fig. 10, where the lottery has been repeated additively.

From now on we will forget about the coin tosses, which are simply a mechanism for selecting one of the possible payouts. In effect, they are just a random number generator. Instead we shall work with the compact definition of the lottery in (Eq. 124) and assume it takes a fixed amount of time, δt, to play.

The rate of change of expected wealth is

$$\frac{\langle \delta x \rangle}{\delta t} = \frac{1}{\delta t} \sum_{k=1}^{\infty} p_k D_k \tag{125}$$

$$= \frac{1}{\delta t} \left(\$ \sum_{k=1}^{\infty} 2^{-k}\, 2^{k-1} - \sum_{k=1}^{\infty} 2^{-k} F \right) \tag{126}$$

$$= \frac{1}{\delta t} \left(\$ \sum_{k=1}^{\infty} \frac{1}{2} - F \right). \tag{127}$$

This diverges for any finite ticket price. Under the expected-wealth paradigm, this means that the lottery is favourable at any price.

This absurd conclusion, which does not accord with human behaviour, exposes the weakness of judging a gamble by its effect on expected wealth. Daniel Bernoulli suggested to resolve the paradox by adopting the expected-utility paradigm. His choice of utility function was the logarithm, $u(x) = \ln x$, which, as we now know, produces a decision rule equivalent to growth-rate optimisation under multiplicative repetition. This correspondence was not appreciated by Bernoulli: indeed, it is unclear whether 18^{th} century mathematics possessed the concepts and language required to distinguish between averages over time and across systems, even if it had the basic arithmetic tools. In any case, the correspondence relies on the choice of a particular utility function, and vanishes the moment something other than the logarithm is chosen.

Unfortunately, Bernoulli made a mathematical error in the implementation of his own paradigm. This was corrected by Laplace in 1814 [16] and then de-corrected by Menger in 1934 [21], who introduced a further error which led to the famous but unjustifiable claim that utility functions must be bounded. We will leave this most chequered part of the paradox's history alone – details, including a rebuttal of Menger's analysis, can be found in [28]. Instead we will focus on what's usually presumed Bernoulli meant to write.

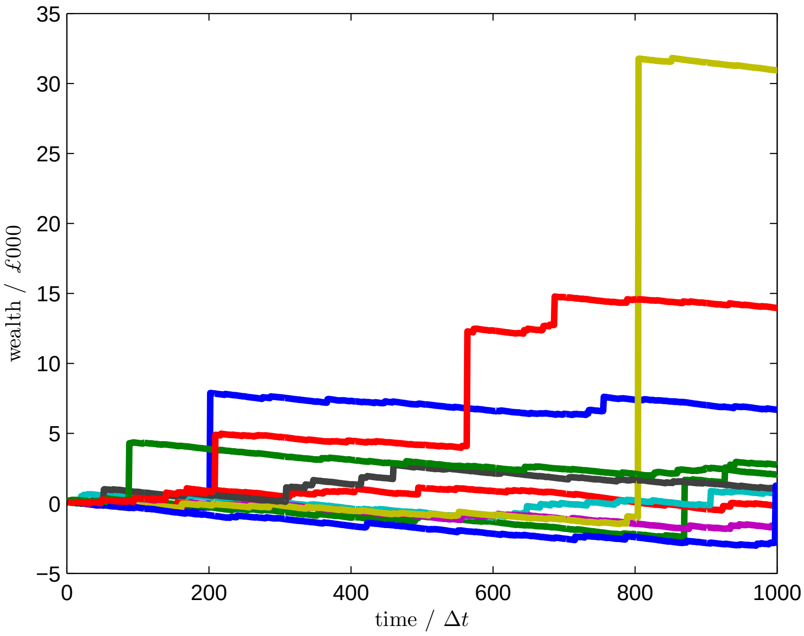


Figure 10: Wealth trajectories for the additively repeated St Petersburg lottery, with starting wealth, $x(0) = \$100$, and ticket price, $F = \$10$. Ten trajectories are plotted over 1,000 rounds.

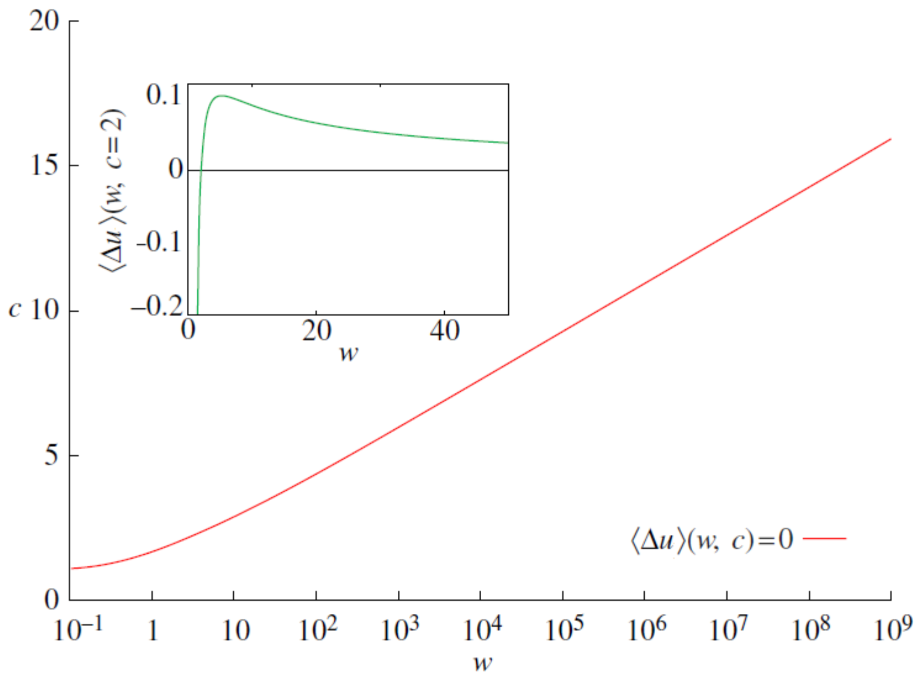


Figure 11: Locus of points in the (x, F)-plane for which the expected change in logarithmic utility is zero. The inset shows the expected change in utility as a function of x for $F = \$2$. Adapted from [25].

Example: Resolution by logarithmic utility

Instead of (Eq. 127), we calculate the rate of change of expected logarithmic utility,

$$\frac{\langle \delta \ln x \rangle}{\delta t} = \frac{1}{\delta t} \sum_{k=1}^{\infty} p_k \left[\ln(x + D_k) - \ln x\right] \tag{128}$$

$$= \frac{1}{\delta t} \sum_{k=1}^{\infty} 2^{-k} \ln \left(\frac{x + \$2^{k-1} - F}{x} \right), \tag{129}$$

where x is the ticket buyer's wealth.

This is finite for all finite ticket prices less than the buyer's wealth plus the smallest prize: $F < x + \$1$. This can be shown by applying the ratio test.[16] It may be positive or negative, depending on the values of F and x. Fig. 11 shows the locus of points in the (x, F)-plane for which the sum is zero.

This resolves the paradox, in that it admits the possibility of the player declining to buy a ticket when his expected change in utility is less than that of the null gamble, *i.e.* zero. Bernoulli argued for this resolution framework in

[16] The ratio of the $(k+1)^{\text{th}}$ term to the k^{th} term in the sum tends to $1/2$ as $k \to \infty$.

plausible terms, which we have already discussed: the usefulness of a monetary gain depends on how much money you already have. He also argued specifically for the logarithm in plausible terms: the gain in usefulness should be proportional to the fractional gain it represents, $du = \delta x/x$. He did not connect this to the idea of repetition over time. If he had, he might have been less willing to accept Cramer's square-root utility function as an alternative.[17]

However, while plausible, the framework relies on a utility function, which must be postulated. It can neither be derived from fundamental considerations nor verified empirically.

Turning to our decision algorithm, we will assume that the lottery is repeated multiplicatively. This means, in effect, that the prizes and ticket price are treated as fractions of the player's wealth, such that the effect of each lottery is to multiply the wealth by a random factor,

$$r_k = \frac{x + \$2^{k-1} - F}{x}, \quad p_k = 2^{-k}. \tag{130}$$

This follows precisely our earlier treatment of a gamble with multiplicative dynamics, and we can apply our results directly. The time-average (exponential) growth rate is

$$\bar{g}_{\mathrm{m}} = \frac{1}{\delta t} \lim_{T\to\infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} \ln r(\tau) \right\} = \frac{1}{\delta t} \sum_{k=1}^{\infty} 2^{-k} \ln r_k, \tag{131}$$

which is identical to the expression for the rate of change of expected log-utility, (Eq. 129). This is, as we've discussed, because $\bar{g}_{\mathrm{m}}$ is the ergodic growth rate for multiplicative dynamics. The result is the same, but the interpretation is different: we have assumed less, only that our player is interested in the growth rate of his wealth and that he gauges this by imagining the outcome of an indefinite sequence of repeated lotteries.

Thus the locus in Fig. 11 also marks the decision threshold *versus* the null gamble under our decision axiom. The player can sensibly decline the gamble, even though it results in a divergent change in expected wealth. This is illustrated by comparing Fig. 12, which shows trajectories of multiplicatively repeated lotteries, with the additively repeated lotteries already seen in Fig. 10. The trajectories are based on the same sequences of lottery outcomes, only the mode of repetition is different. The simulation shows us visually what we have already gleaned by analysis: that what appears favourable in the expected-wealth paradigm (corresponding to additive repetition) results in a disastrous decay of the player's wealth over time under a realistic dynamic.

As $F \to x + \$1$ from above in (Eq. 131), $\bar{g}_{\mathrm{m}}$ diverges negatively, since the first term in the sum is the logarithm of a quantity approaching zero. This corresponds to a lottery which can make the player bankrupt. The effect is also shown in the inset of Fig. 12.

Treatments based on multiplicative repetition have appeared sporadically in the literature, starting with Whitworth in 1870 [35, App. IV].[18] It is related to

[17] Although there exists a dynamic for which $u = \sqrt{x}$ gives the appropriate stationarity mapping, it is rather more esoteric than the multiplicative repetition with which most of humanity is familiar.

[18] Whitworth skewered the field of utility theory thus: "The result at which we have arrived

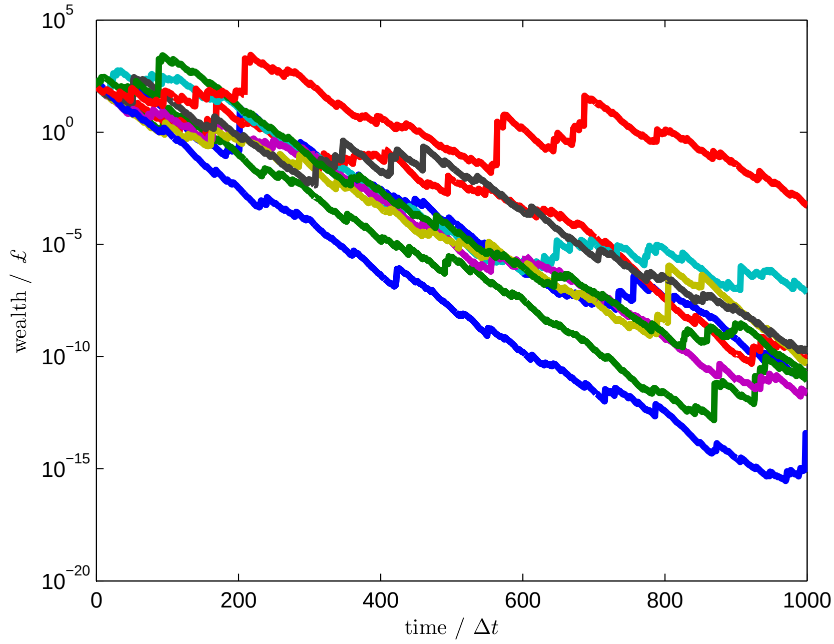


Figure 12: Wealth trajectories for the multiplicatively repeated St Petersburg lottery, with starting wealth, $x(0)$ = \$100, and ticket price, F = \$10. Ten trajectories are plotted over 1,000 rounds. The realisations of the individual lotteries are the same as in Fig. 10 but the mode of repetition is different.

the famous Kelly Criterion [14][19], although Kelly did not explicitly treat the St Petersburg game, and tangentially to Itô's lemma [12]. It appears as an exercise in a well-known text on information theory [6, Ex. 6.17]. Mainstream economics has ignored all this. A full and rigorous resolution of the paradox, including the epistemological significance of the shift from ensemble to time averages, was published recently by one of the present authors [25].

2.8 The insurance puzzle

The insurance contract is an important and ubiquitous type of economic transaction, which can be modelled as a gamble. However, it poses a puzzle [27]. In the expected-wealth paradigm, insurance contracts shouldn't exist, because buying insurance would only be rational at a price at which it would be irrational to sell. More specifically:

1. To be viable, an insurer must charge an insurance premium of at least the expectation value of any claims that may be made against it, called the "net premium" [13, p. 1].

2. The insurance buyer therefore has to be willing to pay more than the net premium so that an insurance contract may be successfully signed.

3. Under the expected-wealth paradigm it is irrational to pay more than the net premium, and therefore insurance contracts should not exist.

In this picture, an insurance contract can only ever be beneficial to one party. It has the anti-symmetric property that the expectation value of one party's gain is the expectation value of the other party's loss.

The puzzle is that insurance contracts are observed to exist.[20] Why? Classical resolutions appeal to utility theory (*i.e.* psychology) and asymmetric information (*i.e.* deception). However, our decision theory naturally predicts contracts with a range of prices that increase the time-average growth rate for both buyer and seller. We illustrate this with an example drawn from maritime trade, in which the use of insurance has a very long history.[21] A similar example was used by Bernoulli [3].

is not to be classed with the arbitrary methods which have been again and again propounded to evade the difficulty of the Petersburg problem. . . . Formulae have often been proposed, which have possessed the one virtue of presenting a finite result. . . but they have often had no intelligible basis to rest upon, or. . . sufficient care has not been taken to draw a distinguishing line between the significance of the result obtained, and the different result arrived at when the mathematical expectation is calculated." Sadly he chose to place these revolutionary remarks in an appendix of a college probability textbook.

[19] Kelly was similarly unimpressed with the mainstream and noted in his treatment of decision theory, which he developed from the perspective of information theory and which is identical to ergodicity economics with multiplicative dynamics, that the utility function is "too general to shed any light on the specific problems of communication theory."

[20] Something of an understatement. The Bank for International Settlements estimated the market value of all the world's derivatives contracts, which are essentially insurance contracts, as $15 trillion in the first half of 2015 (see `http://www.bis.org/statistics/d5_1.pdf`). That's six times the gross domestic product of the United Kingdom.

[21] Contracts between Babylonian traders and lenders were recorded around 1750 BC in the Code of Hammurabi. Chinese traders practised diversification by spreading cargoes across multiple vessels even earlier than this, in the third millennium BC.

Example: A shipping contract

We imagine a shipowner sending a cargo from St Petersburg to Amsterdam, with the following parameters:

- owner's wealth, $x_{\text{own}} = \$100,000$;
- gain on safe arrival of cargo, $G = \$4,000$;
- probability ship will be lost, $p = 0.05$;
- replacement cost of the ship, $C = \$30,000$; and
- voyage time, $\delta t = 1$ month.

An insurer with wealth $x_{\text{ins}} = \$1,000,000$ proposes to insure the voyage for a fee, $F = \$1,800$. If the ship is lost, the insurer pays the owner $L = G + C$ to make him good on the loss of his ship and the profit he would have made.

We phrase the decision the owner is facing as a choice between two gambles.

Definition The owner's gambles

Sending the ship uninsured corresponds to gamble o1

$$D_1^{(\text{o1})} = G, \qquad p_1^{(\text{o1})} = 1 - p; \tag{132}$$
$$D_2^{(\text{o1})} = -C, \qquad p_2^{(\text{o1})} = p. \tag{133}$$

Sending the ship fully insured corresponds to gamble o2

$$D_1^{(\text{o2})} = G - F \qquad p_1^{(\text{o2})} = 1. \tag{134}$$

This is a trivial "gamble" because all risk has been transferred to the insurer.

We also model the insurer's decision whether to offer the contract as a choice between two gambles

Definition The insurer's gambles

Not insuring the ship corresponds to gamble i1, which is the null gamble

$$D_1^{(\text{i1})} = 0 \qquad p_1^{(\text{i1})} = 1. \tag{135}$$

Insuring the ship corresponds to gamble i2

$$D_1^{(\text{i2})} = +F, \qquad p_1^{(\text{i2})} = 1 - p; \tag{136}$$
$$D_2^{(\text{i2})} = -L + F, \qquad p_2^{(\text{i2})} = p. \tag{137}$$

We ask whether the owner should sign the contract, and whether the insurer should have proposed it.

Example: Expected-wealth paradigm

In the expected-wealth paradigm (corresponding to additive repetition under the time paradigm) decision makers maximise the rate of change of the expectation values of their wealths, according to (Eq. 121): Under this paradigm the owner collapses gamble o1 into the scalar

$$\begin{aligned} \bar{g}_a^{(\mathrm{o1})} &= \frac{\langle \delta x \rangle}{\delta t} && (138)\\ &= \frac{\left\langle D^{(\mathrm{o1})} \right\rangle}{\delta t} && (139)\\ &= \frac{(1-p)G + p(-C)}{\delta t} && (140)\\ &= \$2,300 \text{ per month} && (141) \end{aligned}$$

and gamble o2 into the scalar

$$\begin{aligned} \bar{g}_a^{\mathrm{o2}} &= \frac{\left\langle D^{(\mathrm{o2})} \right\rangle}{\delta t} && (142)\\ &= \frac{(G-F)}{\delta t} && (143)\\ &= \$2,200 \text{ per month} && (144) \end{aligned}$$

The difference, $\delta\bar{g}_a^{\mathrm{o}}$, between the expected rates of change in wealth with and without a signed contract is the expected loss minus the fee per round trip,

$$\delta\bar{g}_a^{\mathrm{o}} = \bar{g}_a^{\mathrm{o2}} - \bar{g}_a^{\mathrm{o1}} = \frac{pL - F}{\delta t}. \tag{145}$$

The sign of this difference indicates whether the insurance contract is beneficial to the owner. In the example this is not the case, $\delta\bar{g}_a^{\mathrm{o}} = -\$100$ per month.

The insurer evaluates the gambles i1 and i2 similarly, with the result

$$\bar{g}_a^{(\mathrm{i1})} = \$0 \text{ per month} \tag{146}$$

and

$$\begin{aligned} \bar{g}_a^{(\mathrm{i2})} &= \frac{F - pL}{\delta t} && (147)\\ &= \$100 \text{ per month} && (148) \end{aligned}$$

Again we compute the difference – the net benefit to the insurer that arises from signing the contract

$$\delta\bar{g}_a^{\mathrm{i}} = \bar{g}_a^{\mathrm{i2}} - \bar{g}_a^{\mathrm{i1}} = \frac{F - pL}{\delta t}. \tag{149}$$

In the example this is $\delta\bar{g}_a^{\mathrm{i}} = \$100$ per month, meaning that in the world of the expected-wealth paradigm the insurer will offer the contract.

Because only one party (the insurer) is willing to sign, no contract will come into existence. We could think that we got the price wrong, and the contract would be sigend if offered at a different fee. But this is not the case, and that's

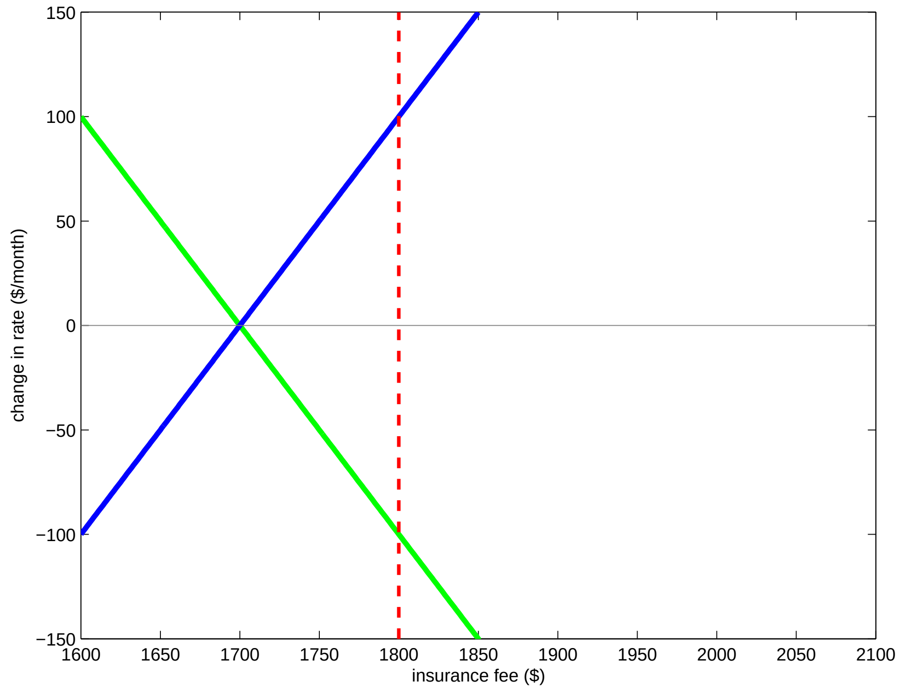


Figure 13: Change in the rate of change of expected wealth for the shipowner (green) and the insurer (blue) as a function of the insurance fee, F.

the fundamental insurance puzzle: in the world created by expected-wealth maximisation no price exists at which both parties will sign the contract.

Looking at (Eq. 145) and (Eq. 149) we notice the anti-symmetric relationship between the two expressions, $\delta\bar{g}_a^{\text{o}} = -\delta\bar{g}_a^{\text{i}}$. By symmetry, there can be no fee at which both expressions are positive. Hence there are no circumstances in the world created by the expected-wealth paradigm under which both parties will sign. Insurance contracts cannot exist in this world.

One party winning at the expense of the other makes insurance an unsavoury business in the expected-wealth paradigm. This is further illustrated in Fig. 13, which shows the change in the rate of change of expected wealth (the decision variable) for both parties as a function of the fee, F. There is no price at which the decision variable is positive for the both parties. The best they can do is to pick the price at which neither of them cares whether they sign or not.

In this picture, the existence of insurance contracts requires some asymmetry between the contracting parties, such as:

- different attitudes to bearing risk;
- different access to information about the voyage;
- different assessments of the riskiness of the voyage;
- one party to deceive, coerce, or gull the other into a bad decision.

It is difficult to believe that this is truly the basis for a market of the size and global reach of the insurance market.

2.8.1 Solution in the time paradigm

Example: Time paradigm

The insurance puzzle is resolved in the 'time paradigm', *i.e.* using the growth-optimal decision theory we have developed in this lecture and multiplicative repetition. Again, we pause to reflect what multiplicative repetition means compared to additive repetition. This is important because additive repetition is equivalent to the expected-wealth paradigm that created the insurance puzzle. Multiplicative repetition means that the ship owner sends out a ship and a cargo whose values are proportional to his wealth at the start of each voyage. A rich owner who has had many successful voyages will send out more cargo, a larger ship, or perhaps a *flotilla*, while an owner to whom the sea has been a cruel mistress will send out a small vessel until his luck changes. Under additive repetition, the ship owner would send out the same amount of cargo on each journey, irrespective of his wealth. Shipping companies of the size of Evergreen or Maersk would be inconceivable under additive repetition, where returns on successful investments are not reinvested.

The two parties seek to maximise

$$\bar{g}_m = \lim_{\Delta t \to \infty} \frac{\Delta v(x)}{\Delta t} = \frac{\langle \delta \ln x \rangle}{\delta t}, \tag{150}$$

where we have used the ergodic property of $\Delta v(x) = \Delta \ln x$ under multiplicative repetition.

The owner's time-average growth rate without insurance is

$$\bar{g}_m^{\text{o1}} = \frac{(1-p)\ln(x_{\text{own}} + G) + p\ln(x_{\text{own}} - C) - \ln(x_{\text{own}})}{\delta t} \tag{151}$$

or 1.9% per month. His time-average growth rate with insurance is

$$\bar{g}_m^{\text{o2}} = \frac{\ln(x_{\text{own}} + G - F) - \ln(x_{\text{own}})}{\delta t} \tag{152}$$

or 2.2% per month. This gives a net benefit for the owner of

$$\delta \bar{g}_m^{o} = \bar{g}_m^{\text{o1}} - \bar{g}_m^{\text{o2}} \approx +0.24\% \text{ per month} \tag{153}$$

The time paradigm thus creates a world where the owner will sign the contract.

What about the insurer? Without insurance, the insurer plays the null gamble, and

$$\bar{g}_m^{\text{i1}} = \frac{0}{\delta t} \tag{154}$$

or 0% per month. His time-average growth rate with insurance is

$$\bar{g}_m^{\mathrm{i2}} = \frac{(1-p)\ln(x_{\mathrm{ins}}+F) + p\ln(x_{\mathrm{ins}}+F-L) - \ln(x_{\mathrm{ins}})}{\delta t} \tag{155}$$

or 0.0071% per month. The net benefit to the insurer is therefore also

$$\delta \bar{g}_m^{\mathrm{i}} = \bar{g}_m^{\mathrm{i2}} - \bar{g}_m^{\mathrm{i1}} \tag{156}$$

i.e. 0.0071% per month. Unlike the expected wealth paradigm, the time paradigm with multiplicative repetition creates a world where an insurance contract can exist – there exists a range of fees F at which both parties gain from signing the contract!

We view this as the

Fundamental resolution of the insurance puzzle:

The buyer and seller of an insurance contract both sign when it increases the time-average growth rates of their wealths.

It requires no appeal to arbitrary utility functions or asymmetric circumstances, rather it arises naturally from the model of human decision-making that we have set out. Fig. 14 shows the mutually beneficial range of insurance fees predicted by our model. Generalizing, the message of the time paradigm is that business happens when both parties gain. In the world created by this model any agreement, any contract, any commercial interaction comes into existence because it is mutually beneficial.

2.8.2 The classical solution of the insurance puzzle

The classical solution of the insurance puzzle is identical to the classical solution of the St Petersburg paradox. Wealth is replaced by a non-linear utility function of wealth, which breaks the symmetry of the expected-wealth paradigm. While it is always true that $\delta \langle r \rangle_{\mathrm{own}} = -\delta \langle r \rangle_{\mathrm{ins}}$, the expected growth rates of non-linear utility functions don't share this anti-symmetry. A difference in the decision makers' wealths is sufficient, though often different utility functions are assumed for owner and insurer, which is a model that can create pretty much any behavior. The downside of a model with this ability is, of course, that it makes no predictions – nothing is ruled out, so the model cannot be falsified.

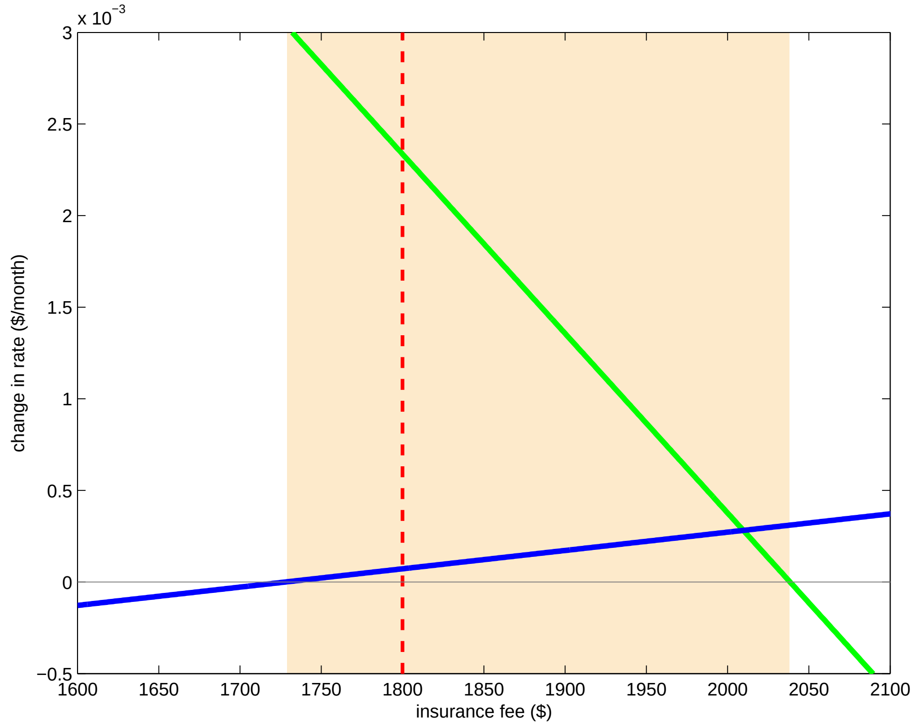


Figure 14: Change in the time-average growth rate of wealth for the shipowner (green) and the insurer (blue) as a function of the insurance fee, F. The mutually beneficial fee range is marked by the beige background.

3 Populations

The previous chapter developed a model of individual behaviour based on an assumed dynamic imposed on wealth. If we know the stochastic process that describes individual wealth, then we also know what happens at population level – each individual is represented by a realisation of the process, and we can compute the dynamics of wealth distributions. We answer questions about inequality and poverty in our model economy. It turns out that our decision criterion generates interesting emergent behaviour – cooperation, the sharing and pooling of resources is often time-average growth optimal. This provides answers to the puzzles why people cooperate, why there is an insurance market and why we see socio-economic structure from the formation of firms to nation states with taxation and redistribution systems.

3.1 Every man for himself

We have seen that risk aversion constitutes optimal behaviour under the assumptions of multiplicative wealth growth, and time scales that are long enough for systematic trends to be significant. In this chapter we will continue to explore our null model, GBM. By "explore" we mean that we will let the model generate its world – if individual wealth was to follow GBM, what kind of features of an economy would emerge? We will see that cooperation and the formation of social structure also constitute optimal behaviour.

GBM is more than a random variable, it's a stochastic process, *i.e.* a set of trajectories $x(t)$ or a family of random variables parameterized by t, depending on how we prefer to look at it. Both perspectives are informative in the context of economic modelling – from the set of trajectories we can judge what is likely to happen to an individual, *e.g.* by following an individual trajectory for a long time, and the PDF of the random variable $x(t^*)$ at some fixed value of t^* is the wealth distributed in our model.

We use the term wealth distribution to refer to the density function $\mathcal{P}_x(x)$ (not to the process of distributing wealth among people). This can be interpreted as follows: if I select a random individual (each individual with uniform probability $\frac{1}{N}$), the probability of the selected individual having wealth greater than x is given by the CDF $F_x(x) = \int_x^\infty ds \mathcal{P}_x(s)$. In a large population of N individuals, $\Delta x \mathcal{P}_x(x) N$ is the approximate number of individuals who have wealth near x. Thus, a broad wealth distribution with heavy tails indicates greater wealth inequality.

Examples:

- Under perfect equality everyone would have the same, meaning that the wealth distribution would be a Dirac delta function centered at the mean of x, that is
$$\mathcal{P}_x(x) = \delta(x - \langle x \rangle_N) \tag{157}$$
- Maximum inequality would mean that one individual owns everything and everyone else owns nothing, that is
$$\mathcal{P}_x(x) = \frac{N-1}{N}\delta(x-0) + \frac{1}{N}\delta(x - N\langle x \rangle_N). \tag{158}$$

3.1.1 Log-normal wealth distribution

GBM is log-normally distributed. If each individual's wealth follows GBM

$$dx = x(\mu dt + \sigma dW) \tag{159}$$

with solution

$$x(t) = x_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right], \tag{160}$$

we will observe a log-normal wealth distribution at each moment in time,

$$\ln x(t) \sim \mathcal{N}\left(\ln x_0 + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right). \tag{161}$$

We notice that the variance of $\ln x$ increases linearly in time – we will develop an understanding of this fact shortly. As we will see (though we cannot conclude this yet), it indicates that any meaningful measure of inequality will grow over time in our simple model. To see what kind of a wealth distribution (Eq. 161) is, it is worth spelling out the lognormal PDF

$$\mathcal{P}(x) = \frac{1}{x\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[\ln(x) - (\mu - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t}\right). \tag{162}$$

It is a well-established empirical observation [23] that the upper tails of real wealth distributions tend to look more like a power law than a log-normal. Our trivial model does not strictly reproduce this feature, but it is instructive to compare the lognormal distribution to a power-law distribution. A power law PDF has the aymptotic form

$$\mathcal{P}_x(x) = x^{-\alpha}, \tag{163}$$

for large arguments x. This implies that the logarithm of the PDF is proportional to the logarithm of its argument, $\ln \mathcal{P}_x(x) = -\alpha \ln x$. Plotting one against the other will yield a straight line, the slope being the exponent $-\alpha$.

Determining whether an empirical observation is consistent with such behaviour is difficult because the behaviour is to be observed in the tail (large x) where data are by definition sparse. A quick-and dirty way of checking for possible power-law behavior is to plot an empirical PDF against its argument on log-log scales, look for a straight line, and measure the slope. However, plotting any distribution on any type of scales results in some line. It may not be a straight line but it will have some slope everywhere. For a known distribution (power law or not) we can interpret this slope as a local apparent power-law exponent.

What is the local apparent power-law exponent of a log-normal wealth distribution near the expectation value $\langle x \rangle$, *i.e.* in the upper tail where approximate power law behavior has been observed empirically? The logarithm of (Eq. 162) is

$$\begin{aligned} \ln \mathcal{P}(x) &= -\ln x\sqrt{2\pi\sigma^2 t} - \frac{([\ln(x) - (\mu - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t} &(164)\\ &= -\ln x - \frac{\ln(2\pi\sigma^2 t)}{2} - \frac{(\ln(x)^2 + [(\mu - \frac{\sigma^2}{2})t]^2 - 2(\mu - \frac{\sigma^2}{2})t \ln x}{2\sigma^2 t} &(165)\end{aligned}$$

Collecting terms in powers of $\ln x$ we find

$$\ln \mathcal{P}(x) = [\ln x]^2 \times \frac{-1}{2\sigma^2 t} + \ln x \times \left(-\frac{3}{2} + \frac{\mu}{\sigma^2}\right) - \frac{\ln(2\pi\sigma^2 t)}{2} - \frac{[(\mu - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t} \tag{166}$$

with local slope, *i.e.* apparent exponent,

$$\frac{d \ln \mathcal{P}(x)}{d \ln x} = -\frac{\ln x}{\sigma^2 t} - \frac{3}{2} + \frac{\mu}{\sigma^2}. \tag{167}$$

If the first term is small compared to the others, this distribution will look like a power law when plotted on double-logarithmic scales. We don't believe that the empirically observed power laws are merely a manifestation of this mathematical feature. Important real-world mechanisms that broaden real wealth distributions, *i.e.* concentrate wealth, are missing from the null model. However, it is interesting that the trivial model of GBM chimes with so many qualitative features of empirical observations.

3.1.2 Inequality measure from two growth rates

In the case of GBM we have just seen how to compute the exact full wealth distribution $\mathcal{P}$. This is intersting, but we often only want summary measures of the distribution. A distributional property of particular interest to economists is inequality. How much inequality is there in a distribution like (Eq. 161), and in what sense does this quantity increase in time under GBM as we have pointed out? Clearly, we should quantify "inequality". In this section we develop a natural way of measuring it, which makes use of the two growth rates we identified for the non-ergodic process. We will see that a particular inequality measure, known to economists as Theil's second index of inequality [33], is the difference between typical wealth (that grows at the time-average growth rate) and average wealth (that grows at the ensemble-average growth rate) in our model. Thus, the difference between the time average and ensemble average, the essence of ergodicity breaking, fundamentally drives the dynamics of inequality.

The two limits of inequality are easily identified: minimum inequality means that everyone has the same wealth, and maximum inequality means that one individual has all the wealth and everyone else has nothing (this assumes that wealth cannot become negative). Quantifying inequality in any other distribution is reminiscent of the gamble problem. Recall that for gambles we wanted make statements of the type "this gamble has more desirability than that gamble". We did this by collapsing a distribution to a scalar. Depending on the question that was being asked the appropriate way of collapsing the distribution and the resulting scalar can be different (the scalar relevant to an insurance company may not be relevant to an individual). In the case of inequality we also have a distribution – the wealth distribution – and we want to make statements of the type "this distribution has more inequality than that distribution". Again, this is done by collapsing the distribution to a scalar, and again many different choices of collapse and resulting scalar are possible. The Gini coefficient is a particularly well-known scalar of this type, the 80/20 ration is another, and many other measures exist.

In this context the expectation value is an important quantity. For instance, if everyone has the same wealth, everyone will own the average $\forall i, x_i = \langle x \rangle_N$, which converges to the expectation value for large N. Also, whatever the distribution of wealth, the total wealth is $N \langle x \rangle_N$ which converges to $N \langle x \rangle$. The growth rate of the expectation value, $g_{\langle\rangle}$, thus tells us how fast the average wealth and the total population wealth grow with probability one in a large ensemble. The time-average growth rate, $\bar{g}$, tells us how fast an individual's wealth grows with probability one in the long run. If the typical individual's wealth grows at a lower rate than the expectation value of wealth then there must be a-typical individuals with very large wealth that account for the difference. This suggests the following measure of inequality.

<u>Definition</u> inequality $J(t)$, is the quantity that grows in time at the rate of the difference between expectation-value and time-average growth rates,

$$\frac{dJ}{dt} = g_{\langle\rangle} - \bar{g}. \tag{168}$$

Equation (168) defines the dynamic of inequality, and inequality itself

is found by integrating over time

$$J(t) = \int_0^t ds[g_{\langle\rangle}(s) - \bar{g}(s)]. \tag{169}$$

This definition may be used for dynamics other than GBM. Whatever the wealth dynamic, typical minus average growth rates are informative of the dynamic of inequality. Within the GBM framework we can write down the two growth rates explicitly and find

$$\frac{dJ}{dt} = \frac{d \ln \langle x \rangle}{dt} - \frac{d \langle \ln x \rangle}{dt}. \tag{170}$$

Integrating over time

$$J(t) = \ln \langle x \rangle - \langle \ln x \rangle, \tag{171}$$

which is known as the mean logarithmic deviation (MLD) or Theil's second index of inequality [33]. This is rather remarkable. Our general inequality measure, (Eq. 169), evaluated for the specific case of GBM, turns out to be a well-known measure of inequality that economists identified independently, without considering non-ergodicity and ensemble average and time average growth rates. Merely by insisting of measuring inequality well, Theil implicitly used the GBM model!

The expectation value $\langle \ln x \rangle = \ln x_0 + \left(\mu - \frac{\sigma^2}{2}\right) t$ is given by (Eq. 161). It differs from the logarithm of the expectation value, $\langle \ln x \rangle \neq \ln \langle x \rangle$, which we now compute. In order to do this we introduce a useful trick that will come in handy again in Sec. 3.3.1 (the general procedure is described in [15, Chapter 4.2]): to compute moments, $\langle x^n \rangle$, of stochastic differential equations for x, like (Eq. 159), we find solvable ordinary differential equations for the moments. For the first moment we do this simply by taking expectations of both sides of (Eq. 159). The noise term disappears, and we turn the SDE for x into an ODE for $\langle x \rangle$

$$\begin{aligned} \langle dx \rangle &= \langle x(\mu dt + \sigma dW) \rangle && (172)\\ d\langle x \rangle &= \langle x \rangle \mu dt + \sigma \overbrace{\langle dW \rangle}^{=0} && (173)\\ &= \langle x \rangle \mu dt. && (174) \end{aligned}$$

This is a (very simple) first order linear differential equation for the first moment (*i.e.* the expectation value) of x. Its solution is

$$\langle x \rangle = \langle x_0 \rangle \exp(\mu t) \tag{175}$$

so that $\ln \langle x \rangle = \ln \langle x_0 \rangle + \mu t$. Now we can carry out the differentiations on the RHS of (Eq. 170) and find the Theil inequality as a function of time

$$J(t) = J(0) + \frac{\sigma^2}{2} t. \tag{176}$$

Inequality increases indefinitely. This result implies an evolution towards wealth condensation. Wealth condensation means that a single individual will own a non-zero fraction of the total wealth in the population in the limit of large N, see *e.g.* [4]. In the present case an arbitrarily large share of total wealth will be owned by an arbitrarily small share of the population.

Over the decades, economists have arrived at many inequality measures, and have drawn up a list of conditions that particularly useful measures of inequality satisfy. Such measures are called "relative measures" [31, Appendix 4], and J is one of them. One of the conditions is that inequality measures should not change when x is divided by the same factor for everyone. Since we are primarily interested in inequality in this section it is useful to remove absolute wealth levels from the analysis and study an object called the rescaled wealth.

Definition **The rescaled wealth,**

$$y = \frac{x}{\langle x \rangle_N}, \tag{177}$$

is the proportion of the population-average wealth owned by an individual.

This quantity is useful, for instance because its numerical value does not depend on the currency used, it is a dimensionless number. Thus if my rescaled wealth, $y = 1/2$, this means that my wealth is half the average wealth, irrespective of whether I measure wealth in Kazakhstani Tenge or in Swiss Francs. Equation (170), may be expressed in terms of y as $\frac{dJ}{dt} = -\frac{d\langle \ln y \rangle}{dt}$.

3.2 Cooperation

Under multiplicative growth, fluctuations are undesirable because they reduce time-average growth rates. In the long run, wealth $x_1(t)$ with noise term σ_1 will outperform wealth $x_2(t)$ with a larger noise term $\sigma_2 > \sigma_1$, in the sense that

$$\bar{g}(x_1) > \bar{g}(x_2) \tag{178}$$

with probability 1.

For this reason it is desirable to reduce fluctuations. One protocol that achieves this is resource pooling and sharing. In Sec. 3.1 we explored the world created by the model of independent GBMs. This is a world where everyone experiences the same long-term growth rate. We want to explore the effect of the invention of cooperation. As it turns out cooperation increases growth rates, and this is a crucial insight.

Suppose two individuals, $x_1(t)$ and $x_2(t)$ decide to meet up every Monday, put all their wealth on a table, divide it in two equal amounts, and go back to their business, *i.e.* they submit their wealth to our toy dynamic (Eq. 159). How would this operation affect the dynamic of the wealth of these two individuals?

Consider a discretized version of (Eq. 159), such as would be used in a numerical simulation. The non-cooperators grow according to

$$\Delta x_i(t) = x_i(t)\left[\mu \Delta t + \sigma\sqrt{\Delta t}\,\xi_i\right], \tag{179}$$

$$x_i(t+\Delta t) = x_i(t) + \Delta x_i(t), \tag{180}$$

where ξ_i are standard normal random variates, $\xi_i \sim \mathcal{N}(0,1)$.

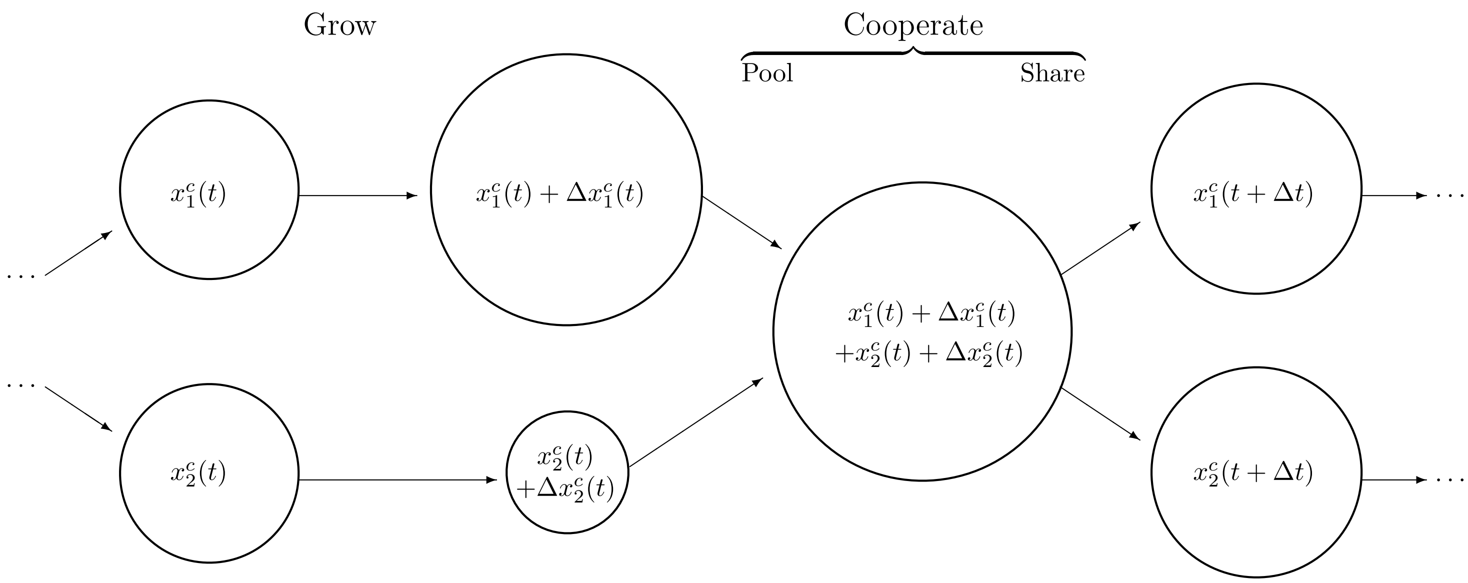


Figure 15: Cooperation dynamics. Cooperators start each time step with equal resources, then they *grow* independently according to (Eq. 182), then they *cooperate* by *pooling* resources and *sharing* them equally, then the next time step begins.

We imagine that the two previously non-cooperating entities, with resources $x_1(t)$ and $x_2(t)$, cooperate to produce two entities, whose resources we label $x_1^c(t)$ and $x_2^c(t)$ to distinguish them from the non-cooperating case. We envisage equal sharing of resources, $x_1^c = x_2^c$, and introduce a cooperation operator, $\oplus$, such that

$$x_1 \oplus x_2 = x_1^c + x_2^c. \tag{181}$$

In the discrete-time picture, each time step involves a two-phase process. First there is a growth phase, analogous to (Eq. 159), in which each cooperator increases its resources by

$$\Delta x_i^c(t) = x_i^c(t) \left[\mu \Delta t + \sigma \sqrt{\Delta t}\, \xi_i \right]. \tag{182}$$

This is followed by a cooperation phase, replacing (Eq. 180), in which resources are pooled and shared equally among the cooperators:

$$x_i^c(t + \Delta t) = \frac{x_1^c(t) + \Delta x_1^c(t) + x_2^c(t) + \Delta x_2^c(t)}{2}. \tag{183}$$

With this prescription both cooperators and their sum experience the following dynamic:

$$(x_1 \oplus x_2)(t + \Delta t) = (x_1 \oplus x_2)(t) \left[1 + \left(\mu \Delta t + \sigma \sqrt{\Delta t}\, \frac{\xi_1 + \xi_2}{2} \right) \right]. \tag{184}$$

For ease of notation we define

$$\xi_{1 \oplus 2} = \frac{\xi_1 + \xi_2}{\sqrt{2}}, \tag{185}$$

which is another standard Gaussian, $\xi_{1\oplus 2} \sim \mathcal{N}(0,1)$. Letting the time increment $\Delta t \to 0$ we recover an equation of the same form as (Eq. 159) but with a different fluctuation amplitude,

$$d(x_1 \oplus x_2) = (x_1 \oplus x_2)\left(\mu dt + \frac{\sigma}{\sqrt{2}} dW_{1\oplus 2}\right). \tag{186}$$

The expectation values of a non-cooperator, $\langle x_1(t)\rangle$, and a corresponding cooperator, $\langle x_1^c(t)\rangle$, are identical. Based on expectation values, we thus cannot see any benefit of cooperation. Worse still, immediately after the growth phase, the better-off entity of a cooperating pair, $x_1^c(t_0) > x_2^c(t_0)$, say, would increase its expectation value from $\frac{x_1^c(t_0)+x_2^c(t_0)}{2}\exp(\mu(t-t_0))$ to $x_1^c(t_0)\exp(\mu(t-t_0))$ by breaking the cooperation. But it would be foolish to act on the basis of this analysis – the short-term gain from breaking cooperation is a one-off, and is dwarfed by the long-term multiplicative advantage of continued cooperation. An analysis based on expectation values finds that there is no reason for cooperation to arise, and that if it does arise there are good reasons for it to end, *i.e.* it will be fragile. Because expectation values are inappropriately used to evaluate future prospects, the observation of widespread cooperation constitutes a conundrum.

The solution of the conundrum comes from considering the time-average growth rate. The non-cooperating entities grow at $g_t(x_i) = \mu - \frac{\sigma^2}{2}$, whereas the cooperating unit benefits from a reduction of the amplitude of relative fluctuations and grows at $g_t(x_1 \oplus x_2) = \mu - \frac{\sigma^2}{4}$, and we have

$$g_t(x_1 \oplus x_2) > g_t(x_i) \tag{187}$$

for any non-zero noise amplitude. Imagine a world where cooperation does not exist, just like in Sec. **??**. Now introduce into this world two individuals who have invented cooperation – very quickly this pair of individuals will be vastly more wealthy than anyone else. To keep up, others will have to start cooperating. The effect is illustrated in Fig. 16 by direct simulation of (Eq. 179)–(Eq. 180) and (Eq. 184).

Imagine again the pair of cooperators outperforming all of their peers. Other entities will have to form pairs to keep up, and the obvious next step is for larger cooperating units to form – groups of 3 may form, pairs of pairs, cooperation clusters of n individuals, and the larger the cooperating group the closer the time-average growth rate will get to the expectation value. For n cooperators, $x_1 \oplus x_2 ... \oplus x_n$ the spurious drift term is $-\frac{\sigma^2}{2n}$, so that the time-average growth approaches expectation-value growth for large n. The approach to this upper bound as the number of cooperators increases favours the formation of social structure.

We may generalise to different drift terms, μ_i, and noise amplitudes, σ_i, for different individual entities. Whether cooperation is beneficial in the long run for any given entity depends on these parameters as follows. Entity 1 will benefit from cooperation with entity 2 if

$$\mu_1 - \frac{\sigma_1^2}{2} < \frac{\mu_1+\mu_2}{2} - \frac{\sigma_1^2+\sigma_2^2}{8}. \tag{188}$$

We emphasize that this inequality may be satisfied also if the expectation value of entity 1 grows faster than the expectation values of entity 2, *i.e.* if $\mu_1 > \mu_2$. An

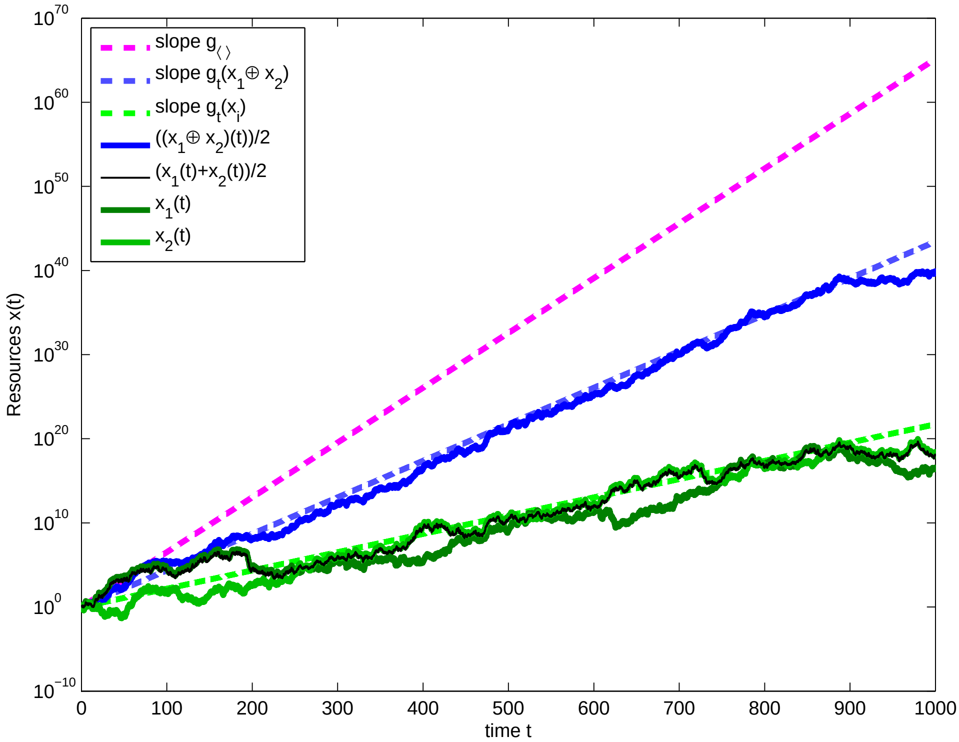


Figure 16: Typical trajectories for two non-cooperating (green) entities and for the corresponding cooperating unit (blue). Over time, the noise reduction for the cooperator leads to faster growth. Even without effects of specialisation or the emergence of new function, cooperation pays in the long run. The black thin line shows the average of the non-cooperating entities. While in the logarithmic vertical scale the average traces the more successful trajectory, it is far inferior to the cooperating unit. In a very literal mathematical sense the whole, $(x_1 \oplus x_2)(t)$, is more than the sum of its parts, $x_1(t) + x_2(t)$. The algebra of cooperation is not merely that of summation.

analysis of expectation values, again, is utterly misleading: the benefit conferred on entity 1 due to the fluctuation-reducing effect of cooperation may outweigh the cost of having to cooperate with an entity with smaller expectation value.

Notice the nature of the Monte-Carlo simulation in Fig. 16. No ensemble is constructed. Only individual trajectories are simulated and run for a time that is long enough for statistically significant features to rise above the noise. This method teases out of the dynamics what happens over time. The significance of any observed structure – its epistemological meaning – is immediately clear: this is what happens over time for an individual system (a cell, a person's wealth, *etc.*). Simulating an ensemble and averaging over members to remove noise does not tell the same story. The resulting features may not emerge over time. They are what happens on average in an ensemble, but – at least for GBM – this is not what happens to the individual with probability 1. For instance the pink dashed line in Fig. 16 is the ensemble average of $x_1(t)$, $x_1(t)$, and $(x_1 \oplus x_2)(t)/2$, and it has nothing to do with what happens in the individual trajectories over time.

When judged on expectation values, the apparent futility of cooperation is unsurprising because expectation values are the result for infinitely many cooperators, and adding further cooperators cannot improve on this.

In our model the advantage of cooperation, and hence the emergence of social structure in the broadest sense – is purely a non-linear effect of fluctuations – cooperation reduces the magnitude of fluctuations, and over time (though not in expectation) this implies faster growth.

Another generalisation is partial cooperation – entities may share only a proportion of their resources, resembling taxation and redistribution. We discuss this in the next section.

3.3 Taxation

In Sec. 3.1 we created a world of independent GBMs; in Sec. 3.2 we introduced to this world the invention of cooperation and saw that it increases long-time growth for those who participate in resource-pooling and sharing. In this section we study what happens if a large number of individuals pool and share a small fraction of their resources, which is reminiscent of taxation and redistribution carried out in a large population. We will find that while cooperation for 2 individuals increases their growth rates, sufficient cooperation in a large population has two related effects. Firstly, everyone's wealth grows asymptotically at a rate close to that of the expectation value. Secondly, wealth condensation and the divergence of inequality no longer occur.

We introduce a model that applies a flat wealth tax rate and every individual, irrespective of his wealth, receives the same benefit from the collected tax, in absolute terms. This mimicks the actions of a central agency that collects each year from everyone 1% of his wealth and pays 1-N^{th} of the total collected amount to each individual. A similar model will be used for income tax, see (Eq. 208) in Sec. 3.3.2.

Of course this isn't how taxation works in reality – wealth taxes are usually only collected in the form of inheritance tax and sometimes property or land tax; often progressive rates are applied, and how tax takings are actually redistributed is very unclear. Who benefits from government activity? Infrastructure is built, benefits payments made, healthcare and education provided, a legal system is maintained of courts that can enforce contracts and enable corporate structures, police and an army may provide security. Individuals will benefit from these different aspects to very different degrees. Our model ignores this and lets everyone benefit equally.

Despite the simplicity of the setup the following important feature emerges: there is a critical tax rate. This qualitative result applies both to income tax and to wealth tax.

Definition Critical tax rate
Below the critical tax rate the variance of rescaled wealth increases indefinitely. Above the critical tax rate it stabilizes to an asymptotic value in the limit $t \to \infty$.

Section 3.2 was concerned with growth, here we are concerned with inequality. We will therefore work with the rescaled wealth, y, introduced in (Eq. 177). Equation (159) defines the dynamic of x. From it we can find the dynamic for $f(x) = y$ using Itô calculus

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dx^2 && (189) \\ = dy &= -\mu y dt + \frac{y}{x}dx && (190) \\ &= y\sigma dW && (191) \end{aligned}$$

3.3.1 Wealth tax

We investigate the situation where each individual's wealth is taxed at a rate of $0 \leq \tau \leq 1$ per unit time, and the total tax thus raised is redistributed equally among the population. This is modelled by the stochastic wealth process,

$$dx = x[(\mu - \tau)\, dt + \sigma\, dW] + \tau \langle x \rangle_N \, dt, \tag{192}$$

which is a modified version of (Eq. 159) – the term $-\tau x dt$ was added to represent tax collection, and the term $+\tau \langle x \rangle_N \, dt$ to represent redistribution of collected tax. To make the model more tractable we consider the case $N \to \infty$, which replaces the finite-ensemble average by the expectation value, $\langle x \rangle_N \to \langle x \rangle$. The finite ensemble size has important effects but we will not discuss them here. Total wealth is conserved by the taxation and redistribution process in this model, and the expectation value is unaffected, $\langle x(t) \rangle = \langle x_0 \rangle e^{\mu t}$, just as for GBM without taxation, (Eq. 175). We are again interested in rescaled wealth, $y = \frac{x}{\langle x \rangle} = x e^{-\mu t}$ ((Eq. 177)), whose dynamic we derive using the chain rule

$$\begin{aligned} dy &= \frac{\partial y}{\partial t}\, dt + \frac{\partial y}{\partial x}\, dx + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}\, dx^2 && (193) \\ &= -\mu y\, dt + \frac{1}{x}\, dx && (194) \\ &= y(-\tau dt + \sigma dW) + \tau dt. && (195) \end{aligned}$$

The first moment of y is trivially 1,

$$\langle y \rangle = \left\langle \frac{x}{\langle x \rangle} \right\rangle = 1. \tag{196}$$

We compute the dynamic of the second moment of y, to first order in dt, using the chain rule again,

$$
\begin{aligned}
d(y^2) &= \frac{\partial(y^2)}{\partial t}\,dt + \frac{\partial(y^2)}{\partial y}\,dy + \frac{1}{2}\frac{\partial^2(y^2)}{\partial y^2}\,(dy)^2 && (197)\\
&= 2ydy + (dy)^2 && (198)\\
&= 2y^2(-\tau dt + \sigma dW) + 2y\tau dt + y^2\sigma^2 dt. && (199)
\end{aligned}
$$

Taking expectation values yields

$$
\begin{aligned}
\left\langle dy^2\right\rangle &= \left\langle -2y^2(-\tau dt + \sigma dW) + 2y\tau dt - y^2\sigma^2 dt\right\rangle && (200)\\
= d\left\langle y^2\right\rangle &= (\sigma^2 - 2\tau)\left\langle y^2\right\rangle dt + 2\tau dt. && (201)
\end{aligned}
$$

This equation is an inhomogeneous first-order ordinary differential equation for the second moment. Perhaps it's more recognizable when written in standard form as

$$\left(\frac{d}{dt} - (\sigma^2 - 2\tau)\right)\left\langle y^2\right\rangle = 2\tau. \tag{202}$$

Such equations are solvable using the method of integrating factors, see *e.g.* [2, Chpater 1.5]. The solution of the dynamic (Eq. 202) is the second moment of the distribution of rescaled wealth as a function of time, namely

$$\left\langle y^2\right\rangle = \frac{2\tau}{2\tau - \sigma^2} + \left(\left\langle y_0^2\right\rangle - \frac{2\tau}{2\tau - \sigma^2}\right)e^{-(2\tau-\sigma^2)t}. \tag{203}$$

This can be rewritten in terms of the variance of rescaled wealth, $V = \left\langle y^2\right\rangle - 1$, as

$$V(t) = V_\infty + (V_0 - V_\infty)e^{-(2\tau-\sigma^2)t}, \tag{204}$$

where V_0 is the initial variance and

$$V_\infty \equiv \frac{\sigma^2}{2\tau - \sigma^2}. \tag{205}$$

V converges in time to the asymptote, V_∞, provided the exponential in (Eq. 204) is decaying. This can be expressed as a condition on τ,

$$\boxed{\tau > \tau_c \equiv \frac{\sigma^2}{2},} \tag{206}$$

which defines the critical tax rate, τ_c. Above this critical tax rate, $\tau > \tau_c$, the variance of the rescaled-wealth distribution stabilises. Below it, the variance grows beyond all bounds. We believe that the divergence or convergence of the variance signals an important change in systemic behavior, but we hasten to point out the following caveat: a finite second moment does not guarantee finiteness of higher moments. A deeper analysis of ODEs of the type of (Eq. 202), which we don't reproduce here, reveals that any finite wealth tax rate implies that all moments of order $n > \frac{2\tau}{\sigma^2} + 1$ diverge. Under the flat wealth tax investigated here, the wealth distribution never fully stabilizes. In the language often used by economists in this debate, an ergodic wealth distribution does not exist for our model.

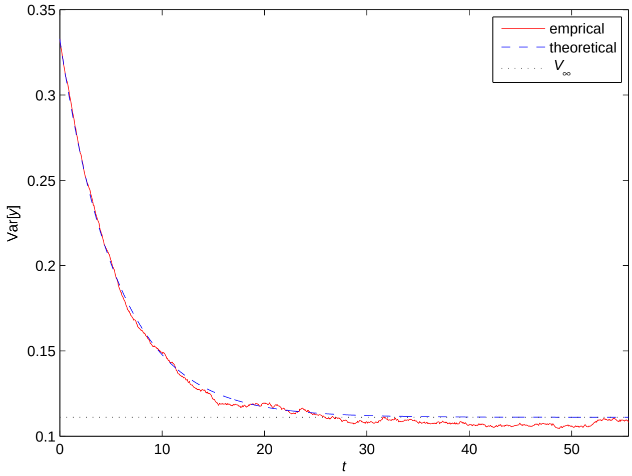


Figure 17: Wealth tax. The empirical variance of the rescaled wealths of $N = 10^4$ realisations of (Eq. 192) with uniformly-distributed initial wealths (red); the theoretical variance for the infinite ensemble, $V(t)$ (blue dashed); and the asymptotic theoretical variance, V_∞ (black dotted). Parameter values are $\mu = 0.05$, $\sigma^2 = 0.02$, and $\tau = 0.1$ per unit time.

Caveats aside, (Eq. 204) also allows us to identify a characteristic timescale over which the variance stabilises for supercritical taxation,

$$T_s = \frac{1}{2\tau - \sigma^2}. \tag{207}$$

τ_c may be viewed as the tax rate at which T_s diverges.

Numerical simulations confirm that the above analytical results are informative for finite ensembles. Fig. 17 compares the evolution of the empirical variance of the rescaled wealths of $N = 10^4$ realisations of the stochastic wealth process in (Eq. 192) with the theoretical result for the infinite ensemble in (Eq. 204). Parameter values were $\mu = 0.05$, $\sigma^2 = 0.02$, and $\tau = 0.1$ per unit time, of which the first two are realistic for a time unit of one year [26] (assuming individual wealth processes share parameters with stock market indices). The differences are finite-sample effects. Fig. 18 shows the initial distribution of rescaled wealths, which was chosen to be uniform, and the final distribution at the end of the period shown in Fig. 17.

The simulated parameter values give a critical tax rate of 1% pa. This is broadly in line with genuine annual wealth and property taxes in the few countries in which they are levied. Under the simulated tax rate of 10% pa, the stabilisation time is $T_s \approx 6$ years. It is hard to imagine a wealth tax of this magnitude being politically feasible in the real world. In our simple model, the tax rate could be set either to achieve convergence of inequality to a desired level, reflected by V_∞, or over a desired timescale, represented by T_s.

It is interesting to connect this with the most widely levied wealth tax: the inheritance tax. In the UK this is levied at 40% of the value of an individual's es-

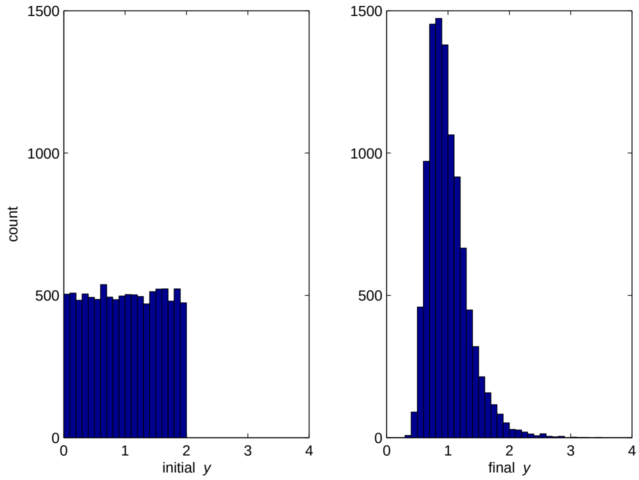


Figure 18: Histograms of the initial (left) and final (right) empirical distributions of the rescaled wealth for the same realisations of (Eq. 192) used in Fig. 17.

tate (above a certain threshold) upon death. We can surmise that an individual will typically hold most of his wealth for the human generation time of around 30 years, this being a sensible estimate of the time between inheriting or otherwise accumulating his wealth and passing it on. Using our plausible parameter values, an inheritance tax of 40% corresonds to an annually compounded wealth tax of $1-(0.6)^{1/30} \approx 1.7\%$ pa and a stabilisation time of around 70 years. The former is close to and, notably, above the critical rate of 1% pa, suggesting that variance stabilisation may be an influential criterion in the determination of our taxes.

3.3.2 Income tax

In our very simple model, we have seen that a flat wealth tax can stabilize the variance of the rescaled-wealth distribution. In this section we show that in a similarly simple model an income tax can achieve the same result. We introduce a model of income tax under which a fraction, $0 \leq \tau \leq 1$, of each individual's determinsitic wealth increment, $\mu x\, dt$, is deducted and the total tax raised is redistributed equally. This is modelled by the stochastic wealth process,

$$dx = x[\mu(1-\tau)\, dt + \sigma\, dW] + \mu\tau \langle x \rangle_N\, dt. \tag{208}$$

Again, we consider the large-population limit $N \to \infty$, corresponding to the replacement $\langle x \rangle_N \to \langle x \rangle$. For positive drift, $\mu > 0$, the deterministic increment, $\mu x\, dt$, is guaranteed to be positive. It can be thought of as the income derived from that individual's activities, such as employment, on which governments typically levy taxes. Note that τ in (Eq. 208) is a dimensionless number, whereas it is a rate of dimension "per unit time" in (Eq. 192). The form of (Eq. 208) is identical to (Eq. 192) with the parameter transformation $\tau \to \mu\tau$. Thus we can immediately deduce the dynamic for the rescaled wealth as

$$dy = y(-\mu\tau\, dt + \sigma\, dW) + \mu\tau\, dt. \tag{209}$$

The variance stabilisation condition analogous to (Eq. 206) becomes

$$\boxed{\tau > \tau_c \equiv \frac{\sigma^2}{2\mu}.} \tag{210}$$

This defines the critical income tax, τ_c, above which the variance converges to its asymptotic value,

$$V_\infty = \frac{\sigma^2}{2\mu\tau - \sigma^2}, \tag{211}$$

according to

$$V(t) = V_\infty + (V_0 - V_\infty)e^{-(2\mu\tau - \sigma^2)t}. \tag{212}$$

Finally, the stabilisation time is

$$T_s = \frac{1}{2\mu\tau - \sigma^2}. \tag{213}$$

Fig. 19 compares the evolution of the empirical variance of the rescaled wealths of 10^4 realisations of the stochastic wealth process in (Eq. 208) with the theoretical result for the infinite ensemble. Parameter values were $\mu = 0.05$ and $\sigma^2 = 0.02$ per unit time, and $\tau = 0.45$. The latter is the UK's limiting income tax rate for large incomes, which will be the determining tax rate for variance stabilisation.

The finite-sample deviations from the infinite-ensemble result are larger in Fig. 19 than in Fig. 17. This is due entirely to the simulated parameter values: (Eq. 192) and (Eq. 208) can be made equivalent by choosing different parameters.

Fig. 20 shows the initial distribution of rescaled wealths, which was chosen to be uniform, and the final distribution at the end of the period shown in Fig. 19. The distribution of wealths under income tax has an appreciably longer tail than under wealth tax. As before this is a function of the parameter choices. The simulated parameter values have a critical income tax rate of $\tau_c = 0.2$ and a stabilisation time of $T_s = 40$ years. Thus the UK sets its income tax at a level which, at least in this simple framework, has a variance stabilising effect.

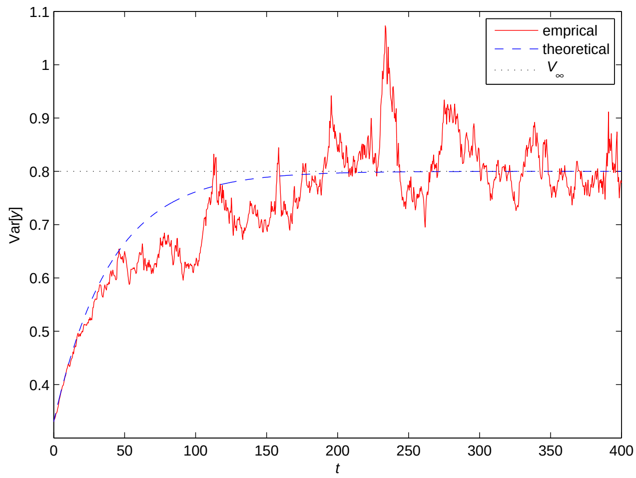


Figure 19: Income tax. The empirical variance of the rescaled wealths of 10^4 realisations of (Eq. 208) with uniformly-distributed initial wealths (red); the theoretical variance for the infinite ensemble, $V(t)$ (blue dashed); and the asymptotic theoretical variance, V_∞ (black dotted). Parameter values are $\mu = 0.05$ and $\sigma^2 = 0.02$ per unit time, and $\tau = 0.45$.

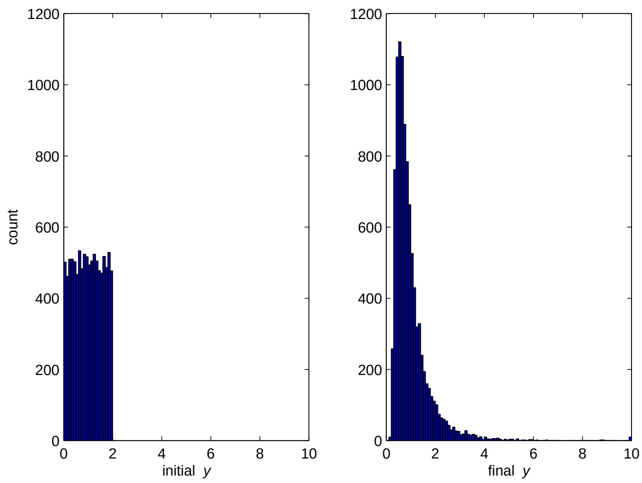


Figure 20: Histograms of the initial (left) and final (right) empirical distributions of the rescaled wealth for the same realisations of (Eq. 209) used in Fig. 19.

4 Markets

This lecture applies the ideas developed in the first two lectures to markets. It is based on two recent papers [24, 26].

Firstly, we set up a simple portfolio selection problem in a market of two assets: one risky, like a stock; and the other riskless, like a bank deposit. We ask how an investor should best allocate his money between the two assets, which we phrase in terms of his leverage. We review the classical approach, which can't answer this question without additional information about the investor's risk preferences. We then use the decision theory we've developed so far to answer the question unambiguously, by deriving the optimal leverage which maximises the investment's time-average growth rate.

Secondly, we consider what this objectively defined optimal leverage might mean for financial markets themselves. If all the participants in a market aim for the same optimal leverage, does this constrain the prices and price fluctuations that emerge from their trading? We argue that it does, we quantify how, and we confirm our prediction using data collected from the American stock markets over almost sixty years.

4.1 Optimal leverage

4.1.1 A model market

We consider assets whose values follow multiplicative dynamics, which we will model using GBM. In general, an amount x invested in such an asset evolves according to the SDE,

$$dx = x(\mu dt + \sigma dW), \tag{214}$$

where μ is the drift and σ is the volatility. By now we are very familiar with this equation and how to solve it.

To keep things simple, we imagine a market of two assets. One asset is riskless: the growth in its value is known deterministically and comes with a cast-iron guarantee.[22] This might correspond in reality to a bank deposit. The other asset is risky: there is uncertainty over what its value will be in the future. This might correspond to a share in a company or a collection of shares in different companies. We will think of it simply as stock.

An amount x_0 invested in the riskless asset evolves according to

$$dx_0 = x_0 \mu_\mathrm{r} dt. \tag{215}$$

μ_r is the riskless drift, known in finance as the riskless rate of return.[23] There is no volatility term. In effect, we have set $\sigma = 0$. We know with certainty what x_0 will be at any point in the future:

$$x_0(t_0 + \Delta t) = x_0(t_0) \exp(\mu_\mathrm{r} \Delta t). \tag{216}$$

An amount x_1 invested in the risky asset evolves according to

$$dx_1 = x_1(\mu_\mathrm{s} dt + \sigma_\mathrm{s} dW), \tag{217}$$

where $\mu_\mathrm{s} > \mu_\mathrm{r}$ is the risky drift and $\sigma_\mathrm{s} > 0$ is the volatility (the subscript s stands for stock). μ_s is also known in finance as the expected return.[24] This equation has solution

$$x_1(t_0 + \Delta t) = x_1(t_0) \exp\left[\left(\mu_\mathrm{s} - \frac{{\sigma_\mathrm{s}}^2}{2}\right)\Delta t + \sigma_\mathrm{s} W(\Delta t)\right], \tag{218}$$

which is a random variable.

The difference

$$\mu_\mathrm{e} = \mu_\mathrm{s} - \mu_\mathrm{r} \tag{219}$$

is known variously as the excess return, the risk premium, and in stock markets as the equity premium. We will refer to it simply as the excess drift. It will be a very important quantity in later discussions. For the moment we can think of it as compensation for accepting the uncertain outcome of (Eq. 218) instead of the guaranteed result of (Eq. 216).

[22]Such guarantees are easy to offer in a model. In the real world, one should be very suspicious of anything that comes with a "cast-iron guarantee".

[23]In general we will eschew financial terminology for rates. Economics has failed to define them clearly, with the result that different quantities, like $\bar{g}$ and $g_{\langle\rangle}$, are often conflated. The definitions developed in these lectures are aimed at avoiding such confusion.

[24]Probably because it's the growth rate of the expected value, see (Eq. 226).

4.1.2 Leverage

Let's turn to the concept of leverage. Imagine a very simple portfolio of value x_ℓ, out of which ℓx_ℓ is invested in stock and the remainder, $(1-\ell)x_\ell$, is put in the bank. ℓ is known as the leverage. It is the fraction of the total investment assigned to the risky asset. $\ell = 0$ corresponds to a portfolio consisting only of bank deposits. $\ell = 1$ corresponds to a portfolio only of stock.

You would be forgiven for thinking that prudence dictates $0 \leq \ell \leq 1$, *i.e.* that we invest some of our money in stock and keep the rest in the bank. However, the financial markets have found all sorts of exciting[25] ways for us to invest almost any amount in an asset. For example, we can make $\ell > 1$ by borrowing money from the bank to buy more stock than we could have bought with only our own money.[26] We can even make $\ell < 0$ by borrowing stock (a negative investment in the risky asset), selling it, and putting the money raised in the bank. In the financial world this practice is called short selling.

Each investment in our portfolio experiences the same relative fluctuations as the asset in which it has been made. The overall fluctuation in the portfolio's value is, therefore,

$$dx_\ell = (1-\ell)x_\ell \frac{dx_0}{x_0} + \ell x_\ell \frac{dx_1}{x_1}. \tag{220}$$

Substituting in (Eq. 215) and (Eq. 217) gives the SDE for a leveraged investment in the risky asset,

$$dx_\ell = x_\ell[(\mu_\mathrm{r} + \ell\mu_\mathrm{e})dt + \ell\sigma_\mathrm{s} dW], \tag{221}$$

with solution,

$$x_\ell(t_0 + \Delta t) = x_\ell(t_0) \exp\left[\left(\mu_\mathrm{r} + \ell\mu_\mathrm{e} - \frac{\ell^2 {\sigma_\mathrm{s}}^2}{2}\right)\Delta t + \ell\sigma_\mathrm{s} W(\Delta t)\right]. \tag{222}$$

We can now see why we labelled investments in the riskless and risky assets by x_0 and x_1: when $\ell = 0$, x_ℓ follows the same evolution as x_0; and when $\ell = 1$, it evolves as x_1.

In our model ℓ, once chosen, is held constant over time. This means that our model portfolio must be continuously rebalanced to ensure that the ratio of stock to total investment stays fixed at ℓ. For example, imagine our stock investment fluctuates down over a short time-step, while our bank deposit accrues a little interest. Immediately we have slightly less than ℓ of the portfolio's value in stock, and slightly more than $1-\ell$ of its value in the bank. To return the leverage to ℓ, we need to withdraw some money from the bank and use it to buy more stock. In (Eq. 221) we are imagining that this happens continuously.[27]

4.1.3 Portfolio theory

Our simple model portfolio parametrised by ℓ allows us to ask the

[25] There are other adjectives.

[26] This doesn't immediately affect the portfolio's value. The bank loan constitutes a negative investment in the riskless asset, whose value cancels the value of the stock we bought with it. Of course, the change in the portfolio's composition will affect its future value.

[27] In reality, of course, that's not possible. We could try to get close by rebalancing frequently. However, every time we buy or sell an asset in a real market, we pay transaction costs, such as broker's fees and transaction taxes. This means that frequent rebalancing in the real world can be costly.

Question:

What is the optimal value of ℓ?

This is similar to choosing between gambles, for which we have already developed a decision theory. The main difference is that we are now choosing from a continuum of gambles, each characterised by a value of ℓ, whereas previously we were choosing between discrete gambles. The principle, however, is the same: we will maximise the time-average growth rate of our investment.

Before we do this, let's review the classical treatment of the problem so that we appreciate the wider context. Intuitively, people understand there is some kind of trade-off between risk and reward. In our model of a generic multiplicative asset, (Eq. 214), we could use σ as a proxy for risk and μ as a proxy for reward. Ideally we want an investment with large μ and small σ, but we also acknowledge the rule-of-thumb that assets with larger μ tend to have larger σ.[28] This is why we model our risky asset as having an excess return, μ_e, over the riskless asset.

Intuition will only take us so far. A rigorous treatment of the portfolio selection problem was first attempted by Markowitz in 1952 [17]. He suggested defining a portfolio with parameters (σ_i, μ_i) as efficient if there exists no rival portfolio with parameters (σ_j, μ_j) for which at least one of the following statements is true:

1. $\mu_j > \mu_i$ and $\sigma_j \leq \sigma_i$;
2. $\sigma_j < \sigma_i$ and $\mu_j \geq \mu_i$.

In other words, if the rival portfolio has higher μ, it had better have higher σ; or if it has lower σ, it had better have lower μ. Markowitz argued that it is unwise to invest in a portfolio which is not efficient.

In our problem, we are comparing portfolios with parameters $(\ell\sigma_\mathrm{s}, \mu_\mathrm{r} + \ell\mu_\mathrm{e})$. These lie on a straight line in the (σ, μ)-plane,[29]

$$\mu = \mu_\mathrm{r} + \left(\frac{\mu_\mathrm{e}}{\sigma_\mathrm{s}}\right)\sigma, \tag{223}$$

shown schematically in Fig. 21. Under Markowitz's classification, all of the leveraged portfolios on this line are efficient.[30] Therefore, any leverage we choose gives a portfolio in which we are not counselled against investing. This does not help answer our question. By itself, Markowitz's approach is agnostic to leverage: it requires additional information to distinguish between portfolios. Markowitz was aware of this limitation and argued that the optimal portfolio could be identified by considering the investor's risk preferences.[31] Typically this is modelled by the introduction of a utility function, with which approach we are now familiar.

[28] A "no such thing as a free lunch" type of rule.

[29] Derived by eliminating ℓ from the equations $\mu = \mu_\mathrm{r} + \ell\mu_\mathrm{e}$ and $\sigma = \ell\sigma_\mathrm{s}$.

[30] Indeed, in finance this line is called the efficient frontier.

[31] "The proper choice among portfolios depends on the willingness and ability of the investor to assume risk." [19].

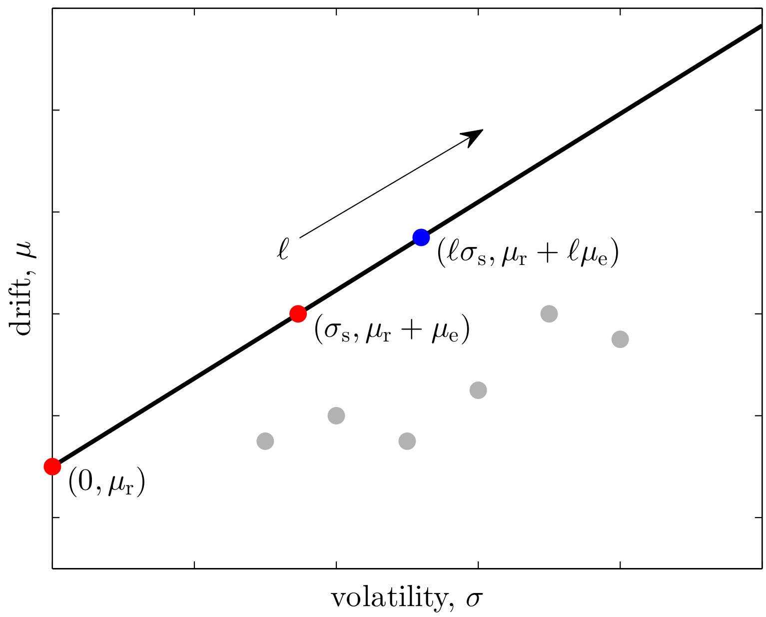


Figure 21: The Markowitz portfolio selection picture. The red dots are the locations in the (σ, μ)-plane of portfolios containing only the riskless (left) or risky (right) asset. The blue dot is one possible leveraged portfolio, in this case with $\ell > 1$. All possible leveraged portfolios lie on the black line, (Eq. 223). The grey dots are hypothetical alternative portfolios, containing different assets excluded from our simple portfolio problem. Their location below and to the right of the black line makes them inefficient under the Markowitz scheme.

4.1.4 Sharpe ratio

The Sharpe ratio [32] for an asset with drift μ and volatility σ is defined as

$$S \equiv \frac{\mu - \mu_\mathrm{r}}{\sigma} \tag{224}$$

It is the gradient of the straight line in the (σ, μ)-plane which passes through the riskless asset and the asset in question. S is often used as a convenient shorthand for applying Markowitz's ideas, since choosing the portfolio with the highest S from the set of available portfolios is equivalent to choosing an efficient portfolio. In our scenario, however, we can immediately see why it sheds no light. All of our leveraged portfolios lie on the same line, (Eq. 223), and so all of them have the same Sharpe ratio, which is simply the line's gradient:

$$S_\ell = \frac{\mu_\mathrm{e}}{\sigma_\mathrm{s}}. \tag{225}$$

This is insensitive to the leverage ℓ, resulting in the same non-advice as the Markowitz approach. Sharpe also suggested considering risk preferences to resolve the optimal portfolio.[32]

4.1.5 Expected return

We noted previously that the growth rate of the expected value of the risky asset is the risky drift, μ_s, also known as the expected return. This is because

$$\langle x_1(t_0 + \Delta t)\rangle = \langle x_1(t_0)\rangle \exp(\mu_\mathrm{s}\Delta t), \tag{226}$$

which, as a multiplicative process, has growth rate

$$g_\mathrm{m}(\langle x_1\rangle) = \frac{\Delta \ln \langle x_1\rangle}{\Delta t} = \mu_\mathrm{s}. \tag{227}$$

It follows immediately from comparison of (Eq. 217) and (Eq. 221) that the expected value of the leveraged portfolio grows at

$$g_\mathrm{m}(\langle x_\ell\rangle) = \mu_\mathrm{r} + \ell\mu_\mathrm{e}. \tag{228}$$

This illustrates why a portfolio theory which is insensitive to leverage, such as that of Markowitz and Sharpe[33], is potentially dangerous. If any leverage is admissible, then an investor misguidedly seeking to maximise his expected return in (Eq. 228) would maximise his leverage, $\ell \to \infty$. This, as we will shortly see, would almost surely ruin him.

4.1.6 Growth rate maximisation

We now understand the classical approach, as applied to a very simple portfolio problem, and we are aware of its limitations. What does our own decision theory have to say?

[32] "The investor's task is to select from among the efficient portfolios the one that he considers most desirable, based on his particular feelings regarding risk and expected return." [32].

[33] Both recipients of the 1990 Alfred Nobel Memorial Prize in Economic Sciences.

The time-average growth rate of the leveraged portfolio is

$$\bar{g}_{\mathrm{m}}(\ell) \equiv \lim_{\Delta t\to\infty} \{g_{\mathrm{m}}(x_\ell, \Delta t)\} = \lim_{\Delta t\to\infty} \left\{\frac{\Delta \ln x_\ell}{\Delta t}\right\}. \tag{229}$$

This will depend on ℓ. Inserting the expression for x_ℓ in (Eq. 222) gives

$$\bar{g}_{\mathrm{m}}(\ell) = \lim_{\Delta t\to\infty} \left\{\frac{1}{\Delta t}\left[\left(\mu_{\mathrm{r}} + \ell\mu_{\mathrm{e}} - \frac{\ell^2{\sigma_{\mathrm{s}}}^2}{2}\right)\Delta t + \ell\sigma_{\mathrm{s}} W(\Delta t)\right]\right\}, \tag{230}$$

which, since $W(\Delta t)/\Delta t \sim \Delta t^{-1/2} \to 0$ as $\Delta t \to \infty$, converges to

$$\boxed{\bar{g}_{\mathrm{m}}(\ell) = \mu_{\mathrm{r}} + \ell\mu_{\mathrm{e}} - \frac{\ell^2{\sigma_{\mathrm{s}}}^2}{2}.} \tag{231}$$

This is a quadratic in ℓ with an unambiguous maximum[34] at

$$\boxed{\ell_{\mathrm{opt}} = \frac{\mu_{\mathrm{e}}}{{\sigma_{\mathrm{s}}}^2}.} \tag{232}$$

ℓ_{opt} is the optimal leverage which defines the portfolio with the highest time-average growth rate. Classical theory is indifferent to where on the line in Fig. 21 we choose to be. Our decision theory, however, selects a particular point on that line,[35] which answers the question we posed at the start of Sec. 4.1.3. This is the key result. We note that it requires no additional knowledge to the parameters μ_{r}, μ_{s}, and σ_{s} of the two assets in our market. In particular, it is defined objectively, with no reference to idiosyncrasies of the investor (other than that we assume him to be a time-average growth rate maximiser).

(Eq. 231) gives the time-average growth rate along the efficient frontier, where all of our leveraged portfolios lie. In fact, it's easy to calculate the growth rate for any point in the (σ, μ)-plane: it is simply $\mu - \sigma^2/2$. Overlaying the Markowitz picture in Fig. 21 on the growth rate landscape is illuminating. In effect, it adds the information missing from the classical model, which was needed to distinguish between portfolios. This is shown in Fig. 22.

That (Eq. 231) defines an inverted parabola means that, even on the efficient frontier, there exist portfolios with $\bar{g}_{\mathrm{m}}(\ell) < 0$. These occur for $\ell < \ell^-$ and $\ell > \ell^+$, where

$$\ell^{\pm} \equiv \ell_{\mathrm{opt}} \pm \sqrt{{\ell_{\mathrm{opt}}}^2 + \frac{2\mu_{\mathrm{r}}}{{\sigma_{\mathrm{s}}}^2}}. \tag{233}$$

This confirms our assertion at the end of Sec. 4.1.5, that an investor maximising his leverage in either direction will, if he is able to apply enough leverage, subject his wealth to negative time-average growth. Indeed, if his leveraging ability is unlimited, he will find to his horror what is easily seen in (Eq. 231), that $\bar{g}_{\mathrm{m}}(\ell)$ diverges negatively as $\ell \to \pm\infty$.

[34] Derived, for example, by setting $\frac{d\bar{g}_{\mathrm{m}}}{d\ell} = 0$.

[35] Subsequently Markowitz became aware of this point, which he called the "Kelly-Latané" point in [18], referring to [14, ?].

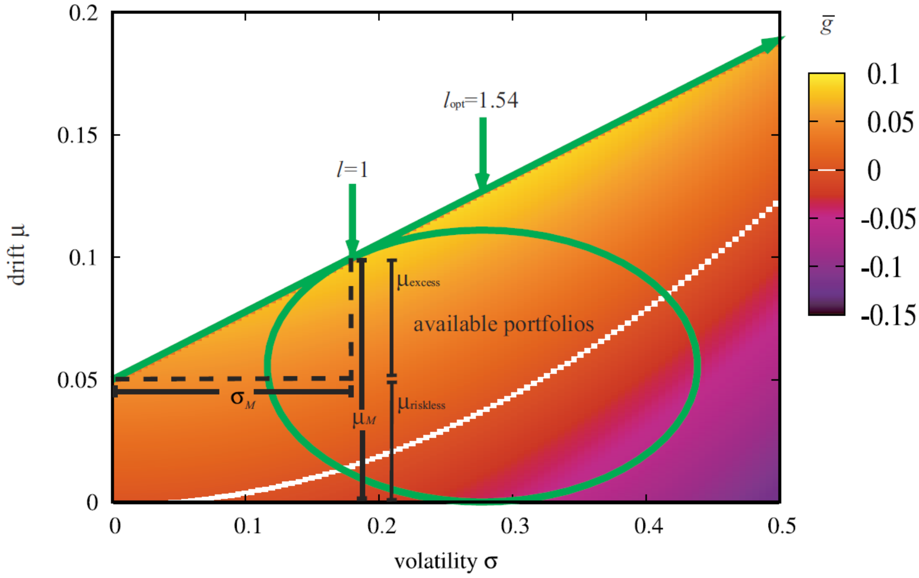


Figure 22: The augmented portfolio selection picture. The Markowitz picture is shown in green, with our model leveraged portfolios on the straight line (the efficient frontier) and hypothetical alternative portfolios within the ellipse (analogous to the grey dots in Fig. 21). This is overlaid on a colour plot of the time-average growth rate, $\bar{g}$. The optimal leverage, $\ell_{\text{opt}} \approx 1.54$, is marked at the location of the highest $\bar{g}$ on the efficient frontier. Portfolios on the white curve have $\bar{g} = 0$. Eventually this will intersect the efficient frontier, at which point applying more leverage will produce a portfolio with negative long-run growth. Parameters are $\mu_{\text{r}} = \mu_{\text{e}} = 0.05$ per unit time and $\sigma_{\text{s}} = 0.18$ per square root time (denoted by μ_{riskless}, μ_{excess}, and σ_{M} in the figure). Adapted from [24].

4.2 Stochastic market efficiency

4.2.1 A fundamental measure

Aside from being insensitive to leverage, the Sharpe ratio, $S = \mu_\text{e}/\sigma_\text{s}$, is a dimensionful quantity. Its unit is (time unit)$^{-1/2}$. This means that its numerical value is arbitrary (since it depends on the choice of time unit) and tells us nothing fundamental about the system under study. For example, a portfolio with $S = 5$ per square root of one year has $S = 5(365)^{-1/2} \approx 0.26$ per square root of one day. Same portfolio, different numbers.

The optimal leverage, $\ell_\text{opt} = \mu_\text{e}/{\sigma_\text{s}}^2$, which differs from the Sharpe ratio by a factor of $1/\sigma_\text{s}$, is a dimensionless quantity. Therefore, its numerical value does not depend on choice of units and has the potential to carry fundamental information about the system.[36] We could view ℓ_opt as the fundamental measure of a portfolio's quality, similarly to how S is viewed in the classical picture. A portfolio with a high optimal leverage must represent a good investment opportunity to justify such a large commitment of the investor's funds.

However, the significance of ℓ_opt runs deeper than this. The portfolio to which ℓ_opt refers is that which optimally allocates money between the risky and riskless assets in our model market. Therefore, it tells us much about conditions in that market and, by extension, in the wider model economy of which we might imagine it is a part. A high ℓ_opt indicates an economic environment in which investors are incentivised to take risks. A low or negative ℓ_opt indicates the converse. This raises a tantalising

Question:

Are there any special numerical values of ℓ_opt which describe different market regimes, or to which markets are attracted?

4.2.2 Relaxing the model

Our model market contains assets whose prices follow GBM with constant drift and volatility parameters.[37] Once specified, μ_r, μ_s, and σ_s are static and, therefore, so is ℓ_opt. This limits the extent to which we can explore the question, since we cannot consider changes in ℓ_opt. To make progress we need to relax the model. We must consider which parts of the model are relevant to the question, and which parts can be discarded without grave loss.

The GBM-based model is useful because it motivates the idea of an objectively optimal leverage which maximises the growth of an investment over time. It also provides an expression for ℓ_opt in terms of parameters which, in essence, describe the market conditions under which prices fluctuate. These parameters have correspondences with quantities we can measure in real markets. All of this is useful.

However, the real markets in which we are ultimately interested contain assets whose prices do not follow GBM (because nothing in nature[38] truly does). In particular, real market conditions are not static. They change over time, albeit on a longer time scale than that of the price fluctuations. In this context,

[36] See Barenblatt's modern classic on scaling [1].
[37] A tautology, since (Eq. 214) only describes a GBM if μ and σ are constant in time.
[38] We consider markets to be part of nature and appropriate subjects of scientific study.

the model assumption of constant μ_r, μ_s, and σ_s is restrictive and unhelpful. We will relax it and imagine a less constrained model market in which these parameters, and therefore ℓ_opt, are allowed to vary slowly. We will not build a detailed mathematical formulation of this, but instead use the idea to run some simple thought experiments.

4.2.3 Efficiency

One way of approaching a question like this is to invoke the concept of efficiency. In economics this has a specific meaning in the context of financial markets, which we will mention imminently. In general terms, however, an efficient system is one which is already well optimised and whose performance cannot be improved upon by simple actions. For example, a refrigerator is efficient if it maintains a cool internal environment while consuming little electrical power and emitting little noise.[39] Similarly, Markowitz's portfolios were efficient because the investor could do no better than to choose one of them.

The "efficient market hypothesis" of classical economics treats markets as efficient processors of information. It claims that the price of an asset in an efficient market reflects all of the publicly available information about it. The corollary is that no market participant, without access to privileged information, can consistently beat the market simply by choosing the prices at which he buys and sells assets. We shall refer to this hypothesis as ordinary efficiency.[40]

We will consider a different sort of efficiency, where we think not about the price at which assets are bought and sold in our model market, but instead about the leverage that is applied to them. Let's run a thought experiment.

Thought experiment: efficiency under leverage

Imagine that $\ell_\mathrm{opt} > 1$ in our model market. This would mean that the simple strategy of borrowing money to buy stock will achieve faster long-run growth than buying stock only with our own money. If we associate putting all our money in stock, $\ell = 1$, with an investment in the market, then it would be a trivial matter for us to beat the market.

Similarly, imagine that $\ell_\mathrm{opt} < 1$. In this scenario, the market could again be beaten very easily by leaving some money in the bank (and, if $\ell_\mathrm{opt} < 0$, by short selling).

It would strain language to consider our market efficient if consistent outperformance were so straightforward to achieve. This suggests a different, fluctuations-based notion of market efficiency, which we call stochastic efficiency. We claim that it is impossible for a market participant without privileged information to beat a stochastically efficient market simply by choosing the *amount*

[39] Under this definition, the refrigerator in the LML library is not efficient.

[40] This is not the most precise hypothesis ever hypothesised. What does it mean for a price to "reflect" information? Presumably this involves some comparison between the of observed price of the asset and its true value contingent on that information. But only the former is observable, while the latter evades clear definition. Similarly, what does it mean to "beat the market"? Presumably something to do with achieving a higher growth rate than a general, naïve investment in the overall market. But what investment, exactly? We will leave these legitimate questions unanswered here, since our focus is a different form of market efficiency. The interested reader can consult the comprehensive review in [?].

he invests in stock, *i.e.* by choosing his leverage.[41] Therefore, we make the following

Hypothesis: stochastic market efficiency (strong form)

Real markets self-organise such that

$$\boxed{\ell_{\mathrm{opt}} = 1} \tag{234}$$

is an attractive point for their stochastic properties.

These stochastic properties are represented by μ_{r}, μ_{s}, and σ_{s} in the relaxed model, in which we permit dynamic adjustment of their values. We call this the strong form of the hypothesis, since it makes a very precise prediction about the attractive value for ℓ_{opt}.

4.2.4 Stability

Another approach to the question in Sec. 4.2.1, whose style owes more to physics than economics, is to consider the stability of the system under study (here, the market) and how this depends on the value of the measure in question (here, ℓ_{opt}). Systems which are stable tend to persist over long time scales and are usually what we observe in nature. Unstable systems tend to last for shorter times,[42] so we observe them less frequently. With this in mind, let's think about the logic of different values of ℓ_{opt}.

Thought experiment: stability under leverage

Imagine that $\ell_{\mathrm{opt}} > 1$ in our relaxed model. Since it is an objectively optimal leverage which does not depend on investor idiosyncrasies, this means that *everyone* in the market should want to borrow money to buy stock. But, if that's true, who's going to lend the money and who's going to sell the stock?

Similarly, imagine that $\ell_{\mathrm{opt}} < 0$. This means that *everyone* should want to borrow stock and sell it for cash. But, if that's true, who's going to lend the stock and who's going to relinquish their cash to buy it?

Unless there are enough market participants disinterested in their time-average growth rate to take the unfavourable sides of these deals – which in our model we will assume there are not – then neither of these situations is globally stable. It's hard to imagine them persisting for long before trading activity causes one or more of the market parameters to change, returning ℓ_{opt} to a stable value.

This thought experiment suggests that we will observe markets where ℓ_{opt} is within its stable range more often than it is not. This motivates the following

[41] This resembles ordinary efficiency except that we have replaced price by amount.

[42] We use the comparative "shorter" here. This does not mean short. It is quite possible for an unstable system to remain in flux for a long time in human terms, perhaps giving the illusion of stability. Indeed, much of classical economic theory is predicated on the idea that economies are at or close to equilibrium, *i.e.* stable. We would argue that economies are fundamentally far-from-equilibrium systems and must be modelled as such, even if their dynamics unfold over time scales much longer than our day-to-day affairs.

Hypothesis: stochastic market efficiency (weak form)

Real markets self-organise such that

$$\boxed{0 \leq \ell_{\text{opt}} \leq 1} \tag{235}$$

is an attractive range for their stochastic properties.

We call this the weak form of the hypothesis, since its prediction about the attractive value for ℓ_{opt} is less precise than in (Eq. 234).

The dynamical adjustment, or self-organisation, of the market parameters takes place through the trading activity of market participants. In particular, this creates feedback loops, which cause prices and fluctuation amplitudes to change, returning ℓ_{opt} to a stable value whenever it strays. To be truly convincing, we should propose plausible trading mechanisms through which these feedback loops arise. We do this in [26]. Since they involve details about how trading takes place in financial markets (in which we assume the typical attendee of these lectures is utterly disinterested) we shall not rehearse them here. The primary drivers of our hypothesis are the efficiency and stability arguments we've just made.

Furthermore, there are additional reasons why we would favour the strong form of the hypothesis over long time scales. The main one is that an economy in which ℓ_{opt} is close to, or even less than, zero gives people no incentive to invest in productive business activity. Such an economy would appear paralysed, resembling perhaps those periods in history to which economists refer as depressions. We'd like to think that economies are not systematically attracted to such states. The other reasons are more technical, to do with the different interest rates accrued on deposits and loans, and the costs associated with buying and selling assets. These are described in [26] and lead to a refined

Hypothesis: stochastic market efficiency (refined)

On sufficiently long time scales, $\ell_{\text{opt}} = 1$ is a strong attractor for the stochastic properties of real markets. Deviations from this attractor over shorter time scales are likely to be confined to the range $0 \leq \ell_{\text{opt}} \leq 1$.

4.2.5 Tests of the hypothesis

We test the stochastic efficiency hypothesis in a real market by simulating leveraged investments in the Standard & Poor's index of 500 leading U.S. companies (the "S&P500") using historical data of its daily returns over 58 years. This index is usually viewed as a proxy for an investment in large American businesses or, more simply, the American economy. We will treat the S&P500 as the real-world equivalent of the risky asset in our model. Bank deposits at historical interest rates will be treated as the real-world equivalent of the riskless asset. Therefore, the dichotomy we set up is: invest generally in business; or put money in the bank.

Data sets

The data used in this study are publicly available from the Federal Reserve Economic Data (FRED) website.[43] We use the daily closing prices of the S&P500 (FRED time series "SP500") from 4$^{\text{th}}$ August 1955 to 21$^{\text{st}}$ May 2013. Additionally, we use two daily bank interest rates: the effective federal funds rate ("DFF"); and the bank prime loan rate ("DPRIME"). Estimates of optimal leverage using these data are likely generous, because the S&P500 represents a well diversified portfolio of large and successful companies, and – since ailing companies are replaced – is positively affected by survivorship bias.

Simulation

An investment of constant leverage over a given time period, or window, is simulated as follows. At the start of the first day we assume an initial net investment, or equity, of \$1. This comprises stock holdings of \$$\ell$ in the S&P500 and bank deposits of \$$(1-\ell)$. At the end of the day the values of these holdings and deposits are updated according to the historical market returns and interest rates. The portfolio is then rebalanced, *i.e.* the holdings in the risky asset are adjusted so that their ratio to the equity remains ℓ. On non-trading days the return of the market is zero, while deposits continue to accrue interest.[44] The simulation proceeds in this fashion until the final day of the window, when the final equity is recorded. If at any time the equity falls below zero, the investment is declared bankrupt and the simulation stops. The procedure is then repeated for different leverages, and the simulated optimal leverage is the leverage for which the final equity is maximised.

Full time series

Fig. 24 shows the simulated multiplicative return as a function of leverage for an investment over the entire time series. The four curves in the figure correspond to four sets of assumptions about interest rates and transaction costs. These are labelled 1–4 in order of increasing complexity and resemblance to actual practices in financial markets. We will not dwell on these here, although brief descriptions are provided in the figure caption.

$\ell_{\text{opt}} = 0.97$ for simulations 1 (which we will refer to as the simple case), 2, and 3. $\ell_{\text{opt}} = 1.00$ for the most realistic simulation, 4 (the complex case). These results appear to lend great support to the stochastic market efficiency hypothesis. Based on simple thought experiments motivated by the theories developed in these lectures, we predicted that ℓ_{opt} in a real market should be close to unity. 58 years of real market data confirm this prediction. We discuss below the statistical significance of these results.

The kinks[45] in the return-leverage curve can be accompanied by a change in sign of the derivative. When this happens, the kink is a global maximum and ℓ_{opt} is fixed at that leverage, either 0 or 1. This makes these special leverages sticky, in that ℓ_{opt} can get trapped there.[46]. This stickiness will tend to promote the likelihood of the hypothesis being confirmed. However, simulation 1 shows that $\ell_{\text{opt}} \approx 1$ even when these effects are neglected.

Parameter estimation

Fig. 24 shows the simulated growth rate as a function of leverage for the simple case (simulation 1) over the entire time series. This is the logarithm of the simulated return of Fig. 24 divided by the window length. Since the window is long,

[43] http://research.stlouisfed.org/fred2

[44] This leads to an unrealistic but negligible rebalancing on those days.

[45] A non-technical term for discontinuities in the derivative.

[46] Although typically only only when the ℓ_{opt} of simulation 1 is already in or very close to the range, $0 \leq \ell_{\text{opt}} \leq 1$.

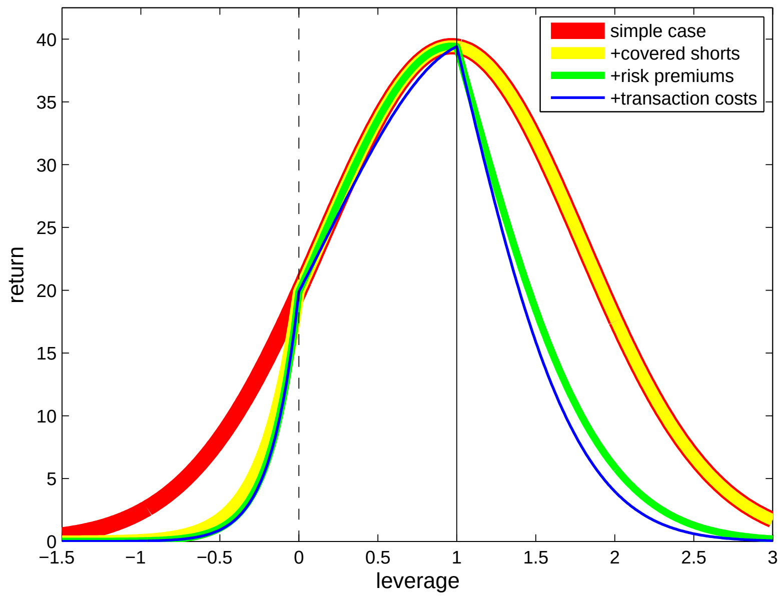


Figure 23: Total return for a constant-leverage investment in the S&P500, starting 4th August 1955 and ending 21st May 2013 as a function of the leverage.
Red line: Simulation 1. Interest at federal funds rate on bank deposits and loans. No stock borrowing costs. No transaction costs.
Yellow line: Simulation 2. As 1, but with interest at federal funds rate on borrowed stock. This introduces a kink at $\ell = 0$, where stock borrowing begins.
Green line: Simulation 3. As 2, but with interest charged at the higher bank prime rate on borrowed cash and stock. This introduces a kink at $\ell = 1$.
Blue line: Simulation 4. As 3, but with a transaction cost of 0.2% of the value of the assets traded on rebalancing.

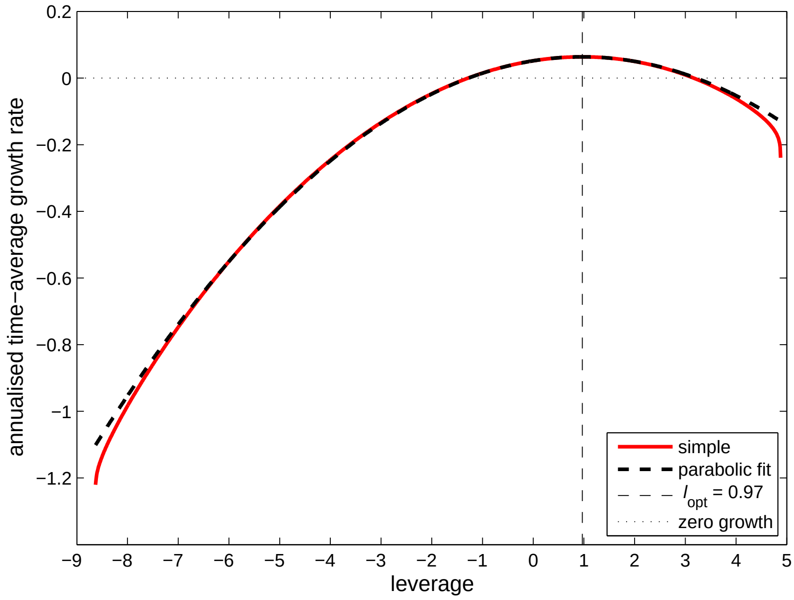


Figure 24: Computed time-average growth rates closely follow a parabola as a function of leverage.

we might expect to approximate the time-average growth rate, $\bar{g}_{\mathrm{m}}(\ell)$, which we know from (Eq. 231) is a parabola in the original GBM model. The black dashed line is a fitted parabola, whose parameters can be taken as meaningful definitions of the effective values of μ_{r}, μ_{e}, and σ_{s} for the S&P500 for the entire time series.[47] A least-squares fit estimates these parameters as $\mu_{\mathrm{r}} = 5.2\%$ *pa*, $\mu_{\mathrm{e}} = 2.4\%$ *pa*, and $\sigma_{\mathrm{s}} = 16\%$ $p\sqrt{a}$.

The deviation from parabolic form for high and low leverages in Fig. 24 is due to extremely large fluctuations in the index, which are much less rare than would be observed in a true GBM. These result in large losses and, indeed, bankruptcy for highly leveraged portfolios. That real returns distributions are observed to have fatter tails than those predicted by GBM is an oft-made criticism of classical theory. In this study it is not especially relevant: indeed, the existence of rogue fluctuations strengthens the stochastic efficiency hypothesis, in that it penalises high leverage strategies.

Equity premium puzzle

These results are relevant to an unexplained phenomenon in economics known as the "equity premium puzzle" [20]. Proponents of the puzzle[48] argue that estimates of the equity premium, represented by μ_{e} in our model, are fundamentally incompatible with classical economic theory – in particular utility theory – be-

[47] We say effective because, although we think of these parameters as being time-varying in the relaxed model, their values from the parabolic fit are those of the original constant-parameter model which produce an almost identical outcome for $\bar{g}_{\mathrm{m}}(\ell)$ as the real data.

[48] Which include another prize winner, Edward C. Prescott, this time of the 2004 Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel (which by then was on its tenth renaming).

cause they imply an implausibly high level of investor risk aversion. A value of 6% *pa* has become established in the literature, which is markedly higher than the $\mu_\mathrm{e} = 2.4\%$ *pa* in our parabolic fit. The source of this discrepancy is unclear because the classical estimates are based on complicated theories of investor behaviour. In our picture, which contains fewer assumptions to question, there is no puzzle: the value of μ_e we find is perfectly consistent with a stochastically efficient market.

Finite time scales

In the real world, of course, we can never truly observe the time-average growth rate of a leveraged investment since this would require an infinite observation time. Instead we observe the finite-time growth rate over a window of length Δt. This is a random variable whose distribution broadens as $\Delta t \to 0$

Likewise, we never truly observe ℓ_opt either, since this is the leverage that maximises a time average-growth rate. Instead we observe simulated optimal leverages which maximise finite-time growth rates. We can guess that these will show larger fluctuations from the underlying ℓ_opt as the simulation window gets shorter, because short periods containing a sequence of almost all positive or negative stock price movements will result in very high (possibly infinite) positive or negative simulated optimal leverages.

More formally, let's denote by $\ell_\mathrm{opt,s}(\Delta t)$ the simulated optimal leverage which maximises the finite-time growth rate, $g_\mathrm{m}(x_\ell, \Delta t)$. As $\Delta t \to 0$, $g_\mathrm{m}(x_\ell, \Delta t)$ becomes a worse estimator of the time-average growth rate, $\bar{g}_\mathrm{m}(\ell)$, because its distribution becomes broader. Thus $\ell_\mathrm{opt,s}(\Delta t)$ becomes a similarly worse estimator of ℓ_opt.

This is important because a single observation of $\ell_\mathrm{opt,s}(\Delta t)$ consistent with our hypothesis (such as the one we just made for $\Delta t \approx 58$ years) is only significant if the uncertainty in $\ell_\mathrm{opt,s}(\Delta t)$ is of the same order of magnitude as $\ell_\mathrm{opt,s}(\Delta t)$ itself. For example, if we knew that $\ell_\mathrm{opt,s}(\Delta t)$ had a distribution that could place it between, say, -10 and 10 with reasonable probability, then we couldn't read much from a single observation either inside or outside the attractive range of our hypothesis. If, on the other hand, most of the distribution's mass were inside the range, then an observation outside would be strong evidence that the hypothesis is flawed, and so an observation inside is significant.

We can quantify these ideas in the original model by discarding the $\Delta t \to \infty$ limit in (Eq. 230). This gives

$$g_\mathrm{m}(x_\ell, \Delta t) = \mu_\mathrm{r} + \ell\mu_\mathrm{e} - \frac{\ell^2 {\sigma_\mathrm{s}}^2}{2} + \frac{\ell\sigma_\mathrm{s} W(\Delta t)}{\Delta t}, \tag{236}$$

which is maximised at

$$\ell_\mathrm{opt,s}(\Delta t) = \ell_\mathrm{opt} + \frac{W(\Delta t)}{\sigma_\mathrm{s}\Delta t}. \tag{237}$$

This is normally distributed with mean ℓ_opt and standard deviation

$$\mathrm{stdev}[\ell_\mathrm{opt,s}(\Delta t)] = \frac{1}{\sigma_\mathrm{s}\sqrt{\Delta t}}. \tag{238}$$

Using the computed volatility of 16% per square root of one year, this standard deviation is approximately 0.83. This means that the uncertainty in $\ell_\mathrm{opt,s}(\Delta t)$

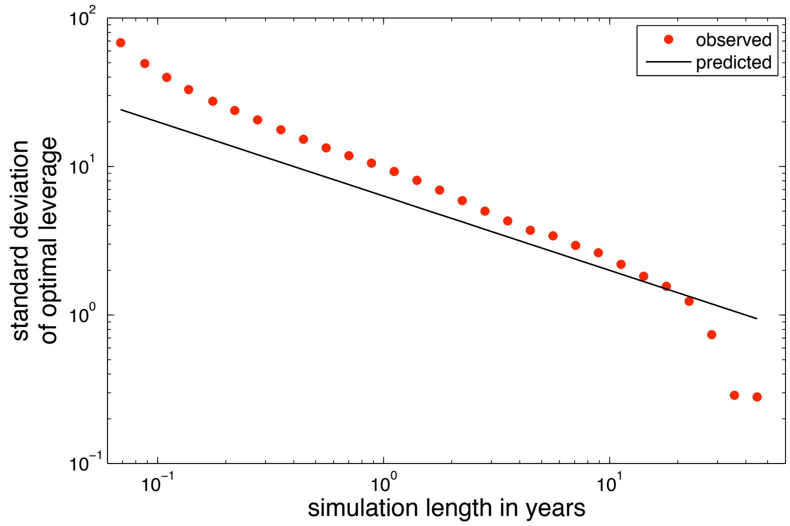


Figure 25: The red dots show the standard deviation of samples of $\ell_{\mathrm{opt,s}}(\Delta t)$ as a function of window length for the simple simulation. The black line shows the prediction of (Eq. 237) using the estimate of σ_{s} from the parabolic fit in Fig. 24.

is about the same size as the hypothesised value of ℓ_{opt}, and so the single observation we made is a significant corroboration of the hypothesis.

We can also test the validity of the relationship in (Eq. 237) by simulating investments of different window lengths in the market data and compiling histograms of the resulting $\ell_{\mathrm{opt,s}}(\Delta t)$. Fig. 25 shows, on double-logarithmic scales, the standard deviation of $\ell_{\mathrm{opt,s}}(\Delta t)$ as a function of Δt for the simple simulation. Good agreement is found with the model-specific prediction.[49]

Fig. 27 shows the simulated $\ell_{\mathrm{opt,s}}(\Delta t)$ for the investment window starting on 4^{th} August 1955 and ending on the date on the horizontal axis. The reduction in fluctuations with increasing window length are broadly consistent with (Eq. 238) and support the hypothesis that ℓ_{opt} – as estimated by $\ell_{\mathrm{opt,s}}(\Delta t)$ – is attracted to the range $0 \leq \ell_{\mathrm{opt}} \leq 1$ and, over long time scales, to $\ell_{\mathrm{opt}} = 1$ in particular.

We can also plot the simulated optimal leverages for investment windows with fixed lengths and moving start date. We do this for the simple and complex simulations in Fig. 27, with window lengths ranging from 5 to 40 years. From the strong fluctuations over short time scales emerges attractive behaviour consistent with our refined hypothesis. The effects of the stickiness of the points $\ell_{\mathrm{opt}} = 0$ and $\ell_{\mathrm{opt}} = 1$ in the complex model are clearly visible. In particular, over the last decade or so optimal leverage for the 20- and 40-year windows

[49] For shorter time scales, the standard deviation is slightly higher than predicted. We suspect this is due to discretisation effects: for investments over a small number of days, very high optimal leverages can be observed when the window contains predominantly positive or negative index movements. These would no longer be optimal if fluctuations and rebalancing were simulated on smaller time scales, *i.e.* intra-day. For longer time scales, the standard deviation drops below the prediction. This is because for window lengths approaching the entire period under study, the number of independent windows in the sample is small, and so the standard deviation of the sample is depressed.

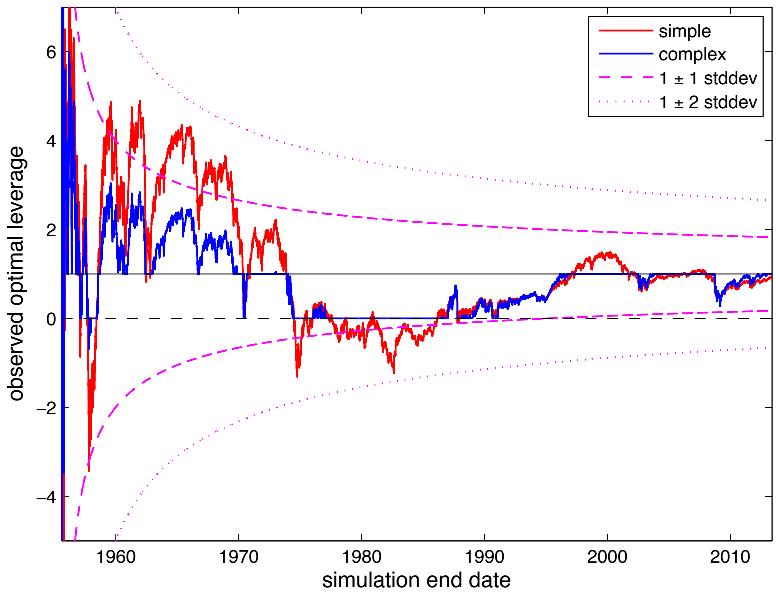


Figure 26: Daily simulated optimal leverages for an expanding window with start date 4th August 1955 and end date on the horizontal axis. Both simple (red line) and complex (blue line) simulations are shown. The broken magenta lines show the one- and two-standard deviation envelopes about $\ell_{\text{opt}} = 1$, based on (Eq. 238) and the estimate of σ_{s} from the parabolic fit in Fig. 24.

remained close to unity.

4.3 Discussion

The primary aim of this final lecture was to demonstrate that the mathematical formalism we have developed to conceptualise randomness in economics, which started with a simple model of wealth evolution by repetition of a gamble, is very powerful. It does more than simply create a rigorous and plausible foundation for economic theory. In particular, because the framework is epistemologically sound, we can make testable predictions – such as the stochastic market efficiency hypothesis – which we can corroborate empirically using real economic data. We could not have guessed from our simple coin-tossing game that we would end up making a prediction about the fluctuations of the American stock market over more than half a century. What other surprising predictions can we make in this formalism? That, dear reader, is a question for you.

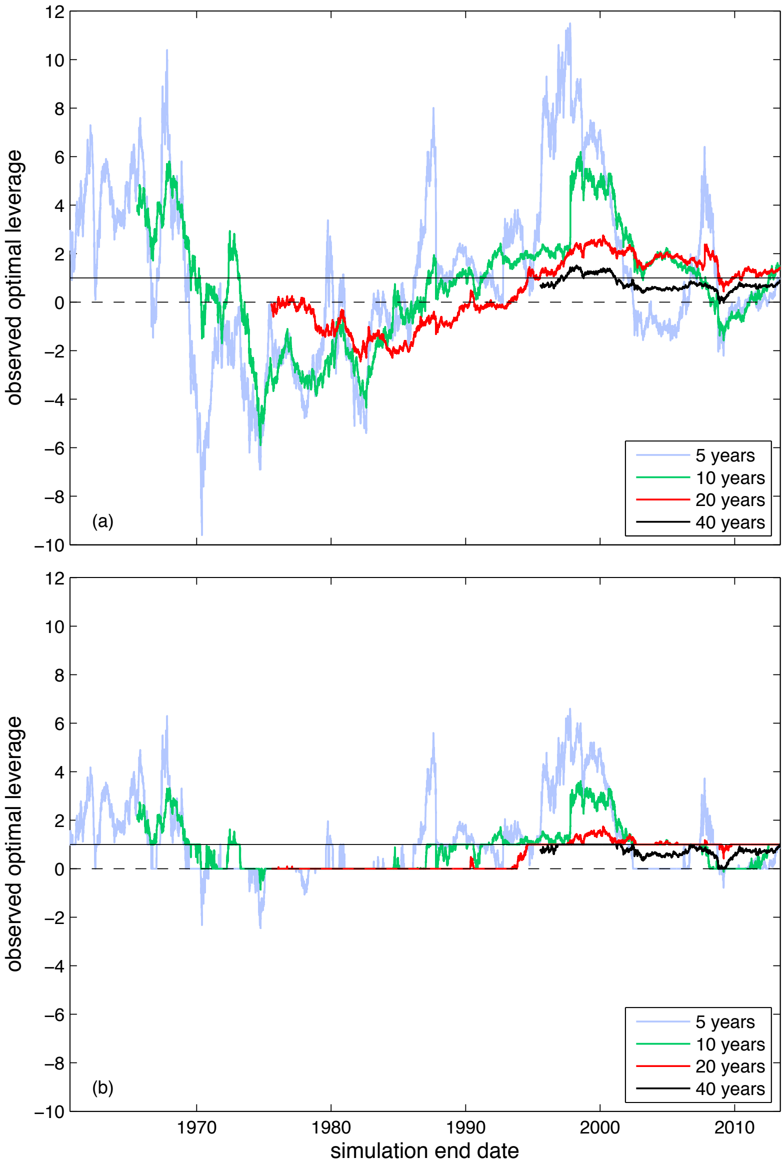


Figure 27:
(a) In the simple simulation, observed optimal leverage fluctuates strongly on short time scales but appears to converge to $\ell_{\text{opt,s}}(\Delta t) = 1$ on long time scales. This constitutes the central result of the study.
(b) In the complex simulation, the kinks in Fig. 24 ensure that $\ell_{\text{opt,s}}(\Delta t) = 0, 1$ are often found exactly. The 40-year simulation supports the strong stochastic efficiency hypothesis, that real markets are attracted to $\ell_{\text{opt}} = 1$, with a dip to $\ell_{\text{opt}} = 0$ only during the financial crisis of 2008.

Acronyms

BM Brownian Motion.

GBM Geometric Brownian motion.

GDP Gross domestic product.

LHS Left-hand side.

LML London Mathematical Laboratory.

PDF Probability density function.

RHS Right-hand side.

SDE Stochastic differential equation.

List of Symbols

a A scalar.

A An observable.

$\overline{A}$ Time average of observable A.

C Cost to participate in a gamble.

d Differential operator in Leibnitz notation, infinitesimal.

D Possible payout for a gamble.

δt Duration of one round of a gamble.

δ δ is most frequently used to express a difference, for instance δx is a difference between two wealths x. It can be the Kronecker delta function, a function of two arguments with properties $\delta(i, j) = 1$ if $i = 1$ and $\delta(i, j) = 0$ otherwise. It can also be the Dirac delta function of one argument, $\int f(x)\delta(x - x_0)dx = f(x_0)$.

Δ Difference operator, for instance Δv is a difference of two values of v, for instance observed at two different times.

Δt A time interval.

η Langevin noise with the properties $\langle\eta\rangle = 0$ and $\langle\eta(t_1)\eta(t_2)\rangle = \delta(t_1 - t_2)$.

f Arbitrary function.

$\mathcal{F}$ Force.

F Insurance fee.

g Growth rate.

G Gain from one round trip of the ship.

$g_{\langle\rangle}$ Exponential growth rate of the expectation value.

$g_{\mathbf{m}}$ Multiplicative growth rate.

$\bar{g}$ Time-average exponential growth rate.

$\bar{g}_{\mathbf{m}}$ time-average multiplicative growth rate.

i Label for a particular realization of a random variable.

j Label of a particular outcome.

J Size of the jackpot.

k dummy.

ℓ Leverage.

L Insured loss.

ℓ^- Smallest leverage for zero time-average growth rate.

$\ell_{\mathbf{opt}}$ Optimal leverage.

$\ell_{\mathbf{opt,s}}$ Simulated optimal leverage.

ℓ^+ Largest leverage for zero time-average growth rate.

$\ell^\pm$ Leverages for zero time-average growth rate.

$\mathbf{m}$ Mass.

m Index specifying a particular gamble.

μ Drift term in BM.

$\mu_{\mathbf{r}}$ Drift of riskless asset.

$\mu_{\mathbf{s}}$ Drift of risky asset.

$\mu_{\mathbf{e}}$ Excess drift.

n n_j is the number of times outcome j is observed in an ensemble.

N Ensemble size, number of realizations.

$\mathcal{N}$ Normal distribution, $x \sim \mathcal{N}(\langle p \rangle, \mathrm{var}(p))$ means that the variable p is normally distributed with mean $\langle p \rangle$ and variance $\mathrm{var}(p)$..

o Little-o notation.

p Probability, p_i is the probability of observing event i in a realization of a random variable.

$\mathcal{P}$ Probability density function.

r Random factor whereby wealth changes in one round of a gamble.

$r_{\langle\rangle}$ Exctation value of growth factor r.

$r_{\mathbf{time}}$ Average growth factor over a long time.

s Dummy variable in an integration.

S Sharpe ratio.

σ Magnitude of noise in a Brownian motion.

$\sigma_{\mathbf{s}}$ Volatility of risky asset.

t Time.

T Number of sequential iterations of a gamble, so that $T\delta t$is the total duration of a repeated gamble.

t_0 Specific value of time t, usually the starting time of a gamble..

τ Dummy variable indicating a specific round in a gamble.

u Utility function.

v Stationarity mapping function, so that $v(x)$ has stationary increments.

var Variance.

W Wiener process, $W(t) = \int_0^t dW$ is continuous and $W(t) \sim \mathcal{N}(0, \bar{g})$.

x Wealth.

$\mathbf{x}$ Position.

x_0 Value of an investment in the riskless asset.

x_1 Value of an invstment in the risky asset.

ξ A standard normal variable, $\xi \sim \mathcal{N}(0, 1)$.

x_ℓ Value of an investment in a leveraged portfolio.

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