Ergodicity Economics

Ole Peters and Alexander Adamou

LONDON MATHEMATICAL LABORATORY

2018/06/27 at 15:23:23

Contents

1 Decision theory

{sec:decision}

Decision theory is a cornerstone of formal economics. As the name suggests, it models how people make decisions. In this chapter we will generalise and formalise the treatment of the coin tossing game to introduce our approach to decision theory. Our central axiom will be that people attempt to maximize the rate at which wealth grows when averaged over time. This is a surprisingly powerful idea. In many cases it eliminates the need for well established but epistemologically troublesome techniques, such as utility functions.

1.1 Models and science fiction

We will do decision theory by using mathematical models, and since this can be done in many ways we will be explicit about how we choose to do it. We will define a gamble, which is a mathematical object, and we will define a decision criterion. The gamble will be reminiscent of real-world situations; and the decision criterion may or may not be reminiscent of how real people make decisions. We will not worry too much about the accuracy of these reminiscences. Instead we will "shut up and calculate" – we will let the mathematical model create its world. Writing down a mathematical model is like laying out the premise for a science-fiction novel. We may decide that people can download their consciousness onto a computer, that medicine has advanced to eliminate ageing and death – these are premises we are at liberty to invent. Once we have written them down we begin to explore the world that results from those premises. A choice of decision criterion implies an endless list of behaviors that will be observed. For example, some criteria will lead to cooperation, others will not, some will lead to the existence of insurance contracts, others will not *etc.* We will explore the worlds created by the different models. Once we have done so we invite you to judge which model you find most useful for your understanding of the world. Of course, having spent many years thinking about these issues we have come to our own conclusions, and we will put them forward because we believe them to be helpful.

To keep the discussion to a manageable volume we will only consider the gamble problem. This is reminiscent of making decisions in an uncertain context – where we have to decide on a course of action now although we don't know with certainty what will happen to us under any of our choices. To limit the debate even further, we will only consider a setup that corresponds to making purely financial decisions. We may bet on a horse or take out personal liability insurance. This chapter will not tell you whom you should marry or even whose economics lectures you should attend.

1.2 Gambles

One fundamental building block of mathematical decision theory is the gamble. This is a mathematical object that resembles a number of situations in real life, namely situations where we face a decision whose consequences will be purely financial and are somewhat uncertain when we make the decision. A real-world example would be buying a lottery ticket. We define the gamble mathematically as follows.

> DEFINITION: **Gamble**
>
> A gamble is a pair of a random variable, D, and a duration, δt.
>
> D is called the payout and takes one of N (mutually exclusive) possible monetary values, $\{D_1, \dots, D_N\}$, associated with probabilities, $\{p_1, \dots, p_N\}$, where $\sum_{i=1}^N p_i = 1$. Payouts can be positive, associated with a monetary gain, or negative, associated with a loss. We order them such that $D_1 < \dots < D_N$.

Everything we need to know about the gamble is contained in the payouts, probabilities, and duration. We relate it to reality through a few examples:

Example: Betting on a fair coin

Imagine betting \$10 on the toss of a fair coin. We would model this with the following payouts and probabilities:

$$D_1 = -\$10, \qquad p_1 = 1/2; \tag{1}$$
$$D_2 = +\$10, \qquad p_2 = 1/2. \tag{2}$$

The duration would be the time between two coin tosses. If you participate in coin tosses once a week it would be $\delta t = 1$ week.

Example: Playing the lottery

We can also imagine a gamble akin to a lottery, where our individual pays an amount, F, for a ticket which will win the jackpot, J, with probability, p. The corresponding payouts and probabilities are:

$$D_1 = -F, \qquad p_1 = 1 - p; \tag{3}$$
$$D_2 = J - F, \qquad p_2 = p. \tag{4}$$

Note that we deduct the ticket price, F, in the payout D_2. The duration would be $\delta t = 1$ week.

Example: Betting at fixed odds

A bet placed at fixed odds, for example on a horse race can also be modelled as a gamble. Suppose we bet on the horse *Ito* to win the 2015 Prix de l'Arc de Triomphe in Paris at odds of 50/1 (the best available odds on 20th September 2015). *Ito* will win the race with unknown probability, p. If we bet F, then this is modelled by payouts and probabilities:

$$D_1 = -F, \qquad p_1 = 1 - p; \tag{5}$$
$$D_2 = 50F, \qquad p_2 = p. \tag{6}$$

The duration would be something like $\delta t = 30$ minutes.

Example: The null gamble

It is useful to introduce the null gamble, in which a payout of zero is received with certainty: $D_1 = \$0$, $p_1 = 1$. This represents the 'no bet' or 'do nothing' option. The duration, δt, has to be chosen appropriately. The meaning of the duration will become clearer later on – often it is the time between two successive rounds of a gamble.

The gamble is a simple but versatile mathematical model of an uncertain future. It can be used to model not only traditional wagers, such as sports bets and lotteries, but also a wide range of economic activities, such as stock market investments, insurance contracts, derivatives, and so on. The gamble we have presented is discrete, in that the payout, D, is a random variable with a countable (and, we usually assume, small) number of possible outcomes. The extension to continuous random variables is natural and used frequently to model real-world scenarios where the number of possible outcomes, *e.g.* the change in a stock price over one day, is large.

Suppose now that you have to choose between two options that you've modeled as two gambles (possibly including the null gamble). Which should you choose, and why? This is the gamble problem, the central question of decision theory, and the basis for much of mainstream economics.

1.3 Repetition and wealth evolution

To solve the gamble problem we must propose a criterion to choose between two gambles. Different criteria will result in different decisions – by writing down a criterion we build a model world of model humans who behave in ways that may seem sensible to us or crazy – if the behavior seems crazy we have probably not chosen a good criterion and we should try a different one.

The wealth process $x(t)$ is connected to the gambles our model humans choose to play. Precisely *how* it is affected remains to be specified.

Considering a single round of a gamble in isolation – the so-called 'one-shot game' of game theory – is relatively uninstructive in this regard. All we know is that one of the possible payouts will be received, leading to the random variable $x(t + \delta t) = x(t) + D$. We don't yet know how accepting this gamble will affect how our wealth, $x(t)$, grows or decays over time, since one time step isn't enough for this to become apparent. The one-shot game takes one random variable, D, and turns it trivially into another, $x(t) + D$. Time has no significance in a one-shot game. An amount δt elapses, but this could be a single heartbeat or the lifetime of the universe, for all the difference it makes to the analysis.

To establish how your wealth evolves, we must imagine that the world does not come to a grinding halt after the gamble. Instead we imagine that the gamble is repeated over many rounds.[1] This does not mean that we actually believe that a real-world situation will repeat itself over and over again, *e.g.* we don't believe that we will bet on the horse *Ito* at odds 50/1 many times in a row. Instead, imagining repetition is a methodological device that allows us to extract tendencies where they would otherwise be invisible. It is the model analogue of the idea that individuals live in time and that their decisions have consequences which unfold over time.

Crucially, *the mode of repetition is not specified in the gamble itself.* It is a second component of the model, which must be specified separately. Initially we shall focus on two modes: *additive* and *multiplicative* repetition. Other dynamics will be considered later on, in Sec. 1.7.

[1] In fact, to make the problem tractable mathematically, it will be necessary to imagine the gamble is repeated indefinitely.

Definition: **Additive repetition**

If a gamble is repeated additively then the random payout, D, is simply added to $x(t)$ at each round. We define the change in wealth occurring over a single round as

$$\delta x(t) \equiv x(t+\delta t) - x(t). \tag{7}$$ {eq:DW_def}

In the additive case, we have

$$\delta x(t) = D. \tag{8}$$ {eq:DW_add}

In other words, under additive repetition, δx is a stationary random variable. Starting at time, t_0, wealth after T rounds is

$$x(t_0 + T\delta t) = x(t_0) + \sum_{\tau=1}^{T} D(\tau), \tag{9}$$ {eq:Wt_add}

where $D(\tau)$ is the realisation of the random variable in round τ. This is an evolution equation for wealth following a noisy additive dynamic. Note that $x(t_0 + T\delta t)$ is itself a random variable.

Example: Additive repetition

We return to our first example of a gamble: a \$10 bet on a coin toss. Under additive repetition, successive bets will always be \$10, regardless of how rich or poor you become. Suppose your starting wealth is $x(t_0) = \$100$. Then, following (Eq. 9), your wealth after T rounds will be

$$\begin{aligned} x(t_0 + T\delta t) &= \$100 + \$10k - \$10(T-k) & (10)\\ &= \$[100 + 10(2k - T)], & (11) \end{aligned}$$

where $0 \le k \le T$ is the number of tosses you've won. Note that we have assumed your wealth is allowed to go negative. If not, then the process would stop when $x < \$10$, since you would be unable to place the next \$10 bet.

An alternative is multiplicative repetition. In the example above, let us imagine that the first \$10 bet were viewed not as a bet of fixed monetary size, but as a fixed fraction of the starting wealth (\$100). Under multiplicative repetition, each successive bet is for the same fraction of wealth which, in general, will be a different monetary amount.

The formalism is as follows. The payout, D, in the first round is expressed instead as a random wealth multiplier,

$$r \equiv \frac{x(t_0) + D}{x(t_0)}. \tag{12}$$ {eq:R_def}

The gamble is repeated by applying another realisation of the same multiplier at all subsequent rounds:

$$x(t+\delta t) = r x(t). \tag{13}$$

From (Eq. 12) we see that r is a stationary random variable, since it depends only on D, which is stationary, and the starting wealth, $x(t_0)$, which is fixed. However, successive changes in wealth,

$$\delta x(t) = (r - 1)x(t), \tag{14}$$ {eq:DW_mult_short}

are not stationary, as they depend on t through $x(t)$. The wealth after T rounds of the gamble is

$$x(t_0 + T\delta t) = x(t_0) \prod_{\tau=1}^{T} r(\tau), \tag{15}$$

where $r(\tau)$ is the realisation of the random multiplier in round τ.

Example: Multiplicative repetition

The \$10 bet on a coin toss is now re-expressed as a bet of a fixed fraction of wealth at the start of each round. Following (Eq. 12), the random multiplier, r, has two possible outcomes:

$$r_1 = \frac{\$100 - \$10}{\$100} = 0.9, \qquad p_1 = 1/2; \tag{16}$$

$$r_2 = \frac{\$100 + \$10}{\$100} = 1.1, \qquad p_2 = 1/2. \tag{17}$$

The wealth after T rounds is, therefore,

$$x(t_0 + T\delta t) = \$100\ (1.1)^k\ (0.9)^{T-k}, \tag{18}$$

where $0 \leq k \leq T$ is the number of winning tosses. In this example there is no need to invoke a 'no bankruptcy' condition, since our individual can lose no more than 10% of his wealth in each round.

The difference between the two modes of repetition might easily be mistaken for a matter of taste. When the \$10 bet was first offered, what difference does it make whether our individual imagined this to be a bet of a fixed size or of a fixed fraction of his wealth? However, the consequences of this choice between imagined situations are enormous. As we saw in the previous lecture, additive and multiplicative dynamics differ as starkly as the linear and exponential functions, and there is evidence that people adjust their behavior to the type of repetition they face. It matters, therefore, that we consider carefully the economic situation we wish to model in order to choose the most realistic mode of repetition. For example, fluctuations in the price of a stock tend to be proportional to the price, *cf.* (Eq. 14), so multiplicativity is the appropriate paradigm here.

Now that we've established how the gamble is related to $x(t)$ we can begin to think about decision criteria. Not surprisingly, appropriate growth rates are useful decision criteria – "pick the gamble that will lead your wealth to grow the fastest" is generally good advice. To be able to follow this advice we will think again about growth rates.

1.4 Growth rates {section:Growth_rates}

In the previous lecture we introduced the concept of a growth rate, g, which is the rate of change of a monotonically increasing function of wealth, $v(x(t))$:

$$g(t, \Delta t) \equiv \frac{\Delta v(x(t))}{\Delta t}. \tag{19}$$ {eq:g_def}

The function, v, is chosen such that the increments, $\Delta v(x(t))$, over the period Δt,[2] are independent instances of a random variable. The growth rate is, therefore, ergodic[3]. We consider g a function of t only inasmuch as this labels a particular realisation of the randomness at a particular point in time.

The statistical irrelevance of the time of measurement is important because we want the distribution of the random growth rate to convey robust information about the underlying process, rather than mere happenstance about when it was sampled.

Under additive repetition, we know from (Eq. 8) that Δx is already ergodic, so we know immediately that the correct mapping is the identity: $v(x) = x$.[4] The ergodic growth rate for an additive process (denoted by the subscript 'a') is therefore:

$$g_{\mathrm{a}}(t, \Delta t) = \frac{\Delta x(t)}{\Delta t}. \tag{20}$$ {eq:g_add}

For a multiplicative dynamic, however, using Δx in the numerator of the rate will not do, as we know from (Eq. 14) that changes in $x(t)$ depend on $x(t)$. Instead we must find the mapping $v(x)$ whose increments are independent instances of a random variable. The correct mapping now is the logarithm, since the increment over a single round is

$$\begin{aligned} \delta \ln x(t) &= \ln x(t+\delta t) - \ln x(t) & (21)\\ &= \ln r x(t) - \ln x(t) & (22)\\ &= \ln r, & (23) \end{aligned}$$

where (Eq. 12) has been used in the second line. This inherits its ergodicity from r. Thus the appropriate growth rate for a multiplicative process (denoted by the subscript 'm') over an arbitrary time period is

$$g_{\mathrm{m}}(t, \Delta t) = \frac{\Delta \ln x(t)}{\Delta t}. \tag{24}$$ {eq:g_mult}

The distribution of the random variable $g(t, \Delta t)$ does not depend on t but it does depend on Δt. Subject to certain conditions on $\Delta v(x(t))$, the distribution of $g(t, \Delta t)$ narrows as Δt increases, converging to a scalar (just a number, no longer a random variable) in the limit $\Delta t \to \infty$. In other words, as the effect of the gamble manifests itself over an increasingly long time, the noise is eliminated to reveal a growth rate reflecting the gamble's underlying tendency.

[2] Note that we use a general time period, Δt, here and not the period of the gamble, δt.

[3] The increments don't necessarily need to be independent, and examples can be constructed where they are drawn from different distributions at different times, but in our case we have indepence and a fixed distribution, and that's a sufficient condition for the growth rate to be ergodic

[4] In fact, any linear function $v(x) = \alpha x + \beta$ has stationary increments and is monotonically increasing provided $\alpha > 0$. However, there is nothing gained by choosing anything other $v(x) = x$.

We define this time-average growth rate, $\overline{g}$, as

$$\overline{g} \equiv \lim_{\Delta t \to \infty} \{g(t, \Delta t)\}. \tag{25}$$

This is the growth rate that an individual will experience almost surely (*i.e.* with probability approaching one) as the number of rounds of the gamble diverges. Indeed, we can express $\overline{g}$ in precisely these terms,

$$\overline{g} = \lim_{T \to \infty} \left\{ \frac{v(x(t + T\delta t)) - v(x(t))}{T\delta t} \right\}, \tag{26}$$

where T is the number of rounds. Expanding the numerator as a sum of increments due to individual rounds of the gamble gives

$$\begin{aligned} \overline{g} &= \lim_{T \to \infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} \frac{\Delta v(x(t + \tau\delta t))}{\delta t} \right\} && (27) \\ &= \lim_{T \to \infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} g(t + \tau\delta t, \delta t) \right\} && (28) \\ &= \langle g(t, \delta t) \rangle , && (29) \end{aligned}$$

where the final line follows from the stationarity and independence of the successive per-round growth rates. This is a restatement of the ergodic property of the previous lecture, namely that the time-average growth rate can be expressed equivalently as the long-time limit and as the ensemble average of the properly chosen ergodic growth rate. For additive and multiplicative dynamics, we obtain the following equivalences:

$$\overline{g}_{\mathrm{a}} = \lim_{\Delta t \to \infty} \left\{ \frac{\Delta x(t)}{\Delta t} \right\} = \left\langle \frac{\Delta x(t)}{\Delta t} \right\rangle; \tag{30}$$

{eq:g_bar_a}

$$\overline{g}_{\mathrm{m}} = \lim_{\Delta t \to \infty} \left\{ \frac{\Delta \ln x(t)}{\Delta t} \right\} = \left\langle \frac{\Delta \ln x(t)}{\Delta t} \right\rangle. \tag{31}$$

{eq:g_bar_m}

These follow the form of the general expression,

$$\overline{g} = \lim_{\Delta t \to \infty} \left\{ \frac{\Delta v(x(t))}{\Delta t} \right\} = \left\langle \frac{\Delta v(x(t))}{\Delta t} \right\rangle. \tag{32}$$

{eq:g_bar_gen}

The value of Δt in the ensemble averages is immaterial. In calculations, it is often set to the period, δt, of a single round of the gamble.

Where we are interested in the value of $\overline{g}$, knowing that it is equal to the value of $\langle g \rangle$ may provide a convenient method of calculating it. However, we will attach no special interpretation to the fact that $\langle g \rangle$ is an expectation value. It is simply a quantity whose value happens to coincide with that of the quantity we're interested in, *i.e.* the time-average growth rate.

Let's take a step back and remark more generally on what we have done so far. We started with a high-dimensional mathematical object, namely the probability distribution of the payout, D, of the gamble. To this we added two model components: the time period, δt, over which the gamble unfolds; and a dynamic, in essence a set of instructions, specifying how the repeated gamble causes your wealth to evolve. We then collapsed all of this information into

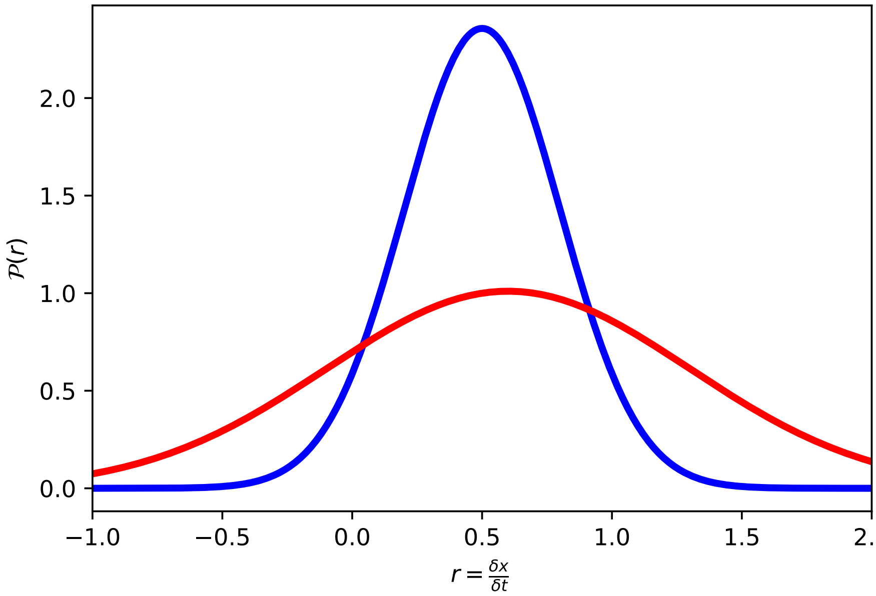

Figure 1: Two possible probability density functions for the per-round multiplier, r, defined in (Eq. 12). The distribution denoted by the blue line has a higher mean and a higher variance than the one in red. How are we to decide which represents the more favourable gamble?fig:dec_dist

a single number, $\overline{g}$, which characterises the effect of the gamble. The collapse from distribution to single number (or, equivalently, from uncertain to certain quantity) allows different gambles to be compared and, in particular, ranked. This permits an unequivocal decision criterion, which would be much harder to formulate for higher-dimensional objects, such as the two distributions shown in Fig. 1.

1.5 The decision axiom

Our model rationale for deciding between two gambles is simple: given a model for the mode of repetition, choose the gamble with the largest time-average growth rate. In other words, choose the gamble which, if repeated indefinitely, causes your wealth to grow fastest.

We are not saying that real repetitions are necessary. We merely create model humans who base decisions on what would happen to their wealth if they were repeated (infinitely many times). It is a conceptual device – in effect a thought experiment – to elicit the underlying tendency of each gamble. A particular choice that can be represented by gambles may be offered only once and, indeed, in the real world this will often be the case. However, in the real world it is also the case that a decision is likely to be followed by many others, a scenario to which indefinite repetition is a plausible approximation.

In our thought experiment this decision rule outperforms any other decision rule almost surely: so, at least in that imagined world, the rationale has a logical basis. If our thought experiment is a good approximation to real-world decision scenarios, then our rationale should be a good model of real-world decisions. Certainly it is parsimonious, based on a single, robust quantity and requiring only the gamble and its mode of repetition to be specified. Unlike some treatments of human decision-making, it contains no arbitrary or hard-to-measure psychological factors.

Having said all this, while we think that the decision axiom is reasonable, we stress that it is an *axiom*, *i.e.* something we assert without empirical justification and which is not deduced from more primitive considerations. It defines a model world where certain types of behaviour will be observed. We feel reminded of reality by this model world, but you may disagree or you may prefer a different model that creates a different model world that also reminds you of reality. Other decision axioms are possible and, indeed, have been proposed. For instance, classical decision theory is defined by the axiom that decision makers maximize expected utility.

Our decision rationale can be expressed as a set of instructions. We denote quantities relating to the m^{th} available gamble with the superscript (m). Each gamble is specified by its random payout, $D^{(m)}$, and per-round period, $\delta t^{(m)}$.

Growth-optimal decision algorithm

1. Specify $D^{(m)}$ and $\delta t^{(m)}$ for the gambles offered;
2. Specify the wealth dynamic, *i.e.* the relationship between $\delta x(t)$, $x(t)$, and D;
3. Find the ergodicity transformation, $v(x)$, of the wealth whose increments are instances of a (time-independent) random variable under this dynamic;
4. Determine the time-average growth rates, $\overline{g}^{(m)}$, either by taking the long-time limit of the growth rates, $g^{(m)}(t, \delta t)$, or by invoking the ergodic property and taking their ensemble averages;
5. Choose the gamble, m, with the largest time-average growth rate.

In examples, we will focus on choices between two gambles, *i.e.* $m \in \{1, 2\}$. Decisions between three or more gambles are straightforward extensions. Often one of the gambles presented is the null gamble, which trivially has a growth rate of zero. In this case, the question then posed is whether the other gamble presented is preferable to doing nothing, *i.e.* bet or no bet?

We will illustrate the decision algorithm by applying it to the coin toss game of the previous lecture. $x(t_0) > \$0$ is the starting wealth and δt is the time between rounds for each of the gambles offered. We recall that the coin toss gamble is specified by the random payout:

$$D_1^{(1)} = -0.4x(t_0), \qquad p_1^{(1)} = 1/2; \tag{33}$$

$$D_2^{(1)} = 0.5x(t_0), \qquad p_2^{(1)} = 1/2. \tag{34}$$

Note that, in our setup, the payouts $D_i^{(m)}$ are always fixed monetary amounts. Here they are expressed as fractions of $x(t_0)$, which is itself a fixed monetary wealth.[5] We shall ask our individual to choose between the coin toss and the null gamble,

$$D_1^{(2)} = \$0, \quad p_1^{(2)} = 1. \tag{35}$$

Both the additive and multiplicative versions of the repeated gamble are analysed.

Example: Additive coin toss game

If the repetition is additive, wealth will evolve over T rounds according to:

$$x^{(1)}(t_0 + T\delta t) = x(t_0) + \sum_{\tau=1}^{T} D^{(1)}(\tau); \tag{36}$$

$$x^{(2)}(t_0 + T\delta t) = x(t_0). \tag{37}$$

Here we have assumed that wealth is free to go negative, *i.e.* that there is no bankrupt state from which the individual can't recover.[6] The ergodicity transformation is the identity, $v(x) = x$, so the growth rates are simply the rates of change of wealth itself. We can express these over the T rounds as:

$$g_{\mathrm{a}}{}^{(1)}(t) = \frac{x^{(1)}(t + T\delta t) - x(t_0)}{T\delta t} = \frac{1}{T}\sum_{\tau=1}^{T}\frac{D^{(1)}(\tau)}{\delta t}; \tag{38}$$ {eq:g1_add}

$$g_{\mathrm{a}}{}^{(2)}(t) = \frac{x^{(2)}(t + T\delta t) - x(t_0)}{T\delta t} = \$0 \tag{39}$$

per unit time. The long-time limit of the growth rate for the null gamble is trivially $g_{\mathrm{a}}{}^{(2)} = \$0$ per unit time. For the coin toss, we calculate it as

$$\overline{g_{\mathrm{a}}}^{(1)} = \lim_{T\to\infty}\left\{\frac{1}{T}\sum_{\tau=1}^{T}\frac{D^{(1)}(\tau)}{\delta t}\right\} \tag{40}$$

$$= \left\langle \frac{D^{(1)}}{\delta t} \right\rangle \tag{41}$$

$$= \frac{p_1^{(1)}D_1^{(1)} + p_2^{(1)}D_2^{(1)}}{\delta t} \tag{42}$$

$$= \frac{0.05x(t_0)}{\delta t}, \tag{43}$$

which is positive (assuming $x(t_0) > \$0$). Therefore, $\overline{g_{\mathrm{a}}}^{(1)} > \overline{g_{\mathrm{a}}}^{(2)}$ and our individual should accept the coin toss gamble under additive dynamics.

We see from this example that the decision rule under additive repetition is to maximise $\langle D/\delta t\rangle$.[7] This is the rate of change of the expected wealth, which, as we know from (Eq. 30), happens to coincide under this dynamic with the

[5] Even though this formulation might appear to encode a multiplicative dynamic (largely because it comes from an imagined multiplicative game), we have arranged things so that formally it does not. Indeed, it encodes no dynamic at all: that is specified separately.

[6] We could model this realistic feature by an absorbing boundary on $x(t)$, if we were so minded.

[7] Too frequently in presentations of decision theory, it is assumed implicitly that δt is the

time-average growth rate. We will see later that humans tend not to act to maximise $\langle D/\delta t\rangle$ in reality. This may not be a great shock: additive repetition without bankruptcy isn't going to win many prizes for the most realistic model of wealth evolution.

Let's try multiplicative repetition instead.

Example: Multiplicative coin toss game

The payout, $D^{(1)}$, is re-expressed as a per-round multiplier,

$$r^{(1)} = \frac{x(t_0) + D^{(1)}}{x(t_0)}, \tag{44}$$

which takes the values:

$$r_1^{(1)} = \frac{x(t_0) + D_1^{(1)}}{x(t_0)} = 0.6, \qquad p_1^{(1)} = 1/2; \tag{45}$$

$$r_2^{(1)} = \frac{x(t_0) + D_2^{(1)}}{x(t_0)} = 1.5, \qquad p_2^{(1)} = 1/2. \tag{46}$$

Under multiplicative dynamics, the wealth evolves according to:

$$x^{(1)}(t_0 + T\delta t) = x(t_0)\prod_{\tau=1}^{T} r^{(1)}(\tau); \tag{47}$$ {eq:W1T_mult}

$$x^{(2)}(t_0 + T\delta t) = x(t_0). \tag{48}$$ {eq:W2T_mult}

The ergodicity transformation is the logarithm, $v(x) = \ln x$. We have already discussed why, but one way of seeing this is to take logarithms of (Eq. 47). This converts the product into a sum of independent instances of a random variable:

$$\ln x^{(1)}(t_0 + T\delta t) = \ln x(t_0) + \sum_{\tau=1}^{T} \ln r^{(1)}(\tau); \tag{49}$$ {eq:lnW1T_mult}

$$\ln x^{(2)}(t_0 + T\delta t) = \ln x(t_0). \tag{50}$$ {eq:lnW2T_mult}

Therefore, $\ln x$ is the desired quantity whose increments inherit their ergodicity from D. The growth rates, expressed over T rounds, are:

$$g_{\text{m}}{}^{(1)}(t) = \frac{\ln x^{(1)}(t + T\delta t) - \ln x(t_0)}{T\delta t} = \frac{1}{T}\sum_{\tau=1}^{T}\frac{\ln r^{(1)}(\tau)}{\delta t}; \tag{51}$$

$$g_{\text{m}}{}^{(2)}(t) = \frac{\ln x^{(2)}(t + T\delta t) - \ln x(t_0)}{T\delta t} = 0 \tag{52}$$

same for all available gambles and the decision algorithm is presented as the maximisation of the expected payout, $\langle D\rangle$. While this is equivalent if all the $\delta t^{(m)}$ are identical, it is not if they aren't. Moreover, the assumption is usually left unstated. This is unhelpful in that it masks the important role of time in the analysis and, in particular, the fact that our individual is maximising a *rate*.

per unit time. As in the additive case, we take the $T \to \infty$ limits. For the null gamble this is trivial: $\overline{g}_{\mathrm{m}}^{(2)} = 0$ per unit time. For the coin toss gamble, we get

$$\overline{g}_{\mathrm{m}}^{(1)} = \left\langle \frac{\ln r^{(1)}}{\delta t} \right\rangle = \frac{\ln\left(\sqrt{0.9}\right)}{\delta t}, \tag{53}$$

which is negative. Thus, $\overline{g_{\mathrm{m}}}^{(1)} < \overline{g_{\mathrm{m}}}^{(2)}$ under multiplicative dynamics and our individual should decline the coin toss. That this is the opposite of his decision under additive repetition highlights the importance of specifying a dynamic that corresponds well to what is happening in reality.

Another way of presenting the repeated coin toss is to express the wealth after T rounds as

$$x^{(1)}(t_0 + T\delta t) = x(t_0)\left(r_T^{(1)}\right)^T, \tag{54}$$

where

$$r_T^{(1)} = (0.6)^{(T-k)/T}(1.5)^{k/T} \tag{55}$$

is the equivalent per-round multiplier after T rounds and $0 \leq k \leq T$ is the number of winning rounds. r_T is a random variable but it converges almost surely to a scalar in the long time limit,

$$r_\infty^{(1)} \equiv \lim_{T\to\infty}\{r_T^{(1)}\} = (0.6)^{1/2}(1.5)^{1/2} = \sqrt{0.9}, \tag{56}$$

since $k/T \to 1/2$ as $T \to \infty$ (the coin is fair). $r_\infty^{(1)} < 1$ so the individual's wealth is sure to decay over time and he should decline the gamble. The two approaches are, of course, linked, in that

$$\overline{g}_{\mathrm{m}}^{(1)} = \frac{\ln r_\infty^{(1)}}{\delta t}. \tag{57}$$

Here we see that our decision rule boils down to maximising

$$\left\langle \frac{\ln r}{\delta t} \right\rangle = \left\langle \frac{\ln(x(t_0) + D) - \ln x(t_0)}{\delta t} \right\rangle. \tag{58}$$

This coincides with the time-average growth rate in (Eq. 31).

1.6 The expected-wealth and expected-utility paradigms

Our decision rule under additive repetition of the gamble is to maximise

$$\left\langle \frac{\delta x}{\delta t} \right\rangle = \left\langle \frac{D}{\delta t} \right\rangle, \tag{59}$$ {eq:ex_crit}

i.e. the rate of change of the expectation value of wealth. This was, in fact, the first decision rule to be suggested when gamble problems were considered in the early days of probability theory in the 17^{th} century. We will call this the 'expected-wealth paradigm'. It was not derived as we have derived it, from a criterion to maximise growth over repetition. Instead, it was essentially proposed as the decision axiom itself, with no reference to dynamics. It is easy to see why: it is a simple rule containing a familiar type of average, which incorporates all the possible outcomes of the game. Indeed, it would be logically sound if we could play the game many times in parallel, thereby accessing all the possible outcomes.

In the language of economics, the expected-wealth paradigm treats humans as 'risk neutral', *i.e.* they have no preference between gambles whose expected changes in wealth are identical (over a given time interval). This treatment has been known to be a flawed model of human decision-making since at least 1713 [29, p. 402], in that it does not accord well with observed behaviour.

The conventionally offered reason for this predictive failure is that the value to an individual of a possible change in wealth depends on how much wealth he already has and his psychological attitude to taking risks. In other words, people do not treat equal amounts of extra money equally. This makes intuitive sense: an extra \$10 is much less significant to a rich man than to a pauper for whom it represents a full belly; an inveterate gambler has a different attitude to risking \$100 on the spin of a roulette wheel than a prudent saver, their wealths being equal.

In 1738 Daniel Bernoulli [7], after correspondence with Cramer, devised the 'expected-utility paradigm' to model these considerations. He observed that money may not translate linearly into usefulness and assigned to an individual an idiosyncratic utility function, $u(x)$, that maps his wealth, x, into usefulness, u. He claimed that this was the true quantity whose rate of change of expected value,

$$\langle r_u \rangle \equiv \left\langle \frac{\delta u(x)}{\delta t} \right\rangle, \tag{60}$$ {eq:euh}

is maximised in a decision between gambles.

This is the axiom of utility theory. It leads to an alternative decision algorithm, which we summarise here:

Expected-utility decision algorithm

1. Specify $D^{(m)}$ and $\delta t^{(m)}$ for the gambles offered;
2. Specify the individual's idiosyncratic utility function, $u(x)$, which maps his wealth to his utility;
3. Determine the rate of change of his expected utility over a single round of the gamble (no dynamic is specified so it's unclear how a gamble would be repeated),
$$\langle r_u \rangle^{(m)} = \left\langle \frac{u\left(x + D^{(m)}\right) - u(x)}{\delta t^{(m)}} \right\rangle; \tag{61}$$
4. Choose the gamble, m, with the largest $\langle r_u \rangle^{(m)}$.

Despite their conceptually different foundations, we note the similarities between the maximands[8] of our growth-optimal decision theory, (Eq. 32), and

[8]The quantities to be maximised.

the expected-utility paradigm, (Eq. 60). Our decision theory contains a mapping which transforms wealth into a variable, v, whose increments are instances of a time-independent random variable. This mapping depends on the wealth dynamic which describes how the gamble is repeated. The expected-utility paradigm contains a function which transforms wealth into usefulness. This is determined by the idiosyncratic risk preferences of the individual. If we identify the ergodicity transformation of the former with the utility transformation of the latter, then the expressions are the same.

But we caution against the conceptual world of expected utility theory. It uses expectation values, where they are inappropriate (because decisions of an individual are considered, not of many parallel systems), and it corrects for this error by introducing a non-linear mapping of wealth (the utility function) whose specific form cannot be pinned down convincingly. Finally, because of this conceptual weakness, utility theory is poorly defined. The very first paper on the topic contains two contradictory definitions of expected utility theory [7], and over the centuries several others have been added.

Nonetheless, with a few assumptions, expected utility theory as we've presented it above is consistent with growth rate optimisation, provided a suitable pair of dynamic and utility function is used. For multiplicative dynamics, the necessary utility function is the logarithm. That this is the most widely used utility function in both theory and practice is a psychological fluke in the classic mindset; from our perspective it indicates that our brains have evolved to produce growth-optimal decisions in a world governed by multiplicative dynamics, *i.e.* where entities produce more of themselves. Incidentally, a common definition of life is "that which produces more of itself," or as Harold Morowitz put it [30, p. 5] "Living systems self-replicate, that is, they give rise to organisms like themselves." From our perspective, the prevalence of logarithmic utility reflects our evolution in an environment of living things.

Thus we can offer a different reason for the predictive failure of the expected-wealth paradigm. In our framework this corresponds to good decisions under additive repetition, which we claim is generally a poor model of how wealth evolves.[9] It fails, therefore, because it corresponds to an unrealistic dynamic.

It's worth being explicit about how our theory differs from expected utility theory. In our model what's stable about human behavior is this: humans optimize wealth over time. Utility theory says humans optimize utility across the ensemble. Given a dynamic, our paradigm predicts human behavior that can be mapped to a predicted utility function. Utility theory believes that behavior, specifically risk preferences, is an idiosyncratic thing: it does not depend on the dynamics but on the individual. This difference (behavior is determined by dynamics vs. behavior is determined by indiosyncrasy) means that in principle we can find out which theory is right. We can change dynamic and see if people change behavior. That would invalidate expected utility theory. We can also take several people and expose them to the same dynamic – the extent to which they display utility functions that are incompatible with the dynamic would invalidate our theory, insofar as it is interpreted to predict human behavior. Our theory provides the optimal behavioral protocol, in terms of long-term wealth growth, but of course real humans will only optimize long-term wealth

[9] It would correspond, for example, to the interest payments in your bank account being independent of your balance!

growth to a certain degree. Nevertheless, we are happy to report that a Danish group of experimentalists has recently carried out experiments where subjects were exposed to different wealth dynamics, to see whether the utility functions implied by their behavior changed as predicted by our theory. At the time of writing, we're still waiting for the results.

1.7 General dynamics

{section:general_dynamics

Before we come to example applications of decision theory, it is time to discuss dynamics that are neither additive nor multiplicative and their relation to general utility functions.

1.7.1 Why discuss general dynamics?

What are we trying to achieve with this? First of all, where the time perspective has come up in history, *e.g.* [49, 21], it was always in the context of multiplicative dynamics, which, as we saw in the previous section, corresponds to logarithmic utility. One persistent criticism from people trained to think in terms of utility has been: what if my utility function is not logarithmic? In other words, the time perspective has seemed overly restrictive to many people.

As a matter of epistemology, we want to be somewhat restrictive. Imagine the opposite: a model that can produce absolutely any behavior. Because it can produce anything, it cannot say "the reason this happens and not that is X." But that's exactly the type of statement we need in order to make sense of the world – why do some structures exist and not others? Consider a specific case of a universal model: the alphabet, including punctuation. That's all the building blocks you need to tell any story you want, true or false, interesting or not. The alphabet is a useful thing, but it's not a good scientific model because it's not restrictive enough. Most sentences allowed by the alphabet are gibberish. Like "flowerbamdoodle zap."

Having said this, it is true that we can also err on the other end of the spectrum and have a model that's too restrictive. Imagine, instead of having the alphabet plus punctuation at your disposal you were only allowed to use sentences of five words, beginning with "heteroglot." That would make it difficult to write an insightful account of the lifecycle of newts, for example.

But to return to the topic at hand, one reason for generalizing the dynamics is to answer to the criticism of utility-theory users who dislike the restriction, as they would phrase it, to logarithmic utility functions.

The second reason for generalizing at this point is that we feel there are important aspects of decision-making that cannot be understood with multiplicative dynamics. Here is an example. Under multiplicative dynamics risk aversion is growth optimal. That means your wealth will grow faster over time if you make decisions that are more cautious than decisions you would make to optimize the expectation value of your wealth. But we know that in reality there are many situations where risk seeking (the opposite of risk aversion) will make your wealth grow faster.

It's worth discussing this because it shows an important epistemological difference between utility theory and time optimization. There's a big difference between the following two life situations.

1. My earned income is much more than what I need to live. I'm sure you know the problem: at the end of each month, having paid the rent for my flat share, bought all the books I was curious about and all the surfwax I needed, I'm left with \$50,000 of my \$52,000 monthly paycheck. I'll put that aside for a rainy day, stick it into bitcoin or buy some Enron shares. Having done this for several years, most of my \$50,000,0000 wealth is just riding the stock market.

2. You won't know about this, but here's another scenario: I don't have \$50,000 to put aside at the end of each month. Actually, a week after payday, everything has been spent, and the remaining 3 weeks have to be bridged. I help you move those boxes, and in return you make lunch. Every couple of months I get a new credit card to max out, until they're onto me and cut me off. That's followed by a few years of handing over everything I earn to my creditors, and after that I declare bankruptcy to start another cycle. I have no wealth riding anything.

We will turn these descriptions into mathematics later on, but for now let's analyse them from the dynamic perspective, and then from the perspective of utility theory to bring out the conceptual differences.

Dynamic perspective: From these situations, I can clearly see that there's a difference between the dynamics my wealth will follow. In situation 1) my wealth is roughly a multiplicative process. Earned income is negligible, and investment income dominates. Not so in situation 2), where earned income is the only income I have. It's so low that I have no savings, I can't pay my health insurance, can't buy a car to get a job in the next town, can't pay for university. I'm stuck at the bottom.

In situation 1) I can relax. I'm happy with the status quo, and there's no reason for me to change it. I should be change-averse, which is called risk-averse in the literature. In situation 2) the status quo is terrible, and I should be change-seeking, or risk-seeking if you prefer that word. It's growth optimal – rational, according to our rationality model – to play the lottery and hope for a big win, even if the expected return is negative. In situation 2, if as little as just \$1,000,000 fell into my lap, my life would change dramatically. Not because I would have all the money I need but because my wealth would enter a different dynamical regime.

Utility perspective: Utility theory would see me in situation 1) and conclude that my utility function is concave. My actions are optimal with respect to my utility function. Where that function comes from is unknown, people will discuss nature vs. nurture, psychology, and brain architecture, something like that. In situation 2) my actions are also optimal with respect to my utility function. It's just that that happens to be convex – I'm the type of person who likes to take risks, again someone will bring up psychology and neuro-imaging. Someone trained to think in terms of expectation values may also conclude that I'm a bit stupid because I accept gambles with negative expected return. Perhaps he (usually it's a he) will perceive the arrow of causality as follows: the reason I'm so poor is that I make such terrible financial decisions (which is, of course, a possibility).

The utility perspective tends to draw our attention on the psychology of the decision maker, whereas the dynamic perspective focuses on the situation of the decision maker.

1.7.2 Technical setup

{section:Technical}

With dynamics generalized beyond additive and multiplicative we have to be careful not to let the scope of our treatment balloon into meaninglessness. We also have to pick a setup where the mathematics won't lead to pages of equations without actually adding much. Here's what we do: we restrict ourselves to wealth dynamics that are expressed as an Itô process, which you may remember from (Eq. **??**) in Sec. **??**. We will further restrict ourselves to coefficient functions $a(x)$ and $b(x)$ without explicit t dependence, meaning wealth will follow

$$dx = a(x)dt + b(x)dW. \tag{62}$$

We can still choose additive dynamics, namely by setting $a = \mu$ and $b = \sigma$ as constants (Brownian motion, (Eq. **??**)). We can also choose multiplicative dynamics, with $a = \mu x$ and $b = \sigma x$ (geometric Brownian motion, (Eq. **??**)).

We are interested in wealth – more is better. We can position ourselves, make one choice or another, and then let time act. Choosing between two repeated gambles is thus a choice between two random sequences of wealths, let's call them $x(t)$ and $x^*(t)$. If life were very simple we could just look at both of them at some moment in the future, t^* say, and choose the bigger one. But that's generally not possible because of noise – both $x(t^*)$ and $x^*(t^*)$ are random variables, and we don't know which realizations we will encounter.

So what do we do? Let's build this up systematically, starting from a probabilistic statement, and let time help us get rid of the randomness later. At each decision time, t_0, we want to maximise subsequent changes in wealth by selecting $x(t)$ so that if we wait long enough wealth will be greater under the chosen process than under the alternative process *with certainty*. Mathematically speaking, there exists a sufficiently large t such that the probability of the chosen $x(t)$ being greater than $x^*(t)$ is arbitrarily close to one,

$$\forall \epsilon, x^*(t) \quad \exists \Delta t \quad \text{s.t.} \quad \mathcal{P}\left(\Delta x > \Delta x^*\right) > 1 - \epsilon, \tag{63}$$ {eq:max_Dx}

where $0 < \epsilon < 1$ specifies how certain we want to be. To keep notation simple we've used

$$\Delta x \equiv x(t_0 + \Delta t) - x(t_0); \tag{64}$$

$$\Delta x^* \equiv x^*(t_0 + \Delta t) - x^*(t_0). \tag{65}$$ {eq:Dx}

The criterion is necessarily probabilistic because the quantities Δx and Δx^* are random variables and it's possible for either of them to exceed the other for any finite Δt. Only in the limit $\Delta t \to \infty$ does the randomness vanish from the system.

Conceptually this criterion is tantamount to maximising $\lim_{\Delta t \to \infty}\{\Delta x\}$ or, equivalently, $\lim_{\Delta t \to \infty}\{\Delta x / \Delta t\}$. However, neither limit is guaranteed to exist. For example, consider a choice between two geometric Brownian motions,

$$dx = x(\mu dt + \sigma dW), \tag{66}$$

$$dx^* = x^*(\mu^* dt + \sigma^* dW). \tag{67}$$

Assuming that both grow over time, meaning $\mu > \sigma^2/2$ and $\mu^* > \sigma^{*2}/2$, the quantities $\Delta x/\Delta t$ and $\Delta x^*/\Delta t$ both diverge in the limit $\Delta t \to \infty$. The growth is exponential, so linear additive changes will diverge over time. A criterion requiring the larger rate of wealth change to be selected fails to yield a decision: comparing ∞ to ∞ is not meaningful.

But what about that idea of transforming wealth? For the moment we're only interested in which wealth will be larger (x or x^*); we don't care by how much. That means we don't have to consider x itself, but any monotonically increasing function of x will also do. Let's again call that $v(x)$. Why do we keep coming back to monotonic functions? Well, monotonicity means that the events $x > x^*$ and $v(x) > v(x^*)$ are identical – whenever one of the inequalities is satisfied, the other is too. So a monotonically increasing function of x works as an indicator and can help us out of that infinity-fix we just found ourselves in. We define:

$$\begin{aligned} Dv &\equiv v(x(t_0 + \Delta t)) - v(x(t_0)); && (68) \\ Dv^* &\equiv v(x^*(t_0 + \Delta t)) - v(x^*(t_0)). && (69) \end{aligned}$$

{eq:Du}

The monotonicity of $v(x)$ means that the events $\Delta x > \Delta x^*$ and $Dv > Dv^*$ are the same. Taking $\Delta t > 0$ allows this event to be expressed as $Dv/\Delta t > Dv^*/\Delta t$, whence the decision criterion in (Eq. 63) becomes

$$\forall \epsilon, x^*(t) \quad \exists \Delta t \quad \text{s.t.} \quad \mathcal{P}\left(\frac{Dv}{\Delta t} > \frac{Dv^*}{\Delta t}\right) > 1 - \epsilon. \tag{70}$$

{eq:max_Du}

Our decision criterion has been recast to focus on the rate of change

$$g_{\mathrm{a}}(u) \equiv \frac{Du}{\Delta t}, \tag{71}$$

As before, it is conceptually similar to maximising

$$\overline{g_{\mathrm{a}}} \equiv \lim_{\Delta t \to \infty} \left\{\frac{Dv(x)}{\Delta t}\right\} = \lim_{\Delta t \to \infty} \{g_{\mathrm{a}}(v)\}. \tag{72}$$

{eq:barr}

If $x(t)$ satisfies certain conditions, to be discussed below, then the function $v(x)$ can be chosen such that this limit exists. We shall see that $\overline{g_{\mathrm{a}}}(v(x))$ is then the appropriately defined time-average growth rate of x. This is quite a powerful bit of mathematics: by insisting on the existence of the limit, we force ourselves to choose $v(x)$ in a certain way. That certain way guarantees that the correct form of growth rate is used. For example, if $x(t)$ is Brownian motion, $v(x)$ will be linear, and if it's geometric Brownian motion, $v(x)$ will be logarithmic. This has nothing to do with psychology and behavior, it's simply imposed on us by the dynamics and our wish to compare long-term performances in a mathematically meaningful way. For the moment we leave our criterion in the probabilistic form of (Eq. 70) but to continue the discussion we assume that the limit (Eq. 72) exists.

Let's connect this back to the general relationship between expected utility theory and ergodicity economics. Perhaps (Eq. 72) is the same as the rate of change of the expectation value of Dv

$$\overline{g_{\mathrm{a}}}(v) = \frac{\langle Dv \rangle}{\Delta t}. \tag{73}$$

{eq:aver}

We could then make the identification of $v(x)$ being the utility function $u(x)$, noting that our criterion is equivalent to maximizing the rate of change in expected utility. We note Dv and hence $g_\mathrm{a}(v)$ are random variables but $\langle Dv \rangle$ and $\overline{g_\mathrm{a}}(v)$ are not. Taking the expectation value is one way of removing randomness from the problem, and taking the long-time limit is another. As we saw in Sec. **??**, (Eq. **??**), the expectation value is simply a different limit: it's an average over N realizations of the random number Dv, in the limit $N \to \infty$. The effect of removing randomness is that the process $x(t)$ is collapsed into the scalar $\langle Dv \rangle$, and consistent transitive decisions are possible by ranking the relevant scalars. In general, maximising $\overline{g_\mathrm{a}}(v)$ does not yield the same decisions as the criterion espoused in (Eq. 70). This is only the case for a particular function $v(x)$ whose shape depends on the process $x(t)$, *i.e.* on the dynamics. Our aim is to find these pairs of processes and functions. When using such $v(x)$ as the utility function, expected utility theory will be consistent with optimisation over time, so long as no one changes the dynamics. It is then possible to interpret behavior consistent with expected utility theory with utility function $u(x)$ in purely dynamical terms: such behavior will lead to the fastest possible wealth growth over time.

We ask what sort of dynamic v must follow so that $\overline{g_\mathrm{a}}(v) = \langle g_\mathrm{a}(v) \rangle$ or, put another way, so that $g_\mathrm{a}(v)$ is an ergodic observable, in the sense that its time and ensemble averages are the same [22, p. 32].

We start by expressing the change Dv as a sum over M equal time intervals,

$$\begin{aligned} Dv &\equiv v(t_0 + \Delta t) - v(t_0) && (74)\\ &= \sum_{m=1}^{M} [v(t_0 + m\delta t) - v(t_0 + (m-1)\delta t)] && (75)\\ &= \sum_{m=1}^{M} \delta v_m(t), && (76) \end{aligned}$$

where $\delta t \equiv \Delta t / M$ and $\delta v_m(t) \equiv v(t_0 + m\delta t) - v(t_0 + (m-1)\delta t)$. From (Eq. 72) we have

$$\begin{aligned} \overline{g_\mathrm{a}} = & \lim_{\Delta t \to \infty} \left\{ \frac{1}{\Delta t} \sum_{m=1}^{M} \delta v_m \right\} && (77) \quad \{\mathtt{eq:barrSum}\}\\ = & \lim_{M \to \infty} \left\{ \frac{1}{M} \sum_{m=1}^{M} \frac{\delta u_m}{\delta t} \right\}, && (78) \quad \{\mathtt{eq:barrSum2}\} \end{aligned}$$

keeping δt fixed. From (Eq. 73) we obtain

$$\langle g_\mathrm{a} \rangle = \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=1}^{N} \frac{Dv_n}{\Delta t} \right\} \qquad (79) \quad \{\mathtt{eq:averSum}\}$$

where each Dv_n is drawn independently from the distribution of Dv.

We now compare the two expressions (Eq. 78) and (Eq. 79). Clearly the value of $\overline{g_\mathrm{a}}$ in (Eq. 78) cannot depend on the way in which the diverging time period is partitioned, so the length of interval δt must be arbitrary and can be set to the value of Δt in (Eq. 79), for consistency we then call $\delta v_m(t) = Dv_m(t)$. Expressions (Eq. 78) and (Eq. 79) are equivalent if the successive additive increments, $Dv_m(t)$, are distributed identically to the Dv_n in (Eq. 79), which requires only that they are independent realizations of a time-independent random variable.

Thus we have a condition on $v(t)$ which suffices to make $\overline{g_\mathrm{a}} = \langle g_\mathrm{a} \rangle$, namely that it be a stochastic process whose additive increments are independent realizations of a time-independent random variable. This means that $v(t)$ is, in general, a Lévy process. If we restrict our attention to processes with continuous paths, then $v(t)$ must be a Brownian motion with drift, as we learned in Sec. **??**. We write this as

$$dv = a_v dt + b_v dW. \tag{80}$$ {eq:bm_u}

By arguing backwards we can address concerns regarding the existence of $\overline{g_\mathrm{a}}$. If v follows the dynamics specified by (Eq. 80), then it is straightforward to show that the limit $\overline{g_\mathrm{a}}$ always exists and takes the value a_v. Consequently the decision criterion (Eq. 70) is equivalent to the optimisation of $\overline{g_\mathrm{a}}$, the time-average growth rate. The process $x(t)$ may be chosen such that (Eq. 80) does not apply for any choice of $u(x)$. In this case we cannot interpret expected utility theory dynamically, and such processes are likely to be pathological.

This gives our central result.

Equivalency criterion

For expected utility theory to be equivalent to optimisation over time, utility must follow a stochastic process with ergodic additive increments.

This is a fascinating general connection. Provided that $v(x)$ is invertible, *i.e.* provided that its inverse, $x(v)$, exists, a simple application of Itô calculus to (Eq. 80) yields directly the stochastic differential equation obeyed by the wealth, x.

Translating into utility language, every invertible utility function is actually an encoding of a unique wealth dynamic which arises as utility performs a Brownian motion. Curiously, a celebrated but erroneous paper by Karl Menger [28] "proved" that all utility functions must be bounded (the proof is simply wrong). Boundedness makes utility functions non-invertible and precludes the developments we present here. Influential economists lauded Menger's paper, including Paul Samuelson [42, p. 49] who called it "a modern classic that [...] stands above all criticism." This is one reason why mainstream economics has failed to use the optimization of wealth growth over time to understand human behavior – a criterion we consider extremely simple and natural. A discussion of Menger's precise errors can be found in [35, p. 7]. Although mainstream economics still considers boundedness of utility to be formally required, it is such an awkward restriction that John Campbell noted recently [11] that "this requirement is routinely ignored."

1.7.3 Dynamic from a utility function

{section:dyn_from_u}

We now use the identification of $v = u$ to illustrate the relationship between utility functions and wealth dynamics. For the reasons discussed above we assume that utility follows a Brownian motion with drift.

If $u(x)$ can be inverted to $x(u) = u^{-1}(u)$, and $x(u)$ is twice differentiable, then it is possible to find the dynamic that corresponds to the utility function

$u(x)$. Equation (80) is an Itô process. Itô's lemma tells us that dx will be another Itô process, and Itô's formula specifies how to find dx in terms of the relevant partial derivatives

$$dx = \underbrace{\left(\frac{\partial x}{\partial t} + a_u \frac{\partial x}{\partial u} + \frac{1}{2} b_u^2 \frac{\partial^2 x}{\partial u^2}\right)}_{a_x(x)} dt + \underbrace{b_u \frac{\partial x}{\partial u}}_{b_x(x)} dW \tag{81}$$ {eq:dx}

We have thus shown that

Invertible utility functions have dynamic interpretations

For any invertible utility function $u(x)$ a class of corresponding wealth processes dx can be obtained such that the rate of change (*i.e.* the additive growth rate) in the expectation value of net changes in utility is the time-average growth rate of wealth.

The utility function is then, simply, the ergodicity mapping, and optimizing its expected changes is equivalent to optimizing time-average wealth growth for the corresponding wealth process.

The origin of optimizing expected utility can be understood as follows: in the 18th century, when utility theory was introduced, the difference between ergodic and non-ergodic processes was unknown, and all stochastic processes were treated by computing expectation values. Since the expectation value of the wealth process is an irrelevant mathematical object to an individual whose wealth is well modelled by a non-ergodic process the available methods failed. The formalism was saved by introducing a non-linear mapping of wealth, namely the utility function. The (failed) expectation value criterion was interpreted as theoretically optimal, and the non-linear utility functions were interpreted as a psychologically motivated pattern of human behavior. Conceptually, this is wrong.

Optimization of time-average growth recognizes the non-ergodicity of the situation and computes the appropriate object from the outset – a procedure whose building blocks were developed beginning in the late 19th century. It does not assume anything about human psychology and indeed predicts that the same behavior will be observed in any growth-optimizing entities that need not be human.

Equation (81), creates pairs of utility functions $u(x)$ and dynamics dx. In Sec. 1.4 we saw in a discrete setting that the ergodicity mapping for additive wealth dynamics is the identity function, and for multiplicative dynamics it is the logarithm. Interpreting utility functions as ergodicity mappings, we can summarize this in continuous time as follows.

- The linear utility function corresponds to additive wealth dynamics (Brownian motion),

$$u(x) = x \quad \leftrightarrow \quad dx = a_u dt + b_u dW, \tag{82}$$

as is easily verified by substituting $x(u) = u$ in (Eq. 81).

- The logarithmic utility function corresponds to multiplicative wealth dynamics (geometric Brownian motion),

$$u(x) = \ln(x) \quad \leftrightarrow \quad dx = x\left[\left(a_u + \frac{1}{2}b_u^2\right)dt + b_u dW\right]. \tag{83}$$

To demonstrate the generality of our procedure, we carry it out for another special case that is historically important.

Example: Square-root (Cramer) utility

The first utility function ever to be suggested was the square-root function $u(x) = x^{1/2}$, by Cramer in a 1728 letter to Daniel Bernoulli, partially reproduced in [7]. This function is invertible, namely $x(u) = u^2$, so that (Eq. 81) applies. We note that the square root, in a specific sense, sits between the linear function and the logarithm: $\lim_{x\to\infty} \frac{x^{1/2}}{x} = 0$ and $\lim_{x\to\infty} \frac{\ln(x)}{x^{1/2}} = 0$. Since linear utility produces additive dynamics and logarithmic utility produces multiplicative dynamics, we expect square-root utility to produce something in between or some mix. Substituting for $x(u)$ in (Eq. 81) and carrying out the differentiations in (Eq. 81) we find

$$u(x) = x^{1/2} \quad \leftrightarrow \quad dx = \left(2a_u x^{1/2} + b_u^2\right)dt + 2b_u x^{1/2} dW. \tag{84}$$ {eq:dx_2}

The drift term contains a multiplicative element (by which we mean an element with x-dependence) and an additive element. We see that the square-root utility function that lies between the logarithm and the linear function indeed represents a dynamic that is partly additive and partly multiplicative.

(Eq. 84) is reminiscent of the Cox-Ingersoll-Ross model [14] in financial mathematics, especially if $a_u < 0$. Similar dynamics, *i.e.* with a noise amplitude that is proportional to $\sqrt{x}$, are also studied in the context of absorbing-state phase transitions in statistical physics [26, 18]. That a 300-year-old letter is related to recent work in statistical mechanics is not surprising: the problems that motivated the development of decision theory, and indeed of probability theory itself are far-from equilibrium processes. Methods to study such processes were only developed in the 20th century and constitute much of the work currently carried out in statistical mechanics.

1.7.4 Utility function from a dynamic

{section:Utility_function

We now ask under what circumstances the procedure in (Eq. 81) can be inverted. When can a utility function be found for a given dynamic? In other words, what conditions does the dynamic dx have to satisfy so that optimization over time can be represented by optimization of expected net changes in utility $u(x)$, or: when does an ergodicity mapping, $v(x)$, exist?

We ask whether a given dynamic can be mapped into a utility whose increments are described by Brownian motion, (Eq. 80).

The dynamic is an arbitrary Itô process

$$dx = a_x(x)dt + b_x(x)dW, \tag{85}$$ {eq:dx_1}

where $a_x(x)$ and $b_x(x)$ are arbitrary functions of x. For this dynamic to translate into a Brownian motion for the utility, $u(x)$ must satisfy the equivalent of (Eq. 81) with the special requirement that the coefficients a_u and b_u in (Eq. 80) be constants, namely

$$du = \underbrace{\left(a_x(x)\frac{\partial u}{\partial x} + \frac{1}{2}b_x^2(x)\frac{\partial^2 u}{\partial x^2}\right)}_{a_u} dt + \underbrace{b_x(x)\frac{\partial u}{\partial x}}_{b_u} dW. \tag{86}$$ {eq:du_2}

To avoid clutter, let's use Lagrange notation, namely a dash – $'$ – to denote a derivative. Explicitly, we arrive at two equations for the coefficients

$$a_u = a_x(x)u' + \frac{1}{2}b_x^2(x)u'' \tag{87}$$ {eq:A}

and

$$b_u = b_x(x)u'. \tag{88}$$ {eq:b_u}

Differentiating (Eq. 88), it follows that

$$u''(x) = -\frac{b_u b_x'(x)}{b_x^2(x)}. \tag{89}$$

Substituting in (Eq. 87) for u' and u'' and solving for $a_x(x)$ we find the drift term as a function of the noise term,

$$a_x(x) = \frac{a_u}{b_u}b_x(x) + \frac{1}{2}b_x(x)b_x'(x). \tag{90}$$ {eq:consistency}

In other words, knowledge of only the dynamic is sufficient to determine whether a corresponding utility function exists. We do not need to construct the utility function explicitly to know whether a pair of drift term and noise term is consistent or not.

Having determined for some dynamic that a consistent utility function exists, we can construct it by substituting for $b_x(x)$ in (Eq. 87). This yields a differential equation for u

$$a_u = a_x(x)u' + \frac{b_u^2}{2u'^2}u'' \tag{91}$$

or

$$0 = -a_u u'^2 + a_x(x)u'^3 + \frac{b_u^2}{2}u''. \tag{92}$$

Overall, then the triplet noise term, drift term, utility function is interdependent. Given a noise term we can find consistent drift terms, and given a drift term we find a consistency condition (differential equation) for the utility function. These arguments may seem a little esoteric when first encountered, using bits and pieces from different fields of mathematics. But they constitute the actual physical story behind the fascinating history of decision theory. Again, we illustrate the procedure with an example.

Example: A curious-looking dynamic

Given a dynamic, it is possible to check whether it can be mapped into a utility function, and the utility function itself can be found. We consider

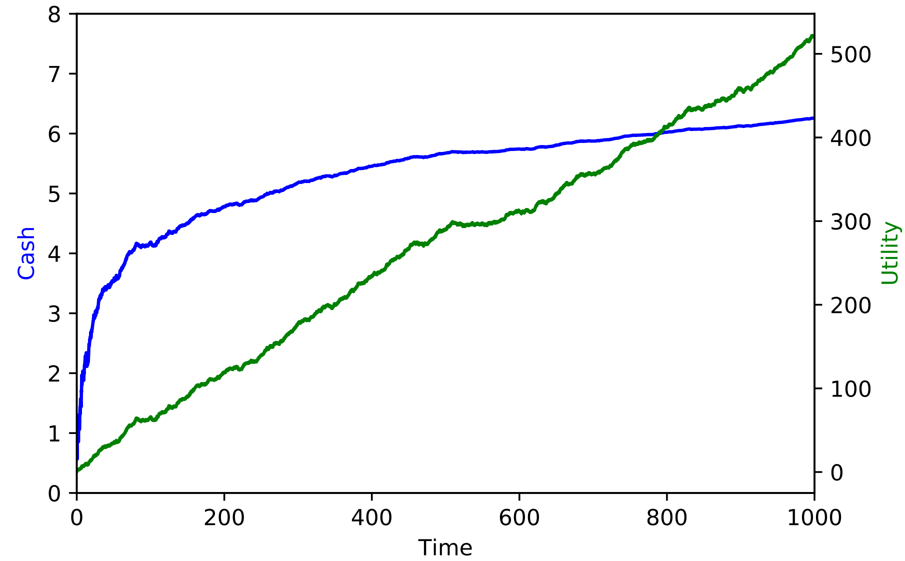

Figure 2: Typical trajectory $x(t)$ of the wealth dynamic (Eq. 93), with parameter values $a_u = 1/2$ and $b_u = 1$, and the corresponding Brownian motion $u(t)$. Note that the fluctuations in $x(t)$ become smaller for larger wealth. {fig:test_dyn}

the wealth dynamic

$$dx = \left(\frac{a_u}{b_u}e^{-x} - \frac{1}{2}e^{-2x}\right)dt + e^{-x}dW. \tag{93}$$ {eq:test_dyn}

We note that $a_x(x) = \frac{a_u}{b_u}e^{-x} - \frac{1}{2}e^{-2x}$ and $b_x(x) = e^{-x}$. Equation (90) imposes conditions on the drift term $a_x(x)$ in terms of the noise term $b_x(x)$. Substituting in (Eq. 90) reveals that the consistency condition is satisfied by the dynamic in (Eq. 93). A typical trajectory of (Eq. 93) is shown in Fig. 2.

Because (Eq. 93) is internally consistent, it is possible to derive the corresponding utility function. Equation (88) is a first-order ordinary differential equation for $u(x)$

$$u'(x) = \frac{b_u}{b_x(x)}, \tag{94}$$ {eq:diff_eq_u}

which can be integrated to

$$u(x) = \int_0^x d\tilde{x}\frac{b_u}{b_x(\tilde{x})} + C, \tag{95}$$

with C an arbitrary constant of integration. This constant, incidentally, implies that only *changes* in utility are meaningful, as was pointed out by von Neumann and Morgenstern [48] – this robust feature is visible whether

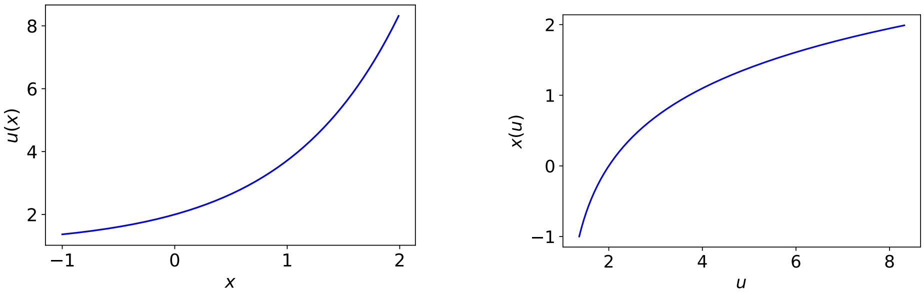

Figure 3: Exponential utility $u(x)$, (Eq. 97) with $b_u = 1$ and $C = 1$, is monotonic and unbounded and therefore invertible. Left panel: $u(x)$. Right panel: inverse $x(u)$. {fig:u_of_x}

one thinks in dynamic terms and time averages or in terms of consistent measure-theoretic concepts and expectation values.

Substituting for $b_x(x)$ from (Eq. 93), (Eq. 94) becomes

$$u'(x) = b_u e^x, \tag{96}$$

which is easily integrated to

$$u(x) = b_u e^x + C, \tag{97}$$ {eq:test_dyn_u}

plotted in Fig. 3. This exponential utility function is monotonic and therefore invertible – we knew that because the consistency condition is satisfied. The utility function is convex. From the perspective of expected-utility theory an individual behaving optimally according to this function would be labelled "risk-seeking." The dynamical perspective corresponds to a qualitatively different interpretation: Under the dynamic (Eq. 93) the "risk-seeking" individual behaves optimally, in the sense that his wealth will grow faster than that of a risk-averse individual. What's optimal is determined by the dynamic, not by the individual. Of course the individual may choose whether to behave optimally. The dynamic (Eq. 93) has the feature that fluctuations in wealth become smaller as wealth grows. High wealth is therefore sticky – an individual will quickly fluctuate out of low wealth and into higher wealth. It will then tend to stay there.

1.8 The St Petersburg paradox

The problem known today as the St Petersburg paradox was suggested by Nicolaus Bernoulli[10] in 1713 in his correspondence with Montmort [29]. It involves a hypothetical lottery for which the rate of change of expected wealth diverges for any finite ticket price. The expected-wealth paradigm would predict, therefore, that people are prepared to pay any price to enter the lottery. However, when the question is put to them, they rarely want to wager more than a few dollars. This is the paradox. It is the first well-documented example of the inadequacy of the expected-wealth paradigm as a model of human rationality. It was the primary motivating example for Daniel Bernoulli's and Cramer's development of the expected-utility paradigm [7].

[10] Daniel's cousin. The Bernoulli family produced a remarkable number of famous mathematicians in the 17$^{\text{th}}$ and 18$^{\text{th}}$ centuries, who helped lay the foundations of applied mathematics and physics.

In some sense it is a pity that this deliberately provocative and unrealistic lottery has played such an important role in the development of classical decision theory. It is quite unnecessary to invent a gamble with a diverging change in expected wealth to expose the flaws in the expected-wealth paradigm. The presence of infinities in the problem and its variously proposed solutions has caused much confusion, and permits objections on the grounds of physical impossibility. Such objections are unhelpful because they are not fundamental: they address only the gamble and not the decision paradigm. Nevertheless, the paradox is an indelible part not only of history but also of the current debate [33], and so we recount it here. We'll start by defining the lottery.

Example: St Petersburg lottery

The classical statement of the lottery is to imagine a starting prize of \$1 (originally the prize was in ducats). A fair coin is tossed: if it lands heads, the player wins the prize and the lottery ends; if it lands tails, the prize is doubled and the process is repeated. Therefore, the player wins \$2, \$4, \$8 if the first head lands on the second, third, fourth toss, and so on. The player must buy a ticket, at price F, to enter the lottery. The question usually posed is: what is the largest F the player is willing to pay?

The lottery can be translated neatly into our gamble formalism:

$$D_k = \$2^{k-1} - F, \quad p_k = 2^{-k}, \tag{98}$$ {eq:lottery_def}

for $k \in \{1, 2, 3, \ldots\}$, *i.e.* the set of positive integers. The vast majority of observed payouts are small, but occasionally an extremely large payout (corresponding to a very long unbroken sequence of tails in the classical description) occurs. This is shown in the example trajectories in Fig. 4, where the lottery has been repeated additively.

From now on we will forget about the coin tosses, which are simply a mechanism for selecting one of the possible payouts. In effect, they are just a random number generator. Instead we shall work with the compact definition of the lottery in (Eq. 98) and assume it takes a fixed amount of time, δt, to play.

The rate of change of expected wealth is

$$\frac{\langle \delta x \rangle}{\delta t} = \frac{1}{\delta t} \sum_{k=1}^{\infty} p_k D_k \tag{99}$$

$$= \frac{1}{\delta t} \left(\$ \sum_{k=1}^{\infty} 2^{-k}\, 2^{k-1} - \sum_{k=1}^{\infty} 2^{-k} F \right) \tag{100}$$

$$= \frac{1}{\delta t} \left(\$ \sum_{k=1}^{\infty} \frac{1}{2} - F \right). \tag{101}$$ {eq:lottery_ex_wealth}

This diverges for any finite ticket price. Under the expected-wealth paradigm, this means that the lottery is favourable at any price.

This implausible conclusion, which does not accord with human behaviour, exposes the weakness of judging a gamble by its effect on expected wealth. Daniel Bernoulli suggested to resolve the paradox by adopting the expected-

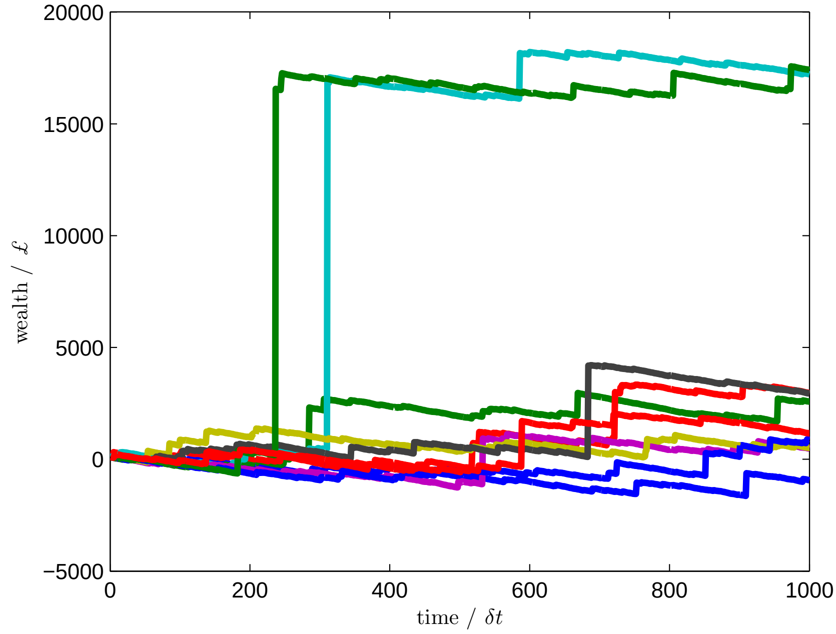

Figure 4: Wealth trajectories for the additively repeated St Petersburg lottery, with starting wealth, $x(0) = \$100$, and ticket price, $F = \$10$. Ten trajectories are plotted over 1,000 rounds. fig:lottery_add_traj

utility paradigm. His choice of utility function was the logarithm, $u(x) = \ln x$, which, as we now know, produces a decision rule equivalent to growth-rate optimisation under multiplicative repetition. This correspondence was not appreciated by Bernoulli: indeed 18$^{\text{th}}$-century mathematics did not possesse the concepts and language required to distinguish between averages over time and across systems, even though it had the basic arithmetic tools.

Unfortunately, Bernoulli made a mathematical error in the implementation of his own paradigm – accidentally he proposed two mutually inconsistent versions of utility theory in the paper that established the paradigm. Initially, the error had little impact, and it was corrected by Laplace in 1814 [23]. But Laplace didn't openly say he'd corrected an error, he just work with what he thought Bernoulli had meant. This politeness had awful consequences. In 1934 Menger [28], keen to get the story right, went back to the original text by Bernoulli. He didn't notice the error but rather got confused by it which led him to introduce a further error. Based on this car crash of scientific communication, Menger derived the infamous (wrong) claim we encountered in Sec. 1.7.2: utility functions must be bounded, with disastrous consequences for the budding neoclassical formalism. We will leave this most chequered part of the paradox's history alone – details can be found in [35]. Instead we will focus on what's usually presumed Bernoulli meant to write.

Example: Resolution by logarithmic utility

Instead of (Eq. 101), we calculate the rate of change of expected logarithmic

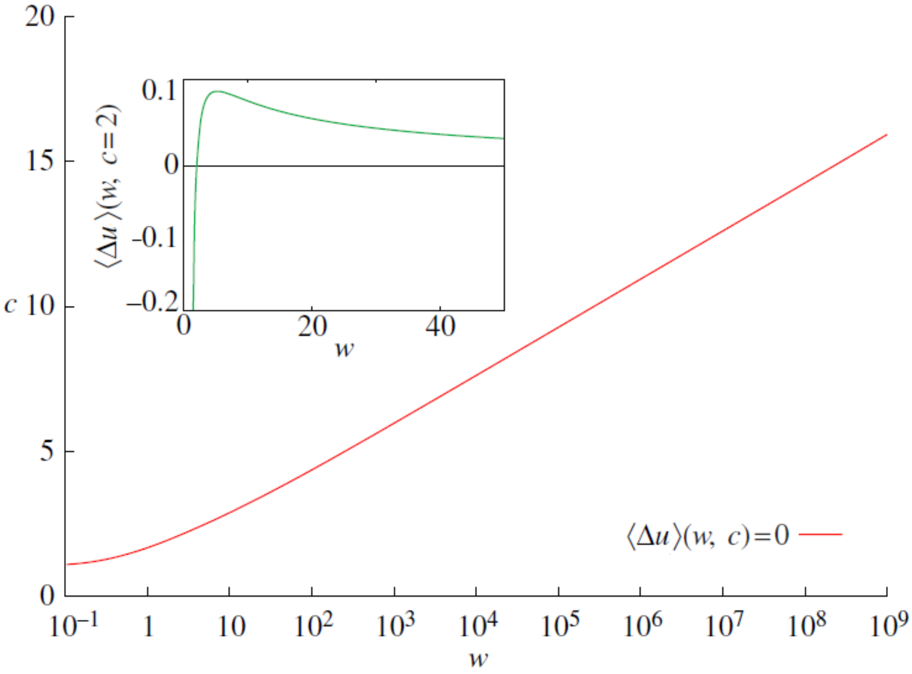

Figure 5: Locus of points in the (x, F)-plane for which the expected change in logarithmic utility is zero. The inset shows the expected change in utility as a function of x for $F = \$2$. Adapted from [33]. fig:gbar_zero

utility,

$$
\begin{aligned}
\frac{\langle \delta \ln x \rangle}{\delta t} &= \frac{1}{\delta t} \sum_{k=1}^{\infty} p_k \left[\ln(x + D_k) - \ln x \right] && (102) \\
&= \frac{1}{\delta t} \sum_{k=1}^{\infty} 2^{-k} \ln \left(\frac{x + \$2^{k-1} - F}{x} \right), && (103)
\end{aligned}
$$
{eq:lottery_ex_util}

where x is the ticket buyer's wealth.

This is finite for all finite ticket prices less than the buyer's wealth plus the smallest prize: $F < x + \$1$. This can be shown by applying the ratio test.[11] It may be positive or negative, depending on the values of F and x. Fig. 5 shows the locus of points in the (x, F)-plane for which the sum is zero.

The utility paradigm is a model that resolves the paradox, in the sense that creates a world where players may decline to buy a ticket. Bernoulli argued for this resolution framework in plausible terms: the usefulness of a monetary gain depends on how much money you already have. He also argued specifically for the logarithm in plausible terms: the gain in usefulness should be proportional to the fractional gain it represents, $du = \delta x / x$. Yet, the framework has left

[11] The ratio of the $(k+1)^{\text{th}}$ term to the k^{th} term in the sum tends to $1/2$ as $k \to \infty$.

many unsatisfied: why does usefulness have this functional form? We provide this deeper reason by connecting the problem to dynamics and time, unlike Bernoulli. Had Bernoulli made the connection, he might have been less willing to accept Cramer's square-root utility function as an alternative, which, as we've seen, corresponds to a rather less intuitive dynamic.

Turning to our decision algorithm, we will assume that the lottery is repeated multiplicatively. This means, in effect, that the prizes and ticket price are treated as fractions of the player's wealth, such that the effect of each lottery is to multiply current wealth by a random factor,

$$r_k = \frac{x + \$2^{k-1} - F}{x}, \quad p_k = 2^{-k}. \tag{104}$$

This follows precisely our earlier treatment of a gamble with multiplicative dynamics, and we can apply our results directly. The time-average (exponential) growth rate is

$$\overline{g}_{\mathrm{m}} = \frac{1}{\delta t} \lim_{T\to\infty} \left\{ \frac{1}{T} \sum_{\tau=1}^{T} \ln r(\tau) \right\} = \frac{1}{\delta t} \sum_{k=1}^{\infty} 2^{-k} \ln r_k, \tag{105}$$

{eq:lottery_gbar}

which is identical to the expression for the rate of change of expected log-utility, (Eq. 103). This is, as we've discussed, because $v(x) = \ln(x)$ is the appropriate ergodicity mapping for multiplicative dynamics. The result is the same, but the interpretation is different: we have assumed less, only that our player is interested in the growth rate of his wealth and that he gauges this by imagining the outcome of an indefinite sequence of repeated lotteries.

Thus the locus in Fig. 5 also marks the decision threshold *versus* the null gamble under our decision axiom. The player can sensibly decline the gamble, even though it results in a divergent change in expected wealth. This is illustrated by comparing Fig. 6, which shows trajectories of multiplicatively repeated lotteries, with the additively repeated lotteries already seen in Fig. 4. The trajectories are based on the same sequences of lottery outcomes, only the mode of repetition is different. The simulation shows us visually what we have already gleaned by analysis: what appears favourable in the expected-wealth paradigm (corresponding to additive repetition) results in a disastrous decay of the player's wealth over time under a realistic dynamic.

As $F \to x + \$1$ from above in (Eq. 105), $\overline{g}_{\mathrm{m}}$ diverges negatively, since the first term in the sum is the logarithm of a quantity approaching zero. This corresponds to a lottery which can make the player bankrupt. The effect is also shown in the inset of Fig. 5.

Treatments based on multiplicative repetition have appeared sporadically in the literature, starting with Whitworth in 1870 [49, App. IV].[12] It is related to

[12] Whitworth was dismissive of early utility theory: "The result at which we have arrived is not to be classed with the arbitrary methods which have been again and again propounded to evade the difficulty of the Petersburg problem. . . . Formulae have often been proposed, which have possessed the one virtue of presenting a finite result. . . but they have often had no intelligible basis to rest upon, or. . . sufficient care has not been taken to draw a distinguishing line between the significance of the result obtained, and the different result arrived at when the mathematical expectation is calculated." Sadly he chose to place these revolutionary remarks in an appendix of a college probability textbook.

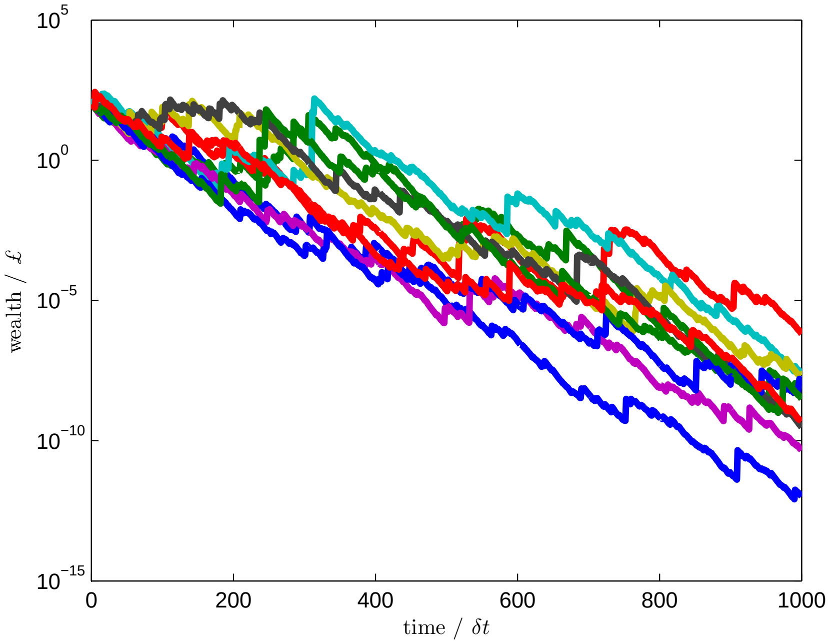

Figure 6: Wealth trajectories for the multiplicatively repeated St Petersburg lottery, with starting wealth, $x(0)$ = \$100, and ticket price, F = \$10. Ten trajectories are plotted over 1,000 rounds. The realisations of the individual lotteries are the same as in Fig. 4 but the mode of repetition is different. fig:lottery_mult_traj

the famous Kelly Criterion [21][13], although Kelly did not explicitly treat the St Petersburg game, and tangentially to Itô's lemma [19]. It appears as an exercise in a well-known text on information theory [13, Ex. 6.17]. Mainstream economics has ignored all this. A full and rigorous resolution of the paradox, including the epistemological significance of the shift from ensemble to time averages, was published recently by one of the present authors [33].

1.9 The insurance puzzle

The insurance contract is an important and ubiquitous type of economic transaction, which can be modelled as a gamble. However, it poses a puzzle [34]. In the expected-wealth paradigm, insurance contracts shouldn't exist, because buying insurance would only be rational at a price at which it would be irrational to sell. More specifically:

1. To be viable, an insurer must charge an insurance premium of at least the expectation value of any claims that may be made against it, called the "net premium" [20, p. 1].

2. The insurance buyer therefore has to be willing to pay more than the net premium so that an insurance contract may be successfully signed.

3. Under the expected-wealth paradigm it is irrational to pay more than the net premium, and therefore insurance contracts should not exist.

In this picture, an insurance contract can only ever be beneficial to one party. It has the anti-symmetric property that the expectation value of one party's gain is the expectation value of the other party's loss.

The puzzle is that insurance contracts are observed to exist.[14] Why? Classical resolutions appeal to utility theory (*i.e.* psychology) and asymmetric information (*i.e.* deception). However, our decision theory naturally predicts contracts with a range of prices that increase the time-average growth rate for both buyer and seller. We illustrate this with an example drawn from maritime trade, in which the use of insurance has a very long history.[15] A similar example was used by Bernoulli [7].

Example: A shipping contract

We imagine a shipowner sending a cargo from St Petersburg to Amsterdam, with the following parameters:

- owner's wealth, $x_{\text{own}} = \$100,000$;

[13] Kelly was similarly unimpressed with the mainstream and noted in his treatment of decision theory, which he developed from the perspective of information theory and which is identical to ergodicity economics with multiplicative dynamics, that the utility function is "too general to shed any light on the specific problems of communication theory."

[14] Something of an understatement. The Bank for International Settlements estimated the market value of all the world's derivatives contracts, which are essentially insurance contracts, as $15 trillion in the first half of 2015 (see `http://www.bis.org/statistics/d5_1.pdf`). That's six times the gross domestic product of the United Kingdom.

[15] Contracts between Babylonian traders and lenders were recorded around 1750 BC in the Code of Hammurabi. Chinese traders practised diversification by spreading cargoes across multiple vessels even earlier than this, in the third millennium BC.

- gain on safe arrival of cargo, $G = \$4,000$;
- probability ship will be lost, $p = 0.05$;
- replacement cost of the ship, $C = \$30,000$; and
- voyage time, $\delta t = 1$ month.

An insurer with wealth $x_{\text{ins}} = \$1,000,000$ proposes to insure the voyage for a fee, $F = \$1,800$. If the ship is lost, the insurer pays the owner $L = G + C$ to make him good on the loss of his ship and the profit he would have made.

We phrase the decision the owner is facing as a choice between two gambles.

Definition The owner's gambles

Sending the ship uninsured corresponds to gamble o1

$$D_1^{(\text{o1})} = G, \qquad p_1^{(\text{o1})} = 1 - p; \tag{106}$$

$$D_2^{(\text{o1})} = -C, \qquad p_2^{(\text{o1})} = p. \tag{107}$$

Sending the ship fully insured corresponds to gamble o2

$$D_1^{(\text{o2})} = G - F \qquad p_1^{(\text{o2})} = 1. \tag{108}$$

This is a trivial "gamble" because all risk has been transferred to the insurer.

We also model the insurer's decision whether to offer the contract as a choice between two gambles

Definition The insurer's gambles

Not insuring the ship corresponds to gamble i1, which is the null gamble

$$D_1^{(\text{i1})} = 0 \qquad p_1^{(\text{i1})} = 1. \tag{109}$$

Insuring the ship corresponds to gamble i2

$$D_1^{(\text{i2})} = +F, \qquad p_1^{(\text{i2})} = 1 - p; \tag{110}$$

$$D_2^{(\text{i2})} = -L + F, \qquad p_2^{(\text{i2})} = p. \tag{111}$$

We ask whether the owner should sign the contract, and whether the insurer should have proposed it.

Example: Expected-wealth paradigm

In the expected-wealth paradigm (corresponding to additive repetition under the time paradigm) decision makers maximise the rate of change of the expectation values of their wealths, according to (Eq. 59): Under this paradigm the owner collapses gamble o1 into the scalar

$$\overline{g}_a^{(o1)} = \frac{\langle \delta x \rangle}{\delta t} \tag{112}$$
$$= \frac{\left\langle D^{(o1)} \right\rangle}{\delta t} \tag{113}$$
$$= \frac{(1-p)G + p(-C)}{\delta t} \tag{114}$$
$$= \$2,300 \text{ per month}, \tag{115}$$

and gamble o2 into the scalar

$$\overline{g}_a^{o2} = \frac{\left\langle D^{(o2)} \right\rangle}{\delta t} \tag{116}$$
$$= \frac{(G-F)}{\delta t} \tag{117}$$
$$= \$2,200 \text{ per month}. \tag{118}$$

The difference, $\delta \overline{g}_a^{o}$, between the expected rates of change in wealth with and without a signed contract is the expected loss minus the fee per round trip,

$$\delta \overline{g}_a^{o} = \overline{g}_a^{o2} - \overline{g}_a^{o1} = \frac{pL - F}{\delta t}. \tag{119}$$ {eq:dro}

The sign of this difference indicates whether the insurance contract is beneficial to the owner. In the example this is not the case, $\delta \overline{g}_a^{o} = -\$100$ per month.

The insurer evaluates the gambles i1 and i2 similarly, with the result

$$\overline{g}_a^{(i1)} = \$0 \text{ per month}, \tag{120}$$

and

$$\overline{g}_a^{(i2)} = \frac{F - pL}{\delta t} \tag{121}$$ {eq:r}
$$= \$100 \text{ per month}. \tag{122}$$

Again we compute the difference – the net benefit to the insurer that arises from signing the contract

$$\delta \overline{g}_a^{i} = \overline{g}_a^{i2} - \overline{g}_a^{i1} = \frac{F - pL}{\delta t}. \tag{123}$$ {eq:dri}

In the example this is $\delta \overline{g}_a^{i} = \$100$ per month, meaning that in the world of the expected-wealth paradigm the insurer will offer the contract.

Because only one party (the insurer) is willing to sign, no contract will come into existence. We could think that we got the price wrong, and the contract would be sigend if offered at a different fee. But this is not the case, and that's the fundamental insurance puzzle: in the world created by expected-wealth maximisation no price exists at which both parties will sign the contract.

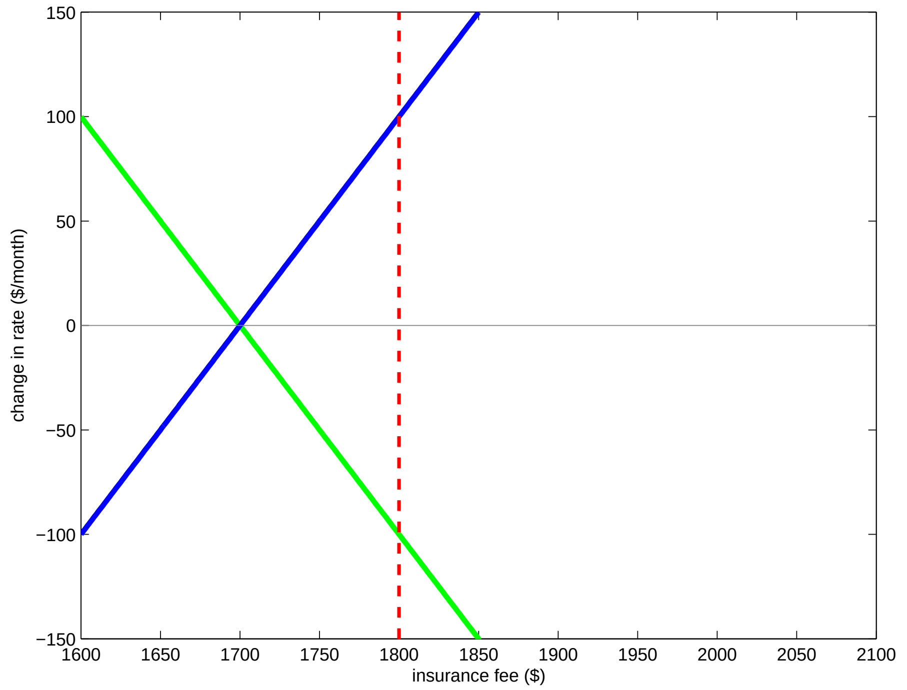

Figure 7: Change in the rate of change of expected wealth for the shipowner (green) and the insurer (blue) as a function of the insurance fee, F. fig:ins_lin

Looking at (Eq. 119) and (Eq. 123) we notice the anti-symmetric relationship between the two expressions, $\delta\overline{g}_a^{\mathrm{o}} = -\delta\overline{g}_a^{\mathrm{i}}$. By symmetry, there can be no fee at which both expressions are positive. Hence there are no circumstances in the world created by the expected-wealth paradigm under which both parties will sign. Insurance contracts cannot exist in this world.

One party winning at the expense of the other makes insurance an unsavoury business in the expected-wealth paradigm. This is further illustrated in Fig. 7, which shows the change in the rate of change of expected wealth (the decision variable) for both parties as a function of the fee, F. There is no price at which the decision variable is positive for the both parties. The best they can do is to pick the price at which neither of them cares whether they sign or not.

In this picture, the existence of insurance contracts requires some asymmetry between the contracting parties, such as:

- different attitudes to bearing risk;
- different access to information about the voyage;
- different assessments of the riskiness of the voyage;
- one party to deceive, coerce, or gull the other into a bad decision.

It is difficult to believe that this is truly the basis for a market of the size and global reach of the insurance market.

1.9.1 Solution in the time paradigm

Example: Time paradigm

The insurance puzzle is resolved in the 'time paradigm', *i.e.* using the growth-optimal decision theory we have developed in this lecture and multiplicative repetition. Again, we pause to reflect what multiplicative repetition means compared to additive repetition. This is important because additive repetition is equivalent to the expected-wealth paradigm that created the insurance puzzle. Multiplicative repetition means that the ship owner sends out a ship and a cargo whose values are proportional to his wealth at the start of each voyage. A rich owner who has had many successful voyages will send out more cargo, a larger ship, or perhaps a *flotilla*, while an owner to whom the sea has been a cruel mistress will send out a small vessel until his luck changes. Under additive repetition, the ship owner would send out the same amount of cargo on each journey, irrespective of his wealth. Shipping companies of the size of Evergreen or Maersk would be inconceivable under additive repetition, where returns on successful investments are not reinvested.

The two parties seek to maximise

$$\overline{g}_m = \lim_{\Delta t \to \infty} \frac{\Delta v(x)}{\Delta t} = \frac{\langle \delta \ln x \rangle}{\delta t}, \tag{124}$$

where we have used the ergodic property of $\Delta v(x) = \Delta \ln x$ under multiplicative repetition.

The owner's time-average growth rate without insurance is

$$\overline{g}_m^{\text{o1}} = \frac{(1-p)\ln(x_{\text{own}} + G) + p\ln(x_{\text{own}} - C) - \ln(x_{\text{own}})}{\delta t} \tag{125}$$

or 1.9% per month. His time-average growth rate with insurance is

$$\overline{g}_m^{\text{o2}} = \frac{\ln(x_{\text{own}} + G - F) - \ln(x_{\text{own}})}{\delta t} \tag{126}$$

or 2.2% per month. This gives a net benefit for the owner of

$$\delta \overline{g}_m^{\text{o}} = \overline{g}_m^{\text{o1}} - \overline{g}_m^{\text{o2}} \approx +0.24\% \text{ per month.} \tag{127}$$

The time paradigm thus creates a world where the owner will sign the contract.

What about the insurer? Without insurance, the insurer plays the null gamble, and

$$\overline{g}_m^{\text{i1}} = \frac{0}{\delta t} \tag{128}$$

or 0% per month. His time-average growth rate with insurance is

$$\overline{g}_m^{\text{i2}} = \frac{(1-p)\ln(x_{\text{ins}} + F) + p\ln(x_{\text{ins}} + F - L) - \ln(x_{\text{ins}})}{\delta t} \tag{129}$$

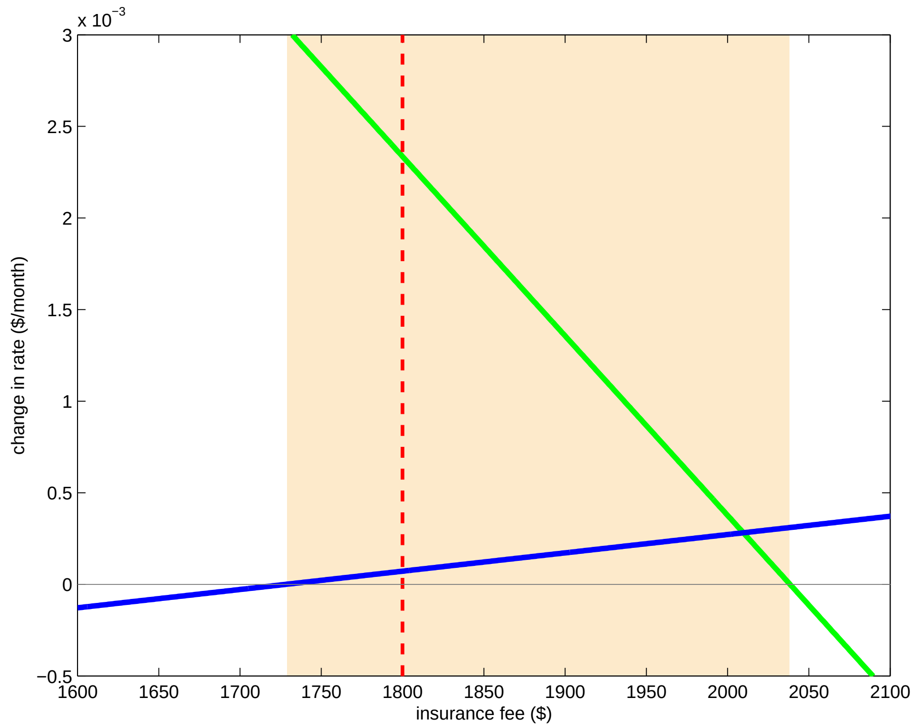

Figure 8: Change in the time-average growth rate of wealth for the shipowner (green) and the insurer (blue) as a function of the insurance fee, F. The mutually beneficial fee range is marked by the beige background. fig:ins_log

or 0.0071% per month. The net benefit to the insurer is therefore also

$$\delta \bar{g}_m^{\mathrm{i}} = \bar{g}_m^{\mathrm{i}2} - \bar{g}_m^{\mathrm{i}1} \tag{130}$$

i.e. 0.0071% per month. Unlike the expected wealth paradigm, the time paradigm with multiplicative repetition creates a world where an insurance contract can exist – there exists a range of fees F at which both parties gain from signing the contract!

We view this as the

Fundamental resolution of the insurance puzzle:

The buyer and seller of an insurance contract both sign when it increases the time-average growth rates of their wealths.

It requires no appeal to arbitrary utility functions or asymmetric circumstances, rather it arises naturally from the model of human decision-making that we have set out. Fig. 8 shows the mutually beneficial range of insurance fees predicted by our model. Generalizing, the message of the time paradigm is that business happens when both parties gain. In the world created by this model any agreement, any contract, any commercial interaction comes into existence because it is mutually beneficial.

1.9.2 The classical solution of the insurance puzzle

{section:The classical so

The classical solution of the insurance puzzle is identical to the classical solution of the St Petersburg paradox. Wealth is replaced by a non-linear utility function of wealth, which breaks the symmetry of the expected-wealth paradigm. While it is always true that $\delta \langle r \rangle_{\text{own}} = -\delta \langle r \rangle_{\text{ins}}$, the expected growth rates of non-linear utility functions don't share this anti-symmetry. A difference in the decision makers' wealths is sufficient, though often different utility functions are assumed for owner and insurer, which is a model that can create pretty much any behavior. The downside of a model with this ability is, of course, that it makes no predictions – nothing is ruled out, so the model cannot be falsified.

2 Populations

The previous chapter developed a model of individual behaviour based on an assumed dynamic imposed on wealth. If we know the stochastic process that describes individual wealth, then we also know what happens at population level – each individual is represented by a realisation of the process, and we can compute the dynamics of wealth distributions. We answer questions about inequality and poverty in our model economy. It turns out that our decision criterion generates interesting emergent behaviour – cooperation, the sharing and pooling of resources, is often time-average growth optimal. This provides answers to the puzzles of why people cooperate, why there is an insurance market, and why we see socio-economic structure from the formation of firms to nation states with taxation and redistribution systems.

2.1 Every man for himself

{section:Every_man}

We have seen that risk aversion constitutes optimal behaviour under the assumption of multiplicative wealth growth and over time scales that are long enough for systematic trends to be significant. In this chapter we will continue to explore our null model, GBM. By "explore" we mean that we will let the model generate its world – if individual wealth was to follow GBM, what kind of features of an economy would emerge? We will see that cooperation and the formation of social structure also constitute optimal behaviour.

GBM is more than a random variable. It's a stochastic process, either a set of trajectories or a family of time-dependent random variables, depending on how we prefer to look at it. Both perspectives are informative in the context of economic modelling: from the set of trajectories we can judge what is likely to happen to an individual, *e.g.* by following a single trajectory for a long time; while the PDF of the random variable $x(t^*)$ at some fixed value of t^* tells us how wealth is distributed in our model.

We use the term wealth distribution to refer to the density function $\mathcal{P}_x(x)$ and not the process of distributing wealth among people. This can be interpreted as follows. Imagine a population of N individuals. If I select a random individual, each having uniform probability $\frac{1}{N}$, then the probability of the selected individual having wealth greater than x is given by the CDF $F_x(x) = \int_x^\infty \mathcal{P}_x(s)\,ds$. If N is large, then $\Delta x \mathcal{P}_x(x) N$ is the approximate number of individuals who have wealth between x and $x + \Delta x$. Thus, a broad wealth distribution with heavy tails indicates greater wealth inequality.

Examples:

- Under perfect equality everyone would have the same, meaning that the wealth distribution would be a Dirac delta function centred at the sample mean of x, that is
$$\mathcal{P}_x(x) = \delta(x - \langle x \rangle_N); \tag{131}$$
- Maximum inequality would mean that one individual owns everything and everyone else owns nothing, that is
$$\mathcal{P}_x(x) = \frac{N-1}{N}\delta(x-0) + \frac{1}{N}\delta(x - N\langle x \rangle_N). \tag{132}$$

2.1.1 Log-normal distribution

{section:Log-normal_wealt

At a given time, t, GBM produces a random variable, $x(t)$, with a log-normal distribution whose parameters depend on t. (A log-normally distributed random variable is one whose logarithm is a normally distributed random variable.) If each individual's wealth follows GBM,

$$dx = x(\mu dt + \sigma dW), \tag{133}$$

{eq:GBM}

with solution

$$x(t) = x(0)\exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right], \tag{134}$$

{eq:GBM_sol}

then we will observe a log-normal distribution of wealth at each moment in time:

$$\ln x(t) \sim \mathcal{N}\left(\ln x(0) + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right). \tag{135}$$

{eq:log-normal}

It will be convenient hereafter to assume the initial condition $x(0) = 1$ (and, therefore, $\ln x(0) = 0$) unless otherwise stated.

Note that the variance of $\ln x(t)$ increases linearly in time. We will develop an understanding of this shortly. As we will see, it indicates that any meaningful measure of inequality will grow over time in our simple model. To see what kind of a wealth distribution (Eq. 135) is, it is worth spelling out the log-normal PDF:

$$\mathcal{P}_x(x) = \frac{1}{x\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[\ln x - (\mu - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t}\right). \tag{136}$$ {eq:PDFx}

This distribution is the subject of a wonderful book [2], sadly out-of-print now. We will find it useful to know a few of its basic properties. Of particular importance is the expected wealth under this distribution. This is

$$\langle x(t) \rangle = \exp(\mu t) \tag{137}$$ {eq:exp_x}

or, equivalently, $\ln \langle x(t) \rangle = \mu t$. We could confirm this result by calculating $\langle x(t) \rangle = \int_0^\infty s\mathcal{P}_x(s)\, ds$, but this would be laborious. Instead we use a neat trick, courtesy of [22, Chapter 4.2], which will come in handy again in Sec. 3.2.6. To compute moments, $\langle x^n \rangle$, of stochastic differential equations for x, like (Eq. 2.1.2), we find ordinary differential equations for the moments, which we know how to solve. For the first moment we do this simply by taking expectations of both sides of (Eq. 2.1.2). The noise term vanishes to turn the SDE for x into an ODE for $\langle x \rangle$:

$$\langle dx \rangle = \langle x(\mu dt + \sigma dW) \rangle \tag{138}$$

$$d\langle x \rangle = \langle x \rangle \mu dt + \sigma \overbrace{\langle dW \rangle}^{=0} \tag{139}$$

$$= \langle x \rangle \mu dt. \tag{140}$$

This is a very simple first-order linear differential equation for the expectation value of x. Its solution with initial condition $x(0) = 1$ is (Eq. 137).

For $\mu > 0$ the expected wealth grows exponentially over time, as do its population median and variance:

$$\text{median}[x(t)] = \exp[(\mu - \sigma^2/2)t]; \tag{141}$$ {eq:median_x}

$$\text{var}[x(t)] = \exp(2\mu t)[\exp(\sigma^2 t) - 1]. \tag{142}$$ {eq:var_x}

2.1.2 Two growth rates

{section:two_rates}

We will recap briefly here one of our key ideas, covered in detail in Sec. **??**, that the ensemble average of all possible trajectories of GBM grows at a different (faster) rate from that achieved by a single trajectory almost surely in the long-time limit. Understanding this difference was the key to developing a coherent theory of individual decision-making. We will see here that it is also crucial in understanding how wealth becomes distributed in a population of individuals whose wealths follow and, in particular, how we can measure the inequality in such a distribution. {eq:GBM}

From (Eq. **??**), we recall that the growth rate of the expected wealth is

$$g_{\langle\rangle} = \frac{d \ln \langle x \rangle}{dt} = \mu, \tag{143}$$

while, from (Eq. **??**), the time-average growth rate of wealth is

$$\overline{g} = \frac{d\langle \ln x \rangle}{dt} = \mu - \frac{\sigma^2}{2}. \tag{144}$$

2.1.3 Measuring inequality

{section:Inequality_measu

In the case of GBM we have just seen how to compute the exact full wealth distribution $\mathcal{P}$. This is interesting but often we want only summary measures of the distribution. One such summary measure of particular interest to economists is inequality. How much inequality is there in a distribution like (Eq. 135)? And how does this quantity increase over time under GBM, as we have suggested?

Clearly, to answer these questions, we must quantify "inequality". In this section, and also in [1], we develop a natural way of measuring it, which makes use of the two growth rates we identified for the non-ergodic process. We will see that a particular inequality measure, known to economists as Theil's second index of inequality [45], is the difference between typical wealth (growing at the time-average growth rate) and average wealth (growing at the ensemble-average growth rate) in our model. Thus, the difference between the time average and ensemble average, the essence of ergodicity breaking, is the fundamental driver of the dynamics of inequality.

The two limits of inequality are easily identified: minimum inequality means that everyone has the same wealth, and maximum inequality means that one individual has all the wealth and everyone else has nothing. (This assumes that wealth cannot become negative.) Quantifying inequality in any other distribution is reminiscent of the gamble problem. Recall that for gambles we wanted make statements of the type "this gamble is more desirable than that gamble". We did this by collapsing a distribution to a scalar. Depending on the question that was being asked the appropriate way of collapsing the distribution and the resulting scalar can be different (the scalar relevant to an insurance company may not be relevant to an individual). In the case of inequality we also have a distribution – the wealth distribution – and we want to make statements of the type "this distribution is more unequal than that distribution". Again, this is done by collapsing the distribution to a scalar, and again many different choices of collapse and resulting scalar are possible. The Gini coefficient is a particularly well-known scalar of this type, the 80/20 ratio is another, and many other measures exist.

In this context the expectation value is an important quantity. For instance, if everyone has the same wealth, everyone will own the average $\forall i, x_i = \langle x \rangle_N$, which converges to the expectation value for large N. Also, whatever the distribution of wealth, the total wealth is $N \langle x \rangle_N$ which converges to $N \langle x \rangle$ as N grows large. The growth rate of the expectation value, $g_{\langle\rangle}$, thus tells us how fast the average wealth and the total population wealth grow with probability one in a large ensemble. The time-average growth rate, $\overline{g}$, on the other hand, tells us how fast an individual's wealth grows with probability one in the long run. If the typical individual's wealth grows at a lower rate than the expectation value of wealth then there must be atypical individuals with very large wealths that account for the difference. This insight suggests the following measure of inequality.

<u>**Definition**</u> **Inequality, J, is the quantity whose growth rate is the**

difference between expectation-value and time-average growth rates,

$$\frac{dJ}{dt} = g_{\langle\rangle} - \overline{g}. \tag{145}$$ {eq:dJ}

Equation (145) defines the dynamic of inequality, and inequality itself is found by integrating over time:

$$J(t) = \int_0^t ds[g_{\langle\rangle}(s) - \overline{g}(s)]. \tag{146}$$ {eq:J}

This definition may be used for dynamics other than GBM. Whatever the wealth dynamic, typical minus average growth rates are informative of the dynamic of inequality. Within the GBM framework we can write the difference in growth rates as

$$\frac{dJ}{dt} = \frac{d \ln \langle x \rangle}{dt} - \frac{d \langle \ln x \rangle}{dt} \tag{147}$$ {eq:J_dyn}

and integrate over time to get

$$J(t) = \ln \langle x(t) \rangle - \langle \ln x(t) \rangle . \tag{148}$$ {eq:J_x}

This quantity is known as the mean logarithmic deviation (MLD) or Theil's second index of inequality [45]. This is rather remarkable. Our general inequality measure, (Eq. 146), evaluated for the specific case of GBM, turns out to be a well-known measure of inequality that economists have identified independently, without considering non-ergodicity and ensemble average and time average growth rates. Merely by insisting on measuring inequality well, Theil used the GBM model without realising it!

Substituting the known values of the two growth rates into (Eq. 145) and integrating, we can evaluate the Theil inequality as a function of time:

$$J(t) = \frac{\sigma^2}{2} t. \tag{149}$$ {eq:J_t}

Thus we see that, in GBM, our measure of inequality increases indefinitely.

2.1.4 Wealth condensation

{section:condensation}

The log-normal distribution generated by GBM broadens indefinitely, (Eq. 142). Likewise, the inequality present in the distribution – measured as the time-integrated difference between ensemble and time average growth rates – grows continually. A related property of GBM is the evolution towards wealth condensation. Wealth condensation means that a single individual will own a non-zero fraction of the total wealth in the population in the limit of large N, see *e.g.* [10]. In the present case an arbitrarily large share of total wealth will be owned by an arbitrarily small share of the population.

One simple way of seeing this is to calculate the fraction of the population whose wealths are less than the mean, *i.e.* $x(t) < \exp(\mu t)$. To do this, we define a new random variable, $z(t)$, whose distribution is the standard normal:

$$z(t) \equiv \frac{\ln x(t) - (\mu - \sigma^2/2)t}{\sigma t^{1/2}} \sim \mathcal{N}(0,1). \tag{150}$$

We want to know the mass of the distribution with $\ln x(t) < \mu t$ or, equivalently, $z < \sigma t^{1/2}/2$. This is

$$\Phi\left(\frac{\sigma t^{1/2}}{2}\right), \tag{151}$$

where Φ is the CDF of the standard normal distribution. This fraction tends to one as $t \to \infty$.

2.1.5 Rescaled wealth

{section:rescaled}

Economists have arrived at many inequality measures, and have drawn up a list of conditions that particularly useful measures of inequality satisfy. Such measures are called "relative measures" [43, Appendix 4], and J is one of them.

One of the conditions is that inequality measures should not change when x is divided by the same factor for everyone. Since we are primarily interested in inequality in this section, we can remove absolute wealth levels from the analysis and study an object called the rescaled wealth.

Definition **The rescaled wealth,**

$$y_i(t) = \frac{x(t)}{\langle x(t)\rangle_N}, \tag{152}$$ {eq:rescaled}

is the proportion of the sample mean wealth – ***i.e.*** **the wealth averaged over the finite population – owned by an individual.**

This quantity is useful because its numerical value does not depend on the currency used: it is a dimensionless number. Thus if my rescaled wealth is $y_i(t) = 1/2$, this means that my wealth is half the average wealth, irrespective of whether I measure it in Kazakhstani Tenge or in Swiss Francs. The sample mean rescaled wealth is easily calculated:

$$\langle y_i(t)\rangle_N = \left\langle \frac{x(t)}{\langle x(t)\rangle_N} \right\rangle_N = 1. \tag{153}$$

If the population size, N, is large enough, then we might expect the sample mean wealth, $\langle x(t)\rangle_N$, to be close to the ensemble average, $\langle x(t)\rangle$, which is simply its $N \to \infty$ limit. We will discuss more carefully when this approximation holds for wealths following GBM in Sec. 2.2. Let's assume for now that it does hold. The rescaled wealth is then well approximated as

$$y_i(t) = \frac{x_i(t)}{\langle x(t)\rangle} = x_i(t)\exp(-\mu t). \tag{154}$$

Now that we have an expression for y in terms of x and t, we can derive the dynamic for rescaled wealth using Itô's formula (just as we did to find the wealth dynamic for a general utility function in Sec. 1.7.3). We start with

$$\begin{aligned} dy &= \frac{\partial y}{\partial t}\,dt + \frac{\partial y}{\partial x}\,dx + \frac{1}{2}\frac{\partial^2 y}{\partial x^2}\,dx^2 && (155)\\ &= -\mu y\,dt + \frac{y}{x}\,dx, && (156) \end{aligned}$$ {eq:ysde}

and then substitute (Eq. 2.1.2) for dx to get

$$dy = y\sigma dW. \tag{157}$$ {eq:GBM_y}

Thus $y(t)$ follows a very simple GBM with zero drift and volatility σ. This means that rescaled wealth, like wealth, has an ever-broadening log-normal distribution:

$$\ln y(t) \sim \mathcal{N}\left(-\frac{\sigma^2}{2}t, \sigma^2 t\right). \tag{158}$$ {eq:log-normal_y}

Finally, noting that $\langle \ln y \rangle = \langle \ln x \rangle - \ln \langle x \rangle$ gives us a simple expression for our inequality measure in (Eq. 148) in terms of the rescaled wealth:

$$J(t) = -\langle \ln y \rangle. \tag{159}$$

2.1.6 u-normal distributions and Jensen's inequality

{section:jensen}

So far we have confined our analysis to GBM, where wealths follow the dynamic specific by (Eq. 2.1.2). However, as we discussed in the context of gambles, other wealth dynamics are possible. In particular, we explored the dynamics corresponding to invertible utility functions, where utility executes a Brownian motion with drift as in (Eq. 80):

$$du = a_u dt + b_u dW. \tag{160}$$

Under this dynamic, utility is normally distributed,

$$u(x(t)) \sim \mathcal{N}\left(a_u t, {b_u}^2 t\right), \tag{161}$$

and we can say that wealth has a "u-normal" distribution. For GBM, the corresponding utility function is, as we know, $u(x) = \ln x$, and u-normal becomes log-normal.

Replacing the logarithm in (Eq. 148) by the general utility function gives a general expression for the wealth inequality measure,

$$J_u(t) = u(\langle x(t) \rangle) - \langle u(x(t)) \rangle. \tag{162}$$ {eq:J_u}

It's not easy to write a general expression for $J_u(t)$ in only t and the model parameters a_u and b_u, because it would involve the solution of the general wealth dynamic in (Eq. 81). However, we can still say something about how inequality evolves. Let's see what happens if we start with perfect equality, $x_i(0) = x_0$ with x_0 fixed, and let wealths evolve a little to $x(\Delta t) = x_0 + \Delta x$, where Δx is a random wealth increment generated by the wealth dynamic. The change in inequality would be

$$\Delta J_u = u(\langle x_0 + \Delta x \rangle) - \langle u(x_0 + \Delta x) \rangle, \tag{163}$$

since $u(\langle x_0 \rangle) = \langle u(x_0) \rangle = u(x_0)$.

We can now appeal to Jensen's inequality: if u is a concave function, like the logarithm, then $\Delta J_u \geq 0$; while if u is convex, like the curious exponential in (Eq. 97), then $\Delta J_u \leq 0$. The only cases for which $\Delta J_u = 0$ are if Δx is non-random or if u is linear. Thus, it is both randomness and the nonlinearity of the utility function (or ergodicity transformation) that creates a difference in growth rates and generates inequality.

2.1.7 Power law resemblance

{section:power_law}

It is an established empirical observation [31] that the upper tails of real wealth distributions look more like a power law than a log-normal. Our trivial model does not strictly reproduce this feature, but it is instructive to compare the log-normal distribution to a power-law distribution. A power law PDF has the asymptotic form

$$\mathcal{P}_x(x) = x^{-\alpha}, \tag{164}$$

{eq:power_law}

for large arguments x. This implies that the logarithm of the PDF is proportional to the logarithm of its argument, $\ln \mathcal{P}_x(x) = -\alpha \ln x$. Plotting one against the other will yield a straight line, the slope being the exponent $-\alpha$.

Determining whether an empirical observation is consistent with such behaviour is difficult because the behaviour is to be observed in the tail (large x) where data are, by definition, sparse. A quick-and-dirty way of checking for possible power-law behaviour is to plot an empirical PDF against its argument on log-log scales, look for a straight line, and measure the slope. However, plotting any distribution on any type of scales results in some line. It may not be a straight line but it will have some slope everywhere. For a known distribution (power law or not) we can interpret this slope as a local apparent power-law exponent.

What is the local apparent power-law exponent of a log-normal wealth distribution near the expectation value $\langle x \rangle = \exp(\mu t)$, *i.e.* in the upper tail where approximate power law behaviour has been observed empirically? The logarithm of (Eq. 136) is

$$\begin{aligned}\ln \mathcal{P}(x) = & \quad -\ln\left(x\sqrt{2\pi\sigma^2 t}\right) - \tfrac{[\ln x - (\mu - \frac{\sigma^2}{2})t]^2}{2\sigma^2 t} && (165)\\ = & \quad -\ln x - \tfrac{\ln(2\pi\sigma^2 t)}{2} - \tfrac{(\ln x)^2 - 2(\mu - \frac{\sigma^2}{2})t \ln x + (\mu - \frac{\sigma^2}{2})^2 t^2}{2\sigma^2 t}. && (166)\end{aligned}$$

Collecting terms in powers of $\ln x$ we find

$$\ln \mathcal{P}(x) = -\frac{(\ln x)^2}{2\sigma^2 t} + \left(\frac{\mu}{\sigma^2} - \frac{3}{2}\right)\ln x - \frac{\ln(2\pi\sigma^2 t)}{2} - \frac{(\mu - \frac{\sigma^2}{2})^2 t}{2\sigma^2} \tag{167}$$

with local slope, *i.e.* apparent exponent,

$$\frac{d \ln \mathcal{P}(x)}{d \ln x} = -\frac{\ln x}{\sigma^2 t} + \frac{\mu}{\sigma^2} - \frac{3}{2}. \tag{168}$$

Near $\langle x \rangle$, $\ln x \sim \mu t$ so that the first two terms cancel approximately. Here the distribution will resemble a power-law with exponent $-3/2$ when plotted on doubly logarithmic scales. (The distribution will also look like a power-law where the first term is much smaller than the others, *e.g.* where $\ln x \ll \sigma^2 t$.) We don't believe that such empirically observed power laws are merely a manifestation of this mathematical feature. Important real-world mechanisms that broaden real wealth distributions, *i.e.* concentrate wealth, are missing from the null model. However, it is interesting that the trivial model of GBM reproduces so many qualitative features of empirical observations.

2.2 Finite populations

{section:finite_populatio

So far we have considered the properties the random variable, $x(t)$, generated by GBM at a fixed time, t. Most of the mathematical objects we have discussed are, strictly speaking, relevant only in the limit $N \to \infty$, where N is the number of realisations of this random variable. For example, the expected wealth, $\langle x(t) \rangle$, is the limit of the sample mean wealth

$$\langle x(t) \rangle_N \equiv \frac{1}{N} \sum_{i=1}^{N} x_i(t), \tag{169}$$ {eq:sample}

as the sample size, N, grows large. In reality, human populations can be very large, say $N \sim 10^7$ for a nation state, but they are most certainly finite. Therefore, we need to be diligent and ask what the effects of this finiteness are. In particular, we will focus on the sample mean wealth under GBM. For what values of μ, σ, t, and N is this well approximated by the expectation value? And when it is not, what does it resemble?

2.2.1 Sums of log-normals

{section:sketch}

In [36] we studied the sample mean of GBM, which we termed the "partial ensemble average" (PEA). This is the average of N independent realisations the random variable $x(t)$, (Eq. 169). Here we sketch out some simple arguments about how this object depends on N and t.

Considering the two growth rates in Sec. 2.1.2, we anticipate the following tension:

A) for large N, the PEA should resemble the expectation value, $\exp(\mu t)$;

B) for long t, all trajectories in the sample – and, therefore, the sample mean – should grow like $\exp[(\mu - \sigma^2/2)t]$.

Situation A – when a sample mean resembles the corresponding expectation value – is known in statistical physics as "self-averaging." A simple strategy for estimating when this occurs is to look at the relative variance of the PEA,

$$R \equiv \frac{\mathrm{var}[\langle x(t) \rangle_N]}{\langle \langle x(t) \rangle_N \rangle^2}. \tag{170}$$ {eq:rel_var}

To be explicit, here the $\langle \cdot \rangle$ and $\mathrm{var}(\cdot)$ operators, without N as a subscript, refer to the mean and variance over all possible PEAs. The PEAs themselves, taken over finite samples of size N, are denoted $\langle \cdot \rangle_N$. Equation (170) simplifies to

$$R = \frac{\frac{1}{N}\mathrm{var}[x(t)]}{\langle x(t) \rangle^2}, \tag{171}$$

into which we insert (Eq. 137) and (Eq. 142) to get an expression for the relative variance in terms of the GBM model parameters:

$$R(N) = \frac{e^{\sigma^2 t} - 1}{N}. \tag{172}$$ {eq:rel_var_N}

If $R \ll 1$, then the PEA will likely be close to its own expectation value, which is equal to the expectation value of the GBM. Thus, in terms of N and t, $\langle x(t)\rangle_N \approx \langle x(t)\rangle$ when

$$t < \frac{\ln N}{\sigma^2}. \tag{173}$$ {eq:short_t}

This hand-waving tells us roughly when the large-sample – or, as we see from (Eq. 173), short-time – self-averaging regime holds. A more careful estimate of the cross-over time in (Eq. 210) is a factor of 2 larger, but the scaling is identical.

For $t > \ln N/\sigma^2$, the growth rate of the PEA transitions from μ to its $t \to \infty$ limit of $\mu - \sigma^2/2$ (Situation B). Another way of viewing this is to think about what dominates the average. For early times in the process, all trajectories are close together and none dominate the PEA. However, as time goes by the distribution broadens exponentially. Since each trajectory contributes with the same weight to the PEA, after some time the PEA will be dominated by the maximum in the sample,

$$\langle x(t)\rangle_N \approx \frac{1}{N} \max_{i=1}^{N}\{x_i(t)\}, \tag{174}$$

as illustrated in Fig. 9.

Self-averaging stops when even the "luckiest" trajectory is no longer close to the expectation value $\exp(\mu t)$. This is guaranteed to happen eventually because the probability for a trajectory to reach $\exp(\mu t)$ decreases towards zero as t grows. We know this from Sec. 2.1.4. Of course, this takes longer for larger samples, which have more chances to contain a lucky trajectory.

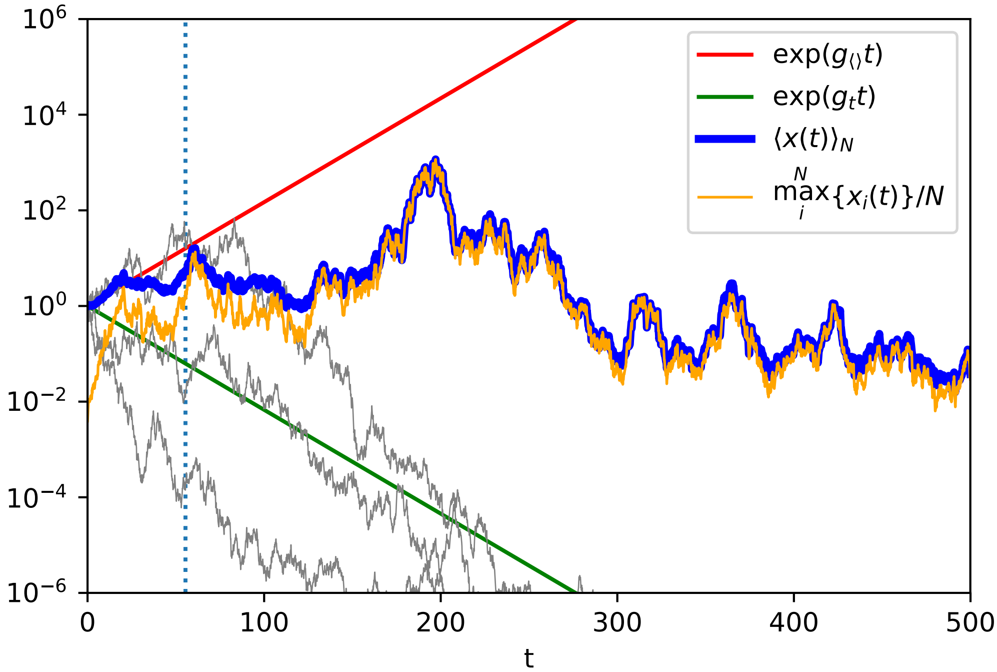

Figure 9: PEA and maximum in a finite ensemble of size $N = 256$. **Red line:** expectation value $\langle x(t)\rangle$. **Green line:** exponential growth at the time-average growth rate. In the $T \to \infty$ limit all trajectories grow at this rate. **Yellow line:** contribution of the maximum value of any trajectory at time t to the PEA. **Blue line:** PEA $\langle x(t)\rangle_N$. **Vertical line:** Crossover – for $t > t_c = \frac{2\ln N}{\sigma^2}$ the maximum begins to dominate the PEA (the yellow line approaches the blue line). **Grey lines:** randomly chosen trajectories – any typical trajectory soon grows at the time-average growth rate. **Parameters:** $N = 256$, $\mu = 0.05$, $\sigma = \sqrt{0.2}$. {fig:trajectories}

In [36] we analysed PEAs of GBM analytically and numerically. Using (Eq. 134) the PEA can be written as

$$\langle x \rangle_N = \frac{1}{N} \sum_{i=1}^{N} \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_i(t)\right], \tag{175}$$

{eq:PEA}

where $\{W_i(t)\}_{i=1\dots N}$ are N independent realisations of the Wiener process. Taking the deterministic part out of the sum we re-write (Eq. 175) as

$$\langle x \rangle_N = \exp\left[\left(\mu - \frac{\sigma^2}{2}\right) t\right] \frac{1}{N} \sum_{i=1}^{N} \exp\left(t^{1/2} \sigma \xi_i\right), \tag{176}$$

{eq:PEA_2}

where $\{\xi_i\}_{i=1\dots N}$ are N independent standard normal variates.

We found that typical trajectories of PEAs grow at $g_{\langle\rangle}$ up to a time t_c that is logarithmic in N, meaning $t_c \propto \ln N$. This is consistent with our analytical sketch. After this time, typical PEA trajectories begin to deviate from expectation-value behaviour, and eventually their growth rate converges to g_t. While the two limiting behaviours $N \to \infty$ and $t \to \infty$ can be computed exactly, what happens in between is less straightforward. The PEA is a random object outside these limits.

A quantity of crucial interest to us is the exponential growth rate experienced by the PEA,

$$g_{\text{est}}(t, N) \equiv \frac{\ln(\langle x(t) \rangle_N) - \ln(x(0))}{t - 0} = \frac{1}{t} \ln(\langle x(t) \rangle_N). \tag{177}$$

{eq:gest}

In [36] we proved that the $t \to \infty$ limit for any (finite) N is the same as for the case $N = 1$,

$$\lim_{t \to \infty} g_{\text{est}}(t, N) = \mu - \frac{\sigma^2}{2} \tag{178}$$

{eq:gest_2}

for all $N \geq 1$. Substituting (Eq. 176) in (Eq. 177) produces

$$g_{\text{est}}(t, N) = \mu - \frac{\sigma^2}{2} + \frac{1}{t} \ln\left(\frac{1}{N} \sum_{i=1}^{N} \exp(t^{1/2} \sigma \xi_i)\right) \tag{179}$$

$$= \mu - \frac{\sigma^2}{2} - \frac{\ln N}{t} + \frac{1}{t} \ln\left(\sum_{i=1}^{N} \exp(t^{1/2} \sigma \xi_i)\right). \tag{180}$$

{eq:gest_4}

We didn't look in [36] at the expectation value of $g_{\text{est}}(t, N)$ for finite time and finite samples, but it's an interesting object that depends on N and t but is not stochastic. Note that this is not g_{est} of the expectation value, which would be the $N \to \infty$ limit of (Eq. 177). Instead it is the $S \to \infty$ limit,

$$\langle g_{\text{est}}(t, N) \rangle = \frac{1}{t} \langle \ln(\langle x(t) \rangle_N) \rangle = f(N, t), \tag{181}$$

{eq:gest_3}

where, as previously, $\langle \cdot \rangle$ without subscript refers to the average over all possible samples, *i.e.* $\lim_{S \to \infty} \langle \cdot \rangle_S$. The last two terms in (Eq. 180) suggest an exponential relationship between ensemble size and time. The final term is a tricky stochastic object on which the properties of the expectation value in (Eq. 181) will hinge. This term will be the focus of our attention: the sum of exponentials of normal random variates or, equivalently, log-normal variates.

2.2.2 The random energy model

{section:REM}

Since the publication of [36] we have learned, thanks to discussions with J.-P. Bouchaud, that the key object in (Eq. 180) – the sum log-normal random variates – has been of interest to the mathematical physics community since the 1980s. The reason for this is Derrida's random energy model [15, 16].

It is defined as follows. Imagine a system whose energy levels are $2^K = N$ normally-distributed random numbers, ξ_i (corresponding to K spins). This is a very simple model of a disordered system, such as a spin glass, the idea being that the system is so complicated that we "give up" and simply model its energy levels as realisations of a random variable. (We denote the number of spins by K and the number of resulting energy levels by N, while Derrida uses N for the number of spins). In this model The partition function is then

$$Z = \sum_{i=1}^{N} \exp\left(\beta J \sqrt{\frac{K}{2}} \xi_i\right), \qquad (182)$$ {eq:Z}

where the inverse temperature, β, is measured in appropriate units, and the scaling in K is chosen so as to ensure an extensive thermodynamic limit [15, p. 79]. J is a constant that will be determined below. The logarithm of the partition function gives the Helmholtz free energy,

$$F = -\frac{\ln Z}{\beta} \qquad (183)$$

$$= -\frac{1}{\beta} \ln \left[\sum_{i=1}^{N} \exp\left(\beta J \sqrt{\frac{K}{2}} \xi_i\right)\right]. \qquad (184)$$ {eq:F}

Like the growth rate estimator in (Eq. 177), this involves a sum of log-normal variates and, indeed, we can rewrite (Eq. 180) as

$$g_{\text{est}} = \mu - \frac{\sigma^2}{2} - \frac{\ln N}{t} - \frac{\beta F}{t}, \qquad (185)$$ {eq:gest_5}

which is valid provided that

$$\beta J \sqrt{\frac{K}{2}} = \sigma t^{1/2}. \qquad (186)$$ {eq:map}

Equation (186) does not give a unique mapping between the parameters of our GBM, (σ, t), and the parameters of the REM, (β, K, J). Equating (up to multiplication) the constant parameters, σ and J, in each model gives us a specific mapping:

$$\sigma = \frac{J}{\sqrt{2}} \quad \text{and} \quad t^{1/2} = \beta\sqrt{K}. \qquad (187)$$ {eq:choice_1}

The expectation value of g_{est} is interesting. The only random object in (Eq. 185) is F. Knowing $\langle F\rangle$ thus amounts to knowing $\langle g_{\text{est}}\rangle$. In the statistical mechanics of the random energy model $\langle F\rangle$ is of key interest and so much about it is known. We can use this knowledge thanks to the mapping between the two problems.

Derrida identifies a critical temperature,

$$\frac{1}{\beta_c} \equiv \frac{J}{2\sqrt{\ln 2}}, \qquad (188)$$ {eq:beta_c}

above and below which the expected free energy scales differently with K and β. This maps to a critical time scale in GBM,

$$t_c = \frac{2 \ln N}{\sigma^2}, \tag{189}$$ {eq:t_c}

with high temperature ($1/\beta > 1/\beta_c$) corresponding to short time ($t < t_c$) and low temperature ($1/\beta < 1/\beta_c$) corresponding to long time ($t > t_c$). Note that t_c in (Eq. 210) scales identically with N and σ as the transition time, (Eq. 173), in our sketch.

In [15], $\langle F \rangle$ is computed in the high-temperature (short-time) regime as

$$\begin{aligned} \langle F \rangle &= E - S/\beta & (190) \\ &= -\frac{K}{\beta} \ln 2 - \frac{\beta K J^2}{4}, & (191) \end{aligned}$$ {eq:F_2}

and in the low-temperatures (long-time) regime as

$$\langle F \rangle = -KJ\sqrt{\ln 2}. \tag{192}$$ {eq:F_3}

Short time

We look at the short-time behavior first (high $1/\beta$, (Eq. 191)). The relevant computation of the entropy S in [15] involves replacing the number of energy levels $n(E)$ by its expectation value $\langle n(E) \rangle$. This is justified because the standard deviation of this number is $\sqrt{n}$ and relatively small when $\langle n(E) \rangle > 1$, which is the interesting regime in Derrida's case.

For spin glasses, the expectation value of F is interesting, supposedly, because the system may be self-averaging and can be thought of as an ensemble of many smaller sub-systems that are essentially independent. The macroscopic behavior is then given by the expectation value.

Taking expectation values and substituting from (Eq. 191) in (Eq. 185) we find

$$\langle g_{\text{est}} \rangle^{\text{short}} = \mu - \frac{\sigma^2}{2} + \frac{1}{t} \frac{KJ^2}{4T^2}. \tag{193}$$ {eq:gest_6}

From (Eq. 186) we know that $t = \frac{KJ^2}{2\sigma^2 T^2}$, which we substitute, to find

$$\langle g_{\text{est}} \rangle^{\text{short}} = \mu, \tag{194}$$ {eq:gest_7}

which is the correct behavior in the short-time regime.

Long time

Next, we turn to the expression for the long-time regime (low temperature, (Eq. 192)). Again taking expectation values and substituting, this time from (Eq. 192) in (Eq. 185), we find for long times

$$\langle g_{\text{est}} \rangle^{\text{long}} = \mu - \frac{\sigma^2}{2} - \frac{\ln N}{t} + \sqrt{\frac{2 \ln N}{t}}\, \sigma, \tag{195}$$ {eq:gest_8}

which has the correct long-time asymptotic behavior. The form of the correction to the time-average growth rate in (Eq. 195) is consistent with [36] and [38], where it was found that approximately $N = \exp(t)$ systems are required for ensemble-average behavior to be observed for a time t, so that the parameter $\ln N/t$ controls which regime dominates – if the parameter is small, then (Eq. 195) indicates that the long-time regime is relevant.

Figure 10 is a direct comparison between the results derived here, based on [15], and numerical results using the same parameter values as in [36], namely $\mu = 0.05, \sigma = \sqrt{0.2}, N = 256$ and $S = 10^5$.

Notice that $\langle g_{\text{est}} \rangle$ is not the (local) time derivative $\frac{\partial}{\partial t} \langle \ln(\langle x \rangle_N) \rangle$, but a time-average growth rate, $\left\langle \frac{1}{t} \ln \left(\frac{\langle x(t) \rangle_N}{\langle x(0) \rangle_N} \right) \right\rangle$. In [36] we used a notation that we've stopped using since then because it caused confusion – $\langle g \rangle$ there denotes the growth rate of the expectation value, which is not the expectation value of the growth rate.

It is remarkable that the expectation value $\langle g_{\text{est}}(N, t) \rangle$ so closely reflects the median, $q_{0.5}$, of $\langle x \rangle_N$, in the sense that

$$q_{0.5}(\langle x(t) \rangle_N) \approx \exp\left(\langle g_{\text{est}}(N, t) \rangle t\right). \tag{196}$$ {eq:quant_ave}

In [35] it was discussed in detail that $g_{\text{est}}(1, t)$ is an ergodic observable for (Eq. 2.1.2), in the sense that $\langle g_{\text{est}}(1, t) \rangle = \lim_{t \to \infty} g_{\text{est}}$. The relationship in (Eq. 196) is far more subtle. The typical behavior of GBM PEAs is complicated outside the limits $N \to \infty$ or $t \to \infty$, in the sense that growth rates are time dependent here. This complicated behaviour is well represented by an approximation that uses physical insights into spin glasses.

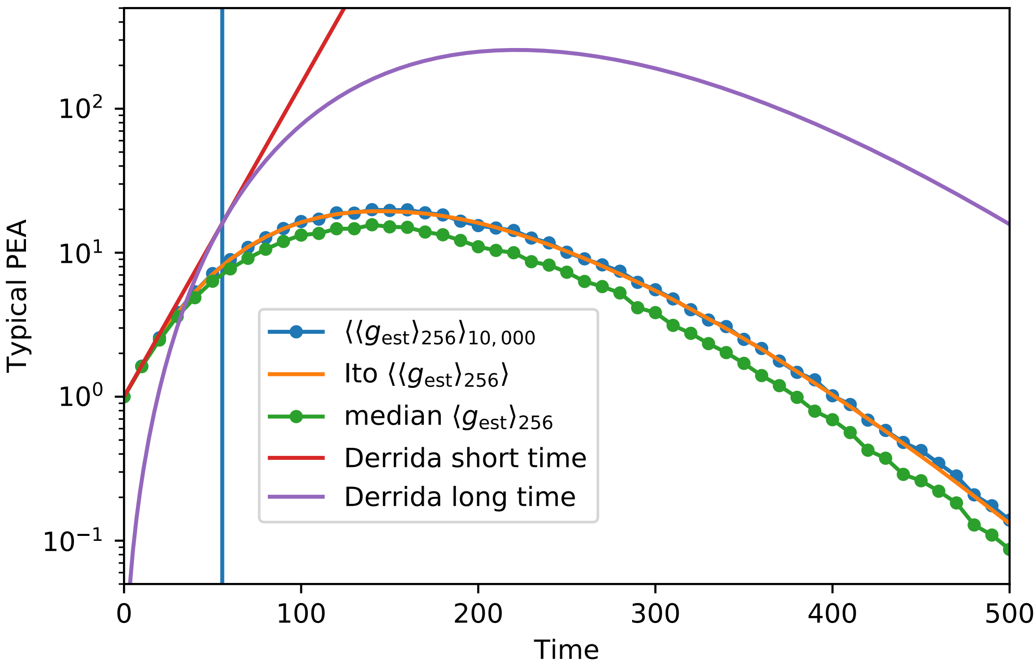

Figure 10: Lines are obtained by exponentiating the various exponential growth rates. **Blue line:** $\langle\langle g_{\mathrm{est}}\rangle_{256}\rangle_{10{,}000}$ is the numerical mean (approximation of the expectation value) over a super-ensemble of $S = 10,000$ samples of g_{est} estimated in sub-ensembles of $N = 256$ GBMs each. **Green line:** median in a super-ensemble of S samples of g_{est}, each estimated in sub-ensembles of size N. **Yellow line:** (Eq. **??**) is an exact expression for $d\,\langle\ln\langle x\rangle_N\rangle$, derived using Itô calculus. We evaluate the expression by Monte Carlo, and integrate, $\langle\ln\langle x\rangle_N\rangle = \int_0^t d\,\langle\ln\langle x\rangle_N\rangle$. Exponentiation yields the yellow line. **Red line:** short-time behavior, based on the random energy model, (Eq. 194). **Purple line:** long-time behavior, based on the random energy model, (Eq. 195). **Vertical line:** Crossover between the regimes at $t_c = \frac{2\ln N}{\sigma^2}$, corresponding to $\beta_c = \frac{2(\ln 2)^{1/2}}{J}$. **Parameters:** $N = 256$, $S = 10,000$, $\mu = 0.05$, $\sigma = \sqrt{0.2}$. {fig:1}

3 Interactions

Insert abstract here.

3.1 Cooperation

{section:Cooperation}

Under multiplicative growth, fluctuations are undesirable because they reduce time-average growth rates. In the long run, wealth $x_1(t)$ with noise term σ_1 will outperform wealth $x_2(t)$ with a larger noise term $\sigma_2 > \sigma_1$, in the sense that

$$\overline{g}(x_1) > \overline{g}(x_2) \tag{197}$$

with probability 1.

For this reason it is desirable to reduce fluctuations. One protocol that achieves this is resource pooling and sharing. In Sec. 2.1 we explored the world created by the model of independent GBMs. This is a world where everyone experiences the same long-term growth rate. We want to explore the effect of the invention of cooperation. As it turns out cooperation increases growth rates, and this is a crucial insight.

Suppose two individuals, $x_1(t)$ and $x_2(t)$ decide to meet up every Monday, put all their wealth on a table, divide it in two equal amounts, and go back to their business, *i.e.* they submit their wealth to our toy dynamic (Eq. 2.1.2). How would this operation affect the dynamic of the wealth of these two individuals?

Consider a discretized version of (Eq. 2.1.2), such as would be used in a numerical simulation. The non-cooperators grow according to

$$\delta x_i(t) = x_i(t)\left[\mu\delta t + \sigma\sqrt{\delta t}\,\xi_i\right], \tag{198}$$

{eq:discrete_nonc_grow}

$$x_i(t+\delta t) = x_i(t) + \delta x_i(t), \tag{199}$$

{eq:discrete_nonc_coop}

where ξ_i are standard normal random variates, $\xi_i \sim \mathcal{N}(0,1)$.

We imagine that the two previously non-cooperating entities, with resources $x_1(t)$ and $x_2(t)$, cooperate to produce two entities, whose resources we label $x_1^c(t)$ and $x_2^c(t)$ to distinguish them from the non-cooperating case. We envisage equal sharing of resources, $x_1^c = x_2^c$, and introduce a cooperation operator, $\oplus$, such that

$$x_1 \oplus x_2 = x_1^c + x_2^c. \tag{200}$$

In the discrete-time picture, each time step involves a two-phase process. First there is a growth phase, analogous to (Eq. 2.1.2), in which each cooperator increases its resources by

$$\delta x_i^c(t) = x_i^c(t)\left[\mu\delta t + \sigma\sqrt{\delta t}\,\xi_i\right]. \tag{201}$$

{eq:discrete_coop_grow}

This is followed by a cooperation phase, replacing (Eq. 199), in which resources are pooled and shared equally among the cooperators:

$$x_i^c(t+\delta t) = \frac{x_1^c(t) + \delta x_1^c(t) + x_2^c(t) + \delta x_2^c(t)}{2}. \tag{202}$$

{eq:discrete_coop_coop}

With this prescription both cooperators and their sum experience the following dynamic:

$$(x_1 \oplus x_2)(t+\delta t) = (x_1 \oplus x_2)(t)\left[1 + \left(\mu\delta t + \sigma\sqrt{\delta t}\,\frac{\xi_1+\xi_2}{2}\right)\right]. \tag{203}$$

{eq:discrete_cooperate}

For ease of notation we define

$$\xi_{1\oplus 2} = \frac{\xi_1+\xi_2}{\sqrt{2}}, \tag{204}$$

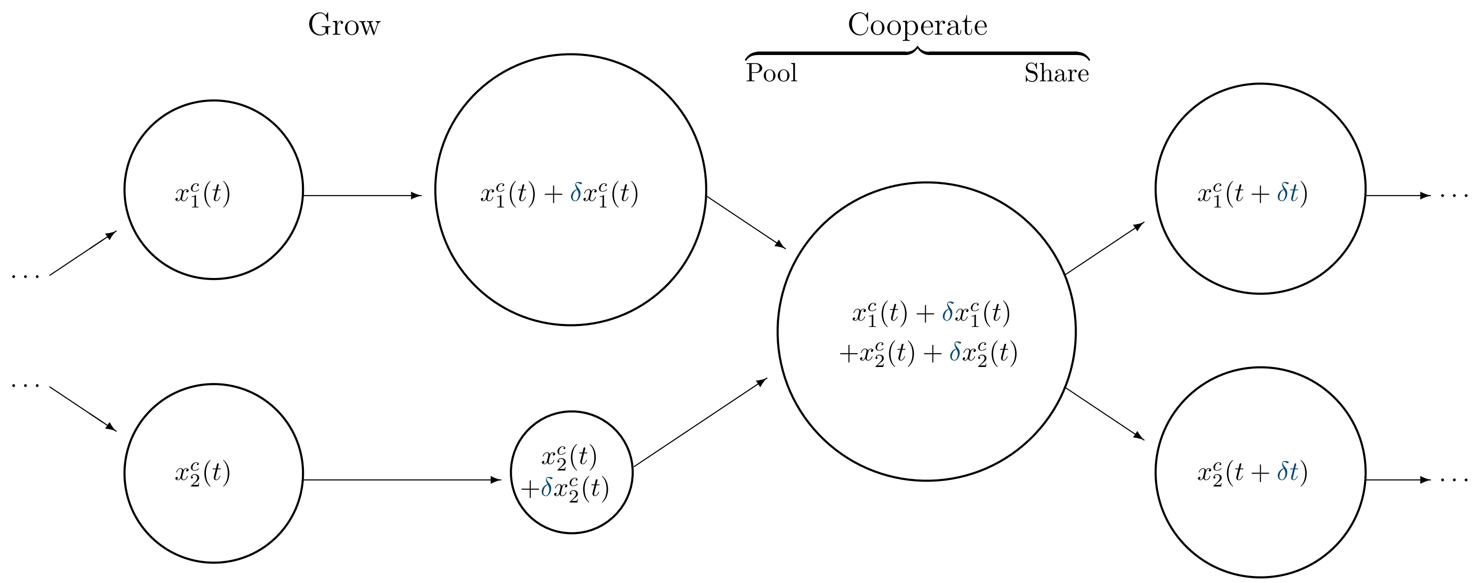

Figure 11: Cooperation dynamics. Cooperators start each time step with equal resources, then they *grow* independently according to (Eq. 201), then they *cooperate* by *pooling* resources and *sharing* them equally, then the next time step begins. {fig:dynamics}

which is another standard Gaussian, $\xi_{1\oplus 2} \sim \mathcal{N}(0,1)$. Letting the time increment $\delta t \to 0$ we recover an equation of the same form as (Eq. 2.1.2) but with a different fluctuation amplitude,

$$d(x_1 \oplus x_2) = (x_1 \oplus x_2)\left(\mu dt + \frac{\sigma}{\sqrt{2}} dW_{1\oplus 2}\right). \tag{205}$$

The expectation values of a non-cooperator, $\langle x_1(t)\rangle$, and a corresponding cooperator, $\langle x_1^c(t)\rangle$, are identical. Based on expectation values, we thus cannot see any benefit of cooperation. Worse still, immediately after the growth phase, the better-off entity of a cooperating pair, $x_1^c(t_0) > x_2^c(t_0)$, say, would increase its expectation value from $\frac{x_1^c(t_0)+x_2^c(t_0)}{2}\exp(\mu(t-t_0))$ to $x_1^c(t_0)\exp(\mu(t-t_0))$ by breaking the cooperation. But it would be foolish to act on the basis of this analysis – the short-term gain from breaking cooperation is a one-off, and is dwarfed by the long-term multiplicative advantage of continued cooperation. An analysis based on expectation values finds that there is no reason for cooperation to arise, and that if it does arise there are good reasons for it to end, *i.e.* it will be fragile. Because expectation values are inappropriately used to evaluate future prospects, the observation of widespread cooperation constitutes a conundrum.

The solution of the conundrum comes from considering the time-average growth rate. The non-cooperating entities grow at $g_t(x_i) = \mu - \frac{\sigma^2}{2}$, whereas the cooperating unit benefits from a reduction of the amplitude of relative fluctuations and grows at $g_t(x_1 \oplus x_2) = \mu - \frac{\sigma^2}{4}$, and we have

$$g_t(x_1 \oplus x_2) > g_t(x_i) \tag{206}$$

for any non-zero noise amplitude. Imagine a world where cooperation does not exist, just like in Sec. 2.1. Now introduce into this world two individuals who have invented cooperation – very quickly this pair of individuals will be vastly more wealthy than anyone else. To keep up, others will have to start cooperating. The effect is illustrated in Fig. 12 by direct simulation of (Eq. 198)–(Eq. 199) and (Eq. 203).

Imagine again the pair of cooperators outperforming all of their peers. Other entities will have to form pairs to keep up, and the obvious next step is for larger cooperating units to form – groups of 3 may form, pairs of pairs, cooperation clusters of n individuals, and the larger the cooperating group the closer the time-average growth rate will get to the expectation value. For n cooperators, $x_1 \oplus x_2 ... \oplus x_n$ the spurious drift term is $-\frac{\sigma^2}{2n}$, so that the time-average growth approaches expectation-value growth for large n. The approach to this upper bound as the number of cooperators increases favours the formation of social structure.

We may generalise to different drift terms, μ_i, and noise amplitudes, σ_i, for different individual entities. Whether cooperation is beneficial in the long run for any given entity depends on these parameters as follows. Entity 1 will benefit from cooperation with entity 2 if

$$\mu_1 - \frac{\sigma_1^2}{2} < \frac{\mu_1 + \mu_2}{2} - \frac{\sigma_1^2 + \sigma_2^2}{8}. \tag{207}$$

We emphasize that this inequality may be satisfied also if the expectation value of entity 1 grows faster than the expectation value of entity 2, *i.e.* if $\mu_1 > \mu_2$. An analysis of expectation values, again, is utterly misleading: the benefit conferred on entity 1 due to the fluctuation-reducing effect of cooperation may outweigh the cost of having to cooperate with an entity with smaller expectation value.

Notice the nature of the Monte-Carlo simulation in Fig. 12. No ensemble is constructed. Only individual trajectories are simulated and run for a time that is long enough for statistically significant features to rise above the noise. This method teases out of the dynamics what happens over time. The significance of any observed structure – its epistemological meaning – is immediately clear: this is what happens over time for an individual system (a cell, a person's wealth, *etc.*). Simulating an ensemble and averaging over members to remove noise does not tell the same story. The resulting features may not emerge over time. They are what happens on average in an ensemble, but – at least for GBM – this is not what happens to the individual with probability 1. For instance the pink dashed line in Fig. 12 is the ensemble average of $x_1(t)$, $x_1(t)$, and $(x_1 \oplus x_2)(t)/2$, and it has nothing to do with what happens in the individual trajectories over time.

When judged on expectation values, the apparent futility of cooperation is unsurprising because expectation values are the result for infinitely many cooperators, and adding further cooperators cannot improve on this.

In our model the advantage of cooperation, and hence the emergence of social structure in the broadest sense – is purely a non-linear effect of fluctuations – cooperation reduces the magnitude of fluctuations, and over time (though not in expectation) this implies faster growth.

Another generalisation is partial cooperation – entities may share only a proportion of their resources, resembling taxation and redistribution. We discuss this in the next section.

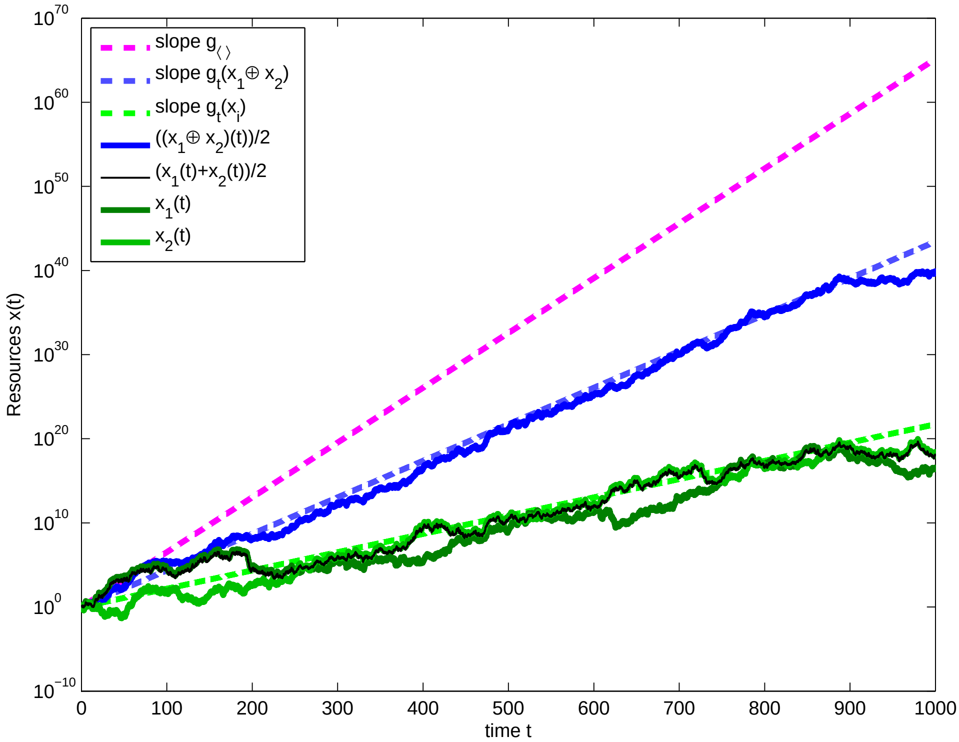

Figure 12: Typical trajectories for two non-cooperating (green) entities and for the corresponding cooperating unit (blue). Over time, the noise reduction for the cooperator leads to faster growth. Even without effects of specialisation or the emergence of new function, cooperation pays in the long run. The black thin line shows the average of the non-cooperating entities. While in the logarithmic vertical scale the average traces the more successful trajectory, it is far inferior to the cooperating unit. In a very literal mathematical sense the whole, $(x_1 \oplus x_2)(t)$, is more than the sum of its parts, $x_1(t) + x_2(t)$. The algebra of cooperation is not merely that of summation. {fig:cooperate}

3.2 Reallocation

{section:reallocation}

3.2.1 Introduction

{section:RGBM_intro}

In Sec. 2.1 we created a model world of independent trajectories of GBM. We studied how the distribution of the resulting random variables evolved over time. We saw that this is a world of broadening distributions, increasing inequality, and wealth condensation. We introduced cooperation to it in Sec. 3.1 and saw how this increases the time-average growth rate for those who pool and share all of their resources. In this section we study what happens if a large number of individuals pool and share only a fraction of their resources. This is reminiscent of the taxation and redistribution – which we shall call "reallocation" – carried out by populations in the real world.

We will find that, while full cooperation between two individuals increases their growth rates, sufficiently fast reallocation from richer to poorer in a large population has two related effects. Firstly, everyone's wealth grows in the long run at a rate close to that of the expectation value. Secondly, the distribution of rescaled wealth converges over time to a stable form. This means that, while wealth can still be distributed quite unequally, wealth condensation and the divergence of inequality no longer occur in our model. Of course, for this to be an interesting finding, we will have to quantify what we mean by "sufficiently fast reallocation."

We will also find that when reallocation is too slow or, in particular, when it goes from poorer to richer – which we will quantify as negative reallocation – no stable wealth distribution exists. In the latter case, the population splits into groups with positive and negative wealths, whose magnitudes grow exponentially.

Finally, having understood how our model behaves in each of these reallocation regimes, we will fit the model parameters to historical wealth data from the real world, specifically the United States. This will tell us which type of model behaviour best describes the dynamics of the US wealth distribution in both the recent and more distant past. You might find the results surprising – we certainly did!

3.2.2 The ergodic hypothesis in economics

{section:RGBM_EH}

Of course, we are not the first to study resource distributions and inequality in economics. This topic has a long history, going back at least as far as Vilfredo Pareto's work in the late 19^{th} century [32] (in which he introduced the power-law distribution we discussed in Sec. 2.1.7). Economists studying such distributions usually assume that they converge in the long run to a unique and stable form, regardless of initial conditions. This allows them to study the stable distribution, for which many statistical techniques exist, and to ignore the transient phenomena preceding it, which are far harder to analyse. Paul Samuelson called this the "ergodic hypothesis" [41, pp. 11-12]. It's easy to see why: if this convergence happens, then the time average of the observable in question will equal its ensemble average over the stable distribution.[16]

Economics is often concerned with growth and a growing quantity cannot be ergodic in Samuelson's sense, because its distribution never stabilises. This

[16] Convergence to a unique and stable distribution is a sufficient but not necessary condition for an ergodic observable, as we have defined it.

suggests the simplifying ergodic hypothesis should never be made. Not so fast! Although rarely stated, a common strategy to salvage these techniques is to find a transformation of the non-ergodic process that produces a meaningful ergodic observable. If such an ergodic observable can be derived, then classical analytical techniques may still be used. We have already seen in the context of gambles that expected utility theory can be viewed as transformation of non-ergodic wealth increments into ergodic utility increments. Expectation values, which would otherwise be misleading, then quantify time-average growth of the decision-maker's wealth.

Studies of wealth distributions also employ this strategy. Individual wealth is modelled as a growing quantity. Dividing by the population average transforms this to a rescaled wealth, as in Sec. 2.1.5, which is hypothesised to be ergodic. For example, [4, p. 130] "impose assumptions ... that guarantee the existence and uniqueness of a limit stationary distribution." The idea is to take advantage of the simplicity with which the stable distribution can be analysed, *e.g.* to predict the effects of policies encoded in model parameters.

There is, however, an elephant in the room. To our knowledge, the validity of the ergodic hypothesis for rescaled wealth has never been tested empirically. It's certainly invalid for the GBM model world we studied previously because, as we saw in Sec. 2.1.5, rescaled wealth has an ever-broadening log-normal distribution. That doesn't seem to say much, as most reasonable people would consider our model world – containing a population of individuals whose wealths multiply noisily and who never interact – somewhat unrealistic! The model we are about to present will not only extend our understanding from this simple model world to one containing interactions, but also will allows us to test the hypothesis. This is because it has regimes, *i.e.* combinations of parameters, for which rescaled wealth is and isn't ergodic. This contrasts with models typically used by economists, which have the ergodic hypothesis "baked in."

If it is reasonable to assume a stable distribution exists, we must also consider how long convergence to would take after a change of parameters. It's no use if convergence in the model takes millennia, if we are using it to estimate the effect of a new tax policy over the next election cycle. Therefore, treating a stable model distribution as representative of the empirical wealth distribution implies an assumption of fast convergence. As the late Tony Atkinson pointed out, "the speed of convergence makes a great deal of difference to the way in which we think about the model" [3]. We will also use our model to discuss this point. Without further ado, let us introduce it.

3.2.3 Reallocating GBM

{section:RGBM_model}

Our model, called Reallocating Geometric Brownian Motion (RGBM), is a system of N individuals whose wealths, $x_i(t)$, evolve according to the stochastic differential equation,

$$dx_i = x_i\left[(\mu - \tau)dt + \sigma dW_i\left(t\right)\right] + tau \left\langle x \right\rangle_N \; dt, \qquad (208)$$ **{eq:rgbm}**

for all $i = 1 \dots N$. In effect, we have added to the GBM model a simple reallocation mechanism. Over a time step, dt, each individual pays a fixed proportion of its wealth, $\tau x_i dt$, into a central pot ("contributes to society") and gets back an equal share of the pot, $\tau \left\langle x \right\rangle_N dt$, ("benefits from society"). We can think of this as applying a wealth tax, say of 1% per year, to everyone's wealth and then redistributing the tax revenues equally. Note that the reallocation parameter, τ, is, like μ, a rate with dimensions per unit time. Note also that when $\tau = 0$, we recover our old friend, GBM, in which individuals grow their wealths without interacting.

RGBM is our null model of an exponentially growing economy with social structure. It is intended to capture only the most general features of the dynamics of wealth. A more complex model would treat the economy as a system of agents that interact with each other through a network of relationships. These relationships include trade in goods and services, employment, taxation, welfare payments, using public infrastructure (roads, schools, a legal system, social security, scientific research, and so on), insurance, wealth transfers through inheritance and gifts, and everything else that constitutes an economic network. It would be a hopeless task to list exhaustively all these interactions, let alone model them explicitly. Instead we introduce a single parameter – the reallocation rate, τ – to represent their net effect. If τ is positive, the direction of net reallocation is from richer to poorer. If negative, it is from poorer to richer.

We will see shortly that RGBM has both ergodic and non-ergodic regimes, characterised to a good approximation by the sign of τ. $\tau > 0$ produces an ergodic regime, in which wealths are positive, distributed with a Pareto tail, and confined around their mean value. $\tau < 0$ produces a non-ergodic regime, in which the population splits into two classes, characterised by positive and negative wealths which diverge away from the mean.

We offer a couple of health warnings. In RGBM, like in GBM, there are no additive changes akin to labour income and consumption. This is unproblematic for large wealths, where additive changes are dwarfed by capital gains. For small wealths, however, wages and consumption are significant and empirical distributions look rather different for low and high wealths [17]. We modelled earnings explicitly in [6] and found this didn't generate insights different from RGBM when fit to real wealth data. We note also, as [27, p. 41] put it, that our agents "do not marry or have children or die or even grow old." Therefore, the individual in our setup is best imagined as a household or a family, *i.e.* some long-lasting unit into which personal events are subsumed.

Having specified the model, we will use insights from Sec. 2.2 to understand how rescaled wealth is distributed in the ergodic and non-ergodic regimes. Then we will show briefly our results from fitting the model to historical wealth data from the United States. The full technical details of this fitting exercise are beyond the scope of these notes – if you are interested, you can find "chapter and verse" in [6]. Fitting τ to data will allow us to answer the important questions:

- What is the net reallocating effect of socio-economic structure on the wealth distribution?
- Are observations consistent with the ergodic hypothesis that the rescaled wealth distribution converges to a stable distribution?
- If so, how long does it take, after a change in conditions, for the rescaled wealth distribution to reach the stable distribution?

3.2.4 Model behaviour

{section:RGBM_behaviour}

It is instructive to write (Eq. 208) as

$$dx_i = \underbrace{x_i \left[\mu dt + \sigma dW_i\left(t\right)\right]}_{\text{Growth}} \underbrace{- \tau(x_i - \langle x \rangle_N)dt}_{\text{Reallocation}}. \tag{209}$$ {eq:rgbm_ou}

This resembles GBM with a mean-reverting term like that of [46] in physics and [47] in finance. It exposes the importance of the sign of τ. We discuss the two regimes in turn.

Positive τ

For $\tau > 0$, individual wealth, $x_i(t)$, reverts to the sample mean, $\langle x(t) \rangle_N$. We explored some of the properties of sample mean in Sec. 2.2 for wealths undergoing GBM. In particular, we saw that a short-time (or large-sample or low-volatility) self-averaging regime exists,

$$t < t_c \equiv \frac{2\ln N}{\sigma^2}, \tag{210}$$ {eq:t_c}

where the sample mean is approximated well by the ensemble average,

$$\langle x(t) \rangle_N \sim \langle x(t) \rangle = \exp(\mu t). \tag{211}$$ {eq:rgbm_self}

(The final equality assumes, as previously, that $x_i(0) = 1$ for all i.) It turns out that the same self-averaging approximation can be made for wealths undergoing RGBM, (Eq. 208), when the reallocation rate, τ, is above some critical threshold:

$$\tau > \tau_c \equiv \frac{\sigma^2}{2\ln N}. \tag{212}$$ {eq:tau_c}

Showing this is technically difficult [8] and we will confine ourselves to sketching the key ideas in Sec. 3.2.5 below. It won't have escaped your attention that $\tau_c = {t_c}^{-1}$ and, indeed, you will shortly have an intuition for why.

Fitting the model to data yields parameter values for which τ_c is extremely small. For example, typical parameters for US wealth data are $N = 10^8$ and $\sigma = 0.2$ year$^{-1/2}$, giving $\tau_c = 0.1\%$ year^{-1} (or $t_c = 900$ years). Accounting for the uncertainty in the fitted parameters makes this statistically indistinguishable from $\tau_c = 0$.

This means we can safely make the self-averaging approximation for the entire positive τ regime. That's great news, because it means we can rescale wealth by the ensemble average, $\langle x(t) \rangle = \exp(\mu t)$, as we did in Sec. 2.1.5 for GBM, and not have to worry about pesky finite N effects. Following the same procedure as there gives us a simple SDE in the rescaled wealth, $y_i(t) = x_i(t)\exp(-\mu t)$:

$$dy_i = y_i \sigma dW_i\left(t\right) - \tau(y_i - 1)dt. \tag{213}$$ {eq:rgbm_ou_re}

Note that the common growth rate, μ, has been scaled out as it was in Sec. 2.1.5.

The distribution of $y_i\left(t\right)$ can found by solving the corresponding Fokker-Planck equation, which we will do in Sec. 3.2.5. For now, we will just quote the result: a stable distribution exists with a power-law tail, to which the distribution of rescaled wealth converges over time. The distribution has a name – the Inverse Gamma Distribution – and a probability density function:

$$\mathcal{P}\left(y\right) = \frac{\left(\zeta - 1\right)^{\zeta}}{\Gamma\left(\zeta\right)} e^{-\frac{\zeta-1}{y}} y^{-(1+\zeta)}. \tag{214}$$ {eq:disti}

$\zeta = 1 + 2\tau/\sigma^2$ is the Pareto tail index (corresponding to $\alpha - 1$ in Sec. 2.1.7) and $\Gamma(\cdot)$ is the gamma function.

Example forms of the stationary distribution are shown in Figure 13. The usual stylised facts are recovered: the larger σ (more randomness in the returns) and the smaller τ (less social cohesion), the smaller the tail index ζ and the fatter the tail of the distribution. Fitted τ values give typical ζ values between 1 and 2 for the different datasets analysed, consistent with observed tail indices between 1.2 to 1.6 (see [6] for details). Not only does RGBM predict a realistic functional form for the distribution of rescaled wealth, but also it admits fitted parameter values which match observed tails. The inability to do this is a known weakness of earnings-based models (again, see [6] for discussion).

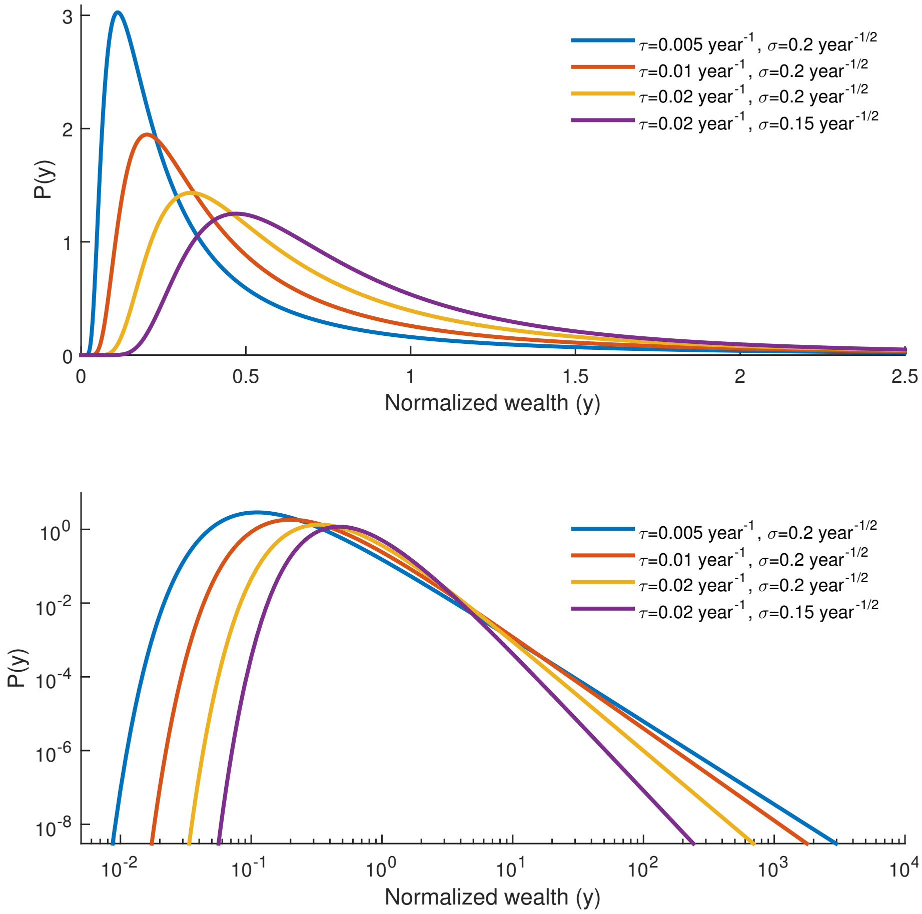

Figure 13: The stationary distribution for RGBM with positive τ. Top – linear scales; Bottom – logarithmic scales. {fig:dist}

For positive reallocation, (Eq. 213) and extensions of it have received much attention in statistical mechanics and econophysics [10, 9]. As a combination of GBM and a mean-reverting process it is a simple and analytically tractable stochastic process. [25] provide an overview of the literature and known results.

Negative τ

For $\tau < 0$ the model exhibits mean repulsion rather than reversion. The ergodic hypothesis is invalid and no stationary wealth distribution exists. The population splits into those above the mean and those below the mean. Whereas in RGBM with non-negative τ it is impossible for wealth to become negative, negative τ leads to negative wealth. No longer is total economic wealth a limit to the wealth of the richest individual because the poorest develop large negative wealth. The wealth of the rich in the population increases exponentially away from the mean, and the wealth of the poor becomes negative and exponentially large in magnitude, see Figure 14.

Such splitting of the population is a common feature of non-ergodic processes. If rescaled wealth were an ergodic process, then individuals would, over long enough time, experience all parts of its distribution. People would spend 99 percent of their time as "the 99 percent" and 1 percent of their time as "the 1 percent". Therefore, the social mobility implicit in models that assume ergodicity might not exist in reality if that assumption is invalid. That inequality and immobility have been linked [12, 24, 5] may be unsurprising if both are viewed as consequences of non-ergodic wealth or income.

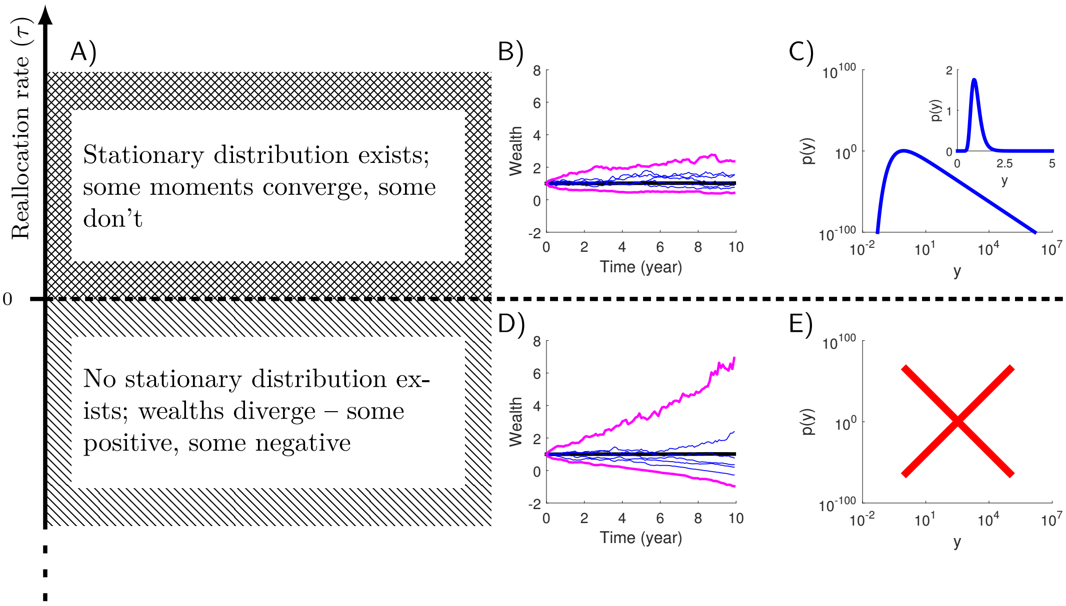

Figure 14: Regimes of RGBM. A) $\tau = 0$ separates the two regimes of RGBM. For $\tau > 0$, a stationary wealth distribution exists. For $\tau < 0$, no stationary wealth distribution exists and wealths diverge. B) Simulations of RGBM with $N = 1000$, $\mu = 0.021$ year^{-1} (presented after rescaling by $\exp(\mu t)$), $\sigma = 0.14$ year$^{-1/2}$, $x_i(0) = 1$, $\tau = 0.15$ year^{-1}. Magenta lines: largest and smallest wealths, blue lines: five randomly chosen wealth trajectories, black line: sample mean. C) The stationary distribution to which the system in B) converges. Inset: same distribution on linear scales. D) Similar to B), with $\tau = -0.15$ year^{-1}. E) in the $\tau < 0$ regime, no stationary wealth distribution exists. {fig:regimes}

3.2.5 Derivation of the stable distribution

{section:RGBM_stable}

In this section we will sketch the argument for why we can make the self-averaging approximation, (Eq. 211), in RGBM with sufficiently fast positive reallocation, (Eq. 212). This is shown rigorously in [8]. Then we will solve the Fokker-Planck equation for the rescaled wealth and derive the inverse gamma distribution, (Eq. 214). If you are happy to believe the quoted results in Sec. 3.2.4, then you can skip the Fokker-Planck bit safely.

We presented arguments in Sec. 2.2.1 for why wealth in GBM is self-averaging, $\langle x(t)\rangle_N \sim \langle x(t)\rangle = \exp(\mu t)$ for short time. By mapping from GBM to the random energy model in Sec. 2.2.2, we showed that "short time" means $t < t_c$, where $t_c = 2\ln N/\sigma^2$. We can think of this as follows: t_c is the timescale over which the inequality-increasing effects of noisy multiplicative growth drive wealths apart, such that a finite sample of wealths stops self-averaging and becomes dominated by a few trajectories.

Let's now think about what happens when we add reallocation to GBM, creating RGBM. τ is the reallocation rate, so τ^{-1} is reallocation timescale, *i.e.* the timescale over which the inequality-reducing effects of reallocation pull wealths together. If $\tau^{-1} > t_c$, then reallocation happens too slowly to prevent the expiry of self-averaging. However, if $\tau^{-1} < t_c$, then reallocation pulls wealths together more quickly than they get driven apart, continually "resetting" the sample and allowing self-averaging to be maintained indefinitely. Converting this condition into a reallocation rate, we get $\tau > {t_c}^{-1}$, as in (Eq. 212). As mentioned in Sec. 3.2.4, this becomes indistinguishable from $\tau > 0$ for realistic parameters, so the self-averaging approximation can be made safely for all positive τ.

We can now approximate the rescaled wealth, $y_i(t) = x_i(t)/\langle x(t)\rangle_N$, as $y_i(t) = x_i(t)\exp(-\mu t)$, which follows the SDE:

$$dy = \sigma\, y\, dW - \tau\,(y-1)\,dt. \tag{215}$$

This is an Itô equation with drift term $A = \tau(y-1)$ and diffusion term $B = y\sigma$. Such equations imply ordinary second-order differential equations that describe the evolution of the PDF, called Fokker-Planck equations. The Fokker-Planck equation describes the change in probability density, at any point in (relative-wealth) space, due to the action of the drift term (like advection in a fluid) and due to the diffusion term (like heat spreading). In this case, we have

$$\frac{d\mathcal{P}(y,t)}{dt} = \frac{\partial}{\partial y}\left[A\mathcal{P}(y,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left[B^2\mathcal{P}(y,t)\right]. \tag{216}$$

The steady-state Fokker-Planck equation for the PDF, $\mathcal{P}(y)$, is obtained by setting the time derivative to zero,

$$\frac{\sigma^2}{2}\left(y^2\mathcal{P}\right)_{yy} + \tau\left[(y-1)\,\mathcal{P}\right]_y = 0. \tag{217}$$

{eq:fokker_planck}

Positive wealth subjected to continuous-time multiplicative dynamics with non-negative reallocation can never reach zero. Therefore, we solve Equation (217) with boundary condition $\mathcal{P}(0) = 0$ to give

$$\mathcal{P}(y) = C(\zeta)\, e^{-\frac{\zeta-1}{y}}\, y^{-(1+\zeta)}, \tag{218}$$

where

$$\zeta = 1 + \frac{2\tau}{\sigma^2} \tag{219}$$

and

$$C(\zeta) = \frac{(\zeta - 1)^\zeta}{\Gamma(\zeta)}, \tag{220}$$

with the gamma function $\Gamma(\zeta) = \int_0^\infty x^{\zeta-1} e^{-x}\, \mathrm{d}x$. The distribution has a power-law tail as $y \to \infty$, resembling Pareto's oft-confirmed observation that the frequency of large wealths tends to decay as a power law. The exponent of the power law, ζ, is called the Pareto parameter and is one measure of economic inequality.

3.2.6 Moments and convergence times

{section:RGBM_moments}

The inverse gamma distribution, (Eq. 214), has a power-law tail. This means that, for positive reallocation, while some of the lower moments of the stable rescaled wealth distribution may exist, higher moments will not. Specifically, the k^{th} moment diverges if $k > \zeta$.

If we find parameters consistent with positive reallocation when we fit our model to data, we will be interested in whether certain statistics – such as the variance – exist. We will also want to know how long it takes the distribution to converge sufficiently to its stable form for them to be meaningful. Here we derive a condition for the convergence of the variance and calculate its convergence time, noting also the general procedure for other statistics.

The variance of y is a combination of the first moment, $\langle y \rangle$ (the average), and the second moment, $\langle y^2 \rangle$:

$$V(y) = \langle y^2 \rangle - \langle y \rangle^2 \tag{221}$$

We thus need to find $\langle y \rangle$ and $\langle y^2 \rangle$ in order to determine the variance. The first moment of the rescaled wealth is, by definition, $\langle y \rangle = 1$.

To find the dynamic of the second moment, we start with the SDE for the rescaled wealth,

$$dy = \sigma y\, dW - \tau(y-1)\, dt, \tag{222}$$ {eq:rescaledSDE}

and follow a now familiar procedure. We insert $f(y,t) = y^2$ into Itô's formula,

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial y} dy + \frac{1}{2}\frac{\partial^2 f}{\partial y^2} dy^2 \tag{223}$$

to obtain

$$d\left(y^2\right) = 2y dy + dy^2. \tag{224}$$ {eq:diff2}

We substitute (Eq. 224) for dy to get terms at orders dW, dt, dW^2, dt^2, and $dW dt$. The scaling of Brownian motion allows us to replace dW^2 by dt and we ignore terms at $o(dt)$. This yields

$$d\left(y^2\right) = 2\sigma y^2\, dW - \left(2\tau - \sigma^2\right) y^2\, dt + 2\tau y\, dt. \tag{225}$$

Taking expectations on both sides and noting that $\langle y \rangle = 1$ gives us an ordinary differential equation for the second moment:

$$\frac{d\langle y^2 \rangle}{dt} = -\left(2\tau - \sigma^2\right)\langle y^2 \rangle + 2\tau \tag{226}$$ {eq:avediff2}

with solution

$$\left\langle y\left(t\right)^2\right\rangle = \frac{2\tau}{2\tau-\sigma^2} + \left(\left\langle y\left(0\right)^2\right\rangle - \frac{2\tau}{2\tau-\sigma^2}\right) e^{-\left(2\tau-\sigma^2\right)t}. \qquad (227)$$ {eq:avediff3}

The variance $V\left(t\right) = \left\langle y\left(t\right)^2\right\rangle - 1$ therefore follows

$$V\left(t\right) = V_\infty + \left(V_0 - V_\infty\right) e^{-\left(2\tau-\sigma^2\right)t}\,, \qquad (228)$$ {eq:var1}

where V_0 is the initial variance and

$$V_\infty = \frac{2\tau}{2\tau-\sigma^2}\,. \qquad (229)$$ {eq:varinf}

V converges in time to the asymptote, V_∞, provided the exponential in (Eq. 228) is decaying. This can be expressed as a condition on τ

$$\tau > \frac{\sigma^2}{2}, \qquad (230)$$

which is the same as the condition we noted previously for the second moment to exist: $\zeta > k$ where $k = 2$.

Clearly, for negative values of τ the condition cannot be satisfied, and the variance (and inequality) of the wealth distribution will diverge. In the regime where the variance exists, $\tau > \sigma^2/2$, it also follows from (Eq. 228) that the convergence time of the variance is $1/\left(2\tau-\sigma^2\right)$.

As τ increases, increasingly high moments of the distribution become convergent to some finite value. The above procedure for finding the second moment (and thereby the variance) can be applied to the k^{th} moment, just by changing the second power y^2 to y^k, and any other cumulant can therefore be found as a combination of the relevant moments. For instance, [25] also compute the third cumulant.

3.2.7 Fitting United States wealth data

{section:RGBM_data}

We have introduced the RGBM model and understood its basic properties. It is a simple model of an interacting population of noisy multiplicative growers. We expect it to be more realistic than GBM ($\tau = 0$) because we know that in the real world people interact. In particular, large populations have over centuries developed public institutions and infrastructure, to which everyone contributes and from which everyone benefits. At first glance, therefore, we might expect to find that RGBM with positive τ fits real wealth data better than GBM or, indeed, RGBM with negative τ.

Additionally, if this is true and if associated convergence times are shorter than the timescales of policy changes, it would indicate that the ergodic hypothesis is warranted and a helpful modelling assumption. If not, then the hypothesis would be unjustified and could be acting as a serious constraint on models of economic inequality, generating misleading analyses and recommendations.

In [6] we fit the RGBM model to historical wealth data from the United States for the last hundred years. We won't include full technical description of this empirical analysis here. It would be too long and our main aim is to communicate the ideas we use to think about problems and build models. If you want to know more, please read the paper (and let us know what you think!) However, the results are interesting and, to us at least, a little shocking, so we include a brief summary.

The basic setup is to fix the values of μ, σ, and N in our RGBM model using data about, respectively, aggregate economic growth, stock market volatility, and population data; and then to find by numerical simulation of (Eq. 208) the time series of $\tau(t)$ values which best reproduces historical wealth shares in the United States. The wealth share, $S_q(t)$, is a type of inequality measure. It is defined as the proportion of total wealth owned by the richest fraction q of the population. So, for example, $S_{0.1} = 0.8$ means that the richest 10% of the population own 80% of the total wealth. Reproducing the historical wealth shares is one way of reproducing approximately the level of inequality in the wealth distribution, and the nice thing is that economists Emmanuel Saez and Gabriel Zucman have estimated around a century's worth of wealth shares for the United States [40].

Fitting the model to these data will address two main questions:

- Is the ergodic hypothesis valid for rescaled wealth in the United States? For it to be valid, fitted values of $\tau(t)$ must be robustly positive.

- If $\tau(t)$ is robustly positive, is convergence of of the distribution to its stable form fast enough for the distribution to be used as a representative of the empirical wealth distribution?

Note that we have relaxed the model slightly. τ is a fixed parameter in (Eq. 208) but we allow it to vary with time in our empirical analysis.

Figure 15 (top) shows the results of fitting the RGBM model to the wealth share data in [40]. There are large annual fluctuations in $\tau_q(t)$ (black line) but we are more interested in longer-term changes in reallocation driven by structural economic and political changes. To show these, we smooth the data by taking a central 10-year moving average, $\widetilde{\tau}_q(t)$ (red line), where the window is truncated at the ends of the time series. We also show the uncertainty in this moving average (red envelope).

To ensure the smoothing does not introduce artificial biases, we reverse the procedure and use $\widetilde{\tau}_q(t)$ to propagate the initially inverse gamma-distributed wealths and determine the wealth shares $S_q^{\text{model}}(t)$. The good agreement with $S_q^{\text{data}}(t)$ suggests that the smoothed $\widetilde{\tau}_q(t)$ is meaningful, see Figure 15 (bottom).

The "take home" is this: while the effective reallocation rate, $\widetilde{\tau}(t)$, was positive for most of the last hundred years, it was negative – *i.e.* reallocation from poorer to richer – from the mid-1980s onwards. Furthermore, even when $\tau(t)$ was positive, associated convergence times (estimated both by numerical simulation and by plugging fitted model parameters into the results of Sec. 3.2.6) were very long compared to the typical times between policy changes – from several decades to several centuries. This makes the answer to both our questions above a resounding "No."

What shocked us most when we first encountered these results was the existence of long periods with negative reallocation. We began the analysis imagining that GBM – *i.e.* $\tau = 0$, no interactions – was a really crazy model of the real world. After all, we personally pay our taxes and use public services, and the London Mathematical Laboratory is supported by generous charitable

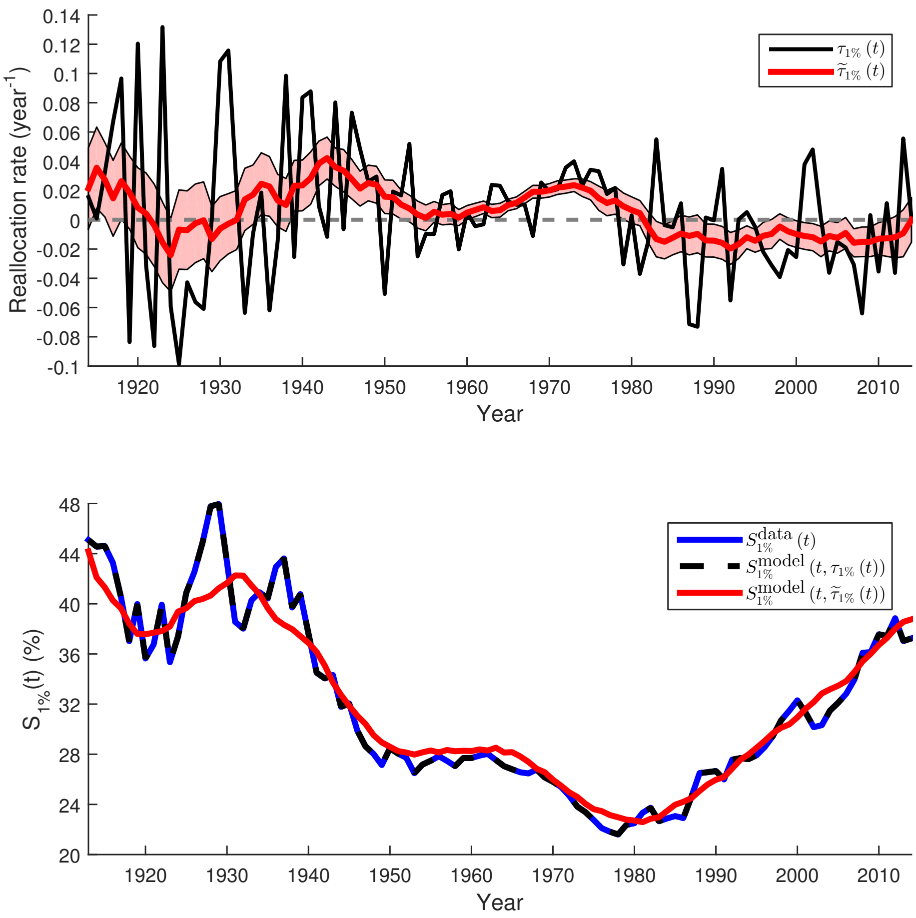

Figure 15: Fitted effective reallocation rates. Calculations done using $\mu = 0.021$ year^{-1} and $\sigma = 0.16$ year$^{-1/2}$. Top: $\tau_{1\%}(t)$ (black) and $\widetilde{\tau}_{1\%}(t)$ (red). Translucent envelopes indicate one standard error in the moving averages. Bottom: $S_{1\%}^{\text{data}}$ (blue), $S_{1\%}^{\text{model}}$ based on the annual $\tau_{1\%}(t)$ (dashed black), based on the 10-year moving average $\widetilde{\tau}_{1\%}(t)$ (red). {fig:tau}

reallocations! We imagined that we would see $\tau > 0$ but that it might be so small that convergence times would be too long for the ergodic hypothesis to be useful. Instead we found that, recently in the Unites States at least, reallocation has been consistently negative. In our model, this corresponds to wealths being driven apart, populations splitting into groups with positive and negative wealths, and no convergence to a stable distribution of rescaled wealth.

In retrospect, perhaps we shouldn't have been so surprised. Qualitatively, our results echo the findings that the rich are experiencing higher growth rates of their wealth than the poor [37, 50] and that the cumulative wealth of the poorest 50 percent of the American population was negative during 2008–2013 [39, 44].

The economic phenomena that trouble theorists most – such as diverging inequality, social immobility, and the emergence of negative wealth – are difficult to reproduce in models that make the ergodic hypothesis. In our simple model, this is easy to see: in the ergodic regime, $\tau > 0$, our model cannot reproduce these phenomena at all. One may be tempted to conclude that their existence is a sign of special conditions prevailing in the real world – collusion and conspiracies. But if we admit the possibility of non-ergodicity, $\tau \leq 0$, it becomes clear that these phenomena can easily emerge in an economy that does not actively guard against them.

Acronyms

BM Brownian Motion.

List of Symbols

C Cost to participate in a gamble.

d Differential operator in Leibnitz notation, infinitesimal.

D Possible payout for a gamble.

δt A time interval corresponding to the duration of one round of a gamble or, mathematically, the period over which a single realisation of the constituent random variable of a discrete-time stochastic process is generated..

δ Most frequently used to express a difference, for instance δx is a difference between two wealths x. It can be the Kronecker delta function, a function of two arguments with properties $\delta(i,j) = 1$ if $i = j$ and $\delta(i,j) = 0$ otherwise. It can also be the Dirac delta function of one argument, $\int f(x)\delta(x - x_0)dx = f(x_0)$.

Δ Difference operator, for instance Δv is a difference of two values of v, for instance observed at two different times.

Δt A general time interval..

F Insurance fee.

g Growth rate.

G Gain from one round trip of the ship.

$g_{\mathbf{est}}$ Growth rate estimator for finite time and finite ensemble size.

$g_{\mathbf{a}}$ Additive growth rate, *i.e.* rate of change.

$\overline{g_{\mathbf{a}}}$ Time-average additive growth rate, *i.e.* long-term rate of change.

$g_{\langle\rangle}$ Exponential growth rate of the expectation value.

$g_{\mathbf{m}}$ Multiplicative growth rate.

$\overline{g}$ Time-average exponential growth rate.

i Label for a particular realization of a random variable.

J Size of the jackpot.

k dummy.

L Insured loss.

m Index specifying a particular gamble.

μ Drift term in Brownian Motion (BM).

N Ensemble size, number of realizations.

$\mathcal{N}$ Normal distribution, $x \sim \mathcal{N}(\langle p \rangle, \text{var}(p))$ means that the variable p is normally distributed with mean $\langle p \rangle$ and variance $\text{var}(p)$..

p Probability, p_i is the probability of observing event i in a realization of a random variable.

$\mathcal{P}$ Probability density function.

r Random factor whereby wealth changes in one round of a gamble.

σ Magnitude of noise in a Brownian motion.

t Time.

T Number of sequential iterations of a gamble, so that $T\delta t$is the total duration of a repeated gamble..

t_0 Specific value of time t, usually the starting time of a gamble..

τ Dummy variable indicating a specific round in a gamble.

u Utility function.

v Stationarity mapping function, so that $v(x)$ has stationary increments.

var Variance.

W Wiener process, $W(t) = \int_0^t dW$ is continuous and $W(t) \sim \mathcal{N}(0, \overline{g})$.

x Wealth.

References

[1] A. Adamou and O. Peters. Dynamics of inequality. *Significance*, 13(3):32–35, 2016.

[2] J. Aitchison and J. A. C. Brown. *The lognormal distribution*. Cambridge University Press, 1957.

[3] A. B. Atkinson. The timescale of economic models: How long is the long run? *The Review of Economic Studies*, 36(2):137–152, 1969.

[4] J. Benhabib, A. Bisin, and S. Zhu. The distribution of wealth and fiscal policy in economies with finitely lived agents. *Econometrica*, 79(1):123–157, 2011.

[5] Y. Berman. Understanding the mechanical relationship between inequality and intergenerational mobility. Available at: http://papers.ssrn.com/abstract=2796563, 2017.

[6] Y. Berman, O. Peters, and A. Adamou. An empirical test of the ergodic hypothesis: Wealth distributions in the United States. January 2017.

[7] D. Bernoulli. Specimen Theoriae Novae de Mensura Sortis. Translation "Exposition of a new theory on the measurement of risk" by L. Sommer (1954). *Econometrica*, 22(1):23–36, 1738.

[8] J.-P. Bouchaud. Note on mean-field wealth models and the random energy model.

[9] J.-P. Bouchaud. On growth-optimal tax rates and the issue of wealth inequalities. *http://arXiv.org/abs/1508.00275*, August 2015.

[10] J.-P. Bouchaud and M. Mézard. Wealth condensation in a simple model of economy. *Physica A*, 282(4):536–545, 2000.

[11] J. Y. Campbell. *Financial decisions and markets, A course in asset pricing.* Princeton University Press, 2017.

[12] M. Corak. Income inequality, equality of opportunity, and intergenerational mobility. *The Journal of Economic Perspectives*, 27(3):79–102, 2013.

[13] T. M. Cover and J. A. Thomas. *Elements of Information Theory.* John Wiley & Sons, 1991.

[14] J. C. Cox, J. E. Ingersoll, and S. A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica*, 53(2):363–384, 1985.

[15] B. Derrida. Random-energy model: Limit of a family of disordered models. *Phys. Rev. Lett.*, 45(2):79–82, July 1980.

[16] B. Derrida. Random-energy model: An exactly solvable model of disordered systems. *Phys. Rev. B*, 24:2613–2626, September 1981.

[17] A. Drăgulescu and V. M. Yakovenko. Exponential and power-law probability distributions of wealth and income in the united kingdom and the united states. *Physica A: Statistical Mechanics and its Applications*, 299(1):213–221, 2001.

[18] H. Hinrichsen. Non-equilibrium critical phenomena and phase transitions into absorbing states. *Adv. Phys.*, 49(7):815–958, 2000.

[19] K. Itô. Stochastic integral. *Proc. Imperial Acad. Tokyo*, 20:519–524, 1944.

[20] R. Kaas, M. Goovaerts, J. Dhaene, and M. Denuit. *Modern Actuarial Risk Theory.* Springer, 2 edition, 2008.

[21] J. L. Kelly Jr. A new interpretation of information rate. *Bell Sys. Tech. J.*, 35(4):917–926, July 1956.

[22] P. E. Kloeden and E. Platen. *Numerical solution of stochastic differential equations*, volume 23. Springer Science & Business Media, 1992.

[23] P. S. Laplace. *Théorie analytique des probabilités.* Paris, Ve. Courcier, 2 edition, 1814.

[24] K. Liu, N. Lubbers, W. Klein, J. Tobochnik, B. Boghosian, and H. Gould. The effect of growth on equality in models of the economy. *arXiv*, 2013.

[25] Z. Liu and R. A. Serota. Correlation and relaxation times for a stochastic process with a fat-tailed steady-state distribution. *Physica A: Statistical Mechanics and its Applications*, 474:301–311, 2017.

[26] J. Marro and R. Dickman. *Nonequilibrium Phase Transitions in Lattice Models.* Cambridge University Press, 1999.

[27] J. E. Meade. *Efficiency, Equality and The Ownership of Property.* Allen & Unwin, London, UK, 1964.

[28] K. Menger. Das Unsicherheitsmoment in der Wertlehre. *J. Econ.*, 5(4):459–485, 1934.

[29] P. R. Montmort. *Essay d'analyse sur les jeux de hazard.* Jacque Quillau, Paris. Reprinted by the American Mathematical Society, 2006, 2 edition, 1713.

[30] H. Morowitz. *Beginnings of cellular life.* Yale University Press, 1992.

[31] M. E. J. Newman. Power laws, Pareto distributions and Zipf's law. *Contemp. Phys.*, 46(5):323–351, 2005.

[32] V. Pareto. *Cours d'économie Politique.* F. Rouge, Lausanne, Switzerland, 1897.

[33] O. Peters. The time resolution of the St Petersburg paradox. *Phil. Trans. R. Soc. A*, 369(1956):4913–4931, December 2011.

[34] O. Peters and A. Adamou. Rational insurance with linear utility and perfect informatio n. *arXiv:1507.04655*, July 2015.

[35] O. Peters and M. Gell-Mann. Evaluating gambles using dynamics. *Chaos*, 26:23103, February 2016.

[36] O. Peters and W. Klein. Ergodicity breaking in geometric Brownian motion. *Phys. Rev. Lett.*, 110(10):100603, March 2013.

[37] T. Piketty. *Capital in the twenty-first century.* Harvard University Press, 2014.

[38] S. Redner. Random multiplicative processes: An elementary tutorial. *Am. J. Phys.*, 58(3):267–273, March 1990.

[39] J.-V. Rios-Rull and M. Kuhn. 2013 Update on the U.S. Earnings, Income, and Wealth Distributional Facts: A View from Macroeconomics. *Quarterly Review*, (April):1–75, 2016.

[40] E. Saez and G. Zucman. Wealth inequality in the united states since 1913: Evidence from capitalized income tax data. Technical report, National bureau of economic research, 2014.

[41] P. Samuelson. What classical and neoclassical monetary theory really was. *Canad. J. Econ.*, 1(1):1–15, February 1968.

[42] P. Samuelson. St. Petersburg paradoxes: Defanged, dissected, and historically described. *J. Econ. Lit.*, 15(1):24–55, 1977.

[43] A. Sen. *On Economic Inequality*. Oxford: Clarendon Press, 1997.

[44] The World Wealth and Income Database. Usa top 10% and 1% and bottom 50% wealth shares, 1913–2014. http://wid.world/data/, 2016. Accessed: 12/26/2016.

[45] H. Theil. *Economics and information theory*. North-Holland Publishing Company, 1967.

[46] G. E. Uhlenbeck and L. S. Ornstein. On the theory of the brownian motion. *Physical Review*, 36(5):823–841, Sep 1930.

[47] O. Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, 1977.

[48] J. von Neumann and O. Morgenstern. *Theory of games and economic behavior*. Princeton University Press, 1944.

[49] W. A. Whitworth. *Choice and chance*. Deighton Bell, 2 edition, 1870.

[50] E. N. Wolff. Household wealth trends in the united states, 1983–2010. *Oxford Review of Economic Policy*, 30(1):21–43, 2014.