Lecture 4: Markets

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Agenda

1 Introduction

2 Optimal leverage

3 Stochastic market efficiency

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Apply our decision theory based on time-average growth rate maximisation to markets.

Based on two recent papers (open access):

- O. Peters. *Optimal leverage from non-ergodicity.* Quant. Fin. 11 (11), 1593–1602, 2011.
- O. Peters and A. Adamou. *Stochastic market efficiency.* Santa Fe Institute working paper 13-06-022, 2013.

1. Portfolio selection problem in simple market of two assets:
   - risky asset, like a stock;
   - riskless asset, like a bank deposit.

How should investor allocate money between these assets (the leverage problem)?

Classical approach $\rightarrow$ need to know investor risk preferences.

Our approach: maximise $\bar{g}_{\mathsf{m}} \rightarrow$ unambiguous optimal leverage.

2. Consider stability and efficiency of markets.

Does existence of objective optimal leverage constrain market prices and fluctuations?

Answer: yes, otherwise markets unstable and easy to beat.

Develop an accountable scientific theory:

- Quantify constraint on stochastic properties of market;
- Make prediction and account for imprecision;
- Test against real market data.

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Consider assets whose values follow multiplicative dynamics, which we model using GBM:

$$dx = x(\mu dt + \sigma dW).$$

μ is the drift and σ is the volatility.

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Simple model market of two assets.

Riskless, *e.g.* bank deposit: $\mu = \mu_r$; $\sigma = 0$.

Investment of x_0 grows deterministically:

$$dx_0 = x_0 \mu_r dt.$$

Future value of x_0 is known with certainty:

$$x_0(t_0 + \Delta t) = x_0(t_0) \exp(\mu_r \Delta t).$$

μ_r is the riskless drift.

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Simple model market of two assets.

Risky, *e.g.* stock: $\mu = \mu_{\mathsf{s}}$; $\sigma = \sigma_{\mathsf{s}}$.

Investment of x_1 grows noisily:

$$dx_1 = x_1(\mu_{\mathsf{s}}dt + \sigma_{\mathsf{s}}dW).$$

Future value of x_1 is a random variable:

$$x_1(t_0 + \Delta t) = x_1(t_0)\exp\left[\left(\mu_{\mathsf{s}} - \frac{\sigma_{\mathsf{s}}^2}{2}\right)\Delta t + \sigma_{\mathsf{s}}W(\Delta t)\right].$$

$\mu_{\mathsf{s}} > \mu_{\mathsf{r}}$ is the risky drift, $\sigma_{\mathsf{s}} > 0$ is the volatility.

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The risky asset has an excess drift over the riskless asset:

$$\mu_e = \mu_s - \mu_r.$$

μ_e is known in finance as:

- excess return;
- risk premium;
- equity premium (in context of stock markets).

We can view μ_e as compensation for accepting an uncertain outcome rather than a guaranteed result.

How to allocate investment between the two assets?

Simple portfolio setup:

- Total value x_ℓ;
- ℓx_ℓ invested in stock;
- $(1 - \ell)x_\ell$ deposited in the bank.

ℓ is known as leverage.

$\ell = 0$ portfolio contains only bank deposits.

$\ell = 1$ portfolio contains only stock.

Not constrained to $0 \leq \ell \leq 1$.

Can make $\ell > 1$ by borrowing money from the bank (a negative deposit) to buy stock.

Can make $\ell < 0$ by borrowing stock, selling it, and putting the proceeds in the bank (known as short selling).

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Each component of the portfolio experiences same relative
fluctuations as the asset in which it was made:

$$dx_\ell = \underbrace{(1-\ell)x_\ell \frac{dx_0}{x_0}}_{\text{in bank}} + \underbrace{\ell x_\ell}_{\text{in stock}} \frac{dx_1}{x_1}.$$

We already know $\frac{dx_0}{x_0}$ and $\frac{dx_1}{x_1}$ from the two asset SDEs.

Substitute in to get the SDE for leveraged portfolio:

$$dx_\ell = x_\ell[(\mu_r + \ell\mu_e)dt + \ell\sigma_s dW].$$

Value of leveraged portfolio evolves according to:

$$x_\ell(t_0+\Delta t) = x_\ell(t_0) \exp\left[\left(\mu_r + \ell\mu_e - \frac{\ell^2 \sigma_s{}^2}{2}\right)\Delta t + \ell\sigma_s W(\Delta t)\right].$$

We chose our notation for the two basic assets carefully:

- when $\ell = 0$, x_ℓ evolves as x_0;
- when $\ell = 1$, x_ℓ evolves as x_1.

In our model ℓ is held constant over time.

Model portfolio must be rebalanced to ensure ratio of stock to total investment remains fixed.

Imagine stock $\downarrow$ and bank deposit $\uparrow$ over a time step.

Result: stock $< \ell x_\ell$; bank deposit $> (1 - \ell)x_\ell$.

To restore leverage to ℓ, withdraw from bank and buy stock.

In the SDE we imagine this happens continuously.

Simple model portfolio setup allows us to ask

Question:

What is the optimal value of ℓ?

Similar to the gamble problem, except we are now choosing from a continuum of gambles parametrised by ℓ.

Apply new decision theory $\rightarrow$ maximise time-average growth rate of investment under appropriate dynamic.

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First let's review classical treatment of problem.

Intuition: there is a trade-off between risk and reward.

In GBM model, we might link risk to σ and reward to μ.

Ideal investment has large μ and small σ, but we acknowledge that larger μ tends to come with larger σ.

Hence excess return of risky over riskless asset: $\mu_{\mathsf{e}} > 0$.

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Markowitz first tried to treat this problem rigorously 1952.

Portfolio with (σ_i, μ_i) is "efficient" if no rival portfolio with (σ_j, μ_j) satisfies at least one of the statements:

- $\mu_j > \mu_i$ and $\sigma_j \leq \sigma_i$;
- $\sigma_j < \sigma_i$ and $\mu_j \geq \mu_i$.
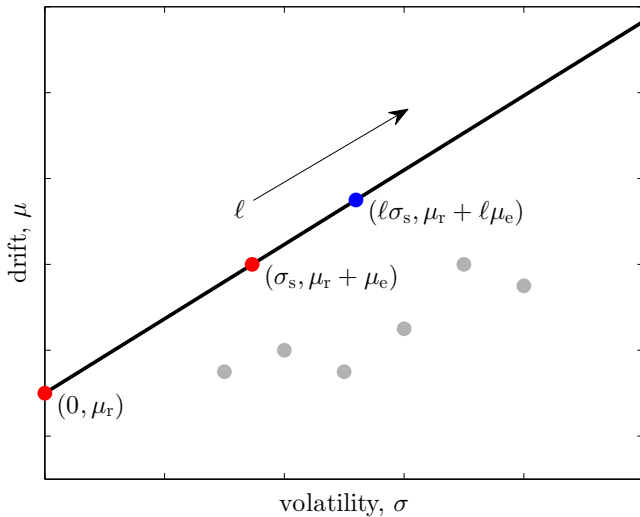
Advice: invest only in efficient portfolios.

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We compare portfolios with $(\ell\sigma_s, \mu_r + \ell\mu_e)$, which lie a on straight line in the (σ, μ)-plane:

$$\mu = \mu_r + \left(\frac{\mu_e}{\sigma_s}\right)\sigma.$$

In Markowitz's classification, all portfolios on this line are efficient wrt to each other (known as "efficient frontier").

All leverages equally investible $\rightarrow$ no answer to our question!

Classical approach requires additional information to distinguish between portfolios, *e.g.* investor risk preferences.

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Gradient of efficient frontier also known as the Sharpe ratio:

$$S = \frac{\mu_{\mathsf{e}}}{\sigma_{\mathsf{s}}}.$$

Shorthand for applying Markowitz's ideas: maximising S is equivalent to selecting an efficient portfolio.

S is insensitive to $\ell \to$ same non-advice as Markowitz approach.

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Expected value of stock grows as

$$\langle x_1(t_0 + \Delta t) \rangle = \langle x_1(t_0) \rangle \exp(\mu_s \Delta t),$$

with ensemble-average growth rate or "expected return":

$$g_m(\langle x_1 \rangle) = \frac{\Delta \ln \langle x_1 \rangle}{\Delta t} = \mu_s.$$

Follows that expected return of leveraged portfolio is:

$$g_m(\langle x_\ell \rangle) = \mu_r + \ell \mu_e.$$

Portfolio theory insensitive to ℓ is potentially dangerous.

If any ℓ is ok, then investor maximising expected return will
seek $\ell \to \infty$, which will surely ruin him.

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We now understand the classical approach and its limitations.

What does our new decision theory have to say?

Time-average growth rate of leveraged portfolio is

$$\bar{g}_{\mathsf{m}}(\ell) \equiv \lim_{\Delta t \to \infty} \{g_{\mathsf{m}}(x_\ell, \Delta t)\} = \lim_{\Delta t \to \infty} \left\{ \frac{\Delta \ln x_\ell}{\Delta t} \right\}.$$

Depends on ℓ. Inserting solution for $x_\ell(t_0 + \Delta t)$ gives

$$\bar{g}_{\mathsf{m}}(\ell) = \lim_{\Delta t \to \infty} \left\{ \frac{1}{\Delta t} \left[\left(\mu_{\mathsf{r}} + \ell\mu_{\mathsf{e}} - \frac{\ell^2 \sigma_{\mathsf{s}}^2}{2} \right) \Delta t + \ell\sigma_{\mathsf{s}} W(\Delta t) \right] \right\}.$$

$$\bar{g}_{\mathsf{m}}(\ell) = \mu_{\mathsf{r}} + \ell\mu_{\mathsf{e}} - \frac{\ell^2 \sigma_{\mathsf{s}}^2}{2}.$$

$\bar{g}_{\mathsf{m}}(\ell)$ is a quadratic $\rightarrow$ unambiguous maximum at:

$$\boxed{\ell_{\mathsf{opt}} = \frac{\mu_{\mathsf{e}}}{\sigma_{\mathsf{s}}{}^2}.}$$

ℓ_{opt} is optimal leverage which defines portfolio with fastest long-run growth $\rightarrow$ single point on the efficient frontier.

Only need to know μ_{r}, μ_{s} and σ_{s} for the market assets.

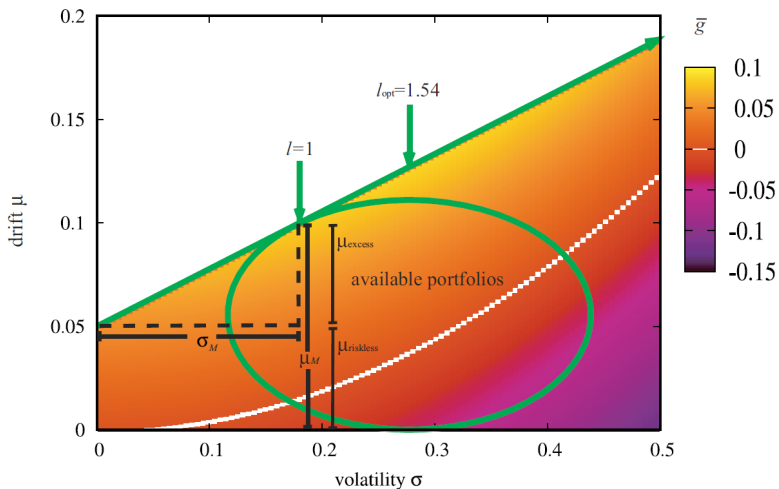No investor preferences (except that he maximises $\bar{g}_{\mathsf{m}}$).

$\bar{g}_{\mathsf{m}}(\ell)$ is time-average growth rate along the efficient frontier.

Time-average growth rate is $\mu - \sigma^2/2$ for arbitrary point in (σ, μ)-plane $\rightarrow$ growth rate landscape.

Overlaying Markowitz picture on landscape gives missing information needed to distinguish portfolios.

Complete portfolio selection picture.

$\bar{g}_{\sf m}(\ell) < 0$ for $\ell < \ell^-$ and $\ell > \ell^+$, where

$$\ell^{\pm} = \ell_{\sf opt} \pm \sqrt{\ell_{\sf opt}{}^2 + \frac{2\mu_{\sf r}}{\sigma_{\sf s}{}^2}}$$

are the roots of $\bar{g}_{\sf m}(\ell) = 0$.

Investor maximising $|\ell|$ in either direction will cause his wealth to decay rapidly:

$$\bar{g}_{\sf m}(\ell) \to -\infty \quad \text{as} \quad |\ell| \to \infty.$$

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$S = \mu_e/\sigma_s$ is a dimensionful quantity: (time unit)$^{-1/2}$.

Value depends on time unit $\rightarrow$ arbitrary, *e.g.* $S = 5$ y$^{-1/2}$
≈ 0.26 d$^{-1/2}$. Same portfolio, different numbers.

Tells us nothing fundamental about system under study.

$\ell_{opt} = \mu_e/\sigma_s{}^2$ is a dimensionless quantity.

Value does not depend units and can carry fundamental
information about system.

View ℓ_{opt} as fundamental measure of portfolio quality.

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However, significance of ℓ_{opt} runs deeper than this.

Optimal allocation of investment between risky and riskless assets in model market $\rightarrow$ tells us about market conditions.

High ℓ_{opt} indicates environment in risk-taking is incentivised. Low/negative ℓ_{opt} indicates the converse.

Question:

Are there any special values of ℓ_{opt} which describe different market regimes, or to which markets are attracted?

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Assets in model market have static μ_r, μ_s, σ_s $\rightarrow$ static ℓ_{opt}.

Hard to explore question if ℓ_{opt} can't vary $\rightarrow$ need to relax model to make progress.

Real market assets don't follow GBM with static parameters.

In particular, real market conditions – represented in model by μ_r, μ_s, σ_s – change slowly (wrt to fluctuations).

Relax model constraint of static parameters and allow them to vary $\rightarrow$ allow ℓ_{opt} to change over time.

In general, an efficient system is one which is well optimised and can't be improved by simple actions.

Classical "efficient market hypothesis" treats markets as information processors (*e.g.* Hayek 1945, Fama 1965).

Claim: price of asset in efficient market accurately reflects all publicly available information.

Corollary: no market participant, without access to privileged information, can consistently beat the market simply by choosing prices at which he buys and sells.

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Consider a different efficiency involving leverage rather than price/information. Start with a thought experiment. . .

Imagine $\ell_{\mathsf{opt}} > 1$ in our model market:

$\rightarrow$ Simple strategy of borrowing money to buy extra stock will achieve faster long-run growth than full investment in stock.

$\rightarrow$ Trivial matter for us to beat the market.

Similarly, imagine $\ell_{\mathsf{opt}} < 1$: again easy to beat market by leaving some money in the bank (and, if $\ell_{\mathsf{opt}} < 0$, short selling).

Hard to call a market efficient if it's so easy to beat.

Suggests different notion of efficiency $\rightarrow$ stochastic efficiency.

Claim: impossible for participant to beat a stochastically efficient market simply by applying leverage.

Hypothesis: stochastic market efficiency (strong form)

Real markets self-organise such that

$$\boxed{\ell_{\mathsf{opt}} = 1}$$

is an attractive point for their stochastic properties (represented by μ_{r}, μ_{s}, σ_{s} in the relaxed model).

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Another approach: consider market stability and its dependence on ℓ_{opt}. Run another thought experiment. . .

Imagine $\ell_{opt} > 1$:

Objectively optimal $\rightarrow$ *everyone* in market should want to borrow money to buy stock.

But, if so, who will lend money and who will sell stock?

Imagine $\ell_{opt} < 0$:

Everyone should want to borrow stock and sell it for cash.

But who will lend stock and who will relinquish cash to buy it?

Neither situation globally stable: $0 \leq \ell_{\text{opt}} \leq 1$ is stable range.

Hard to imagine deviations persisting for long before trading changes market parameters to return ℓ_{opt} to stable value.

Hypothesis: stochastic market efficiency (weak form)

Real markets self-organise such that

$$0 \leq \ell_{\text{opt}} \leq 1$$

is an attractive range for their stochastic properties.

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Self-organisation of parameters occurs *via* trading activity.

Feedback loops dynamically adjust prices and fluctuations to pull ℓ_{opt} back to stability when it strays.

Detailed mechanisms can be found in Peters & Adamou 2013.

Strong hypothesis favoured over weak as expect $\ell = 0$ a weaker attractor than $\ell = 1$ (to avoid economic paralysis).

Hypothesis: stochastic market efficiency (refined)

On sufficiently long time scales, $\ell_{\text{opt}} = 1$ is a strong attractor for the stochastic properties of real markets. Deviations from this attractor over shorter time scales are likely to be confined to the range $0 \leq \ell_{\text{opt}} \leq 1$.

Test by simulating leveraged investments in a real stock market.

Treat U.S. S&P500 index as real-world equivalent of risky asset.

Treat bank deposits at historical interest rates as real-world
equivalent of riskless asset.

Historical period: August 1955 to May 2013.

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FRED

ECONOMIC DATA | ST. LOUIS FED

Use FRED time series:

- SP500 – daily closing prices of S&P500;
- DFF – effective federal funds rate;
- DPRIME – bank prime loan rate.

S&P500 $\rightarrow$ well diversified portfolio of large and successful companies $\rightarrow$ generous estimate of ℓ_{opt}.

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Simulate investment of constant leverage over time window:

- Assume net value (equity) of \$1 at start of window: $\$\ell$ in S&P500; $\$(1 - \ell)$ in bank;

- Update values of each component daily according to historical returns and rebalance portfolio;

- If equity falls below zero, investment is bankrupt and simulation stops;

- Proceed to last day of the window and record final equity.

Repeat simulation for different ℓ to find simulated optimal leverage, $\ell_{\text{opt,s}}$, which maximises final equity.

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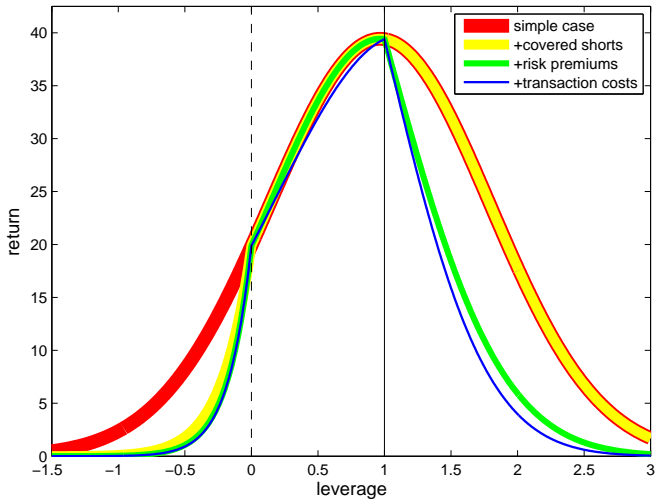
Simulated return-leverage curves correspond to different sets of assumptions about interest rates and transaction costs.

Labelled 1–4 in increasing complexity and resemblance to real market practices (call 1 "simple" and 4 "complex").

$\ell_{\mathsf{opt}} \approx 0.97$ for simulations 1–3.

$\ell_{\mathsf{opt}} = 1.00$ for simulation 4!

Results consistent with stochastic efficiency – discuss statistical significance shortly.
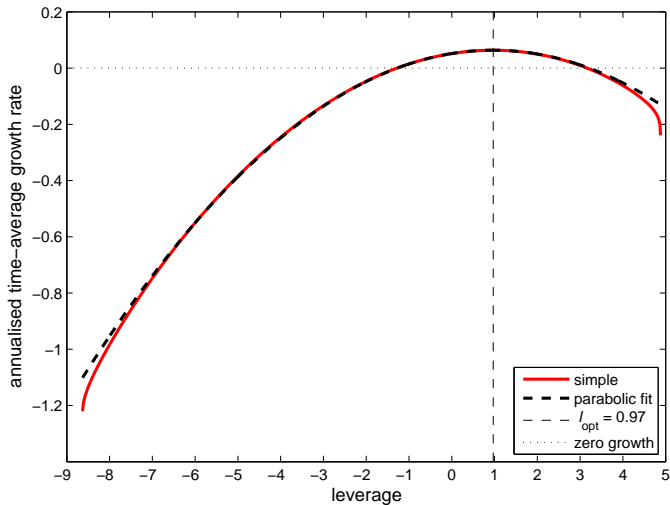
For $\Delta t = 58$ years, expect simulated growth rate $\approx \bar{g}_\mathsf{m}(\ell)$.

Parabolic form of GBM-based model clearly visible in plot.

Parameters of fitted parabola give effective values of μ_r, μ_e, σ_s for S&P500 over time series.

Least-squares fit:

- $\mu_\mathsf{r} = 5.2\%$ *pa*;
- $\mu_\mathsf{e} = 2.4\%$ *pa*;
- $\sigma_\mathsf{s} = 16\%$ $p\sqrt{a}$.

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Deviation from parabolic form for high and low leverages due to extreme fluctuations.

$\rightarrow$ Large losses and bankruptcy for high leverage portfolios.

Common criticism is that: real return distributions have fatter tails than in models.

Not relevant here: large deviations strengthen stochastic efficiency hypothesis because they penalise high leverage.

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Results relevant to unexplained phenomenon in economics: the "equity premium puzzle".

Claim: observed equity premium (μ_e in model) incompatible with theory as it implies investors implausibly risk averse.

6% *pa* established in literature *vs.* 2.4% *pa* in our model.

Source of discrepancy unclear because classical estimates based on complicated behavioural models.

No puzzle in our model: μ_e consistent with stochastic efficiency.

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Can't observe $\bar{g}_m(\ell)$ as this requires infinite observation time.

Can only observe finite-time growth rate $g_m(x_\ell, \Delta t)$ over simulation window Δt.

Random variable whose distribution broadens as $\Delta t \to 0$.

Also can't observe ℓ_{opt} but only simulated optimal leverage $\ell_{opt,s}(\Delta t)$ which maximises $g_m(x_\ell, \Delta t)$.

$g_m(x_\ell, \Delta t)$ worse estimator of $\bar{g}_m(\ell)$ as $\Delta t \to 0$, so $\ell_{opt,s}(\Delta t)$ worse estimator of ℓ_{opt}.

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When is a single observation of $\ell_{\text{opt,s}}(\Delta t)$ consistent with our hypothesis significant?

Depends on width of distribution of $\ell_{\text{opt}}(\Delta t)$:

- if width much larger than hypothesised stable range of ℓ_{opt}, then can't read much into one observation;
- if width similar to or smaller than stable range, then observation outside is strong evidence that hypothesis is flawed, so observation inside significant.

Quantify these ideas in original model:

$$g_{\mathsf{m}}(x_\ell, \Delta t) = \mu_{\mathsf{r}} + \ell\mu_{\mathsf{e}} - \frac{\ell^2\sigma_{\mathsf{s}}{}^2}{2} + \frac{\ell\sigma_{\mathsf{s}}W(\Delta t)}{\Delta t}.$$

$$\Rightarrow \quad \ell_{\mathsf{opt,s}}(\Delta t) = \ell_{\mathsf{opt}} + \frac{W(\Delta t)}{\sigma_{\mathsf{s}}\Delta t}.$$

Normally distributed with mean ℓ_{opt} and standard deviation:

$$\mathsf{stdev}[\ell_{\mathsf{opt,s}}(\Delta t)] = \frac{1}{\sigma_{\mathsf{s}}\sqrt{\Delta t}}.$$

For $\Delta t \approx 58$ years and fitted σ_{s}, get $\mathsf{stdev}[\ell_{\mathsf{opt,s}}(\Delta t)] \approx 0.83$.

So single observation is significant corroboration of hypothesis.

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Test model prediction by running simulations with different window lengths and compiling histograms of $\ell_{\text{opt,s}}(\Delta t)$:

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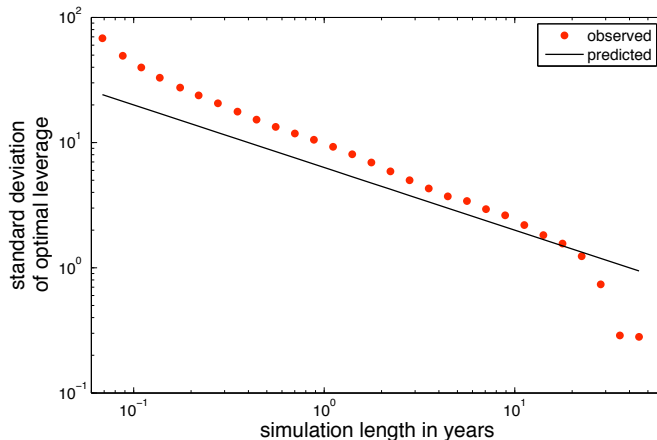
Run simulations starting on first day of time series with increasing window lengths.

Reduced fluctuations in $\ell_{\mathrm{opt,s}}(\Delta t)$ with increasing window length consistent with model.

Evolution of $\ell_{\mathrm{opt,s}}(\Delta t)$ supports hypothesis that ℓ_{opt} attracted to range $0 \leq \ell_{\mathrm{opt}} \leq 1$ and, over long time scales, to $\ell_{\mathrm{opt}} = 1$.

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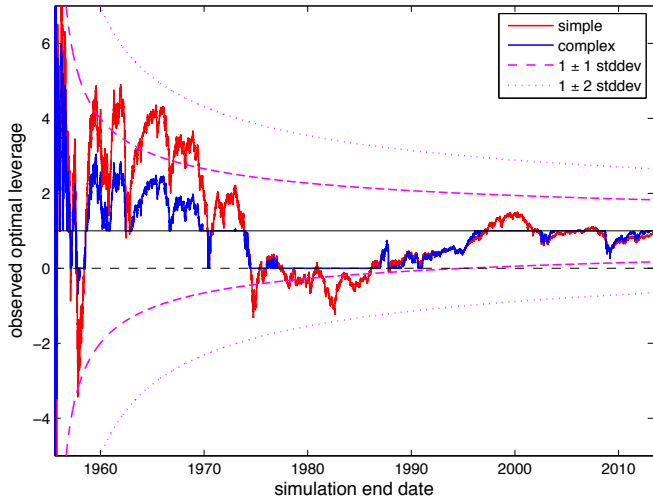
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Can also plot $\ell_{\mathrm{opt,s}}(\Delta t)$ for windows with fixed lengths and moving start date.

Run simple and complex simulations with window lengths ranging from 5 to 40 years. . .
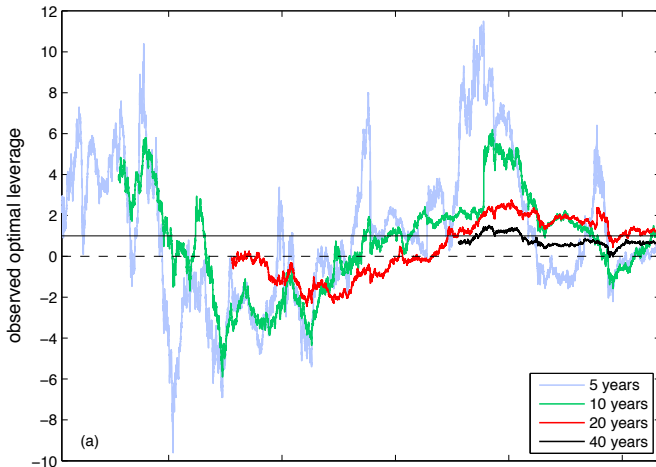
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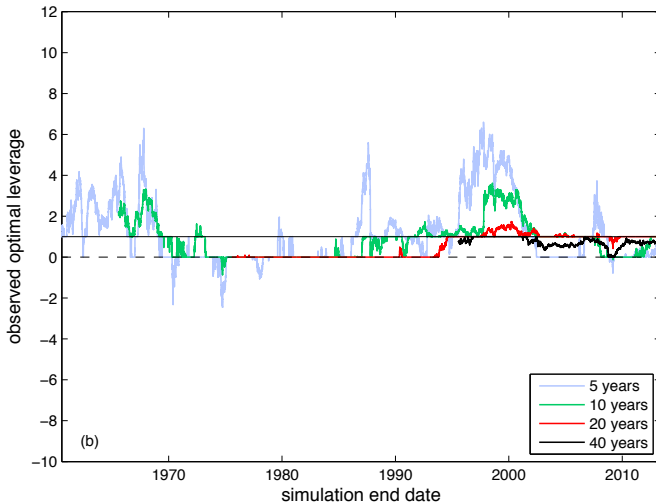Introduction

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Simple simulation:

(a)

Complex simulation:

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Can also plot $\ell_{\mathrm{opt,s}}(\Delta t)$ for windows with fixed lengths and moving start date.

Run simple and complex simulations with window lengths ranging from 5 to 40 years. . .

Observe:

- Strong fluctuations over short time scales;
- Weak fluctuations over long time scales;
- Attractive behaviour consistent with refined hypothesis.

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Mathematical formalism developed to conceptualise randomness in economics is very powerful.

Does more than just create a foundation for economic theory.

Framework is epistemologically sound $\rightarrow$ make testable predictions, *e.g.* stochastic efficiency.

Hard to guess from coin-tossing game that we could predict stochastic properties of U.S. stock market over 58 years!

What other surprising predictions can we make in this formalism?