Introducing *sspaneltvp*: a code to estimating state-space time-varying parameter models in panels. An application to Okun's law.[*]

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Abstract

This paper introduces a new code that provides researchers with a complete toolbox for estimating state-space time-varying parameter models. Our proposal extends the simple seminal framework into a panel-data one, combining both fixed (either common or country-specific) and varying components. Under specific conditions, this setting becomes a mean-reverting model, where the fixed mean parameter may include a deterministic trend. Regarding the transition equation, we allow for estimating different autoregressive alternatives and control instruments whose coefficients can be set up either common or idiosyncratic (this is particularly interesting for detecting asymmetries among individuals, i.e., countries, to common shocks). Furthermore, the GAUSS code allows for restrictions to the variances of both the transition and measurement equations. Finally, we illustrate our proposal with an empirical application to explore Okun's Law for a panel of EU peripheral countries during the period 1965-2021.

Keywords: State Space models, Kalman Filter, Time-varying parameters, Okun's law.
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1 Introduction

This paper presents a fully-fledged state-space framework for modeling panel time series and a dedicated code (`sspaneltvp`) to implement it. The code is written in Aptech Gauss, providing researchers with a complete toolbox to deal with different specifications of time-varying parameters in time series panels.

The rationale for time-varying parameter modeling has gained increasing importance at the theoretical and empirical levels. Time-varying parameters regressions are a type of regression analysis used in economics to model a variable that changes over time. These models allow regression parameters, such as the slope and intercept, to vary as a function of time. This type of analysis has several advantages over traditional regression models, which assume that the parameters are fixed over time.

One advantage of time-varying parameter (TVP hereafter) regressions is that they provide a more accurate representation of reality. In many economic situations, the relationship between variables is not constant. For example, the relationship between unemployment and growth may be different during periods of economic expansion compared to periods of recession. By allowing the regression parameters to vary over time, time-varying parameter regressions can capture the changes in the relationship between variables.

Another advantage of TVP regressions is that they can help identify structural breaks in the data. Structural breaks occur when there is a sudden change in the relationship between variables, such as a shift in the slope of a regression line. Traditional regression models are not well-equipped to identify these breaks, as they assume that the relationship between variables is constant over time. On the other hand, time-varying parameter regressions can identify these breaks and provide a more accurate representation of the data. Granger (2008) underlines that a time-varying parameter linear model can approximate any non-linear model.

Third, TVP regressions can provide more accurate forecasts. Traditional regression models assume that the relationship between the variables is constant over time and may not be well-suited for forecasting future values. TVP regressions, on the other hand, can take into account changes in the relationship between the variables over time and provide more accurate forecasts.

According to Hamilton (1994a) among others, the state-space representation provides a flexible framework for TVP modeling. In particular, the proposal presented here extends the most straightforward state-space representation of a single equation model, typically employed in the empirical literature, into a fully-fledged panel-data time-varying parameters framework.

To illustrate the potential use of the package, we present an application for the estimation of Okun's law, where time and cross-section dimensions may play a crucial role[1]. Okun's Law states that there is a relationship between the unemployment rate and the economic growth rate. Overall, the empirical evidence supports a relationship as described by Okun's Law. However, the strength of this relationship may vary across countries Blanchard and Wolfers (2000) and overtime Stock and Watson (1996). As a result, it is important for economists to carefully consider these potential differences when using Okun's Law in their analysis.

Our proposed state-space framework can be easily estimated using the so-called Kalman Filter (KF hereafter) algorithm (Kalman, 1960), which permits the estimation of the underlying states, in our case, the time-varying parameters, and also the hyper-parameters driving their evolution along time. We contribute to previous work in three directions: first, we develop a seminal GAUSS code written by Hamilton to fit a time-varying multiparameter model; second, we extend the KF estimation of a single-country model to panel data; third, we adapt the transition equation to include control inputs. Both fixed parameters in the measurement equation and control variables can be common or idiosyncratic (country-specific) parameters.

While some standard software packages used by econometricians, like Stata, RATS, MATLAB, or E-views, partially cover the topics included in our code[2], a complete set of functions is not embedded in one single code. Indeed, to the best of our knowledge, some recent theoretical contributions are missing in the available commercial econometric software. Our `sspaneltvp`

[1]Other applications using earlier versions of this code can be found in Camarero et al. (2021), Camarero et al. (2020), Paniagua et al. (2017a) and Paniagua et al. (2017b), where this methodology is applied to the velocity of money, fiscal and external sustainability, and the Feldstein-Horioka puzzle.

[2]TVP models can be estimated using either the Kalman filter (as it is our case) or by Bayesian techniques. We have found two complete packages following this second modeling approach. In Stata, the module `xtnptimevar` (Amadou, 2014) performs estimations of panel data models of nonparametric time-varying coefficients with fixed effects without imposing a specific functional form for the analysis of coefficients that vary over time for panel data models, following estimators proposed by Li et al. (2011). More recently, Casas et al. (2021) have created the R-package `tvReg` to estimate time-varying coefficients in multi equation regressions and available for download from CRAN. The six basic functions in this package cover the kernel estimation of semiparametric panel data, seemingly unrelated equations, vector autoregressive, impulse response, and linear regression models whose coefficients may vary with time or any random variable, following the Sun et al. (2009) approach, as implemented in Casas et al. (2021).

toolbox aims to fill this gap, allowing for greater flexibility in the formulations. In addition, in line with recent contributions to non-commercial statistical software, primarily in R, our toolbox includes hypotheses testing that is almost nonexistent in older alternatives. Finally, this paper aims to illustrate the potential use of the code with an example of Okun's Law estimation.

The remainder of the paper is organized as follows. Section 2 presents the model we will consider. Section 3 describes the `sspaneltvp` code. Section 4 contains the empirical analysis of Okun's Law, and section 5 concludes the paper. Finally, Section 6 includes instructions for installing `sspaneltvp`.

2 A state-space representation for time-varying parameter panel time series models

2.1 Time-varying parameter models in State-Space framework.

State-space representation of a linear system constitutes a statistical framework for modeling the dynamics of a $(n \times 1)$ vector of variables observed at regular time intervals t, y_t, in terms of a possibly unobserved (or state) $(r \times 1)$ vector ξ_t. [3].

The origin of state-space modeling is intimately linked with the Kalman filter (see Kalman, 1960), a recursive algorithm for generating minimum mean square error estimations and forecasts of the underlying states. The state-space representation consists of two equations. The measurement equation models the dynamics of the observable vector y_t, possibly measured with noise, which is assumed to be related to the state vector, ξ_t. It takes the following general form:

$$\underset{(n\times 1)}{y_t} = \underset{(n\times k)}{\mathbf{A}^\top}\underset{(k\times 1)}{x_t} + \underset{(n\times r)}{\mathbf{H}^\top}\underset{(r\times 1)}{\xi_t} + \underset{(n\times 1)}{w_t} \tag{2.1}$$

In (2.1), $y_t \in \mathbb{R}^n$ represents an $(n \times 1)$ vector of variables observed at date t, $x_t \in \mathbb{R}^{k\times n}$ represents a $(k \times n)$ vector of exogenous determinants, their coefficients being included in the $(k \times n)$ matrix A. H is an $(r \times n)$ matrix of coefficients for the $(r \times 1)$ vector of unobserved components ξ_t.

[3]Excellent textbook treatments of state-space models are provided in Harvey (1993, 1989), Hamilton (1994b,a), West and Harrison (1997), or Kim and Nelson (1999), among others. Authors use different conventions, but the notation used here is based on James Hamilton's, with slight variations.

Finally, the measurement or observational error, w_t, is an (n x 1) vector where its components are distributed $N(0, \sigma^2_{w_i})$. w_t is assumed to be i.i.d., independent of the unobserved vector ξ_t and its disturbances vector ν_t and for $t = 1, 2, ...$ so that

$$E(w_{i,t} w_{i,\tau}^{\top}) = \begin{cases} \sigma^2_{w_i} & for \quad t = \tau \\ 0 \quad for \quad t \neq \tau \end{cases} \tag{2.2}$$

and the variance-covariance matrix is diagonal,

$$\underset{(n \times n)}{R} = \begin{bmatrix} \sigma^2_{w_1} & 0 & \cdots & 0 \\ 0 & \sigma^2_{w_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma^2_{w_n} \end{bmatrix} \tag{2.3}$$

The evolution of the underlying unobserved states that determine the time-series behaviour in (2.1) is described by the state transition equation, generated by a linear stochastic difference representation through a first-order Markov process, such as :

$$\underset{(r \times 1)}{\xi_{t+1}} = \underset{(r \times r)}{F} \underset{(r \times 1)}{\xi_t} + \underset{(r \times s)}{B} \underset{(s \times 1)}{Z_{t+1}} + \underset{(r \times 1)}{\nu_{t+1}} \tag{2.4}$$

In (2.4), F denotes an $(r \times r)$ state-transition matrix, which applies the effect of each system state parameter at time $t-1$ to the system state at time t, ξ_t. Some specifications of the state transition equation also include a vector containing control inputs, denoted Z_t with dimension $(s \times 1)$, either deterministic (a drift and/or a deterministic trend) or stochastic variables. When present, control inputs affect the state through the $(r \times s)$ control input matrix, B, which applies the effect of each control input parameter in the vector to the state vector.

Control inputs are frequently employed in the literature on control engineering. The intuition behind refers to simulating the effect of changes in a control variable on a system, namely the state vector[4]. But, despite their potential uses, stochastic control inputs haven't been employed very often in empirical economic research, as most of the models include simple state-transition

[4]A very simple example proposed by Faragher (2012) is what happens to the trajectory of a rocket when fuel injection is activated during flight.

equations where the unobserved vector follows a random walk with noise.

Finally, ν_t represents the $(r \times 1)$ vector of serially uncorrelated disturbances containing the process noise terms for each parameter, $\nu_{i,kv,t}$, assumed to be distributed $N(0, \sigma^2_{\nu_{i,kv}})$. ν_t is assumed to be i.i.d., so the variance-covariance matrix is also diagonal, so that

$$E(\nu_{i,kv,t+1}\nu^{\top}_{i,kv\tau+1}) = \begin{cases} \sigma^2_{\nu_{i,kv}} & for \quad t = \tau \\ 0 & for \quad t \neq \tau \end{cases} \tag{2.5}$$

and

$$\underset{(r\times r)}{Q} = \begin{bmatrix} \sigma^2_{\nu_{1,1}} & 0 & \cdots & 0 \\ 0 & \sigma^2_{\nu_{1,2}} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma^2_{\nu_{n,kvmax}} \end{bmatrix} \tag{2.6}$$

Once both the measurement and transition equations have been specified, one needs to set-up the system by assuming initial conditions on the state vector, as in (2.7):

$$\xi_1 \sim N\left(\xi_{1|0}, P_{1|0}\right). \tag{2.7}$$

Writing a model in state-space form means imposing certain values (such as zero or one) on some of the elements of its matrices: F, Q, B, A, H and R, and interpreting the other elements as particular parameters of interest. Typically we will not know the values of these other elements, but need to estimate them based on the observation of $(y_1, y_2, ..., y_T)$ and $(x_1, x_2, ..., x_T)$. In its basic form, the model assumes that the values of F, (and even B), Q, A, H and R are all fixed and known, but (some of them) could be functions of time.

State-Space representation of dynamic models is particularly useful for measuring expectations that cannot be observed directly. If these expectations are formed rationally, there are certain implications for the time-series behaviour of the observed series that can help to model them. According to Harvey (1989), it does not exist a unique representation of a state-space formulation of a model. That is why the state variables obtained internally in the system have to be specified according to the nature of the problem with the ultimate goal of containing all the information necessary to determine the behaviour of the period-to-period system with the minimum number of parameters. Time-varying parameters regression models constitute an interesting application of the state-space representation.

In this case, the measurement equation can be written as follows:

$$\underset{(n\times 1)}{y_t} = \underset{(n\times k)}{\mathbf{A}^\top} \times \underset{(k\times 1)}{x_t} + \underset{(n\times r)}{\mathbf{H}^\top(x_t)} \times \underset{(r\times 1)}{\xi_t} + \underset{(n\times 1)}{w_t} \tag{2.8}$$

where A represents a matrix of fixed parameters $\bar{\beta}$.

Compared to the general model where the elements of the matrices F, Q, A, H and R are treated as constants, in this model H depends on the observed regressors, as $[H(x_t)]^\top = x_t$.

The vector of unobserved coefficients, ξ_t, evolves along time according to the expression:

$$\underset{(r\times 1)}{\xi_{t+1}} = \underset{(r\times r)}{F} \cdot \underset{(r\times 1)}{\xi_t} + \underset{(r\times s)}{B} \cdot \underset{(s\times 1)}{Z_t} + \upsilon_{t+1} \tag{2.9}$$

As stated in Hamilton (1994a), assuming that the eigenvalues of F in (2.4) are all inside the unit circle, the elements of the vector of unobserved coefficients, ξ_t, can be interpreted as the average or steady-state coefficient vector,

$$y_t = x_t^\top \bar{\beta} + x_t^\top \xi_t + \omega_t \tag{2.10}$$

where the vector of unobserved coefficients, $\xi_t = \left(\beta_t - \bar{\beta}\right)$, evolves along time according to the expression:

$$\underset{(r\times 1)}{\left(\beta_{t+1} - \bar{\beta}\right)} = \underset{(r\times r)}{F} \cdot \underset{(r\times 1)}{\left(\beta_t - \bar{\beta}\right)} + \underset{(r\times s)}{B}\underset{(s\times 1)}{Z_t} + \upsilon_{t+1} \tag{2.11}$$

Equation (2.11) represents a simple transition equation to be estimated through the Kalman Filter, where $\left(\beta_t - \bar{\beta}\right) = \xi_t$ is the unobserved component of our time-varying parameter, while the fixed component is also included at the measurement equation as $\bar{\beta}$.

2.2 A Panel Time-Varying State-Space extension

In this subsection, we extend the previous time-varying parameters model to a panel setting. Our main goal is to explore the use of the state-space modelling and the Kalman filter algorithm as an effective method for combining time series in a panel. This flexible structure allows the model specification to be affected by different potential sources of cross-section heterogeneity. This approach can be a superior alternative to the estimation of the model in unstacked form, commonly employed when there is a small number of cross-sections.

The general model can be written as follows:

$$y_{i,t} = x_{i,t}^{\top}\bar{\beta} + x_{i,t}^{\top}\xi_{i,t} + \omega_t \tag{2.12}$$

or in matrix form:

$$\underset{(n\times t)}{y} = \underset{(n\times n*k)}{\mathbf{A}^{\top}} \times \underset{(n*k\times t)}{x} + \underset{(n\times r)}{\mathbf{H}^{\top}(x)} \times \underset{(r\times t)}{\xi} + \underset{(n\times t)}{w} \tag{2.13}$$

representing the measurement equation for a $y_t \in \mathbb{R}^n$ vector containing the dependent variable for a panel of countries. $x_t \in \mathbb{R}^{k\times n}$ is a vector of k exogenous variables, including either (or both) stochastic and deterministic components. The unobserved vector $\xi_t \in \mathbb{R}^r$ influences the dependent variable through a varying $H^{\top}(x_t)$ $(n \times r)$ matrix, whose simplest form is $H^{\top}(x_t) = x_t$. Finally, $w_t \in \mathbb{R}^n$ represents the $(n \times 1)$ vector of N measurement errors.

The specification of the model in Equation (2.13) relies on a Mean-Reverting-type modelization of the measurement equation, which also allows for the inclusion of fixed parameters in matrix $\mathbf{A}$. Each of the fixed parameters can be modeled, either as a common parameter for all the agents in the panel, $\bar{\beta}$, or, alternatively, as a country-specific coefficient, $\bar{\beta}_i$. The model also includes time-varying parameters (ξ_t) for some of the regressors, that eventually can be interpreted as deviations from the mean parameters ($\left(\beta_{it} - \bar{\beta}_i\right) = \xi_t$).

The measurement equation for each i-th element in the t-th period ($y_{i,t}$) in the vector of the dependent variable, can be expressed as follows:

$$
\begin{aligned}
y_{i,t} = & \sum_{ks=ksmin}^{ksmax} \bar{\beta}_{ks,i} x_{ks,i,t} \quad + \sum_{kc=kcmin}^{kcmax} \bar{\beta}_{kc} x_{kc,i,t} \\
& + \sum_{kv=kvmin}^{kvmax} \xi_{kv,it} x_{kv,i,t} \quad + \quad h_i \xi_{r,it} + \underset{(n \times 1)}{w_{it}}
\end{aligned}
\tag{2.14}
$$

In Equation (2.14), the measurement equation for each of the individuals (countries) included in the panel allows for the potential inclusion of both fixed (mean) and/or time-varying parameters for the regressors included in the model to be estimated. For the fixed-parameters case, different partitions can be considered. First, one can choose a subset of $ksnum = ksmax - ksmin + 1$ regressors $(x_{ksmin}, ..., x_{ks}, ..., x_{ksmax})$ in the interval $[0, k]$, whose coefficients will be modeled as country-specific or idiosyncratic. A second subset of $kcnum = kcmax - kcmin + 1$ regressors $(x_{kcmin}, ..., x_{kc}, ..., x_{kcmax})$, also defined in the interval $[0, k]$, is possible, but related, in this case, to the dependent vector through a common coefficient for all the countries included in the panel. Last, the varying parameters are associated to a third subset of $kvnum = kvmax - kvmin + 1$ regressors $(x_{kvmin}, ..., x_{kv}, ..., x_{kvmax})$, also in the interval $[0, k]$.

Regressors with varying parameters can eventually enter the measurement equation with fixed parameters, if any of the following conditions hold:

$$
(x_{kvmin}, ..., x_{kv}, ..., x_{kvmax}) \bigcap (x_{ksmin}, ..., x_{ks}, ..., x_{ksmax}) \neq \varnothing \tag{2.15}
$$

$$
(x_{kvmin}, ..., x_{kv}, ..., x_{kvmax}) \bigcap (x_{kcmin}, ..., x_{kc}, ..., x_{kcmax}) \neq \varnothing \tag{2.16}
$$

Condition at equation (2.15) presents a combination of varying and fixed idiosyncratic parameters, while equation (2.16) presents the combination of varying parameters with common ones[5].

The panel specification presented in Equations (2.14) can also be enriched to allow for the potential inclusion, as in Broto and Pérez-Quirós (2015), of a dynamic common-factor in the measurement equation driving $y_{i,t}$, the dependent variable vector. This common factor is modeled simply as an additional unobserved state in the state-vector (whose number of rows increases to $n * kv + 1$) that enters each country's measurement equation with a country-specific loading

[5]Camarero et al. (2020) also consider the possibility of the fixed parameters being affected by a deterministic time trend, that is one of the options of the present version of the code.

parameter, h_i[6].

Finally, some restrictions can be imposed to the $(n \times n)$ variance-covariance matrix R of the measurement vector:

$$\underset{(n\times n)}{R} = \begin{bmatrix} \sigma^2_{w_1} & 0 & \cdots & 0 \\ 0 & \sigma^2_{w_2} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \sigma^2_{w_n} \end{bmatrix} \tag{2.17}$$

The general assumption implies uncorrelated errors with idiosyncratic variances; hence, it is a diagonal matrix with idiosyncratic components, as in equation (2.17). To reduce the number of hyperparameters, one could impose a common variance for the uncorrelated errors, $\sigma^2_{w_1} = \cdots = \sigma^2_{w_n} = \sigma^2_w$.

Regarding the state-transition Equation, the $\xi_t \in \mathbb{R}^r$ vector of unobserved state evolves according to the equation (2.18):

$$\underset{(r\times 1)}{\xi_{t+1}} = \underset{(r\times r)}{F} \underset{(r\times 1)}{\xi_t} + \underset{(r\times s)}{B} \underset{(s\times 1)}{Z_t} + \underset{(r\times 1)}{\nu_t} \tag{2.18}$$

In equation (2.18), the vector of unobserved components ξ_t follows an autoregressive process where F denotes an $(r \times r)$ state-transition matrix[7]:

$$\underset{(r\times r)}{F} = \begin{bmatrix} \phi_{1,1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \phi_{2,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & \vdots & \vdots \\ \vdots & 0 & 0 & \phi_{kv,1} & 0 & 0 \\ 0 & \vdots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \phi_{kv,n} \end{bmatrix} \tag{2.19}$$

$Z_{i,t}$ represents the vector containing any control inputs affecting the state through the control

[6]Note that our Gauss code allows for multiple common-factors as well as for the inclusion of potential restrictions on them.

[7]As suggested by one of the reviewers, this matrix could potentially have a more complex structure. However, the corresponding increase in the number of hyper-parameters would make convergence difficult.

input matrix B, which applies the effect of each control input on the state vector. These control variables are frequently employed in engineering but are not so commonly applied to state-space models in economics. Their use could be interpreted as the "coefficient-drivers" of the second-generation TVP models, described in Swamy and Tavlas (2003) and related work. As stated in Gourieroux and Monfort (1997), with the introduction of an input in the "transition equation" or in the "measurement equation", all the formulas of the filter remain valid except for the introduction of the variable in the update equation.

Finally, ν_t represents the $(r \times 1)$ vector containing the terms of the noise process for each parameter in the state vector, and it is assumed to be i.i.d.. To reduce the number of hyperparameters, one could impose restrictions on the variance for the uncorrelated errors, $\sigma^2_{\nu_{kv,i}}$, assuming a common value for the individuals in the panel, and/or for the noise related to the different unobserved varying parameters.

Each one of the $(r = kv \times n)$ first components of ξ_{t+1} are driven by the following expression:

$$
\begin{aligned}
\xi_{kv,i,t+1} = \\
\phi_{kv,i} \cdot \xi_{kv,i,t} \\
+ \sum_{js=jsmin}^{jsmax} \bar{\mu}_{js,i} \cdot z_{j1,i,t} \\
+ \sum_{jc=jcmin}^{jcmax} \bar{\mu}_{jc} \cdot z_{jc,i,t} \quad + \nu_{kv,i,t+1}
\end{aligned}
\tag{2.20}
$$

Equation (2.20) describes the autoregressive process followed by every unobserved component whose coefficients $\phi_{kv,i}$ are to be estimated. The state component is also influenced by the evolution of a vector containing s observed variables or control instruments, z_t.

Our framework allows us to define the parameter for each control instrument in the state transition equation, either as common or as specific for every individual considered.

Thus, a subset of $jsnum = jsmax - jsmin + 1$ control inputs $(z_{jsmin}, ..., z_{js}, ..., z_{jsmax})$ in the interval $[0, s]$, whose coefficients $\bar{\mu}_{js,i}$ are modelled as idiosyncratic.

Similarly, for a second subset of $jcnum = jcmax - jcmin + 1$ regressors $(z_{jcmin}, ..., z_{jc}, ..., z_{jcmax})$, also defined in the interval $[1, s]$, the coefficient $\bar{\mu}_{jc}$ is estimated as common for all the countries included in the panel. This specification is particularly helpful in detecting heterogeneity or asymmetries among the countries when estimating the impact of any particular variable.

If a dynamic common factor is introduced in the model, its transition equation can be conveniently restricted so that it would not require the inclusion of potentially idiosyncratic control variables.

In contrast to most of the literature, the autoregressive parameters in the state-transition equation are estimated, restricting their process to following a random walk. According to Hamilton (1994a), if the eigenvalues of matrix F remain inside the unit circle, then the system is stable and the vector process defined by (2.18) is stationary. In this case, the inclusion of fixed and varying parameters for the regressors can be interpreted as a mean-reverting model. The fixed parameter, either idiosyncratic ($\bar{\beta}_i$) or common ($\bar{\beta}$) should be interpreted as the average or steady-state coefficient vector, and the varying parameter as the deviation from this mean in a mean-reverting framework:

$$\xi_{kv,i,t} = \left(\beta_{kv,i,t} - \bar{\beta}_{kv,i}\right) \tag{2.21}$$

Equation (2.21) presents the case of a mean reverting parameter with a specific fixed mean for each individual.

$$\xi_{kv,i,t} = \left(\beta_{kv,i,t} - \bar{\beta}_{kv}\right) \tag{2.22}$$

In contrast, in Equation (2.22) the mean reverting parameter exhibits a common mean for all the individuals in the panel.

As for the measurement equation, the above specification of the transition equation can be adapted to introduce different restrictions, that can be tested using likelihood ratio tests, for example. Regarding the variances of the noise process for the varying components, they can be restricted to be zero or identical for all countries in the panel and/or regressors with varying parameters in the model. It can also be defined as a value for the signal-to-noise ratio (ratio between the two variances)[8].

[8] Increasing signal-to-noise ratio would also increase the weight of the observations in the correction equations of the Kalman filter.

A value for the signal-to-noise ratio (ratio between the two variances) can also be defined. Another set of restrictions has to deal with the autoregressive parameters in the diagonal of F, which can also be restricted to one (the random walk), or their value to be identical among all the countries in the panel, and/or regressors included in the model with varying parameters. Further restrictions can also be introduced on any of the components of the hyper-parameters vector and checked accordingly.

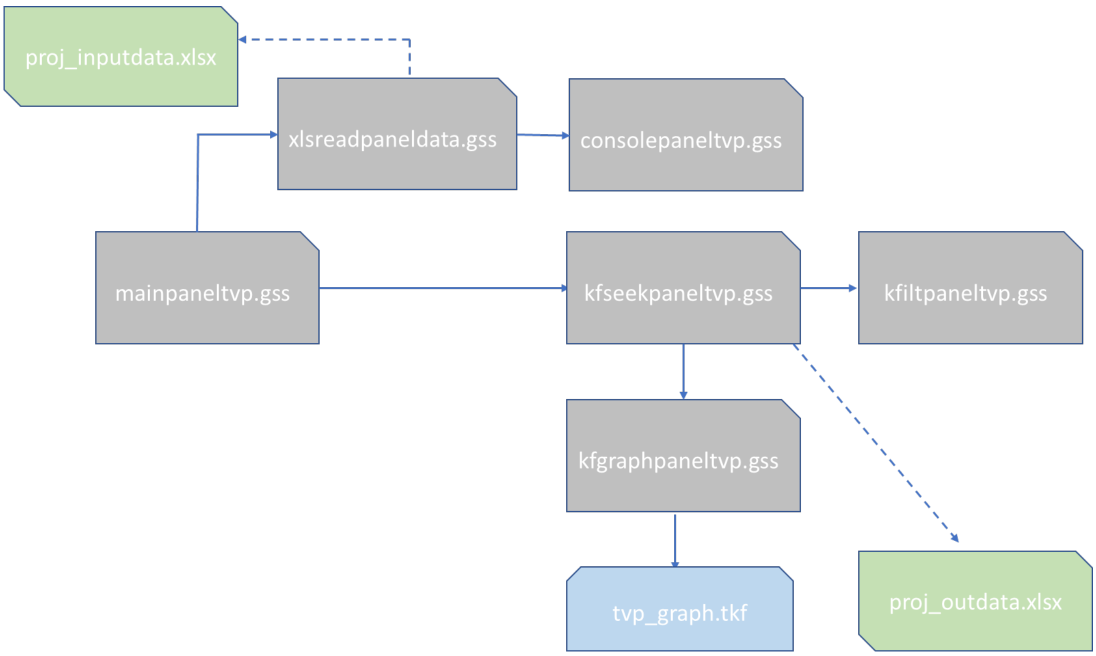
3 The `sspaneltvp` GAUSS code

The `sspaneltvp.gss` **GAUSS** code performs time-varying parameter estimation for panel time series models using the Kalman filter algorithm. The code includes the following files: `mainpaneltvp.gss`; `xlsreadpaneldata.gss`; `kfseekpaneltvp.gss`; `kfiltpaneltvp.gss`; `kfgrappaneltvp.gss`.

Additionally, some tools are embedded in the bundle: `datasplitnegpos.gss`; `genpurinputdata.gss`; `newvar.gss`.

The flowchart below shows the relationship among the files:

Figure 1: Code files in `sspaneltvp` (GAUSS)



3.1 The executable `mainpaneltvp.gss` file.

The executable file `mainpaneltvp.gss` is the main file, that needs to be adapted to each specific project. In what follows, we use Okun's law, applied to a group of five peripheral European countries, as an illustration.

Figure 2: `mainpaneltvp.gss`

```
1  ////////////////////////////////////////////////////////////////////////////////
2  //                PANELTVP                                                    //
3  // APTECH GAUSS code to perform calculation of Time-Varying-Parameter models  //
4  //                        for Time Series Panel data                          //
5  ////////////////////////////////////////////////////////////////////////////////
6
7  /*
8  As performed in:
9  Camarero, M., Sapena, J., & Tamarit, C. (2019).
10 Modelling Time-Varying Parameters in Panel Data State-Space Frameworks:
11 An Application to the Feldstein-Horioka Puzzle. Computational Economics, 1-28.
12 */
13
14 /*  This Draft : JULY 15, 2020 by Juan Sapena (juan.sapena@ucv.es)        */
15
16 new;
17
18
19 library optmum, pgraph;
20
21 graphset;
22
23 /* ====================================================================== */
24 /* Creates a short project identifier using the variable proj             */
25
26 proj="okun";
27
28 outfile = proj$+"_output"$+".txt";
29 output file = outfile reset;
30
31 inputdatafile = "/data/"$+proj$+"_data"$+".xlsx";
32
33 /*
34 cellid = xlsMakeRange(row, col);
35 */
36
37 /* ====================================================================== */
38 /* Replace this first section with a section to read in your data         */
39
```

We should just indicate the name of the project (line 26, "okun") and the output file (line 26). Then, `mainpaneltvp.gss` calls `xlsreadpaneldata.gss` that reads input data from an excel workbook located in subfolder "data", and named after the project as well (line 31).

3.2 Reading input data using xlsreadpaneldata.gss.

The main file `mainpaneltvp.gss` invokes `xlsreadpaneldata.gss`. This file doesn't need to be modified to read the excel workbook data file, if the latter is located and named following the format indicated in 2. `xlsreadpaneldata.gss` invokes the input data excel workbook following a straight-forward format[9].

Figure 3: Reading data

```
45
46 //////////////////////////////////////////
47 // THEN READ DATA FROM INPUT EXCEL FILE //
48 //////////////////////////////////////////
49
50 #include xlsreadpaneldata.gss;
51
52
53 ////////////////////////////////////////////////
54 // DEFINE THE VARIABLES TO BE POTENTIALLY USED//
55 ////////////////////////////////////////////////
56
57 /* GENERATE PERIODS AND INTERCEPT */
58
59 INTERCEPT = ones(capt,n);
60
61 tempo = period;
62
63 //ind = seqa(2000.25,1/12,capt); @ an alternative way  @
64
65 /* GENERATE A DETERMINISTIC TREND FOR THE PANEL */
66
67 trendcol = seqa(1,1,capt);
68 trend = trendcol + zeros(capt,n);
69
70   x0 = intercept;
71
72 //////////////////////////////////////////////////////////////
73
74 //#include newvar_okun.gss;
75
76 //////////////////////////////////////////////////////////////
77
78 OILG=LOG(OIL./OIL_LAG);
79 OIL_LOG=LOG(OIL);
80 OILLAG_LOG=LOG(OIL_LAG);
81
82 URATE_GAP = URATE-NAWRU;
83
84 RGDPG_VAR=RGDPG-RGDPG_LAG;
85
86
87 //////////////////////////////////////////////////////////////
88 // CODE TO SPLIT POSITIVE AND NEGATIVE VALUES OF A VARIABLE //
89 //////////////////////////////////////////////////////////////
90
91 // #include datasplitnegpos.gss
92
93 //////////////////////////
94 //  MODEL SPECIFICATION //
95 //////////////////////////
96
97 // HERE THE DEPENDENT VECTOR. Candidates: URATE_DIF ///
98
99     y = URATE_DIF;
100
```

If the variables have been already read (as it is our case), we can define them from line 54 onward as shown if Figure 3. This includes the time-period, the intercept (x_0 in line 70), as well as the other variables to include in the model that, in our example, are oil prices (line 78, with lags), unemployment gap “URATE GAP" (from NAIRU, in line 82) or the first difference of real

[9]In particular, the first sheet contains the index of variables included in files 2 to the last one; sheet number 2 and following contain the variables to be imported from `xlsreadpaneldata.gss`. In each of them, (1) cell $A1$ states the sheet number; (2) $A2$ contains the variable name; (3) row 3, starting from $B3$ includes country codes; (4) row 4, starting from $B4$ includes country names; (5) row A starting in cell $A5$ includes the time periods. The data is organised in rows (time) and columns (countries).

GDP (line 84). The dependent variable is called y in the code and it is the first difference of the unemployment rate (line 99 in figure 3).

3.3 Selecting options from `consolepaneltvp.gss`.

This file is invoked from `xlsreadpaneldata.gss` to ask the practitioner for the main elements of the model. We extract the following code from `consolepaneltvp.gss`, although the user will only see the questions that appear in Figure 4 in magenta, starting from the number and identification of the countries that integrate the panel (lines 9 and 13, respectively), as well as the time dimension (lines 18 and following). This choice permits the creation of a matrix per variable, with the time-dimension in columns. The name of the variables is obtained from the excel file.

Figure 4: `consolepaneltvp.gss` (1)

```
1  ////////////////////////////////////////////////////////////////////
2  // THIS PART NEEDS TO BE ADAPTED TO THE CONCRETE IMPLEMENTATION //
3  ////////////////////////////////////////////////////////////////////
4
5  "Countries to be analised:"
6
7   allcountryconsole;
8
9  "Number of countries? ";;
10
11 countnum = con(1,1);
12
13 "Countries identifyers?
14 ";;;
15
16 countriesid = con(1,countnum);
17
18 @  CHOOSE STARTING YEAR @
19
20    "SELECT STARTING PERIOD:"
21     period5in;
22
23     "Initial Period? ";;
24
25       timeri = con(1,1);
26
27 @  CHOOSE ENDING YEAR @
28
29    "SELECT ENDING PERIOD:"
30    period5fin;
31
32     "Final Period? ";;
33
34       timerf = con(1,1);
35
36
37 timer = timerf-timeri+1;
38
39 timer =  (timerf-timeri)+1;
40
```

Moreover, the program automatically creates lags and first-differences for the variables in the dataset. Log-transformation is not automatically performed, as some values of the series could eventually exhibit negative values. If needed, one can create the log-transformed series adding this feature manually. After that, the console asks the researcher the number of variables for each one of the categories.

Figure 5: `consolepaneltvp.gss` (2)

```
@  Set Kalman filter control parameters @

// n is dimension of observed vector, in this case the NUMBER OF COUNTRIES

    n =  countnum;

// k is dimension of exogenous (IDIOSYNCRATIC-MEAN) variables (NUMBER OF REGRESSORS)

    "Dimension of fixed-parameter (IDIOSYNCRATIC-MEAN) exogenous variables (number)?";;

    FIXEDREGRESSORS = con(1,1);

    k=FIXEDREGRESSORS;

@ nn=n when single-country parameters allowed else nn=n/n=1 @

    "Single-country parameters allowed? (y or n)?";;

    SINGLEPAR = cons;

    if singlepar $== "y";

    nn=n;

    else;

    nn=1;

    endif;


// kc is dimension of exogenous (COMMON-MEAN) variables (NUMBER OF REGRESSORS)

    "Dimension of fixed-parameter COMMON-MEAN) exogenous variables (number)?";;

    COMMONFIXEDREGRESSORS = con(1,1);

    kc=COMMONFIXEDREGRESSORS;

// k1 is dimension of TVP variables

    "Dimension of TVP exogenous variables (number)?";;

    TVPREGRESSORS = con(1,1);

    k1=TVPREGRESSORS;

// rx1 is dimension of country-specific unobservable vector

    rx1 = n*k1;

// rx2 is dimension of common factor rx2=1 if it exists

"Common Unobservable Factor? (y or n)?";;
```

Regarding the state transition equation, the main elements prompted are the following:

- The nature of the auto-regressive component in the transition equation for each time-varying-parameter can be defined, either country-specific or common for the panel (line 123 in Figure 6).
- Similarly, the variance of the error term in the transition equation can be restricted to be common for the countries included in the panel or idiosyncratic (line 151).
- When multiple time-varying-components are included in the measurement equation, their auto-regressive parameter in the transition equation can also be restricted to be identical for all explanatory variables (line 136).
- Additionally, the number of elements to be estimated can be reduced by restricting the variance of the error terms for multiple time-varying parameters to be identical for all regressors included (line 164).

Figure 6: `consolepaneltvp.gss` (3)

```
123 "Common PHI for all countries? (y or n)?";;
124     PHIN = cons;
125
126     if PHIN $== "y";
127
128     rxpn = 1;
129
130     else;
131
132     rxpn = n;
133
134     endif;
135
136 "Common PHI for all regressors? (y or n)?";;
137     PHIK1 = cons;
138
139     if PHIK1 $== "y";
140
141     rxpk = 1;
142
143     else;
144
145     rxpk = k1;
146
147     endif;
148
149     rxp=rxpn*rxpk;
150
151 "Common SIGV for all countries? (y or n)?";;
152     SIGVN = cons;
153
154     if SIGVN $== "y";
155
156     rxvn = 1;
157
158     else;
159
160     rxvn = n;
161
162     endif;
163
164 "Common SIGV for all regressors? (y or n)?";;
```

- Next, the user is asked about the number of control inputs with country-specific parameter (that is, the choice of idiosyncratic parameters in line 182, Figure 7).
- A potential way to simplify the hyperparameter vector is to assume equal parameter for the countries in the panel (that is, common JOTA, in line 186)
- When more than one TVP is included, one can assume that the parameter on the control input is identical for the different unobserved elements to estimate (common JOTA for the regressors in line 199).

Figure 7: `consolepaneltvp.gss` (4)

```
179
180 // s is dimension of state-equation IDIOSYNCRATIC-PARAMETER exogenous variables
181
182     "State Transition IDIOSYNCRATIC-PARAMETER Variables (number)?";;
183     STATEV = con(1,1);
184     s=STATEV;
185
186 "Common JOTA for all countries? (y or n)?";;
187     JOTAN = cons;
188
189     if JOTAN $== "y";
190
191     rxsn = 1;
192
193     else;
194
195     rxsn = n;
196
197     endif;
198
199 "Common JOTA for all regressors? (y or n)?";;
200     JOTAK1 = cons;
201
202     if JOTAK1 $== "y";
203
204     rxsk = 1;
205
206     else;
207
208     rxsk = k1;
209
210     endif;
211
212     rxs=rxsn*rxsk;
213
214
215 // sc is dimension of state-equation COMMON-PARAMETER exogenous variables
216
```

- The user is asked about the number of control inputs with common parameter for the panel countries (that is, the common parameters in the exogenous variables, line 215 in Figure 8).
- When more of one TVP is included, a simplification could be to have a common JOTA for all the regressors (line 221).
- Finally, one can assume the unobserved vector follows a random walk (that is, the parameter $\phi_1 = 1$) in line 239 of the script.

The choice of all the elements above allows the definition of the hyperparameter vector.

Figure 8: `consolepaneltvp.gss` (5)

```
// sc is dimension of state-equation COMMON-PARAMETER exogenous variables

    "State Transition COMMON-PARAMETER Variables (number)?";;
    STATEVC = con(1,1);
    sc=STATEVC;

"Common JOTACOM for all regressors? (y or n)?";;
    JOTACOMK1 = cons;

    if JOTACOMK1 $== "y";

    rxsck = 1;

    else;

    rxsck = k1;

    endif;

    rxsc=rxsck;


// phi is the autoregressive parameter of unobserved variables

    "Set PHI to 1? (y or n)";;

     autoone = cons;

   if autoone $== "y";

    phione=0;

    phi=1*ones(rxp+rx2,1);

   else;

    phione=1;

   endif;

```

3.4 Model specification in `mainpaneltvp.gss`.

`mainpaneltvp.gss` permits to implement the model specification and includes the following elements (see Figure 9):

- Dependent variable for the measurement equation, y (line 99).
- Measurement equation regressors with (a potentially) idiosyncratic fixed coefficient, **measdep** in line 103.
- Measurement equation regressors with a common fixed coefficient, **measdepc** in line 107.
- Measurement equation regressors with a time-varying coefficient, **measdepv**, line 111.
- State transition equation control inputs with a (potentially) idiosyncratic fixed parameter, **statedep** (line 116).
- State transition equation control inputs with a (potentially) common fixed parameter, **statedepc** in line 120.

Figure 9: Model specification

```
92
93  /////////////////////////
94  //  MODEL SPECIFICATION //
95  /////////////////////////
96
97  // HERE THE DEPENDENT VECTOR. Candidates: URATE_DIF ///
98
99      y = URATE_DIF;
100
101 // HERE THE MEASUREMENT INDEPENDENTS WITH IDIOSYNCRATIC MEAN (FIXED) PARAMETER
102
103     measdep = RGDPG;
104
105 // HERE THE MEASUREMENT INDEPENDENTS WITH COMMON MEAN (FIXED) PARAMETER
106
107     measdepc = INTERCEPT;
108
109 // HERE THE MEASUREMENT INDEPENDENTS WITH UNOBSERVED (VARYING) PARAMETER
110
111     measdepv = RGDPG;
112
113 // HERE THE IDIOSYNCRATIC-PARAMETER CONTROL VARIABLES IN STATE EQUATION
114 // BAAFFM BAA10YM
115
116     statedep = RGDPG_VAR;
117
118 // HERE THE COMMON-PARAMETER CONTROL VARIABLES IN STATE EQUATION
119
120     statedepc = URATE_GAP;
121
```

3.5 Maximum Likelihood estimation and output files

`mainpaneltvp.gss` calls then `kfseekpaneltvp.gss` that creates the hyper-parameter vector using the starting values. This GAUSS file controls the Kalman filter estimation, performed in kfiltpaneltvp.gss.

Once the maximum likelihood estimation is obtained, the output is stored in an excel workbook, together with the graphs for both the original data and the estimated unobserved time varying parameters. The information about the variables should be included in mainpaneltvp.gss and the graphs are obtained after answering “yes” to the console’s request.

Figure 10: Graphs

```
//////////////////////////////////////////////
//  PREPARES DATA (IDIOSYNCRATIC) FOR PLOT //
//////////////////////////////////////////////

DATATOPLOT = RGDPG~URATE~URATE_DIF~NAWRU~NAWRU_DIF;
seriesnum = cols(DATATOPLOT)/n;

serieslab1="Real GDP growth ";
serieslab2="Unemployment Ratio ";
serieslab3="Unemployment ratio variation, % ";
serieslab4="NAWRU ";
serieslab5="NAWRU variation ";


datatoplotlab = serieslab1$~serieslab2$~serieslab3$~serieslab4$~serieslab5;

//////////////////////////////////////
//  PREPARES DATA (COMMON) FOR PLOT //
//////////////////////////////////////

DATATOPLOT2 = BAA10YM;
series2num = cols(DATATOPLOT2)/n;

series2lab1="BAA spread ";

datatoplot2lab = series2lab1;

```

4 Empirical application: estimating a time-varying Okun’s relationship

Okun’s Law states that there is a relationship between the unemployment rate and the economic growth rate. Specifically, the law states that for every 1% increase in the unemployment rate, there is a corresponding 2% decrease in the economic growth rate. This relationship is typically modeled using regression analysis, with the unemployment rate as the independent variable and the economic growth rate as the dependent variable.

Traditionally, regression analysis has been used to model this relationship using fixed parameters, assuming that the relationship between unemployment and economic growth is constant over time. However, more recent studies have used time-varying parameter regressions that allow some of them, such as the slope and intercept, to vary as a function of time.

4.1 Brief literature review and econometric specification

After Okun (1962) found a negative short-run relationship between unemployment and output, its empirical support has made Okun's Law a fixture in standard macroeconomics and a simple rule of thumb frequently used by economic forecasters.

In its "gap specification", Okun's Law postulates an inverse relationship between the deviation of the unemployment rate, ur_t, from its natural level, ur_t^*, and (log-transformed) actual GDP ($\ln y_t$) gap from trend or potential output ($\ln y_t^*$), or percent deviation from trend output.

$$(ur_t - ur_t^*) = \beta \left(\ln y_t^* - \ln y_t\right) \tag{4.1}$$

where the parameter β is usually referred to as Okun's coefficient, and the natural unemployment rate, ur_t^*, can also be interpreted as the unemployment rate associated with full employment.

For this relation to be fulfilled, either the two gaps must be stationary, or they have to be cointegrated. Moreover, the problem with potential output and full employment is that none of them are directly observable macroeconomic variables, and their calculation relies on the researcher's choice.

An alternative specification is the "first-differences" version of Okun's Law, which relies on observable data, although its validity requires additional conditions. To obtain the "first-differences" specification, we must differentiate (4.1) to the analogous expression for $t-1$.

Taking annual differences on both sides on (4.1), we obtain:

$$(ur_t - ur_{t-1}) - \left(ur_t^* - ur_{t-1}^*\right) = \beta \left[\left(\ln y_t^* - \ln y_{t-1}^*\right) - \left(\ln y_t - \ln y_{t-1}\right)\right] \tag{4.2}$$

$$(ur_t - ur_{t-1}) = \left(ur_t^* - ur_{t-1}^*\right) + \beta \left(\ln y_t^* - \ln y_{t-1}^*\right) - \beta \left(\ln y_t - \ln y_{t-1}\right) \tag{4.3}$$

Or, in a simplified form,

$$\Delta ur_t = \Delta ur_t^* + \beta \; g_t^* - \beta \; g_t \tag{4.4}$$

where

$$\Delta ur-_t = ur_t - ur_{t-1} \tag{4.5}$$

$$\Delta ur^*_t = ur^*_t - ur^*_{t-1} \tag{4.6}$$

and

$$g^*_t = \ln y^*_t - \ln y^*_{t-1} \tag{4.7}$$

$$g_t = \ln y_t - \ln y_{t-1} \tag{4.8}$$

To obtain the basic "first difference" version of Okun's law at Okun (1962), we need to make two additional assumptions, i.e., the constancy of the natural unemployment rate, $\Delta ur^*_t = 0$; and also a constant average annual growth rate for potential full-employment output, $g^*_t = g^*$ Then, we rename the corresponding parameters so that $\beta\ g^* = \beta_0$ and $-\beta = \beta_1$, and equation (4.4) becomes the static first-differences version:

$$(ur_t - ur_{t-1}) = \beta_0 + \beta_1 (\ln y_t - \ln y_{t-1}) \tag{4.9}$$

or in a simplified form

$$\Delta ur_t = \beta_0 + \beta_1 g_t \tag{4.10}$$

The main advantage of the first-difference version of Okun's Law is that it avoids using controversial assumptions about the definition and computation of potential output and full employment. Still, strong assumptions are required, particularly on the constancy of the natural unemployment rate, making its basic formulation less realistic for empirical applications. Besides, for this specification to be correct, the series between brackets must be stationary or, if nonstationary, cointegrated to avoid spurious regressions.

The parameter β_1 is expected to be negative so that, while rapid output growth is associated with a falling unemployment rate, slow or negative output growth is associated with a rising unemployment rate, while the ratio $-\beta_0/\beta_1$ gives the output growth rate consistent with a stable unemployment rate, or how quickly the economy would typically need to grow to maintain a given level.

Nevertheless, as pointed out by Knotek II (2007), Okun's Law's "static version" captures only the contemporaneous correlation. This means that it ignores the rich dynamics between Δur_t and g_t, posited by the literature on unemployment persistence, such as, for instance, the effect of past GDP growth on the current unemployment rate or the impact of the past unemployment rate on the current unemployment rate. (i.e. Barro, 1988; Mortensen and Pissarides, 1994). This fact would recommend using a dynamic version of Okun's Law, including past real output growth and changes in the unemployment rate on the right side of the first-differences version, contributing to the explanation of the current change in the unemployment rate on the left side.

The baseline dynamic model has evolved to include additional variables in the specification. For instance, an alternative approach has led to production-function versions of Okun's Law, combining a theoretical production function with the gap-based version of Okun's Law to try to capture the economy's idle resources. The main drawback to this approach stems from difficulties associated with measuring inputs such as capital and technology.

Moreover, other authors have stressed the importance of the hysteresis hypothesis (Blanchard and Summers, 1987; León-Ledesma, 2002; Lang and de Peretti, 2009) for a correct specification of Okun's Law. It crucially determines the time evolution of the unemployment rate and, in its first-difference specification, refutes the above assumption of a constant equilibrium unemployment rate assumption.

Knotek II (2007) re-estimates the original Okun's Law first-differences version, extending the period to the second quarter of 2007 and obtaining similar results with the only minor difference lying in the estimated constant term. Since the (negative sign of the) constant term divided by Okun's coefficient gives the rate of output growth consistent with a stable unemployment rate, this implies that the economy required slightly more rapid growth to maintain a given level of unemployment in Okun's time than it has over a longer time span.

Recently, Ball et al. (2017) have shown that while Okun's Law fits the data in most countries, the coefficient in the relationship, i.e., the effect of a 1% change in output on the unemployment rate, varies substantially across them.

Moreover, linearity in Okun's Law has also been criticized in the literature, suggesting that nonlinearities and asymmetries characterize the relationship (e.g. Virén, 2001; Crespo-Cuaresma, 2003; Silvapulle et al., 2004).

These problems can be addressed by using a time-varying parameter model representation:

$$\Delta ur_{i,t} = \left(\bar{\beta}_{0,i} + \xi_{0,i,t}\right) + \left(\bar{\beta}_{1,i} + \xi_{1,i,t}\right) g_t \tag{4.11}$$

$$\xi_{k,i,t} = \phi \xi_{k,i,t-1} + \mu_{k,1,i} \Delta ur_t^* + \mu_{k,2,i} \Delta g_t^* \tag{4.12}$$

Cœuré (2017) highlights that potential output estimates are chronically unreliable and often subject to substantial revisions ex-post. Moreover, recessions usually tend to affect the level of potential output permanently. Whether the 2008 financial crisis will also leave a permanent footprint on the trend potential growth rate is still to be determined– something that Ball (2014) has called "super-hysteresis" effects. Are these effects present? Several arguments are being made. First, in most countries, the loss of potential output is almost as large as the shortfall of actual output from its pre-crisis trend. Second, in those countries more severely hit by the recession, the estimated growth rate of potential output is significantly lower today than before 2008.

Since Friedman (1968), economists use NAIRU (non-accelerating inflation rate of unemployment) as a synonym for the natural rate because the natural rate is the unemployment level consistent with stable inflation. However, the value of NAIRU is hard to measure, mainly because it changes over time and, as highlighted by Krugman (1994) among others, the shocks and institutions produce heterogeneous impacts on employment affected by technological change.

In this same line of research, Virén (2001) allows for non-linearities in estimating Okun's Law by a threshold model. The choice of the threshold is crucial in this methodology. An alternative is to separate increases from decreases in the unemployment rate, as in:

$$y_t^c = \alpha + \beta_{1,i,t} u_t^{c+} + \beta_{2,i,t} u_t^{c-} + \nu_{i,t} \tag{4.13}$$

Following Abel et al. (2017), our specification of Okun's Law departs from the textbook one to consider a time-varying-parameter. We employ a state-space approach, estimated through the Kalman filter, to model a time-varying Okun's Law specification for a panel of 5 peripheral EMU countries from 1965 to 2021. Our structural model allows estimating both fixed-mean (idiosyncratic) coefficients (intercept) and their time-varying evolution. Control inputs (also defined as country-specific or common) drive the evolution of the time-varying parameter. According to this, our measurement equation can be written as follows:

$$\begin{aligned} \Delta ur_{i,t} &= \bar{\beta}_{0,i} \\ &+ \left(\bar{\beta}_1 + \xi_{1,i,t}\right) g_{i,t} \\ &+\omega_{i,t} \end{aligned} \tag{4.14}$$

where the dependent variable is $\Delta ur_{i,t}$, the first-difference of unemployment rate; the explanatory variables are the intercept (constant for the sample period but idiosyncratic) and GDP growth, $g_{i,t}$ (country-specific parameter); and $\xi_{i,t}$ stands for the time-varying parameter for $g_{i,t}$.

Thus, our state-space model incorporates heterogeneity by allowing for common and country-specific parameters.

Concerning the transition equation, it can be written as:

$$\begin{aligned} \xi_{k,t} &= \phi \xi_{k,t-1} \\ +\mu_{k,1,i}\left(ur_{i,t} - ur^*_{i,t}\right)^+ + \mu_{k,2a}\Delta g_t^+ + \mu_{k,2b}\Delta g_t^- &+ \upsilon_{i,t+1} \end{aligned} \tag{4.15}$$

where an autoregressive component drives the transition of the unobserved vector with parameter ϕ together with two additional control instruments. The first one is $\left(ur_{i,t} - ur^*_{i,t}\right)$, the deviation from actual to natural unemployment rate. The second, Δg_t, is GDP growth's first difference. Finally, $\upsilon_t \sim N\left(0, Q\right)$.

The model is estimated using maximum likelihood. The extension from time series to panel data can dramatically increase the number of parameters in the model, reducing the model's degrees of freedom and making the vector of hyperparameters likely to be unique. In this context, the choice and nature of parameters, either common or idiosyncratic, both in the measurement and the state transition equations, becomes particularly important. We recommend sequentially applying likelihood-ratio tests to choose between competing alternatives nested in the general unrestricted model, where all hyperparameters are defined as idiosyncratic. Finally, the choice of initial values is frequently critical to achieving convergence. We employ a naive approach to determine the initial values for the fixed parameters, setting them close to zero. Then, if convergence is achieved, the optimization process drives the values towards the maximum likelihood estimates.

4.2 TVP Estimates

In this section, we estimate the empirical specification described in Equations 4.14 and 4.15 for our panel. We use a state-space model, estimated through the Kalman filter, which extends the traditional approach by Hamilton (1994a).

Table 1: State-Space Okun's Law estimates 1965-2021. Peripheral EMU ountries.

	IRE	**GRC**	**ESP**	**ITA**	**PRT**
Measurement Equation					
intercept	0.010***	0.001	0.012***	0.002**	0.006***
	(0.002)	(0.002)	(0.003)	(0.001)	(0.001)
g_t	-0.105**				
	(0.045)				
σ_{ω_i}	0.010***	0.009***	0.014***	0.005***	0.007***
	(0.001)	(0.001)	(0.002)	(0.001)	(0.001)
State Transition equation					
ξ_{t-1}	0.775***				
	(0.056)				
$\left(ur_{i,t} - ur^*_{i,t}\right)^+$	-5.355*	-0.200	-4.004***	-0.670	-6.188***
	(3.239)	(0.596)	(1.208)	(1.841)	(2.432)
Δg_t^+	0.834				
	(0.614)				
Δg_t^-	-1.152				
	(0.876)				
σ_ν	0.106***				
	(0.017)				
Log likelihood: 1054.728					
Number of Observations: 57					

Notes: Standard errors in parentheses. * $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$

The results for the maximum likelihood estimation of the elements of the hyper-parameter vector are reported in Table 1. In Figure 11, we show how each of the estimated parameters can be found in the output of our Gauss code.

Figure 11: Estimation output

```
MLE as parameterized for numerical optimization
Coefficients:
    0.0097116343     0.0088285430      0.013686689     0.0046268074     0.0070830275
    0.0095885059    0.00076224502      0.012495464     0.0017105655     0.0064896971
    -0.10451807       0.77513454       0.10591989       -5.3553955      -0.20022844
    -4.0036664      -0.66962875       -6.1883705       0.83440294      -1.1518700

sigw:
    0.0097116343
    0.0088285430
     0.013686689
    0.0046268074
    0.0070830275

mua:
    0.0095885059
   0.00076224502
     0.012495464
    0.0017105655
    0.0064896971

mub:    -0.10451807

phi:     0.77513454

sigv:     0.10591989

jota1:
     -5.3553955
    -0.20022844
     -4.0036664
    -0.66962875
     -6.1883705

jotacom:
     0.83440294
     -1.1518700

Value of log likelihood:      1054.7285
```

Sigma w, third row of Table 1

Estimated idiosyncratic intercepts.
First row of Table 1.

Common estimated parameter.
Second row Table 1.

Measurement equation

Common autoregressive component.
First row transition equation. Table 1.

Sigma v. Transition equation.

Unemployment positive deviations from its equilibrium value.

Positive and negative first differences of GDP growth

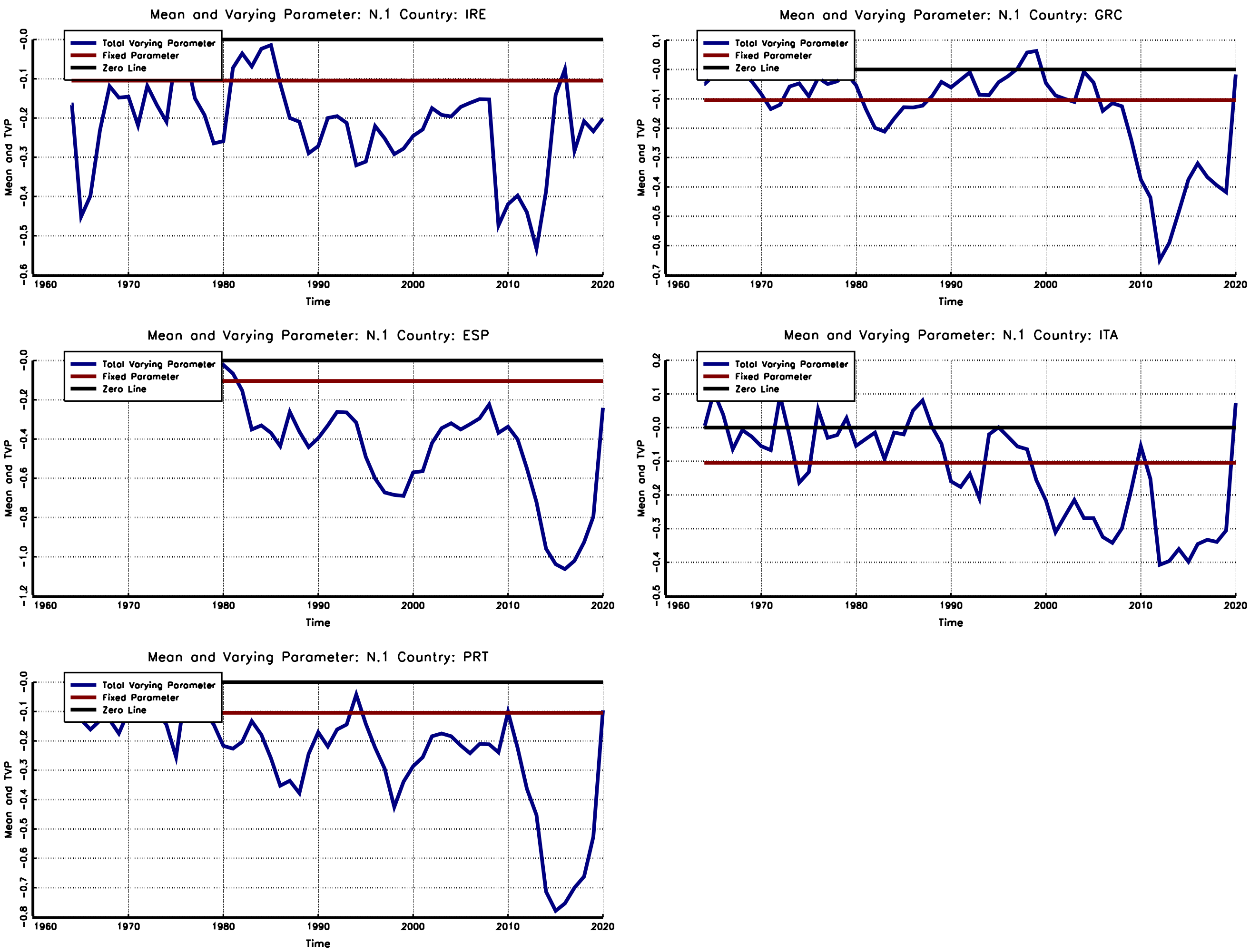
Transition equation

According to our model, unemployment and growth for all the countries considered have a negative dynamic relationship. In the empirical model, we allow for estimating both a fixed-mean (idiosyncratic) coefficient and its time-varying counterpart. Concerning the measurement equation, an idiosyncratic intercept is significant for all the countries analyzed except Greece. This idiosyncratic intercept represents the GDP growth level that would maintain the unemployment rate unchanged, and it ranges from 0.2% for Italy to 1.2% for Spain. We have also specified the empirical model, including a fixed Okun coefficient, common for all countries. The parameter has the expected negative sign and a value of -0.105, i.e., an increase of 1% in GDP growth reduces the unemployment rate by 10%. Finally, an error term completes the panel's measurement equation, exhibiting a country-specific variance.

Regarding the transition equation, the autoregressive component coefficient is 0.775. This implies stationarity but strong persistence. In the equation, positive deviations of the unemployment rate from its equilibrium value are also significant, but negative shocks on unemployment have a heterogeneous and asymmetric impact. However, neither positive nor negative first differences in GDP growth seem to affect the evolution of the unobserved component. Finally, the transition equation includes an error term common to all panel members.

In Figure 12, we present the estimated time-varying parameters for each country in the panel. The red horizontal line is the common mean value for the Okun coefficient, which is negative, as predicted by theory. The blue line results from adding the estimated mean-reverting varying component to the fixed one. The joint evolution of the varying Okun coefficient exhibits a gradual reduction over time, implying a greater response of unemployment to growth. This process has been interrupted by the concatenation of crises at the end of the period.

Figure 12: Okun's Law Time-Varying estimation, 1965-2020.



5 Conclusions

In this paper, we introduce `sspaneltvp`, a new code developed for Gauss to estimate fully-fledged panel-TVP state-space models with a long T dimension. We contribute to previous work in four directions. First, we modify Hamilton's seminal GAUSS code (Hamilton, 1994a) to fit a time-varying multi-parameter model; second, we extend the Kalman Filter estimation from a single-country model to a panel setting; third, we adapt the transition equation to include control inputs; finally, both fixed parameters in the measurement equation and the control variables are allowed to be either common or idiosyncratic.

As an example, we implement the code to a time-varying parameter specification of Okun's Law for five peripheral EMU countries for the period 1965-2021. The encompassing framework nests the conditions to determine the properties of the Okun coefficient. Once specified, the state-space model is estimated using the Kalman filter, producing both the hyperparameters and the evolution of the states. Our econometric model allows the Okun coefficient to be time-varying and thus to evolve endogenously and asymmetrically over contractions and expansions. However, the framework is flexible enough to nest various possibilities for the Okun coefficient, including when it is a constant and not asymmetric.

The time-varying estimate of Okun's Law helps us to disentangle the variations in the output-unemployment relationship. Our results reveal the existence of heterogeneity among the countries. As acknowledged in the literature, there is an evident variation across countries, ranging from -0.0106 in the case of Greece to -0.241 in the case of Spain. More importantly, the unemployment-output responsiveness has changed over time. The heterogeneity of the results at the country level highlights the advantages of estimating country-specific functions when possible. Most coefficients evolve around a fixed parameter with some degree of persistence. We conjecture that some (structural) characteristics must be behind such a phenomenon. Our approach allows the Okun coefficient to be constant, time-varying, symmetric, or asymmetric. Moreover, we add some control instruments that govern the dynamics of the transition equation, namely, the deviation between the actual and the natural unemployment rate and the economy's acceleration along the business cycle.

Finally, we use this example of Okun's Law to show how to implement the estimation using `sspaneltvp`. We show how to organize the data, use the available functions in the toolbox, and interpret results. We illustrate all the models in the toolbox. This positions the new toolbox as a valid self-contained package for this estimation technique in GAUSS. Since the code is freely available in an open-source repository, users will benefit from the collaboration and review from the community.

6 How to install

The latest version of the sspaneltvp bundle can be obtained from:

`https://github.com/JuanSapena/sspaneltvp_gauss` Updates and further documentation will be found on the GitHub repository.

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Appendix

A Measurement and State transition equations

$$
\begin{aligned}
\begin{pmatrix} y_{1,t} \\ \vdots \\ \vdots \\ y_{n,t} \end{pmatrix} = {} & \begin{pmatrix}
\beta_{1,1} & 0 & \cdots & 0 & \beta_{1,2} & 0 & \cdots & \cdots & \beta_{1,k} & 0 & 0 \\
\vdots & \ddots & 0 & \vdots & 0 & \ddots & 0 & \cdots & 0 & \ddots & \vdots \\
0 & \cdots & \beta_{n,1} & 0 & \cdots & 0 & \beta_{n,2} & \cdots & 0 & 0 & \beta_{n,k}
\end{pmatrix} \times \begin{pmatrix} x^s_{1,1,t} \\ \vdots \\ x^s_{n,1,t} \\ x^s_{1,2,t} \\ \vdots \\ x^s_{1,k,t} \\ \vdots \\ x^s_{n,k,t} \end{pmatrix} \\
& + \begin{pmatrix}
\beta^c_{1} & 0 & \cdots & 0 & \beta^c_{2} & 0 & \cdots & \cdots & \beta^c_{kc} & 0 & 0 \\
\vdots & \ddots & 0 & \vdots & 0 & \ddots & 0 & \cdots & 0 & \ddots & \vdots \\
0 & \cdots & \beta^c_{1} & 0 & \cdots & 0 & \beta^c_{2} & \cdots & 0 & 0 & \beta^c_{kc}
\end{pmatrix} \times \begin{pmatrix} x^c_{1,1,t} \\ \vdots \\ x^c_{n,1,t} \\ x^c_{1,2,t} \\ \vdots \\ x^c_{1,kc,t} \\ \vdots \\ x^c_{n,kc,t} \end{pmatrix} \\
& + \begin{pmatrix}
x^v_{1,1,t} & \cdots & x^v_{1,k,t} & 0 & \cdots & 0 & \cdots & \cdots & & 0 \\
0 & \cdots & 0 & x^v_{2,1,t} & \cdots & x^v_{2,kv,t} & 0 & \cdots & \cdots & \vdots \\
\vdots & \ddots & \vdots & 0 & 0 & 0 & x^v_{3,1,t} & \cdots & \cdots & \vdots \\
\vdots & 0 & 0 & \vdots & \ddots & \vdots & \ddots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \ddots & x^v_{n,1,t} & \cdots & x^v_{n,kv,t}
\end{pmatrix} \times \begin{pmatrix} \zeta_{1,1,t} \\ \zeta_{1,2,t} \\ \vdots \\ \zeta_{1,k,t} \\ \vdots \\ \zeta_{n,1,t} \vdots \\ \zeta_{n,k,t} \end{pmatrix} \\
& + \begin{pmatrix} h_{1,1} \\ \vdots \\ \vdots \\ h_{n,1} \end{pmatrix} \times f_1 + \begin{pmatrix} \omega_{1t} \\ \vdots \\ \vdots \\ \omega_{nt} \end{pmatrix}
\end{aligned} \tag{A.1}
$$

$$
\begin{aligned}
\begin{pmatrix} \zeta_{1,1,t+1} \\ \zeta_{1,2,t+1} \\ \vdots \\ \zeta_{1,k,t+1} \\ \vdots \\ \zeta_{n,k,t+1} \\ f_{1,t+1} \end{pmatrix}
&=
\begin{pmatrix}
\phi_{1,1} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \phi_{1,2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & \vdots & \vdots & \vdots \\
 & \vdots 0 & 0 & \phi_{1,k} & 0 & 0 & 0 \\
0 & \vdots & 0 & 0 & \ddots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \phi_{n,k} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \phi_{nk+1}
\end{pmatrix}
\times
\begin{pmatrix} \zeta_{1,1,t} \\ \zeta_{1,2,t} \\ \vdots \\ \zeta_{1,k,t} \\ \vdots \\ \zeta_{n,k,t} \\ f_{1,t} \end{pmatrix}
+
\begin{pmatrix} \upsilon_{1,1,t} \\ \upsilon_{1,2,t} \\ \vdots \\ \upsilon_{1,k,t} \\ \vdots \\ \upsilon_{n,k,t} \\ \upsilon_{nk+1,t} \end{pmatrix} \\
&+
\begin{pmatrix}
\mu_{1,1,1} & 0 & \cdots & \cdots & 0 & \mu_{1,1,2} & 0 & \cdots & 0 & 0 & \mu_{1,1,s} & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\mu_{1,k,1} & \cdots & \cdots & \cdots & 0 & \mu_{1,k,s} & 0 & \cdots & 0 & 0 & \mu_{1,k,s} & 0 & \cdots & 0 \\
0 & \mu_{2,1,1} & 0 & \cdots & \cdots & 0 & \mu_{2,1,2} & \cdots & \cdots & \vdots & 0 & \mu_{2,1,s} & \ddots & \vdots \\
\vdots & \cdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \cdots & 0 \\
0 & \mu_{2,k,1} & 0 & \cdots & \cdots & 0 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & \cdots & 0 & \cdots & \mu_{n,1,1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \mu_{n,1,s} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & \cdots & \mu_{n,k,1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \mu_{n,k,s} \\
0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\times
\begin{pmatrix} z_{1,1} \\ \vdots \\ z_{n,1} \\ z_{1,2} \\ z_{2,2} \\ \vdots \\ z_{n,2} \\ \vdots \\ z_{1,s} \\ \vdots \\ z_{n,s} \end{pmatrix} \\
&+
\begin{pmatrix}
\mu^c_{1,1} & 0 & \cdots & \cdots & 0 & \mu^c_{1,2} & 0 & \cdots & 0 & 0 & \mu^c_{1,s} & 0 & \cdots & 0 \\
\vdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\mu^c_{k,1} & \cdots & \cdots & \cdots & 0 & \mu^c_{k,2} & 0 & \cdots & 0 & 0 & \mu^c_{k,s} & 0 & \cdots & 0 \\
0 & \mu^c_{1,1} & 0 & \cdots & \cdots & 0 & \mu^c_{1,2} & \cdots & \cdots & \vdots & 0 & \mu^c_{1,s} & \ddots & \vdots \\
\vdots & \cdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & 0 \\
0 & \mu^c_{k,1} & 0 & \cdots & \cdots & 0 & \mu^c_{k,2} & \vdots & \vdots & \vdots & \vdots & \mu^c_{k,s} & \cdots & \vdots \\
0 & \cdots & 0 & \cdots & \mu^c_{1,1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \mu^c_{1,s} \\
\vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & \cdots & 0 & \cdots & \mu^c_{k,1} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \mu^c_{k,s} \\
0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\times
\begin{pmatrix} z^c_{1,1} \\ \vdots \\ z^c_{n,1} \\ z^c_{1,2} \\ z^c_{2,2} \\ \vdots \\ z^c_{n,2} \\ \vdots \\ z^c_{1,s} \\ \vdots \\ z^c_{n,s} \end{pmatrix}
\end{aligned}
\tag{A.2}
$$

Appendix A. The Kalman Filter

The objective of the state-space formulation is to define the state vector ξ_t in a way that guarantees the minimization of the number of elements and the comprehension of all the available information at time t. To estimate the model we use the Maximum Likelihood technique. Essentially, the Kalman[10] filter (hereafter KF) is, in fact, an algorithm composed by a set of equations, which, performed sequentially, allows to obtain in a forward prediction procedure (called "filtering"), the sequence of linear least squares forecasts of the state vector, $\hat{\xi}_{t|t-1} = \mathrm{E}[\xi_t|\zeta_{t-1}]$ (and hence for the vector of dependents, $\hat{y}_{t|t-1}$), on the basis of the information available at $t-1$, summarized by the vector $\zeta_{t-1} \equiv (y_{t-1}, y_{t-2}, ..., y_1, x_{t-1}, x_{t-2}, ..., x_1)$:

In general, the KF, described in Zadeh and Desoer (1963), Harvey (1989), or Hamilton (1994a)[11], is particular suited for Bayesian analysis where, at each time t, there are prior distributions for state variables ξ_t and parameters produced at time $t-1$, from which posterior distributions at time t are calculated as observation y_t becomes available.

The iteration is started by assuming that the initial value of the state vector ξ_1 is drawn from a normal distribution with mean denoted $\hat{\xi}_{1|0}$ and unconditional variance $P_{1|0}$,

$$\xi_1 \sim N\left(\hat{\xi}_{1|0}, P_{1|0}\right) \tag{A.3}$$

In this case, the distribution of ξ_t conditional on ζ_{t-1} turns out to be normal for $t = 2, 3, , .., T$. The mean of this conditional distribution is represented by the $(r \times 1)$ vector $\hat{\xi}_{t|t-1}$ and the variance of this conditional distribution is represented by the $(r \times r)$ matrix $P_{t|t-1} = \mathrm{E}\left(\xi_t - \hat{\xi}_{t|t-1}\right) \cdot \left(\xi_t - \hat{\xi}_{t|t-1}\right)'$.

When all elements of the state vector ξ_t defined by 2.4 are stationary, i.e. if the eigenvalues of F are all inside the unit circle, the initial means, variances and covariances of these initial state elements can be derived from the model parameters, the system is stable, and $\hat{\xi}_{1|0}$ would be the unconditional mean[12] of this stationary vector, while $P_{1|0}$ would be the unconditional variance, that can be calculated from:

[10] The Kalman Filter owes its name to the Hungarian Rudolf E. Kalman and his contributions in Kalman (1960), Kalman and Bucy (1961), and Kalman (1963), although similar algorithms had been developed earlier by Thiele (1880) and Swerling (1959), and also at Zadeh and Desoer (1963), constituting an efficient way to formulate the likelihood (usually Gaussian) for many complex econometric models for estimation and prediction purposes.

[11] This section, and the notation employed, draws heavily on the exposition in chapter 13 of Hamilton (1994).

[12] When $\hat{\xi}_{1|0}$ is covariance stationary, a candidate value for $\hat{\xi}_{1|0}$ is zero so that all state variables are initially in steady state.

$$vec\left(P_{1|0}\right) = \left[I_{r^2} - (F \otimes F)\right]^{-1} .vec\,(Q) \tag{A.4}$$

where $vec\left(P_{1|0}\right)$ is the $(r_2 \times 1)$ vector formed by stacking the columns of $P_{1|0}$, one on top of the other, ordered from left to right, I_{r^2} represents a r^2 dimension identity matrix, and the $\otimes$ operator represents the Kronecker product.

On the contrary, if (at least some of) the elements of the state vector ξ_t are non-stationary, the process of starting up the series is said to be diffuse initialization of the filter, as at least some of the elements of $\hat{\xi}_{1|0}$ and $P_{1|0}$ are unknown. Hence, for time-variant or non-stationary systems, $\hat{\xi}_{1|0}$ represents a guess as to the value of ξ_1 based on prior information, while $P_{1|0}$ measures the uncertainty associated with this guess [13]. This prior cannot be based on the data, since it is assumed in the derivations to follow that ν_{t+1} and w_t, are independent of ξ_1 for $t = 1, 2, ..., T$. Harvey and Phillips (1979) propose a simple approximate technique for the diffuse initialization of the filter, that consists in to initialize non-stationary components of the state vector by any value (say ,$\hat{\xi}_{1|0} = 0$), and an arbitrary large variance, $P_{1|0}$, relative to the magnitude of the series, and then use the standard Kalman filter. The larger the variance, the lesser informative the initialization is for the filter. Koopman (1997) and Koopman and Durbin (2003) propose a more transparent treatment of diffuse filtering and smoothing based on Ansley and Kohn (1985).

Departing from the initial conditions, the algorithm works sequentially in a two-step process. Having described the values of $\hat{\xi}_{t|t-1}$ and $P_{t|t-1}$ for $t = 1$, once the outcome of the next measurement (possibly corrupted with some amount of error, including random noise) is observed, these estimates are updated into the *a posteriori* estimate, as in (A.5) and (A.6):

$$\hat{\xi}_{t|t} = \hat{\xi}_{t|t-1} + \cdot P_{t|t-1} H (H' P_{t|t-1} H + R)^{-1} (y_t - A' x_t - H' \hat{\xi}_{t|t-1}) \tag{A.5}$$

$$P_{t|t} = \cdot P_{t|t-1} - \cdot P_{t|t-1} H (H' P_{t|t-1} H + R)^{-1} H' P_{t|t-1} \tag{A.6}$$

The predictive phase uses the state estimate from the previous timestep to produce an estimate of the state at the current timestep. This predicted state estimate is also known as the *a priori* state estimate, along with their uncertainties:

As $\hat{\xi}_{t+1|t} = F\hat{\xi}_{t|t}$ and $P_{t+1|t} = F \cdot P_{t|t} F' + Q$, to calculate the sequence $\left\{\hat{\xi}_{t+1|t}\right\}_{t=1}^{T}$ and $\left\{P_{t+1|t}\right\}_{t=1}^{T}$,

[13] The greater our prior uncertainty, the larger the diagonal elements of $P_{1|0}$.

one has simply to iterate on equations (A.7) and (A.8) for $t = 1, 2, ..., T$.

$$\hat{\xi}_{t+1|t} = F\hat{\xi}_{t|t-1} + F \cdot P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}(y_t - A'x_t - H'\hat{\xi}_{t|t-1}) \tag{A.7}$$

$$P_{t+1|t} = F \cdot P_{t|t-1}F' - F \cdot P_{t|t-1}H(H'P_{t|t-1}H + R)^{-1}H'P_{t|t-1}F' + Q \tag{A.8}$$

The output obtained for step t is used sequentially as the input for the step t + 1.

When the values of the matrices F, Q, B, A, H and R are unknown, we collect the unknown elements of these matrices in a hyper-parameter vector, θ, and obtain their maximum likelihood estimates. Casual starting values are assigned to the ξ_t vector and to the unknown elements of matrices included in the vector of parameters θ and the estimation procedure maximizes the likelihood function. In general, Q and R are assumed to be positive semi-definite (which includes the possibility that some of the error terms may be zero).

In some cases it is desirable to use information through the end of the sample (date T) to help improving the inference about the historical value that the state vector took on at any particular date t in the middle of the sample. Such an inference is known as a smoothed estimate, $\hat{\xi}_{t|T} = \mathrm{E}[\xi_t|\zeta_T]$. The mean squared error of this estimate is denoted $P_{t|T} = \mathrm{E}\left(\xi_t - \hat{\xi}_{t|T}\right)\left(\xi_t - \hat{\xi}_{t|T}\right)'$. The estimation of the sequence $\left\{\hat{\xi}_{t|T}\right\}_{t=1}^{T}$ can be calculated by iterating in reverse order for for $t = T-1, T-2, ..., 1$ on:

$$\hat{\xi}_{t|T} = \widehat{\xi_{t|t}} + J_t(\hat{\xi}_{t+1|T} - \hat{\xi}_{t+1|t}) \tag{A.9}$$

$$P_{t|T} = P_{t|t} + J_t(P_{t+1|T} - P_{t+1|t})J_t' \tag{A.10}$$

where

$$J_t = P_{t|t}F'P_{t+1|t}^{-1} \tag{A.11}$$