

Discrete 2 (Chapter 7)

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B.Sc. (Hons)(Melit.) Computing Science and Mathematics (Second Year)

April 7, 2023

1 Relations

Totality: $r \in X \leftrightarrow Y$ is said to be total iff $\text{total}(r) \stackrel{\text{df}}{=} \forall x \in X, \exists y \in Y \cdot (x, y) \in r$ is true.

Functionality: $r \in X \leftrightarrow Y$ is said to be functional iff $\text{functional}(r) \stackrel{\text{df}}{=} \forall y_1, y_2 \in Y, \forall x \in X \cdot (x, y_1) \in r \wedge (x, y_2) \in r \Rightarrow y_1 = y_2$ is true.

Surjectivity: $r \in X \leftrightarrow Y$ is said to be surjective iff $\text{surjective}(r) \stackrel{\text{df}}{=} \forall y \in Y, \exists x \in X \cdot (x, y) \in r$ is true.

Injectivity: $r \in X \leftrightarrow Y$ is said to be injective iff $\text{injective}(r) \stackrel{\text{df}}{=} \forall x_1, x_2 \in X, \forall y \in Y \cdot (x_1, y) \in r \wedge (x_2, y) \in r \Rightarrow x_1 = x_2$ is true.

Set of Functional Relations: $X \rightarrow Y \stackrel{\text{df}}{=} \{r \in X \leftrightarrow Y : \text{functional}(r)\}.$

2 Multisets

2.1 Definitions

Set of Multisets Over X : $\mathbb{M}X \stackrel{\text{df}}{=} \{r \in X \rightarrow \mathbb{N} : \text{total}(r)\}.$

Empty Multiset: $\varnothing \stackrel{\text{df}}{=} \lambda x \in X \cdot 0.$

Multiset Element Of: $x \in T \stackrel{\text{df}}{=} T(x) > 0$

Multiset Subset: $T \sqsubseteq S \stackrel{\text{df}}{=} \forall x \in X \cdot T(x) \leq S(x)$.

Multiset Equality: $T \stackrel{\text{mst}}{=} S \stackrel{\text{df}}{=} \forall x \in X \cdot T(x) = S(x)$.

Multiset Union: $T \sqcup S \stackrel{\text{df}}{=} \lambda x \in X \cdot T(x) + S(x)$.

Multiset Intersection: $T \sqcap S \stackrel{\text{df}}{=} \lambda x \in X \cdot \min(\{T(x), S(x)\})$.

2.2 Theorems

Theorem 7.4: $T \sqcup S \stackrel{\text{mst}}{=} S \sqcup T$.

Theorem 7.6: $T \sqcap S \sqsubseteq T \sqcup S$.

2.3 Exercises

Exercise 7.1: $T \sqsubseteq T$ (reflexive) and $T \sqsubseteq R \wedge R \sqsubseteq S \Rightarrow T \sqsubseteq S$ (transitive).

Exercise 7.2: $\emptyset \sqcap T \stackrel{\text{mst}}{=} T \sqcap \emptyset \stackrel{\text{mst}}{=} \emptyset$ and $\emptyset \sqcup T \stackrel{\text{mst}}{=} T \sqcup \emptyset \stackrel{\text{mst}}{=} T$.

Exercise 7.3: $T \sqcap T \stackrel{\text{mst}}{=} T$ (idempotent), $T \sqcap S \stackrel{\text{mst}}{=} S \sqcap T$ (commutative) and $(T \sqcap S) \sqcap R \stackrel{\text{mst}}{=} T \sqcap (S \sqcap R)$ (associative).

Exercise 7.4: $(T \sqcup S) \sqcup R \stackrel{\text{mst}}{=} T \sqcup (S \sqcup R)$ (associative).

Exercise 7.5: $T - S \stackrel{\text{df}}{=} \lambda x \in X \cdot \max(\{0, T(x) - S(x)\})$ (multiset difference), $T - T \stackrel{\text{mst}}{=} \emptyset$ and $T - \emptyset \stackrel{\text{mst}}{=} T$.

3 Sequences

3.1 Definitions

Finite Subsets of \mathbb{N} : $\text{upto}(N) \stackrel{\text{df}}{=} \{n \in \mathbb{N} : n < N\} = \{0, 1, \dots, N-1\}$.

Set of Sequences Over X : $\text{seq } X = \{s_X \in \mathbb{N} \rightarrow X : \exists N \in \mathbb{N} \cdot \text{dom}(s_X) = \text{upto}(N)\}$.

Empty Sequence: $\langle \rangle \stackrel{\text{df}}{=} \lambda n \in \mathbb{N} \cdot \text{undefined}$.

Head of a Sequence: $\text{head}(s_X) \stackrel{\text{df}}{=} s_X(0)$.

Tail of a Sequence: $\text{tail}(s_X) \stackrel{\text{df}}{=} \lambda n \in \mathbb{N} \cdot s_X(n+1).$

Cons Operator:

$$\text{cons}(x, s_X) \stackrel{\text{df}}{=} \lambda n \in \mathbb{N} \cdot \begin{cases} x & \text{if } n = 0 \\ s_X(n-1) & \text{otherwise} \end{cases}.$$

Length of a Sequence:

$$\begin{aligned} \text{len}(s_X) &\stackrel{\text{df}}{=} \begin{cases} 0 & \text{if } s_X \stackrel{\text{seq}}{=} \langle \rangle \\ 1 + \text{len}(\text{tail}(s_X)) & \text{otherwise} \end{cases} \\ &= \lambda n \in \mathbb{N} \cdot \begin{cases} 0 & \text{if } s_X \stackrel{\text{seq}}{=} \langle \rangle \\ 1 + \max(\text{dom}(s_X)) & \text{otherwise} \end{cases}. \end{aligned}$$

Last Element of a Sequence: $\text{last}(s_X) \stackrel{\text{df}}{=} s_X(\text{len}(s_X) - 1).$

Sum of a Sequence of Natural Numbers:

$$\text{sum}(s_{\mathbb{N}}) \stackrel{\text{df}}{=} \begin{cases} 0 & \text{if } s_{\mathbb{N}} \stackrel{\text{seq}}{=} \langle \rangle \\ \text{head}(s_{\mathbb{N}}) + \text{sum}(\text{tail}(s_{\mathbb{N}})) & \text{otherwise} \end{cases}.$$

Sequence Concatenation:

$$\begin{aligned} t_X ++ s_X &\stackrel{\text{df}}{=} \begin{cases} s_X & \text{if } t_X \stackrel{\text{seq}}{=} \langle \rangle \\ \text{cons}(\text{head}(t_X), \text{tail}(t_X) ++ s_X) & \text{otherwise} \end{cases} \\ &= \lambda n \in \mathbb{N} \cdot \begin{cases} t_X(n) & \text{if } n \in \text{dom}(t_X) \\ s_X(n - \text{len}(t_X)) & \text{otherwise} \end{cases}. \end{aligned}$$

Prefix of a Sequence: $t_X \preceq_p s_X \stackrel{\text{df}}{=} \exists r_X \in \text{seq } X \cdot t_X ++ r_X \stackrel{\text{seq}}{=} s_X.$

Suffix of a Sequence: $t_X \preceq_s s_X \stackrel{\text{df}}{=} \exists r_X \in \text{seq } X \cdot r_X ++ t_X \stackrel{\text{seq}}{=} s_X.$

Exact Subsequence: $t_X \preceq s_X \stackrel{\text{df}}{=} \exists r_{1_X}, r_{2_X} \in \text{seq } X \cdot r_{1_X} ++ t_X ++ r_{2_X} \stackrel{\text{seq}}{=} s_X.$

3.2 Exercises

Exercise 7.7: $\text{len}(s_X) = 1 \Rightarrow \text{head}(s_X) = \text{last}(s_X)$.

Exercise 7.8: $x \lessdot s_X \stackrel{\text{df}}{=} \exists n \in \mathbb{N} \cdot n \in \text{dom}(s_X) \wedge s_X(n) = x$.

Exercise 7.9:

$$\text{sorted}(s_X) \stackrel{\text{df}}{=} \begin{cases} \mathbf{True} & \text{if } \text{len}(s_X) \leq 1 \\ \text{head}(s_X) \leq_X \text{head}(\text{tail}(s_X)) \wedge \text{sorted}(\text{tail}(s_X)) & \text{otherwise} \end{cases}.$$

Exercise 7.10: $\text{items} \stackrel{\text{df}}{=} \lambda s_X \in \text{seq } X \cdot (\lambda x \in X \cdot \#\{n \in \mathbb{N} : s_X(n) = x\})$.

Exercise 7.11: $\text{sorts} \stackrel{\text{df}}{=} \{(s_X, s'_X) \in \text{seq } X \times \text{seq } X : \text{items}(s_X) \stackrel{\text{mst}}{=} \text{items}(s'_X) \wedge \text{sorted}(s_X)\}$.
Hence, $(s_X, s'_X) \in \text{sorts}$ iff s_X is a sorted version of s'_X .

Exercise 7.12: $s_X \stackrel{\text{seq}}{=} s'_X \stackrel{\text{df}}{=} \text{len}(s_X) = \text{len}(s'_X) \wedge \forall n \in \mathbb{N} \cdot n \in \text{upto}(\text{len}(s_X)) \Rightarrow s_X(n) = s'_X(n)$.

Exercise 7.13:

$$\begin{aligned} \text{(i)} \quad \text{reverse}(s_X) &\stackrel{\text{df}}{=} \begin{cases} s_X & \text{if } \text{len}(s_X) \leq 1 \\ \text{reverse}(\text{tail}(s_X)) ++ \text{cons}(\text{head}(s_X), \langle \rangle) & \text{otherwise} \end{cases} \\ \text{(ii)} \quad \text{reverse}(s_X) &\stackrel{\text{df}}{=} \lambda n \in \mathbb{N} \cdot s_X(\text{len}(s_X) - (n + 1)). \end{aligned}$$

4 Graph Theory