Discrete 2 (Chapter 7)

Noti għal ~~Qabel~~ Waqt l-Eżami

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1 Relations

Totality: $r \in X \leftrightarrow Y$ is said to be total iff $\text{total}(r) \stackrel{\text{def}}{=} \forall\, x \in X, \exists\, y \in Y \cdot (x, y) \in r$ it true.

Functionality: $r \in X \leftrightarrow Y$ is said to be functional iff $\text{functional}(r) \stackrel{\text{def}}{=} \forall\, y_1, y_2 \in Y, \forall\, x \in X \cdot (x, y_1) \in r \wedge (x, y_2) \Rightarrow y_1 = y_2$ is true.

Surjectivity: $r \in X \leftrightarrow Y$ is said to be surjective iff $\text{surjective}(r) \stackrel{\text{def}}{=} \forall\, y \in Y, \exists\, x \in X \cdot (x, y) \in r$ is true.

Injectivity: $r \in X \leftrightarrow Y$ is said to be injective iff $\text{injective}(r) \stackrel{\text{def}}{=} \forall\, x_1, x_2 \in X, \forall\, y \in Y \cdot (x_1, y) \in r \wedge (x_2, y) \Rightarrow x_1 = x_2$ is true.

Set of Total and Functional Relations: $X \to Y \stackrel{\text{def}}{=} \{r \in X \leftrightarrow Y : \text{total}(r) \wedge \text{functional}(r)\}$

2 Multisets

2.1 Definitions

Set of Multisets Over X: $\mathbb{M}\, X \stackrel{\text{def}}{=} X \to \mathbb{N}$.

Empty Multiset: $\not\square \stackrel{\text{def}}{=} \lambda\, x \in X \cdot 0$.

Multiset Element Of: $x \sqsubseteq T \stackrel{\text{def}}{=} T(x) > 0$

Multiset Subset: $T \sqsubseteq S \stackrel{\text{def}}{=} \forall\, x \in X \cdot T(x) \leq S(x)$.

Multiset Equality: $T \stackrel{\text{mst}}{=} S \stackrel{\text{def}}{=} \forall\, x \in X \cdot T(x) = S(x)$.

Multiset Union: $T \sqcup S \stackrel{\text{def}}{=} \lambda\, x \in X \cdot T(x) + S(x)$.

Multiset Intersection: $T \sqcap S \stackrel{\text{def}}{=} \lambda\, x \in X \cdot \min(\{T(x), S(x)\})$.

2.2 Theorems

Theorem 7.4: $T \sqcup S \stackrel{\text{mst}}{=} S \sqcup T$.

Theorem 7.6: $T \sqcap S \sqsubseteq T \sqcup S$.

2.3 Exercises

Exercise 7.1: $T \sqsubseteq T$ (reflexive) and $T \sqsubseteq R \wedge R \sqsubseteq S \Rightarrow T \sqsubseteq S$ (transitive).

Exercise 7.2: $\not\sqsubset \sqcap T \stackrel{\text{mst}}{=} T \sqcap \not\sqsubset \stackrel{\text{mst}}{=} \not\sqsubset$ and $\not\sqsubset \sqcup T \stackrel{\text{mst}}{=} T \sqcup \not\sqsubset \stackrel{\text{mst}}{=} T$.

Exercise 7.3: $T \sqcap T \stackrel{\text{mst}}{=} T$ (idempotent), $T \sqcap S \stackrel{\text{mst}}{=} S \sqcap T$ (commutative) and $(T \sqcap S) \sqcap R \stackrel{\text{mst}}{=} T \sqcap (S \sqcap R)$ (associative).

Exercise 7.4: $(T \sqcup S) \sqcup R \stackrel{\text{mst}}{=} T \sqcup (S \sqcup R)$ (associative).

Exercise 7.5: $T - S \stackrel{\text{def}}{=} \lambda\, x \in X \cdot \max(\{0, T(x) - S(x)\})$ (multiset difference), $T - T \stackrel{\text{mst}}{=} \not\sqsubset$ and $T - \not\sqsubset \stackrel{\text{mst}}{=} T$.

3 Sequences

Finite Subsets of $\mathbb{N}$: $\text{upto}(N) \stackrel{\text{def}}{=} \{n \in \mathbb{N} : n < N\}$ $(= \{0, 1, \ldots, N-1\})$.

Set of Sequences Over X: $\text{seq}\ X = \{s_X \in \mathbb{N} \rightarrow X : \exists\, N \in \mathbb{N} \cdot \text{dom}(s_X) = \text{upto}(N)\}$.

Empty Sequence: $\langle\rangle \stackrel{\text{def}}{=} \lambda\, n \in \mathbb{N} \cdot \text{undefined}$.

Head of a Sequence: $\text{head}(s_X) \stackrel{\text{def}}{=} s_X(0)$.

Tail of a Sequence: $\text{tail}(s_X) \stackrel{\text{def}}{=} \lambda\, n \in \mathbb{N} \cdot s_X(n+1)$.

Cons Operator:

$$\operatorname{cons}(x, s_X) \stackrel{\text{def}}{=} \lambda\, n \in \mathbb{N} \cdot \begin{cases} x & \text{if } n = 0 \\ s_X(n-1) & \text{otherwise} \end{cases}.$$

Length of a Sequence:

$$\begin{aligned} \operatorname{len}(s_X) &\stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } n = 0 \\ 1 + \max(\operatorname{dom}(s_X)) & \text{otherwise} \end{cases}, \quad \text{(Non-recursive)}. \\ &\stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } n = 0 \\ 1 + \operatorname{len}(\operatorname{tail}(s_X)) & \text{otherwise} \end{cases}, \quad \text{(Recursive)}. \end{aligned}$$

$$\operatorname{len}(s_X) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } n = 0 \\ 1 + \max(\operatorname{dom}(s_X)) & \text{otherwise} \end{cases}, \quad \text{(Non-recursive)}.$$

$$\operatorname{len}(s_X) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } n = 0 \\ 1 + \operatorname{len}(\operatorname{tail}(s_X)) & \text{otherwise} \end{cases}, \quad \text{(Recursive)}.$$

4 Graph Theory