Discrete 2 (Chapter 7)

Noti għal ~~Qabel~~ Waqt l-Eżami

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1 Relations

Totality: $r \in X \leftrightarrow Y$ is said to be total iff $\text{total}(r) \overset{\text{df}}{=} \forall\, x \in X, \exists\, y \in Y \cdot (x, y) \in r$ it true.

Functionality: $r \in X \leftrightarrow Y$ is said to be functional iff
$\text{functional}(r) \overset{\text{df}}{=} \forall\, y_1, y_2 \in Y, \forall\, x \in X \cdot (x, y_1) \in r \wedge (x, y_2) \Rightarrow y_1 = y_2$ is true.

Surjectivity: $r \in X \leftrightarrow Y$ is said to be surjective iff $\text{surjective}(r) \overset{\text{df}}{=} \forall\, y \in Y, \exists\, x \in X \cdot (x, y) \in r$ is true.

Injectivity: $r \in X \leftrightarrow Y$ is said to be injective iff
$\text{injective}(r) \overset{\text{df}}{=} \forall\, x_1, x_2 \in X, \forall\, y \in Y \cdot (x_1, y) \in r \wedge (x_2, y) \Rightarrow x_1 = x_2$ is true.

Set of Functional Relations: $X \to Y \overset{\text{df}}{=} \{r \in X \leftrightarrow Y : \text{functional}(r)\}$.

2 Multisets

2.1 Definitions

Set of Multisets Over X: $\mathbb{M}\, X \overset{\text{df}}{=} \{r \in X \to \mathbb{N} : \text{total}(r)\}$.

Empty Multiset: $\not\sqsubset \overset{\text{df}}{=} \lambda\, x \in X \cdot 0$.

Multiset Element Of: $x \sqsubset T \overset{\text{df}}{=} T(x) > 0$

Multiset Subset: $T \sqsubseteq S \stackrel{\text{df}}{=} \forall\, x \in X \cdot T(x) \leq S(x)$.

Multiset Equality: $T \stackrel{\text{mst}}{=} S \stackrel{\text{df}}{=} \forall\, x \in X \cdot T(x) = S(x)$.

Multiset Union: $T \sqcup S \stackrel{\text{df}}{=} \lambda\, x \in X \cdot T(x) + S(x)$.

Multiset Intersection: $T \sqcap S \stackrel{\text{df}}{=} \lambda\, x \in X \cdot \min(\{T(x), S(x)\})$.

2.2 Theorems

Theorem 7.4: $T \sqcup S \stackrel{\text{mst}}{=} S \sqcup T$.

Theorem 7.6: $T \sqcap S \sqsubseteq T \sqcup S$.

2.3 Exercises

Exercise 7.1: $T \sqsubseteq T$ *(reflexive)* and $T \sqsubseteq R \land R \sqsubseteq S \Rightarrow T \sqsubseteq S$ *(transitive)*.

Exercise 7.2: $\not\sqsubset \sqcap T \stackrel{\text{mst}}{=} T \sqcap \not\sqsubset \stackrel{\text{mst}}{=} \not\sqsubset$ and $\not\sqsubset \sqcup T \stackrel{\text{mst}}{=} T \sqcup \not\sqsubset \stackrel{\text{mst}}{=} T$.

Exercise 7.3: $T \sqcap T \stackrel{\text{mst}}{=} T$ *(idempotent)*, $T \sqcap S \stackrel{\text{mst}}{=} S \sqcap T$ *(commutative)* and $(T \sqcap S) \sqcap R \stackrel{\text{mst}}{=} T \sqcap (S \sqcap R)$ *(associative)*.

Exercise 7.4: $(T \sqcup S) \sqcup R \stackrel{\text{mst}}{=} T \sqcup (S \sqcup R)$ *(associative)*.

Exercise 7.5: $T - S \stackrel{\text{df}}{=} \lambda\, x \in X \cdot \max(\{0, T(x) - S(x)\})$ *(multiset difference)*, $T - T \stackrel{\text{mst}}{=} \not\sqsubset$ and $T - \not\sqsubset \stackrel{\text{mst}}{=} T$.

3 Sequences

3.1 Definitions

Finite Subsets of $\mathbb{N}$: $\text{upto}(N) \stackrel{\text{df}}{=} \{n \in \mathbb{N} : n < N\} = \{0, 1, \ldots, N-1\}$.

Set of Sequences Over X: $\text{seq}\ X = \{s_X \in \mathbb{N} \to X : \exists\, N \in \mathbb{N} \cdot \text{dom}(s_X) = \text{upto}(N)\}$.

Empty Sequence: $\langle\rangle \stackrel{\text{df}}{=} \lambda\, n \in \mathbb{N} \cdot \text{undefined}$.

Head of a Sequence: $\text{head}(s_X) \stackrel{\text{df}}{=} s_X(0)$.

Tail of a Sequence: $\mathrm{tail}(s_X) \stackrel{\mathrm{df}}{=} \lambda\, n \in \mathbb{N} \cdot s_X(n+1)$.

Cons Operator:

$$\mathrm{cons}(x, s_X) \stackrel{\mathrm{df}}{=} \lambda\, n \in \mathbb{N} \cdot \begin{cases} x & \text{if } n = 0 \\ s_X(n-1) & \text{otherwise} \end{cases}.$$

Length of a Sequence:

$$\mathrm{len}(s_X) \stackrel{\mathrm{df}}{=} \begin{cases} 0 & \text{if } s_X \stackrel{\mathrm{seq}}{=} \langle\rangle \\ 1 + \mathrm{len}(\mathrm{tail}(s_X)) & \text{otherwise} \end{cases}$$

$$= \lambda\, n \in \mathbb{N} \cdot \begin{cases} 0 & \text{if } s_X \stackrel{\mathrm{seq}}{=} \langle\rangle \\ 1 + \max(\mathrm{dom}(s_X)) & \text{otherwise} \end{cases}.$$

Last Element of a Sequence: $\mathrm{last}(s_X) \stackrel{\mathrm{df}}{=} s_X(\mathrm{len}(s_X) - 1)$.

Sum of a Sequence of Natural Numbers:

$$\mathrm{sum}(s_{\mathbb{N}}) \stackrel{\mathrm{df}}{=} \begin{cases} 0 & \text{if } s_{\mathbb{N}} \stackrel{\mathrm{seq}}{=} \langle\rangle \\ \mathrm{head}(s_{\mathbb{N}}) + \mathrm{sum}(\mathrm{tail}(s_{\mathbb{N}})) & \text{otherwise} \end{cases}.$$

Sequence Concatenation:

$$t_X +\!\!+\, s_X \stackrel{\mathrm{df}}{=} \begin{cases} s_X & \text{if } t_X \stackrel{\mathrm{seq}}{=} \langle\rangle \\ \mathrm{cons}(\mathrm{head}(t_X), \mathrm{tail}(t_X) +\!\!+\, s_X) & \text{otherwise} \end{cases}$$

$$= \lambda\, n \in \mathbb{N} \cdot \begin{cases} t_X(n) & \text{if } n \in \mathrm{dom}(t_X) \\ s_X(n - \mathrm{len}(t_X)) & \text{otherwise} \end{cases}.$$

Prefix of a Sequence: $t_X \preceq_p s_X \stackrel{\mathrm{df}}{=} \exists\, r_X \in \mathrm{seq}\ X \cdot t_X +\!\!+\, r_X \stackrel{\mathrm{seq}}{=} s_X$.

Suffix of a Sequence: $t_X \preceq_s s_X \stackrel{\mathrm{df}}{=} \exists\, r_X \in \mathrm{seq}\ X \cdot r_X +\!\!+\, t_X \stackrel{\mathrm{seq}}{=} s_X$.

Exact Subsequence: $t_X \preceq s_X \stackrel{\mathrm{df}}{=} \exists\, r_{1_X}, r_{2_X} \in \mathrm{seq}\ X \cdot r_{1_X} +\!\!+\, t_X +\!\!+\, r_{2_X} \stackrel{\mathrm{seq}}{=} s_X$.

3.2 Exercises

Exercise 7.7: $\text{len}(s_X) = 1 \Rightarrow \text{head}(s_X) = \text{last}(s_X)$.

Exercise 7.8: $x \lessdot s_X \overset{\text{df}}{=} \exists\, n \in \mathbb{N} \cdot n \in \text{dom}(s_X) \wedge s_X(n) = x$ *(sequence element of).*

Exercise 7.9:

$$\text{sorted}(s_X) \overset{\text{df}}{=} \begin{cases} \textbf{True} & \text{if } \text{len}(s_X) \leq 1 \\ \text{head}(s_X) \leq_X \text{head}(\text{tail}(s_X)) \wedge \text{sorted}(\text{tail}(s_X)) & \text{otherwise} \end{cases}.$$

Exercise 7.10: $\text{items} \overset{\text{df}}{=} \lambda\, s_X \in \text{seq } X \cdot (\lambda\, x \in X \cdot \#\{n \in \mathbb{N} : s_X(n) = x\})$.

Exercise 7.11: $\text{sorts} \overset{\text{df}}{=} \{(s_X, s'_X) \in \text{seq } X \times \text{seq } X : \text{items}(s_X) \overset{\text{mst}}{=} \text{items}(s'_X) \wedge \text{sorted}(s_X)\}$.
Hence, $(s_X, s'_X) \in \text{sorts}$ iff s_X is a sorted version of s'_X.

Exercise 7.12: $s_X \overset{\text{seq}}{=} s'_X \overset{\text{df}}{=} \text{len}(s_X) = \text{len}(s'_X) \wedge \forall\, n \in \mathbb{N} \cdot n \in \text{upto}(\text{len}(s_X)) \Rightarrow s_X(n) = s'_X(n)$.

Exercise 7.13:

(i) $$\text{reverse}(s_X) \overset{\text{df}}{=} \begin{cases} s_X & \text{if } \text{len}(s_X) \leq 1 \\ \text{reverse}(\text{tail}(s_X)) +\!\!+\, \text{cons}(\text{head}(s_X), \langle\rangle) & \text{otherwise} \end{cases}.$$

(ii) $\text{reverse}(s_X) \overset{\text{df}}{=} \lambda\, n \in \mathbb{N} \cdot s_X(\text{len}(s_X) - (n+1))$.

4 Graph Theory

4.1 Definitions

Directed Multigraph: G is said to be a directed multigraph iff $G = (V, L, E)$ where $E \subseteq V \times L \times V$.

A Clique in G: $V' \subseteq V$ is said to be a clique iff
$\text{clique}(V') \overset{\text{df}}{=} \forall\, v, v' \in V' \cdot v \neq v' \Rightarrow (\exists\, l \in L \cdot (v, l, v') \in E)$ is true.

A Walk in G: $s_V \in V$ is said to be a walk iff
$\text{walk}(s_V) \overset{\text{df}}{=} \forall\, n \in \mathbb{N} \cdot n \in \text{upto}(\text{len}(s_V) - 1) \Rightarrow (\exists\, l \in L \cdot (s_V(n), l, s_V(n+1) \in E)$ is true.

Set of Walks in G: $\text{walks}(G) \overset{\text{df}}{=} \{s_V \in \text{seq } V : \text{walk}(s_V)\}$.

A Cycle in G: $s_V \in V$ is said to be a cycle iff $\text{cycle}(s_V) \stackrel{\text{df}}{=} \text{walk}(s_V) \wedge \text{head}(s_V) = \text{last}(s_V)$ is true.

Set of Cycles in G: $\text{cycles}(G) \stackrel{\text{df}}{=} \{s_V \in \text{seq } V : \text{cycle}(s_V)\}$.

Directed Acyclic Graph (DAG): G is said to be a DAG iff $\text{dag}(G) \stackrel{\text{df}}{=} \text{cycles}(G) = \emptyset$ is true.

Set of Predecessors of v: $\text{predecessors}(v) \stackrel{\text{df}}{=} \{v' \in V : \exists\, l \in L \cdot (v', l, v) \in E\}$.

Set of Vertices Reachable from v:
$\text{reachable}(v) \stackrel{\text{df}}{=} \{v' \in V : \exists s_V \in \text{seq } V \cdot \text{walk}(s_V) \wedge \text{head}(s_V) = v \wedge \text{last}(s_V) = v'\}$.

Tree: G is said to be a tree iff $\text{tree}(G) \stackrel{\text{df}}{=} (\forall\, v \in V \cdot \#\,\text{predecessors}(v) \leq 1) \wedge (\exists_1\, v \in V \cdot \#\,\text{predecessors}(v) = 0) \wedge (\forall\, v \in V \cdot \#\,\text{predecessors}(v) = 0 \Rightarrow \text{reachable}(v) = V)$ is true.

A Label Walk in G: $s_L \in \text{seq } L$ is said to be a label walk iff
$\exists\, s_V \in \text{seq } V \cdot \text{labelwalk}(s_L, s_V) \stackrel{\text{df}}{=} \text{walk}(s_V) \wedge \text{len}(s_V) = \text{len}(s_L) + 1 \wedge (\forall\, n \in \mathbb{N} \cdot n \in \text{upto}(\text{len}(s_V) - 1) \Rightarrow (s_V(n), s_L(n), s_V(n+1)) \in E)$ is true.

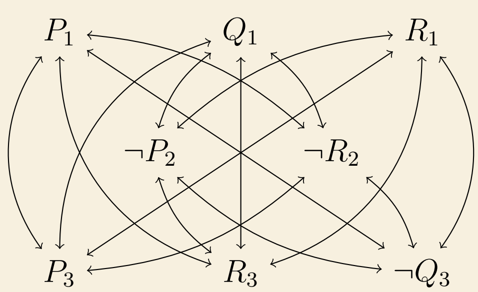
Set of Label Walks in G: $\text{labelwalks}(G) \stackrel{\text{df}}{=} \{(s_L, s_V) \in \text{seq } L \times \text{seq } V : \text{labelwalk}(s_L, s_V)\}$.

Travelling Salesman Problem (TSP): Let $G = (V, \mathbb{N}, E)$ and suppose a buget B, is the following predicate true? $\exists\,(s_L, s_V) \in \text{labelwalks}(G) \cdot \text{ran}(s_V) = V \wedge \text{sum}(s_L) \leq B$.

Boolean Satisfiability Problem (SAT): Consider the following boolean expression:

$$\begin{aligned} & (P \vee Q \vee R) \\ \wedge\ & (\neg P \vee \neg R) \\ \wedge\ & (P \vee R \vee \neg Q) \end{aligned}$$

$$\downarrow$$



Vertices on the same level e.g. P_1 and Q_1 are not connected. Additionally, complementary vertices e.g. P_1 and $\neg P_2$ are not connected.

Finding a clique corresponds to a solution for the boolean expression e.g. Q_1, $\neg P_2$ and R_3. This means SAT is as hard as finding cliques.

4.2 Exercises

Exercise 7.14: $\text{hamiltonian}(s_V) \stackrel{\text{df}}{=} \forall\, v \in V, \exists_1\, n \in \mathbb{N} \cdot s_V(n) = v.$

Exercise 7.15: $\text{simplecycle}(s_V) \stackrel{\text{df}}{=} \text{cycle}(s_V) \wedge \#\,\text{rng}(s_V) + 1 = \text{len}(s_V),$
$\text{simplecycles}(G) \stackrel{\text{df}}{=} \{s_V \in \text{seq}\ V : \text{simplecycle}(s_V)\}.$

Exercise 7.16: $\text{path}(s_V) \stackrel{\text{df}}{=} \text{walk}(s_V) \wedge \#\,\text{rng}(s_V) = \text{len}(s_V),$
$\text{diameter}(G) \stackrel{\text{df}}{=} \max(\{s_V \in \text{seq}\ V : \text{path}(s_V) \cdot \text{len}(s_V)\}).$

Exercise 7.17:

(i) $\hat{E} \stackrel{\text{df}}{=} \{(v, l, v') \in V \times L \times V : (v, l, v') \in E \cdot (v', l, v)\}$, $\hat{G} \stackrel{\text{df}}{=} (V, L, E \cup \hat{E})$ *(undirected G).*

(ii) $\text{connected}(G) \stackrel{\text{df}}{=} \forall\, v, v' \in V, \exists\, s_V \in \text{seq}\ V \cdot \text{path}(s_V) \wedge \text{head}(s_V) = v \wedge \text{last}(s_V) = v',$
$\text{weaklyconnected}(G) \stackrel{\text{df}}{=} \text{connected}(\hat{G}).$

(iii) $\text{sharedstructure}(G) \stackrel{\text{df}}{=} \exists\, v, v' \in V, \exists\, s_V, s'_V \in \text{seq}\ V \cdot \text{path}(s_V) \wedge \text{path}(s'_V) \wedge \neg(s_V \stackrel{\text{seq}}{=} s'_V) \wedge ((\text{head}(s_V) = \text{head}(s'_V) = v \wedge \text{last}(s_V) = \text{last}(s'_V) = v') \vee (\text{head}(s_V) = \text{last}(s'_V) = v \wedge \text{last}(s_V) = \text{head}(s'_V) = v'))$ *(at least two vertices are joined by two distinct paths).*

(iv) $\text{tree}(G) \stackrel{\text{df}}{=} \text{dag}(G) \wedge \text{weaklyconnected}(G) \wedge \neg\,\text{sharedstructure}(G).$