To find Fibonacci sums very quickly, the two results listed below **?** can be used.

$\mathrm{F}_n = \frac{\phi^n}{\sqrt{5}}, \quad \phi = \frac{1+\sqrt{5}}{2}.$

$\sum_{i=1}^{n} F_i = F_{n+2} - 1.$

The original equation for the Fibonacci numbers is a recurrence relation defined as

$\mathrm{F}_n = F_{n-1} + F_{n-2}, \qquad F_1 = F_2 = 1.$

However, there exists a closed-form expression called Binet's Formula **?** which is the following

$\mathrm{F}_n = \frac{\phi^n - (-\phi)^n}{\sqrt{5}}.$

One notices that,

$\forall$ n $\in$N : (-ϕ)$^n \overline{\sqrt{5} < \frac{1}{2},}$

which implies

$\forall$ n $\in$N : $\mathrm{F}_n - \frac{\phi^n}{\sqrt{5}} < \frac{1}{2}.$

Visually, this can be represented as follows.

push0 g 0 G pop [thick] (-4, 0) – (4, 0); [thick] (0, -0.2) – (0, 0.2);
at (-1.5, 0) (; at (1.5, 0));
[below] at (0, -0.25) F_n; [below] at (-1.5, -0.25) $F_n - \frac{1}{2}$; [below] at (1.5, -0.25) $F_n + \frac{1}{2}$;
[thick, ->] (-1, 0.5) – (-1, 0);
[above] at (-1, 0.5) $\frac{\phi^n}{\sqrt{5}}$;

This allows to conclude that $\frac{\phi^n}{\sqrt{5}}$ is always within rounding error of the actual Fibonacci number. Hence, by rounding $\frac{\phi^n}{\sqrt{5}}$ we get the n-th Fibonacci number.

This completes our derivation for **??**.

We can prove **??** by using an inductive argument.

Argument.

Base case ($n = 1$):

$$LHS = \sum_{i=1}^{1} F_i = F_1 = 1$$

$$RHS = F_{1+2} - 1 = F_3 - 1 = F_2 + F_1 - 1 = 1 + 1 - 1 = 1$$

Hence, the base case holds since $LHS = RHS$.

Inductive case ($n = k$):
Suppose that sum holds for $n = k - 1$.
$\sum_{i=1}^{k-1} F_i = F_{k-1+2} - 1 = F_{k+1} - 1$
We are required to show that the sum holds for $n = k$.
$\sum_{i=1}^{k} F_i = F_k + \sum_{i=1}^{k-1} F_i$
$= F_{k+1} + F_k - 1$
$= F_{k+2} - 1$
Therefore, by induction, **??** holds for all natural numbers.