Graph Theory

Personal Research

Juan Scerri
B.Sc. (Hons)(Melit.) Computing Science and Mathematics (Second Year)

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Conjecture 1 (Erdős–Gyàrfàs Conjecture). *Every simple graph G with minimum degree 3 contains a simple cycle whose length is a power of two.*

At first glance there does not seem any clear way to approach the problem. So instead we will define a weaker version of the conjecture. Hopefully, the machinery developed for solving the weaker conjecture will prove to be useful when tackling the full conjecture.

Conjecture 2. *Every simple graph G with minimum degree 3 contains a simple cycle whose length is even.*

We will try to approach this by assuming that the all cycles are of odd length. Now we need to look at the most general way cycles of odd length can interact with each other.

Definition 1. *Let C be a cycle. C is said to be a k–mod–cycle if and only if the length of C is divisible by k.*

Definition 2. *A 2–mod–cycle is said to be an even cycle. And a non–2–mod–cycle is a said to be an odd cycle.*

Proposition 1. *If G and H are two odd cycles connected by at least two edges (let this graph be F), then F contains an even cycle.*

Proof. Let $V(G) = \{g_1, g_2, \ldots, g_k\}$ and $V(H) = \{h_1, h_2, \ldots, h_l\}$ where k and l are odd numbers greater than 3. Pick any two vertices in $V(G)$ say g_a and g_b and pick any two vertices in $V(H)$ say h_c and h_d such that $h_c \neq h_d$. h_c and h_d cannot be equal because if they where and $g_a = g_b$, then we would have connected the cycles with only one unique edge but we want do do so with two distinct edges.

Since, the length of G is odd picking any two vertices (even the same vertex) will split the cycle into two, a segment having even length and a segment having odd length. Similarly, this will happen on H as well. Starting from h_c, traverse the even path to h_d. Crossover, to H using the edge connected to h_d. Say this is connected to g_b. From g_b traverse the even path to g_a. Then crossover again to h_c. This completes the cycle and the cycle is even since both paths have the same parity and we are adding 2 because of the two crossovers. □

Proposition 2. *If G and H are two odd cycles and they share at least 2 and at most $\max\{V(G), V(H)\} - 1$ vertices (let this graph be F), then F contains an even cycle.*

Proposition 3. *If G and H are two odd cycles connected by at least two paths of length greater than 1 (let this graph be F), then F contains an even cycle.*

TODO: The above proof should be almost as the proof for proposition simple to prove as the proof for proposition 1. The basic idea is as follows look at the graph F if it has two paths of the same parity connecting G and H this is reduced to the case for proposition 1 since adding two lengths of same parity and another two lengths of same parity will result in a final length of even parity. The only difficulty is considering the case when one cannot pick two paths which have identical parity. In that case one is forced to traverse the segment of one of the cycles which has opposite parity to the segment walk on the other. Hence giving us two length which are of same parity and another two lengths which are of same parity. Hence, the total is also guaranteed to be even.

So it is true for odd cycles which are connected by at least two paths. What about shared vertices. This is a more complicated endeavour. Additionally, it will also be more complicated to show this for when a mixture of the two types is present. But let us suppose for second that we have shown that said statements are true. Using this knowledge we can try to show that our weak conjecture is true.

So suppose that we have a connected minimum degree 3 graph and that all cycles is said graph are odd. Now what we want to show is that any about our contradiction

TODO: Add exposition as to why I arrived at this point.

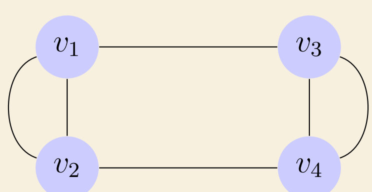


Figure 1: Smallest non–trivial case

So all the different paths in the above configuration are the following:

1. $v_1v_2v_4v_3$

2. $v_1v_2v_4v_3$

3. $v_1v_2v_4v_3$

4. $v_1v_2v_4v_4$

TODO: Introduce labels for this case only because technically this is a multigraph. However, it is still a valid case for analysis in our case.

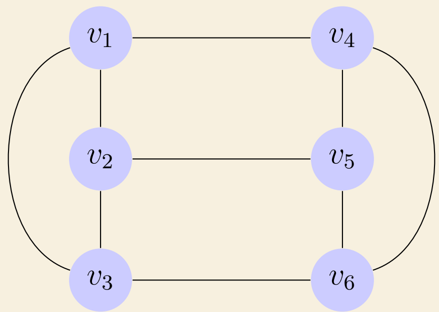


Figure 2: The 3 case

Again we will manually count all of the unique cycles. (From 3 onwards we do not need to worry about the ambiguity of our notation since these are not multigraphs).

1. $v_1v_2v_5v_4v_1$

2. $v_2v_3v_6v_5v_2$

3. $v_1v_4v_6v_3v_1$

4. $v_1v_2v_5v_6v_4v_2$

5. $v_1v_2v_3v_6v_4v_1$

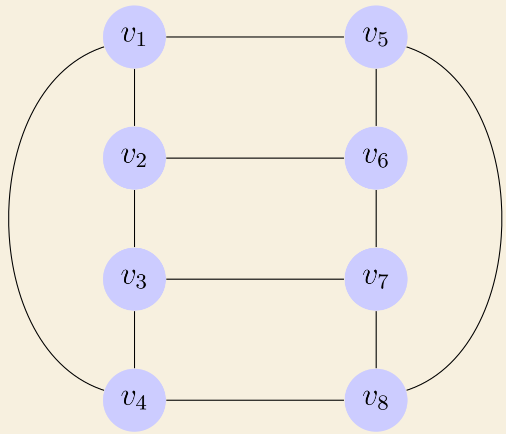
6. $v_2v_3v_1v_4v_5v_2$

7. $v_2v_3v_1v_4v_5v_6v_2$

8. $v_1v_2v_3v_6v_5v_4v_1$

9. $v_4v_5v_2v_1v_3v_6v_4$

Note: From what I recall you can only have an even number of crossings across the bridges formed. An odd number of crossings is impossible. Since you would connect back to something which you already visited which is not actually your returning node.


v_1
v_5
v_2
v_6
v_3
v_7
v_4
v_8