Multiphase Flow: An Introduction

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1 Introduction

Multiphase flow concerns flows with more than one phase (component). For example, it could be gas-liquid, liquid-liquid, or solid-liquid flow. Multiphase flow is naturally highly relevant in science and technology – boiling of water, oil extraction, blood flow, bacteria suspension and the cell cytoplasm.

There are two general topologies of multiphase flow: dispersive flows and separate flows. In dispersive flows, finite particles, drops or bubbles are distributed in a connected volume of the continuous phase. In separated flows, two or more continuous streams of different fluids are separated by interfaces (Fig. 1).

1.1 Navier-Stokes equation

To study fluid flow, the starting point is the Navier-Stokes (NS) equation, which is of the form [1]:

$$\begin{aligned} \rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) &= -\nabla p + \nabla\cdot\left[\mu\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right)\right] + \nabla\left(-\frac{2\mu}{3}\nabla\cdot\mathbf{v}\right) + \mathbf{f}_{ext} \quad (1) \\ \partial_t\rho + \nabla\cdot(\rho\mathbf{v}) &= 0 \; , \quad (2) \end{aligned}$$

where $\mathbf{v}$ is the fluid velocity, ρ is the density, p is the pressure, μ is the viscosity, $\mathbf{f}_{ext}$ is any other external forces acting on the system, such as due to gravity or thermal fluctuations [2]. For an incompressible fluid ($\nabla\cdot\mathbf{v} = 0$), the equation of motion (EOM) simplifies to (Exercise 1.3.1)

$$\rho\left(\partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) = -\nabla p + \mu\nabla^2\mathbf{v} + \mathbf{f}_{ext} \; . \quad (3)$$

Throughout the course, we will assume that the liquid phase is always incompressible.

To study a multiphase flow involving two fluids, we will have two NS equations with potentially different parameters that we will need to solve with the appropriate boundary conditions at the interface. If we ignore mass and heat exchange at the interface (which we will do in this course), the boundary condition on $\mathbf{v}$ is the no-slip boundary at a fluid-solid interface, while at a gas-liquid interface, the boundary condition is the zero tangential stress condition in the liquid phase, which comes about because we typically assume that the viscosity of the gas phase is much smaller than the viscosity of the liquid phase so that the gas phase cannot sustain any nonzero tangential stress.

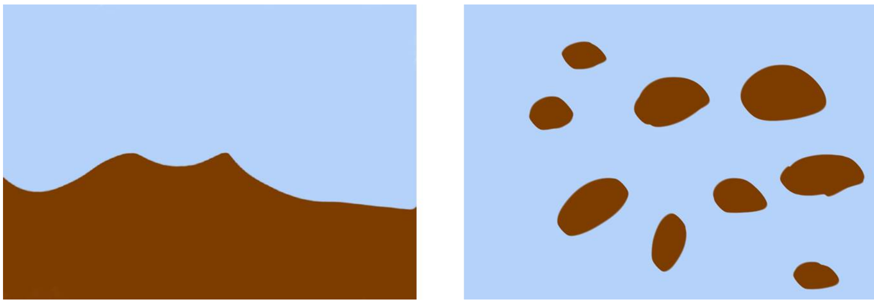

Figure 1: Two topologies of two-phase flows. Separate flows (left) and dispersive flows (right).

1.2 An alternative derivation of the incompressible Navier-Stokes equation

Before we start introducing objects into the fluid, let us see how the incompressible NS could be derived by considering the structure of the equation along, with a bit of physics.[1]

Since we are interested in how an incompressible fluid flows, the variable of interest is naturally the velocity $\mathbf{v}$, whose dynamics can be generally written as:

$$\partial_t v_i = F_i(\mathbf{v}) \quad (4)$$

[1]This chapter follows closely the exposition in [3] where the authors investigate "active" fluids

where the vector field $\mathbf{F}$ depends on $\mathbf{v}$ and its derivatives. To the zeroth order in $\mathbf{v}$, we can in principle have a constant vector α_i as a term in $\mathbf{F}$. But since we are dealing with an isotropic fluid, there is no reason why a particular direction, as set by the vector $\vec{\alpha}$, should be preferred. We therefore conclude that a constant term is ruled out. To the first order, we can add the term $\beta_1 v_i + \beta_2 \partial_j \partial_j v_i + \beta_3 \partial_i(\partial_j v_j) + \mathcal{O}(\partial^4 \mathbf{v})$ (why can't we have $\partial^3 \mathbf{v}$ terms?). But since $\partial_j v_j = 0$ due to the incompressibility condition, we can ignore the β_3 term. One can continue like this and arrived at the following general equation of motion (EOM), to order $\mathcal{O}(v^2, \partial^2)$:

$$\partial_t v_i = \beta_1 v_i + \beta_2 \nabla^2 v_i + \gamma_1 v_i \partial_j v_i + \gamma_2 v_j \partial_j v_i + \lambda_i \tag{5}$$

where $\vec{\lambda}$ is there purely to enforce the incompressibility condition $\nabla \cdot \mathbf{v} = 0$.

Now, let's add a bit more physics to the problem, namely, the Galilean invariance of the form: $\mathbf{v}'(\mathbf{r}', t) = \mathbf{v}(\mathbf{r} - \mathbf{u}t, t) + \mathbf{u}$. Then $\partial_t v'_i = \partial_t v_i - \mathbf{u} \cdot \nabla v_i$, $v'_i = v_i + u_i$, $\nabla'^2 v'_i = \nabla^2 v_i$, $v'_i \partial'_j v'_i = u_i \partial_j v_i + v_i \partial_j v_i$, $v'_j \partial'_j v'_i = u_j \partial_j v_i + v_j \partial_j v_i$. From this, we can conclude $\beta_1 = 0$, $\gamma_1 = 0$ and $\gamma_2 = -1$, the EOM thus becomes

$$\partial_t v_i + v_j \partial_j v_i = \nu \nabla^2 v_i + \rho^{-1} \partial_i p \tag{6}$$

where we have replace γ_3 by ν and $\vec{\lambda}$ by $\rho^{-1} \nabla p$ and call p the pressure, which can be determined by solving a Poisson equation.

Note that strictly speaking, our equation is only true to order $\mathcal{O}(v^2, \partial^2)$. We can argue that higher order terms are unimportant since we are interested in coarse-grained (long wavelength) behaviour (hence higher order derivatives become small) and slow velocity compared to the speed of sound as the incompressibility condition breaks if the speed gets close to the speed of sound.

1.3 Exercises

Exercise 1.3.1 *Show that the Navier-Stokes in Eq. (1) reduces to Eq. (3) if the flow is incompressible.*

Exercise 1.3.2 *Show that an incompressible flow does not necessarily imply constant density throughout the system.*

2 Solid spheres in liquid

A simple multiphase scenario concerns solid particles flowing in a continuous liquid phase. As a preliminary consideration, let us study a solid sphere of radius R held in place and is subject to a uniform flow of speed U. For *steady* flows, i.e., flows that are not time dependent ($\partial_t \mathbf{v} = 0$), the flow field can be solved in the inviscid and inertia-free Reynolds number's limits.

2.1 Stead flow

2.1.1 Inviscid limit

In this limit, we ignore the viscosity term. The EOM is thus

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p \ , \tag{7}$$

together with the following conditions:

$$\begin{aligned} \nabla \cdot \mathbf{v} &= 0 && (8) \\ v_r|_{r=R} &= 0 && (9) \\ \lim_{r\to\infty} \mathbf{v} &= U\mathbf{x} && (10) \\ \lim_{r\to\infty} p &= \text{constant} \ . && (11) \end{aligned}$$

To solve this problem, we immediately encounter two problems: 1) The nonlinear advective term $(\mathbf{v} \cdot \nabla \mathbf{v})$ makes it difficult to solve; 2) We don't seem to have an equation to determine the pressure p.

Let's forget about these two difficulties and try to simplify the problem a bit first. We all know about the Kelvin's theorem, which says that an initially vorticity-free (i.e., $\nabla \times \mathbf{v} = 0$) inviscid flow remains vorticity-free forever. In our case, since the flow far ahead of the sphere is certainly vorticity-free, it will remain vorticity-free around the sphere. Therefore, there exists a scalar function ϕ such that $\nabla \phi = \mathbf{v}$. This is great because instead of having to deal with a vector field $\mathbf{v}$, we now only need to worry about a scalar field ϕ. Note again that this reduction of degrees of freedom is due to the vorticity-free condition $\nabla \times \mathbf{v}$ which is guaranteed by the Kelvin's theorem.

Now, instead of tackling the EOM in Eq. (7), let us tackle the incompressibility condition first, which leads to the Laplace equation: $\nabla^2 \phi = 0$. The general solution with azimuthal symmetry is of the form [4]:

$$\phi = \sum_{n=0}^{\infty} \left(A_n r^n + B_n r^{-n-1} \right) P_n(\cos\theta) \tag{12}$$

where $P_n(\cos\theta)$ are the Legendre polynomials.

The solution that satisfies the boundary conditions in Eqs (9) & (10) is

$$\phi = U \cos\theta \left(r + \frac{R^3}{2r^2} \right) \tag{13}$$

and the corresponding flow field is (see Fig. 2):

$$\begin{aligned} v_r &= \partial_r \phi = U\cos\theta - U \left(\frac{R}{r} \right)^3 \cos\theta & (14) \\ v_\theta &= \frac{1}{r} \partial_\theta \phi = -U \sin\theta - \frac{U}{2} \left(\frac{R}{r} \right)^3 \sin\theta\ . & (15) \end{aligned}$$

There we have it. We have solved the flow field without ever needing the EOM in Eq. (7). But we can now use the EOM to get the pressure (try it). Here, instead of doing that, let us use the Bernoulli's theorem, which dictates that

$$p = \frac{\rho}{2} \left(U^2 - v_r^2 - v_\theta^2 \right) \tag{16}$$

where we have set $p|_{r\to\infty}$ to zero (see Fig. 2). In particular, at the surface of the sphere

$$p(R,\theta) = \frac{\rho}{2} U^2 \left(1 - \frac{9\sin^2\theta}{4} \right)\ . \tag{17}$$

Knowing the pressure on the surface of the sphere allows us to calculate the force it experiences, which amounts to

$$\mathbf{f} = -\oint_A p \hat{n} \mathrm{d}S \tag{18}$$

where A denotes the surface of the sphere and $\hat{n}$ the normal vector of the surface pointing radially outward from the centre of the sphere. Since the pressure variation goes like $\sin^2\theta$, the pressure distribution in the front half of the sphere is exactly the mirror image of the other half of the sphere. In other words, $\mathbf{f} = 0$ and so the sphere feels no force (or *drag*) even though fluid is rushing past it. This is the *d'Alembert's paradox*! This result is of course nonsense as you can readily check by holding your hand out of the window of a moving car. The paradox can be resolved by putting the viscosity term $(\mu \nabla^2 \mathbf{v})$ back in. With the viscosity term, we need one more boundary condition to solve the NS equation, which is $u_\theta|_{r=R} = 0$. This is due to the fact that the steady NS equation is now one order of derivative higher than the Euler equation. This new *no-slip* boundary condition induces vorticity into the flow close to the sphere's surface. As a result, the pressure is no longer symmetric with respect to the mid-plane of the sphere. In addition, there is further drag coming from the stresses experienced by the sphere's surface due to the shearing of the liquid. These effects combined induce a drag force on the sphere.

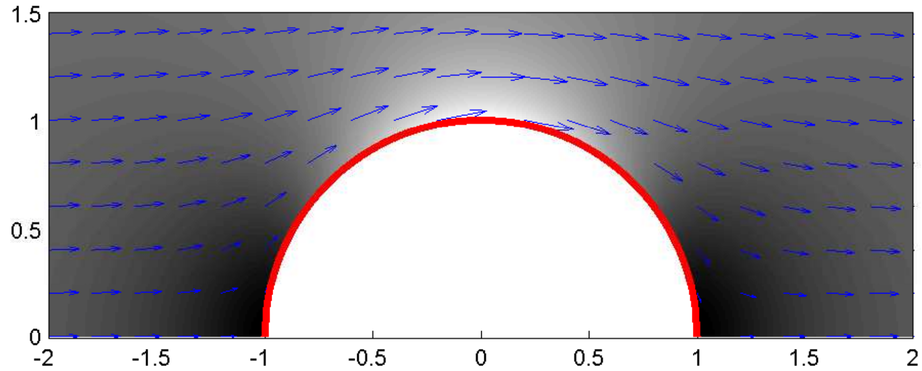

Figure 2: Inviscid flow past a sphere with velocity vector field and pressure field depicted.

2.1.2 Inertia-free limit

In this limit, we ignore the terms $\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}$ in the NS equation. We are thus left with

$$\frac{1}{\rho}\nabla p = \nu \nabla^2 \mathbf{v} \ . \tag{19}$$

By taking the curl on both sides of the above equation, we can get rid of the pressure term to arrive at

$$0 = \nabla \times \nabla^2 \mathbf{v} \ . \tag{20}$$

Instead of using the potential function ϕ in the inviscid flow (such a potential doesn't exist here because the flow is not vorticity-free), we employ the *streamline function* ψ defined as follows

$$v_r = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta} \quad , \quad v_\theta = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} \ . \tag{21}$$

Note that the incompressibility condition is automatically satisfied. Indeed, it is the incompressibility condition that guarantees the existence of ψ. Eq. (20) becomes:

$$\mathcal{L}^2 \psi = 0 \tag{22}$$

where

$$\mathcal{L} \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\right) \ . \tag{23}$$

Knowing that at $r \to \infty$, $\psi = Ur^2 \sin^2\theta/2$, let us try $\psi(r,\theta) = f(r)\sin^2\theta$. Eq. (22) then implies that

$$\psi = \left(Ar^4 + Br^2 + Cr + \frac{D}{r}\right)\sin^2\theta \ . \tag{24}$$

With the boundary conditions $v_\theta|_{r=R} = 0$, $v_r|_{r=R} = 0$, the solution is thus

$$\psi = U\sin^2\theta\left[\frac{R^3}{4r} - \frac{3Rr}{4} + \frac{r^2}{2}\right] \tag{25}$$

and so (see Fig. 3).

$$v_r = U\cos\theta\left[1 - \frac{3R}{2r} + \frac{R^3}{2r^3}\right] \tag{26}$$

$$v_\theta = -U\sin\theta\left[1 - \frac{3R}{4r} - \frac{R^3}{4r^3}\right] \ . \tag{27}$$

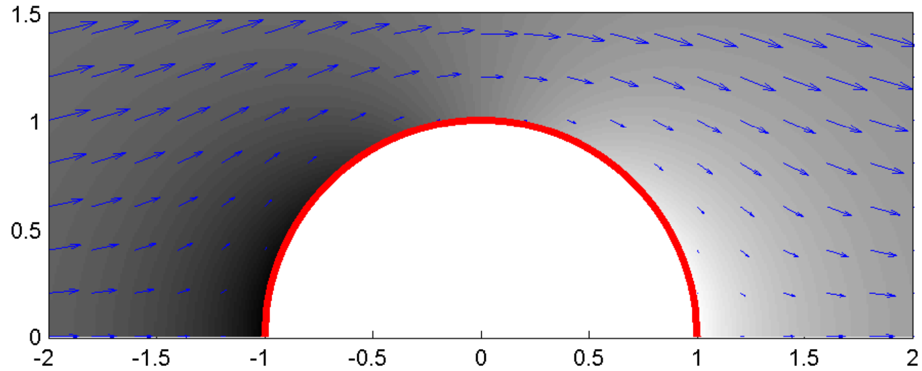

Figure 3: Stokes flow past a sphere with velocity vector field and pressure field depicted.

Substituting the above expressions into Eq. (19) enables us to obtain the pressure distribution, which is

$$p = -\frac{3\mu U R \cos\theta}{2r^2} \ . \tag{28}$$

To find the drag on the sphere, we calculate the force experienced by the sphere. Besides the pressure term, there is also the contributions from the tangential stress on the surface. As a result,

$$f_x = \oint_A (-p\cos\theta - \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\mathrm{d}S \tag{29}$$

where the viscous stress tensor σ is:

$$\sigma_{rr} = 2\mu\frac{\partial v_r}{\partial r} \quad , \quad \sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) \ . \tag{30}$$

Doing the surface integral gives that the drag coefficient is given by the Stokes' formula $6\pi\mu R$.

2.1.3 Viscosity of dilute suspensions

Imagine having lots of solid spheres in a viscous fluid, which constitutes a suspension. The viscosity of the suspension is modified due to the presence of the spheres. If the concentration of spheres is low enough, we can make the assumption that the increase of the viscosity is linear in the sphere's concentration. Let us now examine how the viscosity is affected by the presence of just one sphere. To do so, we consider how an extensile flow along the x-axis is affected by the presence of one sphere at the origin. The extensile flow without the sphere is of the form (for $r \gg 0$):

$$v_x = ax \quad , \quad v_y = -\frac{a}{2}y \quad , \quad v_z = -\frac{a}{2}z \ , \tag{31}$$

or in polar coordinates:

$$v_r = \frac{ar(3\cos^2\theta - 1)}{2} \quad , \quad v_\theta = -\frac{3ar\sin\theta\cos\theta}{2} \ . \tag{32}$$

Inserting a sphere of radius R at the origin and solving Eq. (22) with this far-field boundary conditions, we have the following solution [5]:

$$v_r = \frac{ar(3\cos^2\theta - 1)}{2}\left[1 - \frac{5}{2}\left(\frac{R}{r}\right)^3 + \frac{3}{2}\left(\frac{R}{r}\right)^5\right] \tag{33}$$

$$v_\theta = -\frac{3ar\sin\theta\cos\theta}{2}\left[1 - \left(\frac{R}{r}\right)^5\right] \ , \tag{34}$$

and the pressure is given by

$$p = -\frac{5\mu a(3\cos^2\theta - 1)}{2}\left(\frac{r}{R}\right)^3 \ . \tag{35}$$

We can now relate how the viscosity is changed by considering how much work is dissipated in setting up this particular fluid flow pattern. The work done a spherical body of fluid of radius L can be found by calculating the work done at the surface, which is of the form

$$2\pi L^2 \int_0^\pi \left[\left(p - 2\mu\frac{\partial v_r}{\partial r}\right) v_r + \mu\left(r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\right]_{r=L} \sin\theta \mathrm{d}\theta \ . \tag{36}$$

The result is to order $\mathcal{O}((R/L)^3)$

$$3\mu a^3\left(\frac{4\pi L^3}{3}\right)\left[1 + \frac{5}{2}\left(\frac{R}{L}\right)^3\right] \equiv 3\mu' a^3\left(\frac{4\pi L^3}{3}\right) \ . \tag{37}$$

where $\mu' = \mu(1 + 5f/2)$ with f being the volume fraction of the solute in the suspension. In other words, the viscosity of the suspension increases linearly with the solute concentration. This result was first derived by Einstein in 1906 [6].

2.2 Unsteady flow

2.2.1 Added mass

If acceleration or deceleration of the particle occurs, then the resulting flow is unsteady, i.e., time varying. Imagine now that the particle begins to accelerate due to some external force, in order to do so the particle has to move the body of fluid around it. Therefore, the external force would have to be higher than expected to achieve the intended acceleration. In terms of Newton's law, we can write

$$\vec{F} = m_{eff}\vec{a} \tag{38}$$

where $m_{eff} = m_p + m_{add}$ with m_p being the particle mass and m_{add} is the *added mass* coming from the additional fluid inertia during the displacement. Indeed, for a spherical particle, the added mass equals half the fluid mass displaced. To see this, let us consider the acceleration a sphere in an inviscid fluid. The flow potential can be written as (see Exercise 2.4.2)

$$\frac{\partial \phi}{\partial t} = \frac{R^3\cos\theta}{2r^2}\frac{\mathrm{d}U}{\mathrm{d}t} \ . \tag{39}$$

Recall that the Bernoulli's theorem for unsteady potential flow is of the form

$$\frac{p}{\rho} + \frac{v^2}{2} + \frac{\partial \phi}{\partial t} = \text{constant} \ . \tag{40}$$

This thus tells us that on the sphere's surface, the pressure is

$$p|_{r=R} = -\frac{1}{2}\rho R\cos\theta\frac{\mathrm{d}U}{\mathrm{d}t} + \text{constant} \ . \tag{41}$$

Integrating over the surface again to get the force due to acceleration, we find

$$f_{add} = m_{add}\frac{\mathrm{d}U}{\mathrm{d}t} \tag{42}$$

where $m_{add} = 2\pi\rho R^3/3$, which is the mass of the fluid occupying half of the sphere's volume. Note that the same result can be obtained for Stokes flow instead of inviscid flow considered here, although the mathematics is more involved [7].

The effects of this virtual mass are in fact dependent on the shape of the object, the direction of its acceleration, and whether the object is close to the wall. These dependencies should be expected

Figure 4: If the bottle on the left is accelerated to the right from rest, the air bubble on top will accelerate faster compared to the fluid in the bottle because of Eq. (45).

since the added mass effect has to do with how the liquid moves around the body as it accelerates. Because of its directional dependency, the added mass force is generally written as

$$\mathbf{f}_{add} = -M \cdot \dot{\mathbf{u}} \tag{43}$$

where M is the *added mass matrix*. The added mass matrices for a few distinct scenarios can be found in chapter 2 of Ref. [7].

2.2.2 Body force

We have seen how accelerating the solid sphere in fluid is harder because of the added mass coming from displacing the fluid around the body. What if the fluid is now accelerating from rest instead. If the sphere is carried along with the fluid in such a way that $\mathbf{u} = \mathbf{v}$, then the force acting on it would be exactly the force acting on the volume of fluid replacing the sphere if the sphere is absent, i.e., $\mathbf{f} = m_f \dot{\mathbf{v}}$ where $m_f = 4\pi\rho R^3/3$. But the sphere may not flow together with the fluid, and if it does not, then there will be the extra contribution from the added mass. In other words, we have

$$\mathbf{f} = m_f \dot{\mathbf{v}} = m_p \dot{\mathbf{u}} + m_{add}(\dot{\mathbf{u}} - \dot{\mathbf{v}}) \ , \tag{44}$$

or

$$\dot{\mathbf{u}} = \frac{m_f + m_{add}}{m_p + m_{add}} \dot{\mathbf{v}} \ . \tag{45}$$

This means that if the sphere is denser than the fluid, then it will accelerate slower than the fluid acceleration, and *vice versa* (see Fig. 4).

2.2.3 Basset "memory" forces

Besides the additional force induced by the added mass, as a particle accelerates, it creates additional vorticity which is another source of drag. To treat this phenomenon mathematically, let us consider a Stokes flow with the $\partial_t \mathbf{v}$ term while ignoring the convective inertial terms:

$$\partial_t \mathbf{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v} \ . \tag{46}$$

For simplicity, we will consider here an infinite plate on the $y = 0$ plane that accelerates to speed U_0 from rest instantaneously along the x-direction. Since the problem is translationally invariant in the x-direction, $v_y = 0$ due to the incompressibility condition and v_x, p depend exclusively on y. The EOM thus becomes

$$\partial_t v_x = \nu \partial_{yy} v_x \ . \tag{47}$$

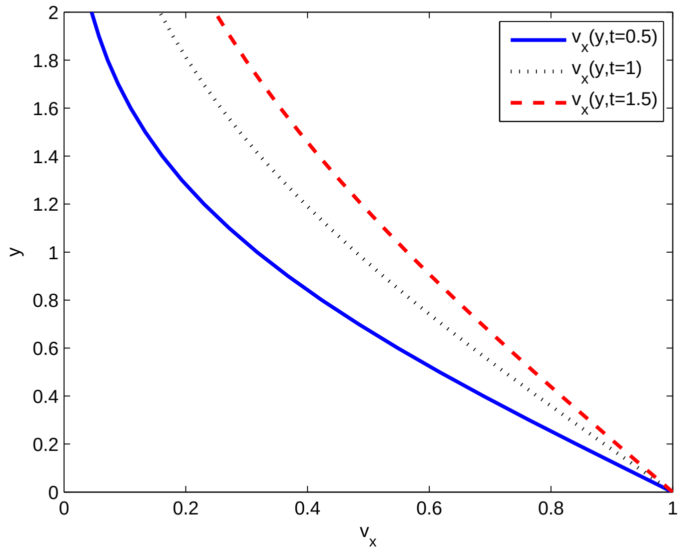

Figure 5: The horizontal velocity profile of at various times after the infinite plate at $y = 0$ is accelerated instantaneously from rest to unit speed along the x direction.

The above equation is just the diffusion equation with the initial condition $v_x(y = 0, t) = U_0\Theta(t)$ and $v_x(y = \infty, t) = 0$. The solution is (Exercise 2.4.4)

$$v_x(y, t) = U_0 \left[1 - \mathrm{erf}\left(\frac{y}{2\sqrt{\nu t}}\right)\right] \tag{48}$$

where $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-y^2} \mathrm{d}y$ is the error function (see Fig. 5).

The tangential stress on the plane is

$$\sigma_{yx} = \mu \left.\frac{\partial v_x}{\partial y}\right|_{y=0} = \sqrt{\frac{\mu\rho}{\pi t}} U_0 \ . \tag{49}$$

Note that the stress is time dependent. In other words, an acceleration in the past will still have an influence on the tangential stress felt by the plate.

In general, the acceleration cannot be instantaneously, so let us partition the acceleration into infinitesimal steps. In particular, if the speed is increased by $\triangle U$ at time t_0, the stress felt by the plate at t is

$$\sqrt{\frac{\mu\rho}{\pi}} \frac{\triangle U}{\sqrt{t - t_0}} \ . \tag{50}$$

By summing over all these changes in speed, we have the following expression for the tangential stress felt by the plate at t:

$$\sqrt{\frac{\mu\rho}{\pi}} \int_0^t \frac{\dot{U}(t')}{\sqrt{t - t'}} \mathrm{d}t' \ . \tag{51}$$

This memory effect due to acceleration in the past was first discussed by Basset and is thus called the Basset force.

2.3 Particle equation of motion: Basset-Boussinesq-Oseen equation

We have so far seen that a moving solid object will generally experience the Stokes drag (in the low Re limit), and if accelerating, additional drags due to the added mass effect and Basset force. For a spherical particle, the equation with all these terms incorporated is called the Basset-Boussinesq-Oseen (BBO) equation, which is of the form [7]:

$$m_p\dot{\mathbf{u}} = -6\pi R\mu(\mathbf{u}-\mathbf{v}) + \frac{4}{3}\pi R^3\rho\dot{\mathbf{v}} - \frac{2}{3}\pi R^3\rho(\dot{\mathbf{u}}-\dot{\mathbf{v}}) - 6R^2\sqrt{\pi\mu\rho}\int_{-\infty}^{t}\frac{\dot{\mathbf{u}}-\dot{\mathbf{v}}}{\sqrt{t-t'}}\mathrm{d}t' + \mathbf{f}_{ext} \tag{52}$$

where $\mathbf{u}$ is the particle's velocity, $\mathbf{v}$ is the fluid flow field and m_p is the particle mass. On the right hand side, we have sequentially the contributions from the Stokes drag, the body force, the added mass, the Basset force, and $\mathbf{f}_{ext}$ denotes any other external forces, such as the gravitational force.

2.4 Exercises

Exercise 2.4.1 *Show that for the Stokes flow past a sphere, the assumption that the inertia term is negligible breaks down as $r \to \infty$.*

Exercise 2.4.2 *Find the potential function for the flow field if a sphere is moving with speed U in a quiescent inviscid liquid.*

Exercise 2.4.3 *Calculate the drag coefficient for an air bubble.*

Exercise 2.4.4 *Obtain the solution in Eq. (48) for the diffusion problem.*

3 Bubbles in liquid

We considered a solid particle (sphere) in liquid in the previous section, we will now consider what happens if the "particle" is now a fluid or a gas, which is deformable and compressible. These effects would naturally modify the drag of the bubble or droplet and as a result its dynamics as well.

3.1 Bubble deformation

A moving bubble may change its shape due to the surrounding moving fluid. The typical force holding a bubble in its spherical shape is of the order γR where γ is the surface tension and R is the length scale of the bubble. The bubble's shape will be deformed if the surrounding fluid exerts a similar force on the bubble. At high Re and at the steady state, we expect that the relevant parameters are ρ, R, U where ρ is the density of the fluid and U is the steady state speed of the bubble relative to the fluid flow. By dimensional analysis, deformation occurs if $\gamma R \sim \rho U^2 R^2$. At low Re, the relevant parameters are μ, U, R, and deformation occurs at $\gamma R \sim \mu U R$.

3.2 Rising bubble

A small rising bubble will be more or less spherical due to its surface tension. For a large bubble, it is found that at the steady state it will adopt an umbrella shape with a spherical top and a flat bottom. So what will the steady-state speed for the rise of a large bubble? We will now estimate that.

In the high Re and large R limits, we can neglect surface tension and viscousity, and we further assume that the pressure inside the bubble is uniform. Bernoulli principle ($v^2/2 + gz + p/\rho =$ constant) then gives

$$\frac{v_s^2}{2} = gR(1-\cos\theta) \tag{53}$$

where v_s is the speed of water at the surface of the bubble (see Fig. 6).

Now, we know from inviscid flow past a bubble that at the surface of the sphere, the speed is $v_s = (3/2)U\sin\theta$. Substituting this into Eq. (53) and expanding with respect to θ on both sides give

$$U = \frac{2}{3}\sqrt{gR}\ . \tag{54}$$

The above relation between the rising speed and the radius of the curvature of the bubble has been verified experimentally [8].

3.3 Bubble ring

Consider a gas bubble produced at $t = 0$ at the bottom of a water tank (see Fig. 7). The pressure at the bottom of the bubble is higher than at the top, so the bottom surface rises more quickly, which creates a fluid jet rising through the center of the bubble. If the surface tension and viscosity is low, this fluid jet may puncture through the bubble and thus a bubble ring is created. The poloidal vorticity generated around the bubble ring generates a "lift" force that expands the ring outwards due to the "Magnus" effect (think a rotating cylinder moving through a fluid or a bending free kick in football.) As the bubble ring rises and expands radially, it becomes thinner and thinner. Eventually it will be broken up into a ring of spherical bubbles due to Plateau-Rayleigh instability. You can see many Youtube videos of these effects by searching "bubble ring".

3.4 Small bubble under pressure oscillation

3.4.1 Primary Bjerknes force

We have seen how a big bubble can be deformed in shape due to pressure and the flow field. Here we will discuss the dynamics of a small bubble under an oscillating pressure field.

Imagine a stable bubble consisting of a gas that is not dissolvable in the surrounding liquid. If the external pressure is increased, the bubble will naturally shrink from its original size. And once the pressure returns to its original value, the bubble will also return to its initial size. If we ignore damping (due to the viscosity in the fluid) and if the change in radius is small, we expect that the bubble will behave more or less like a harmonic oscillator. Therefore, if we now assume that the pressure is of the form $p_0 + P_a \sin \omega t$ and write the bubble radius as $R(t) = R_0 + r(t)$, we expect that the dynamical equation for r is of the form

$$\ddot{r} + \omega_R^2 r = -\frac{P_a}{\rho R_0} \sin \omega t \tag{55}$$

where (if γ is high)

$$\omega_R^2 \sim \frac{\gamma}{\rho R_0^3} \ . \tag{56}$$

One could obtain the form of the resonance frequency ω_R and the prefactor in front of $\sin \omega t$ in the above equation by dimensional analysis. In Exercise 3.6.1, you will derive the Rayleigh-Plesset (RP) equation and you can then obtain the dynamical equation above properly by linearising the RP equation with respect to r.

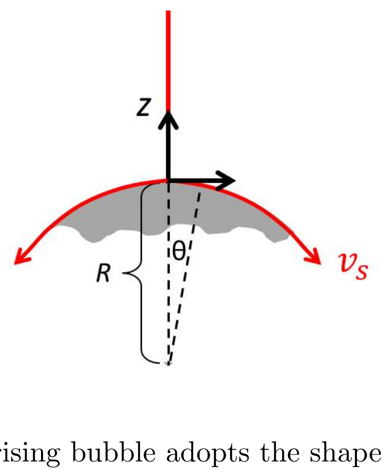

Figure 6: A large rising bubble adopts the shape of a spherical cap.

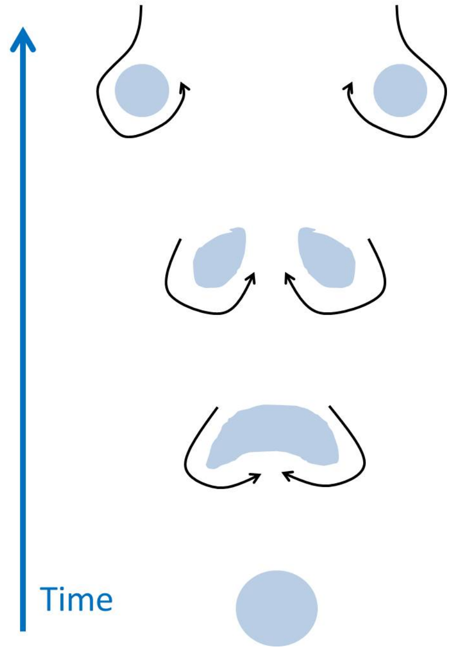

Figure 7: Schematics of the evolution of a bubble ring starting from a single bubble at the bottom. The fluid flow field is depicted by the black arrows. As the bubble tube gets thinner and thinner due to radial expansion, it will eventually be broken up into small bubbles due to Plateau-Rayleigh instability.

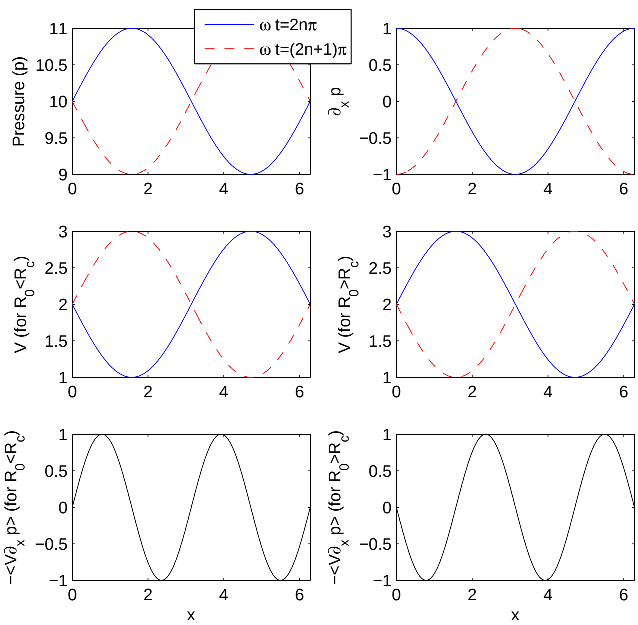

Figure 8: Pictorial explanation of the primary Bjerknes forces. [Adapted from [9].]

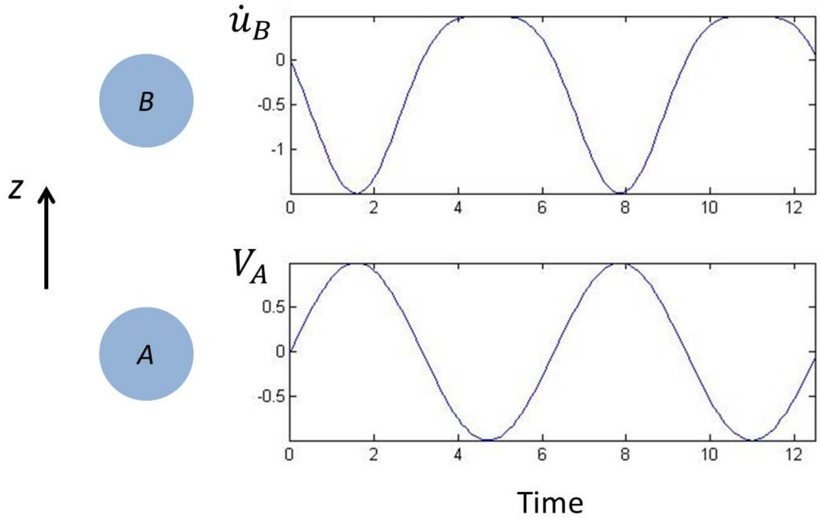

Figure 9: The effect of pulsation in bubble A on the acceleration of bubble B.

Since Eq. (55) describes a forced harmonic oscillator (see Exercise 3.6.2), there exists a critical radius $R_c \sim (\gamma/\rho\omega)^{1/3}$ such that if $R_0 < R_c$, the radius oscillation will be in antiphase with P, i.e., high pressure together with small radius. On the other hand, if $R_0 > R_c$, pressure and radius will be in phase.

Now let us see how the standing pressure wave will affect the position of the bubble. By assumption, the pressure is $p(x,t) = p_0 + P_a \sin(kz)\cos(\omega t)$ and we are in the limit of $R_0 \ll 1/k$, $R_0 < R_c$ and $P_A \ll 1$. So we expect that the bubble's volume is:

$$V(t) = V_0 - V_a \sin(kz)\cos(\omega t) \tag{57}$$

The force acting on the bubble is of the form

$$\mathbf{f}(z,t) = -\oint_A p\hat{n}\mathrm{d}S = -V\vec{\nabla}p \; . \tag{58}$$

Time averaging the above force over one period ($0 \le t \le 2\pi/\omega$) gives

$$\langle \mathbf{f}(z) \rangle = \frac{kV_a P_a}{4} \sin(2kz)\hat{z} \; . \tag{59}$$

The significance here is that the wavelength of the time-averaged force is half of the wavelength of the pressure wave. As a result of this force, a bubble smaller than the critical radius will be driven to the antinodes of the pressure wave (see Fig. 8). The same analysis applied to a bubble bigger than the critical radius will lead to the conclusion that the bubble will be driven to the nodes of the pressure wave. The force underlying this phenomenon is called the primary Bjerknes force.

3.4.2 Secondary Bjerknes force

Besides being pushed around by a pressure wave, bubbles will also interact *via* their cycles of expansion and contraction. It turns out that two bubbles pulsating in phase will attract each other while two bubbles pulsating out of phase will repel each other. To see this, we imagine two bubbles close by and consider how the pulsation of one bubble affects the other (see Fig. 9). If we assume that the bubble A pulsates as $R_A(t) = R_0 + K \sin\omega t$, we expect that fluid above the bubble also goes like $v_z(t) \sim \dot{R}_A \sim \cos\omega t$, which means that the fluid acceleration is $\dot{v}_z(t) = -K' \sin\omega t$ for some constant K'. This fluid acceleration will impact a body force on bubble B and as a result, bubble B will have an accelerate of the form (see Eq. (45) and Fig. 9) :

$$\dot{u}_z = 3\rho V_B(t)\dot{v}_z(t) = -3\rho(R_0 + K \sin\omega t)K' \sin\omega t \tag{60}$$

where we have assumed that the mass of the gas in the bubble is negligible. Averaging over one period leads to

$$\langle \dot{u}_z \rangle = -\frac{3\rho K K'}{2} \tag{61}$$

which means that bubble B is on average accelerating towards bubble A. The same analysis applies also to bubble A and so the two bubbles will move towards each other. One can go through the same calculation to conclude that if two bubble are pulsating out of phase, they will then repel each other.

3.5 Bubble migration under a thermal gradient

As surface tensions generally decrease with an increase in temperature, a bubble under a temperature gradient will have higher surface tension on the side with lower temperature. Fluid close to the surface is thus pulled towards that side (Marangoni effect) and as a result, the bubble migrates to the direction of higher temperature.

To determine qualitatively the speed of migration, let us assume that the temperature gradient is of the form

$$T(x) = T_0 + \alpha x \ , \tag{62}$$

and the surface tension depends on T as

$$\gamma(T) = \gamma_0 - \beta T \ . \tag{63}$$

In the low Re limit, we expect that the relevant parameters are μ, R, α and β. By dimensional analysis, we find that the migration speed is

$$U \sim \frac{\alpha\beta R}{\mu} \ . \tag{64}$$

The above derivation assumes that the viscosity does not vary substantially with temperature, which seems to be true in some experiments [10].

3.6 Exercises

Exercise 3.6.1 *Go through the derivation of the Rayleigh-Plesset equation in [9].*

Exercise 3.6.2 *Describe the dynamics of a periodically driven harmonic oscillator described by the following equation:*

$$\ddot{x} + \omega_c x = A \cos\omega t \ . \tag{65}$$

Exercise 3.6.3 *Use dimensional analysis to determine the temperature gradient needed to make a bubble stationary in a vertical pipe filled with liquid.*

4 Multiphase flow patterns

Consider a two-phase gas-liquid flow along a horizontal pipe, many different flow patterns occur depending on the liquid and gas volumetric fluxes (see Fig. 10). We will now try to understand these patterns.

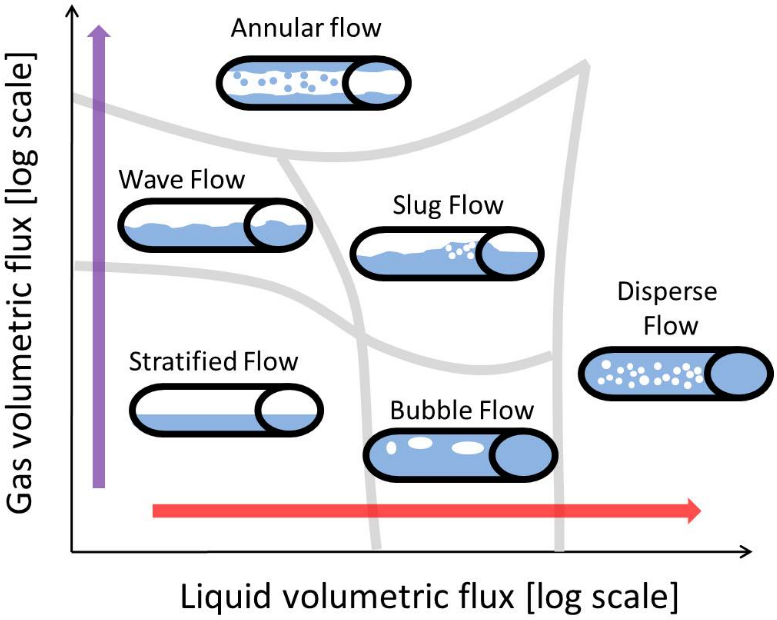

Figure 10: Generic patterns of a two-phase gas-liquid flow in a horizontal pipe. The transitions indicated by the purple (red) arrow are described in Section 4.2 (4.3).

4.1 Low gas–low liquid flow

At low gas–low liquid flow (lower left region in Fig. 10(b)), we expect that the configuration is similar to that of a static water tank partly filled with water. This corresponds to the stratified flow pattern.

4.2 Higher gas flow

As gas flow increases from the stratified flow region, it triggers Kelvin-Helmhotz instability and thus the flow will become wavy [5]. In inviscid flow, Kelvin-Helmhotz instability occurs when

$$U_g > \frac{2(\rho_g + \rho_l)\sqrt{(\rho_g - \rho_l)g\gamma}}{\rho_g \rho_l} \tag{66}$$

where $\rho_g(\rho_l)$ is the density of the gas (liquid) phase, U_g is the speed of the gas flow and γ is the surface tension.

As the gas flow increases even further, the two-phase flow becomes annular, with liquid forming a layer sandwiched by the gas flow in the core and the inner surface of the pipe. This phenomenon is still not fully understood. One proposed mechanism is that the liquid wave continuously pumps liquid onto the top surface to balance out the drainage of the liquid layer due to gravity [11]. A rough estimate when wavy flow-annular flow transition occurs is proposed in [12], which suggests that the transition happens when the liquid level is below the mid point of the pipe so that the wavy pattern cannot be sustainable.

4.3 Higher liquid flow

Starting again at the low gas flow-low liquid flow region, if the liquid volumetric flow rate increases, Kevin-Helmholtz instability occurs. If the water level is higher than the mid-point of the pipe, the resulting wave can encompass the whole cross section of the pipe and thus the gas phase is separated into bubbles. As liquid flow rate increases further, fission of bubbles occurs and we arrive at the disperse flow regime.

4.4 Exercises

Exercise 4.4.1 *Consider the laminar flow of two immiscible liquids down a (infinitely wide) channel of height* $2h$*. The two fluids have the same density but liquid 1 is more viscous. Given a pressure gradient of* $-\triangle P$ *and the steady flow is such that the cross section of the channel is always filled equally by liquid 1 and 2, what is the combined volumetric flow rate for the following configuration?*

a) Liquid 1 on top of liquid 2;
b) Liquid 1 sandwiched by layers of liquid 2;
c) Liquid 2 sandwiched by layers liquid 1.

Which configuration gives the highest flow rate if $\mu_1 \ll \mu_2$*?*

References

[1] L.D. Landau and E.M. Lifshitz. *Fluid Mechanics.* Elsevier Science, 2013.

[2] E.M. Lifshitz and L.P. Pitaevskii. *Statistical Physics. Part 2.* Pergamon Press, 1980.

[3] John Toner and Yuhai Tu. Flocks, herds, and schools: A quantitative theory of flocking. *Physical Review E*, 58(4):4828–4858, 1998.

[4] M.L. Boas. *Mathematical Methods in the Physical Sciences.* Wiley, 2005.

[5] T.E. Faber. *Fluid Dynamics for Physicists.* Cambridge University Press, 1995.

[6] A. Einstein. Eine neue bestimmung der molekueldimensionen. *Annalen der Physik*, 324(2):289–306, 1906.

[7] C.E. Brennen. *Fundamentals of Multiphase Flow.* Cambridge University Press, 2005.

[8] R. M. Davies and Geoffrey Taylor. The mechanics of large bubbles rising through extended liquids and through liquids in tubes. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 200(1062):375–390, 1950.

[9] T. G. Leighton, A. J. Walton, and M. J. W. Pickworth. Primary bjerknes forces. *European Journal of Physics*, 11(1):47, 1990.

[10] Martin E. R. Shanahan and Khellil Sefiane. Recalcitrant bubbles. *Scientific Reports*, 4, 2014.

[11] T. Fukano and A. Ousaka. Prediction of the circumferential distribution of film thickness in horizontal and near-horizontal gas-liquid annular flows. *International Journal of Multiphase Flow*, 15(3):403–419, 1989.

[12] Yemada Taitel and A. E. Dukler. A model for predicting flow regime transitions in horizontal and near horizontal gas-liquid flow. *AIChE Journal*, 22(1):47–55, 1976.