HOMEWORK 1

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0.1. Suppose that S is a set and V is a vector space. Show that the set V^S of all maps from S to V has a unique structure of a vector space such that for any $s \in S$ the evaluation map $\mathtt{ev}_s \colon V^S \to V$ is linear. $\square$

1. Subspaces of $\mathbb{R}^{\mathbb{N}}$

An element of $\mathbb{R}^{\mathbb{N}}$ is often called a sequence (of real numbers). In that case, for a function $f \colon \mathbb{N} \to \mathbb{R}$ one usually (say, in analysis) writes f_n for $f(n)$ and denotes the function f by its values, i.e. by (f_n). The set $\mathbb{R}^{\mathbb{N}}$ of sequences is a vector space over $\mathbb{R}$ in the usual way.

1.1. A sequence f is called *bounded* if there exist a number $M \in \mathbb{R}$ such that $|f_n| \leq M$ for all $n \in \mathbb{N}$. Let $\mathbb{R}^{\mathbb{N}}_b \subset \mathbb{R}^{\mathbb{N}}$ denote the subset of bounded sequences.

Show that $\mathbb{R}^{\mathbb{N}}_b$ is a subspace of $\mathbb{R}^{\mathbb{N}}$.

1.2. A sequence f is said to be *eventually zero* if there exists $N \in \mathbb{N}$ such that $f_n = 0$ for all $n \geq N$. Let $\mathbb{R}^{\mathbb{N}}_{fin} \subset \mathbb{R}^{\mathbb{N}}$ denote the subset of sequences which are eventually zero.

Show that $\mathbb{R}^{\mathbb{N}}_{fin}$ is a subspace of $\mathbb{R}^{\mathbb{N}}$.

2. More linear maps

Suppose that V is a vector space

2.1. Show that the addition map (part of the vector space structure)

$$+ \colon V \times V \to V$$

is linear.

2.2. Show that for any $\lambda \in \mathbb{R}$ multiplication by λ (part of the vector space structure)

$$\lambda \cdot (-) \colon V \to V$$

is linear.

2.3. From the previous problem we deduce that scalar multiplication give rise to the map

$$\mathbb{R} \to \mathrm{Hom}(V, V)$$

which sends λ to "multiplication by λ". Show that this map is linear.

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