From symplectic geometry and group actions to singular symplectic structures

Juan Brieva Ramírez

University of Oxford
Mathematical Institute

Universitat Politècnica de Catalunya
Centre de Formació Interdisciplinària Superior (CFIS)

Supervisors:
Andrew Dancer (UO)
Eva Miranda (UPC)

May 2025

A thesis submitted in partial fulfillment of the requirements for
Bachelor's Degree in Mathematics
Bachelor's Degree in Engineering Physics

"Because I'm stuck like this, my thoughts are crazy, perfidious tripe: anyone shoots badly through a crooked blowpipe.
My painting is dead. Defend it for me, Giovanni, protect my honour. I am not in the right place—I am not a painter."

Talking about the Sistine Chapel,
Michelangelo, To Giovanni da Pistoia

"It's like everyone tells a story about themselves inside their own head. Always. All the time. That story makes you what you are. We build ourselves out of that story"

Patrick Rothfuss, The Name of the Wind

Abstract

Symplectic reduction [33] is a well-studied construction that takes advantage of symmetries on symplectic manifolds to reduce their dimension. This procedure is not always well-behaved, and the construction for non-free actions or over singular values results in singular symplectic structures, such as orbifolds [27], or more generally, stratified symplectic spaces [42]. This reduction procedure has been generalized and studied over other multiple classes of spaces (see [9, 11, 24, 32, 35, 36]). In this text we first review the singular generalization of the Marsden-Weinstein reduction [33] by Sjamaar-Lerman [42], and we later study the reduction of the cotangent bundle of a Lie group G by subgroups of its natural $G \times G$-action. Moreover, we take a look at the state of the symplectic reduction on b^m-symplectic manifolds [35], where the singular reduction is yet to be studied, and give an approach to generalize the singular construction for this category of spaces.

Keywords: Symplectic Geometry, Symplectic Reduction, Singular Geometry, b-Symplectic Geometry, Lie Groups, Hamiltonian Actions, Singular Symplectic Manifolds, Marsden-Weinstein Reduction.

AMS2020 Code: 53D20, 53D05

Resum

La reducció simplèctica [33] és una construcció àmpliament reconeguda que aprofita les simetries d'una varietat simplèctica per reduir la seva dimensió. Aquest procés no té sempre un bon comportament, i la construcció dona com a resultat estructures simplèctiques singulars per a accions no lliures o en valors singulars. Aquestes poden ser orbifolds [27] en el cas d'accions localment lliures, o més generalment espais simplèctics estratificats [42]. Aquest process the reducció ha estat estudiat i generalitzat per multituds d'altres classes d'espais (vegeu [9, 11, 24, 32, 35, 36]). En aquest text primer revisarem la generalització de la reducció de Marsden-Weinstein [33], realitzada per Sjamaar-Lerman [42], i posteriorment estudiarem el cas particular de la reducció del fibrat cotangent d'un grup de Lie G per subgrups de l'acció natural de $G \times G$. També repassarem l'estat actual de la reducció simplèctica sobre b^m-varietats simplèctiques [35], on la reducció singular no ha estat estudiada encara, i proposem un procediment per a generalitzar la construcció singular a aquesta categoria d'espais.

Paraules clau: Geometria Simplèctica, Reducció Simplèctica, Geometria Singular, Geometria b-simplèctica, Grups de Lie, Accions Hamiltonianes, Varietats Simplèctiques Singulars, Reducció de Marsden-Weinstein Codi AMS2020: 53D20, 53D05

Resumen

La reducción simpléctica es una construcción ampliamente conocida que aprovecha las simetrías de una variedad simpléctica para reducir su dimensión. Este proceso no siempre tiene un buen comportamiento, y la construcción tiene como resultado estructuras simplécticas singulares para acciones no libres o en valores singulares. Estas puedes ser orbifolds [27] en el caso de acciones localmente libres, o más generalmente espacios simplécticos estratificados. Este proceso de reducción ha sido estudiado y generalizado para multitud de otras clases de espacios (véase [9, 11, 24, 32, 35, 36]). En este texto primero revisamos la generalización de la reducción de Marsden-Weinstein [33], realizada por Sjamaar-Lerman [42], y posteriormente estudiamos el caso particular de la reducción del fibrado cotangente de un grupo de Lie G por subgrupos de la acción natural de $G \times G$. También repasamos el estado actual de la reducción simpléctica sobre b^m-variedades simplécticas [35], donde la reducción singular no ha sido estudiada todavía, y proponemos un procedimiento para generalizar la construcción singular a esta categoría de espacios.

Palabras clave: Geometría Simpléctica, Reducción Simpléctica, Geometría Singular, Geometría b-Simpléctica, Grupos de Lie, Acciones Hamiltonianas, Variedades Simplécticas Singulares, Reducción de Marsden-Weinstein Código AMS2020: 53D20, 53D05

Acknowledgements

This thesis is the culmination of the last five years of my life, during which I have grown and changed, during which I have lived some of the best moments of my life, during which I have lived some of the worst. It has been an incredible journey, and if I have been able to arrive at where I am now, it is thanks to all of the people that I have met and who have accompanied me along the way. There are too many of you, for which I am incredibly grateful. So even if you are not listed here, please be sure I had you in mind.

This project, as well as the rest of my university studies, has been supported by the CFIS and the Mir-Puig foundation. I am deeply grateful to these institutions, whose generous support has made everything possible.

I would like to thank my supervisors for their guidance and support. Thank you, Andrew, for giving this thesis a direction and for letting me peek into the academic life at Oxford. Thank you, Eva, for all the opportunities you have provided me, and for being a beacon in a moment when I felt lost.

Gràcies a tots els meus amics de l'Aléu, per fer de la resi una llar per a mi, i gràcies als matfis (et al.), els meus companys de batalla, que han fet que la facultat sigui més que una llar. No sé què hauria estat de mi sense vosaltres.

Gracias a todos los grandes profesores de Matemáticas que he tenido a lo largo de mi vida, y en especial a Francisco Bellot, quien me introdujo al maravilloso mundo de las olimpiadas y sin el cual mi trayectoria no tendría un inicio.

Gracias a mis amigos de Arroyo, con los que he vivido tantas vidas, que me han hecho quien soy en todos los sentidos. Sobre todo a Irene, Jorge, Gonzalo, Natalia y Matilla, que han sido mis mejores consejeros y confidentes.

Gracias a toda mi familia. A mis padres, por darme apoyo y aceptarme como soy, y a mis hermanas por soportarme y hacerme compañía. Os quiero más que a nada.

Por último, gracias a toda la gente que ha pasado por mi vida y que ya no está aquí. A Marc. A mi tía Julia. A mis abuelas Api y Celia. Porque aunque no estéis con nosotros, siempre os llevo conmigo.

Contents

Abstract **ii**

Acknowledgements **iv**

Table of Contents **iv**

Introduction **1**

1 Preliminaries **3**

1.0 Notation commentaries 3
1.1 Smooth Manifolds 3
1.2 Symplectic Manifolds 4
1.3 Poisson Structures 6
1.4 Lie Groups and Lie Algebras 7
1.4.1 Lie groups and classical examples 7
1.4.2 Lie algebras 9
1.4.3 Lie groups and Lie algebras 10
1.4.4 Root systems, Weyl groups and regular subalgebras 13
1.4.5 Classification of Semisimple Lie algebras 15
1.5 Hamiltonian Actions 16
1.5.1 Marsden-Weinstein Reduction 17

2 Singular reduction **19**

2.1 Stratified spaces 19
2.2 Sjamaar-Lerman Reduction 23
2.2.1 Decomposition of the reduced phase space 24
2.2.2 Dynamics on the reduced space 25
2.2.3 Reduction in stages 26
2.2.4 A normal form and a Whitney embedding 26
2.2.5 Examples of singular reduction 28
2.2.6 Some useful lemmas 28

3 **Singular Reduction of the Cotangent Bundle of a Lie Group** **31**
3.1 Reduction by $T \times T$ . . . 32
3.1.1 The stratification only depends on the Lie algebra $\mathfrak{g}$ . . . 32
3.1.2 The stratification poset . . . 33
3.1.3 Description of the strata . . . 35
3.1.4 A coarser stratification . . . 36
3.1.5 Examples of $\mathcal{D}(\mathfrak{g})$ . . . 38
3.2 Reduction by the Diagonal Group . . . 41
3.2.1 The model of the stratification . . . 41
3.2.2 The stratification poset . . . 44

4 **b^m-Symplectic Geometry** **46**
4.1 Singular Symplectic Structures . . . 46
4.1.1 b^m-Symplectic manifolds . . . 47
4.1.2 Folded symplectic manifolds . . . 50
4.1.3 Relations between singular symplectic manifolds . . . 50
4.1.4 b^m-Symplectic group actions . . . 51
4.2 b^m-Symplectic Reduction . . . 55
4.2.1 b^m-Cotangent lift . . . 55
4.2.2 A b^m-symplectic slice theorem . . . 56
4.2.3 The b^m-symplectic reduction . . . 57
4.2.3.1 The case of non-vanishing highest modular weight . . . 57
4.2.3.2 The case of vanishing modular weights . . . 58
4.2.3.3 Reduction commutes with desingularization . . . 59
4.3 The singular b^m-symplectic reduction . . . 59

5 **Methodology and planning** **61**

6 **Sustainability Report** **63**
6.1 Environmental impact . . . 63
6.2 Economic impact . . . 64
6.3 Social impact . . . 65
6.4 Ethical implications . . . 66
6.5 Relationship with the sustainability development goals . . . 66

7 **Conclusion and future work** **67**

Bibliography **69**

Introduction

Symplectic geometry lies at the intersection of Mathematics and Physics. It is the language of Classical Mechanics, and can even be generalized to quantum models. At its core lies the idea that the evolution of a physical system—be it a planet tracing its orbit around the sun, or a pendulum swinging back and forth—can be encoded geometrically through the interaction of position and momentum. The space formed by these two coordinates combined forms what is known as phase space: the stage on which dynamics unfold. The structure that governs this space is called a symplectic form, which will help us identify the trajectories followed by a particle with a given energy function.

Moreover, symmetries are the passion of physicists. They are what makes complex problems simple, what makes quantities be conserved. Isotropy, a concept in physics that states the property where all directions in space can be considered equal, that there are no privileged directions, is a basic fact of our space-time that no one, nowadays, would dare to challenge. These symmetries allow us to reduce, to erase directions and degrees of freedom, obtaining simpler systems and enabling us to do calculations. This procedure is in mathematics formalized and generalized through *symplectic reduction*.

Marsden-Weinstein (MW) reduction is the foundational result in this theory. It tells us precisely how, given a manifold with a nice symmetry, a Hamiltonian action, endowed with a moment map, a generalization of physical momenta, can be simplified by the symmetry. This reduction results in another symplectic manifold, where an equal quantity of position and momentum coordinates have been removed, reducing the dimension of the phase space by twice the dimensions of the acting group. However, this theory has gaps, as only free actions—symmetries that never leave fixed points—yield a well-behaved reduced space.

In those cases, the Sjamaar-Lerman (SL) reduction tells us that the space is no longer a manifold. It is a stratified space that can be divided into symplectic manifolds that are nicely glued together. In particular, we will be interested in looking at this construction over the phase space of symmetry groups themselves, which, of course, are spaces with rich symmetries that can be studied with this reduction.

Moreover, multiple more singular structures exist in symplectic geometry, and we will focus on a particular kind, b^m-manifolds. They are a generalization of manifolds with boundary, and as a lot of other categories of spaces, they have their own reduction theory. However, this theory is only limited to the MW reduction, and we will be interested in how a SL theory can be adapted to such kinds of spaces.

organizar esto mejor. Explorar la estructura del trabajo y sus contenidos, incluido distinguir mis aportaciones. Posiblemente mover lemma ¡s propios a la seccion de resultados

Chapter 1

Preliminaries

1.0 Notation commentaries

To begin, we clarify some conventions and terminology used throughout this text:

- The terms differentiable, smooth, and C^∞ are used interchangeably.
- The expressions moment map, momentum map, and momentum application are used interchangeably.
- The symbol $\cong$ will denote an isomorphism between spaces in their respective categories.
- The symbol $\simeq$ will denote an homeomorphism of topological spaces.

1.1 Smooth Manifolds

In this section we will recall some important definitions regarding smooth manifolds. These definitions will follow from [29] and [44].

Definition 1.1. A (*smooth*) **fibre bundle** is a space that locally is a product space. Formally, it is a structure (E, B, π, F), with E, B, F (*smooth*) manifolds, and $\pi : E \longrightarrow B$ a (*surjective submersion*) continuous surjective map. For a point $x \in B$, there is an open neighbourhood U of x such that the space $\pi^{-1}(U)$ is (*diffeomorphic*) homeomorphic via ϕ to $U \times F$, in such a way that the diagram

$$\begin{tikzcd}
\pi^{-1}(U) \arrow[r, "\phi"] \arrow[d, "\pi"'] & U \times F \\
U \arrow[ur, "\pi_U"'] &
\end{tikzcd}$$

commutes. B is called the *base space*, E the *total space*, and F is the *fibre*. The map π is the *bundle projection*.

The prototypical example is the vector bundle of a smooth manifold.

Definition 1.2. Let M and X be smooth manifolds. A map $i : X \longrightarrow M$ is called an **immersion** if $di_p : T_pX \longrightarrow T_{i(p)}M$ is injective for all points $p \in X$. An **embedding** is an immersion which is an homeomorphism onto its image, and a **closed embedding** is a proper injective immersion. In particular, a closed embedding is an embedding whose image $i(X)$ is closed on M.

Definition 1.3. A manifold X is called a submanifold of M if its inclusion function is a closed embedding.

1.2 Symplectic Manifolds

For this section we will follow [3], [12], and [29]. We will introduce symplectic manifolds, other related definitions, and provide examples. We will also explain one of the main theorems in symplectic geometry, Darboux's theorem, which provides a local normal form for symplectic geometry.

Definition 1.4. A *symplectic manifold* (M, ω) is a manifold M with a closed non-degenerate 2-form $\omega \in \Omega^2(M)$.

Definition 1.5. A smooth map $F : M \longrightarrow M'$ between symplectic manifolds (M, ω), (M', ω') is **symplectic**, or a morphism of symplectic manifolds, if $F^*\omega' = \omega$.

If F is a symplectic diffeomorphism, then F^{-1} is also symplectic and F is called a **symplectomorphism**.

We now present some classical examples of symplectic manifolds.

Example 1.6. The cotangent bundle of a manifold M is a symplectic manifold (T^*M, ω) with the 2-form $\omega = -d\theta$, where θ is the tautological one form of the cotangent bundle.

Example 1.7. The Euclidean space $\mathbb{R}^n \times \mathbb{R}^n$ with coordinates $\{x_i, y_i\}_{1\leq i\leq n}$ is symplectic with the symplectic form $\omega = \sum_{i=1}^n dx_i \wedge dy_i$.

Since ω is non-degenerate, a symplectic manifold M must be even-dimensional. However, this does not mean that every even-dimensional manifold can be given a symplectic form. For example, $M = S^4$ has no symplectic forms. In addition, the non-degenerate nature of ω also implies that $\omega^n \neq 0$ is a volume form, and therefore M is orientable. As we will now see, the dimension of a symplectic manifold is its only local invariant, which implies that all symplectic manifolds of the same dimension are locally the symplectomorphic.

Theorem 1.8 (Darboux)**.** *Let (M^{2n}, ω) be a symplectic manifold. For all points $p \in M$ there exists an open neighbourhood $U \subseteq M$, with local coordinates $\{x_i, y_i\}_{1\leq i\leq n}$, in which the symplectic form can be written as*

$$\omega|_U = \sum_{i=1}^n dx_i \wedge dy_i.$$

Definition 1.9. For a smooth function f, we define its **Hamiltonian vector field** as a unique vector field such that

$$\iota_{X_f}\omega = df.$$

This vector field can be seen as the velocity that a particle with Hamiltonian f, that is, a particle with energy given by the function f, would have at each of the points of the manifold.

Definition 1.10. Let (M^{2n}, ω) be a symplectic manifold and let $X \hookrightarrow M$ be a submanifold. For each point $p \in X$, define the **symplectic complement** of T_pX in T_pM by

$$(T_pX)^\omega = \{v \in T_pM \mid \omega(v, w) = 0 \text{ for all } w \in T_pX\}.$$

Then, the submanifold X is called **isotropic** if $T_pX \subseteq (T_pX)^\omega$ for all $p \in M$, which is to say, the symplectic form ω restricts to zero on X: $\omega|_X = 0$. In particular, it is called **Lagrangian** if it is isotropic and $\dim X = n$. X is called **coisotropic** if $(T_pX)^\omega \subseteq T_pX$ for all $p \in X$.

1.3 Poisson Structures

Poisson manifolds are a generalization of symplectic manifolds, and arise naturally in classical mechanics, giving a relation between observables and Hamiltonian flows. We will follow [18], [35], and[45].

Definition 1.11. A **Poisson bracket** on a smooth manifold M is an $\mathbb{R}$-bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \longrightarrow C^\infty(M).$$

For $f, g, h \in C^\infty(M)$, it satisfies

- Anti-symmetry
$$\{f, g\} = -\{g, f\},$$
- Leibniz rule
$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\},$$
- Jacobi identity:
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

A pair $(M, \{\cdot,\cdot\})$ of a smooth manifold endowed with a Poisson bracket is called a **Poisson Manifold**.

Definition 1.12. A smooth map $F : M \longrightarrow N$ between Poisson manifolds $(M, \{\cdot,\cdot\}_M)$, $(N, \{\cdot,\cdot\}_N)$ is a **Poisson map** if for $f, g \in C^\infty(N)$

$$F^*\{f, g\}_N = \{F^*f, F^*g\}_M.$$

Example 1.13. All manifolds M are Poisson manifolds if we consider the trivial Poisson bracket $\{\cdot,\cdot\} = 0$.

Example 1.14. For a symplectic manifold (M, ω), one can construct a Poisson bracket as follows. For $f, g \in C^\infty(M)$, let X_f, X_g be their corresponding Hamiltonian vector fields. We can define $\{f, g\} = \omega(X_f, X_g)$. It can be proved that this bracket satisfies the properties of the Poisson bracket.

However, the relation between symplectic and Poisson manifolds does not only go in one direction. For a Poisson manifold, there exists a natural partition into regularly immersed symplectic manifolds, possibly of different dimension, which are called the symplectic leaves [46].

Definition 1.15. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold, and $f \in C^\infty(M)$. As in symplectic geometry, one can obtain a related vector field defined as

$$X_f = \{f, \cdot\}$$

called the **Hamiltonian vector field** associated with the Hamiltonian f.

By definition, for a symplectic manifold this definition of a Hamiltonian vector field coincides with the definition in symplectic geometry.

These vector fields give the mechanics followed by a particle with Hamiltonian (energy) defined by f, analogous to the symplectic case. Moreover, the flow of the vector field X_f is tangent to the symplectic leaves, and therefore it preserves them.

1.4 Lie Groups and Lie Algebras

In the study of symplectic reduction, Lie groups and their associated Lie algebras play a central role. Symmetries in mechanics are encoded by smooth group actions, which are described using Lie groups. The infinitesimal counterparts of these actions are given by Lie algebras, which provide an algebraic framework for understanding the structure of the symmetries. This section briefly reviews the essential concepts and properties of Lie groups and Lie algebras, along with some of the standard examples, while following [7, 8, 12, 14, 41].

1.4.1 Lie groups and classical examples

Definition 1.16. A **Lie group** is a smooth manifold G that is also a group, in such a way that the multiplication of the group $\mu : G \times G \longrightarrow G$ and the inverse map $\mathrm{inv} : G \longrightarrow G$ (the map sending g to g^{-1}) are smooth maps. An **homomorphism of Lie groups** is a differentiable group homomorphism between Lie groups.

We will now review some of the most common Lie groups.

Example 1.17. The complex circumference $S^1 \subset \mathbb{C}$ is a Lie group acting on itself by multiplication, as well as the torus $\mathbb{T}^n = (S^1)^n$.

The classical example of Lie groups are the matrix groups.

Example 1.18. The group $GL(n, \mathbb{R})$ of real $n \times n$ real matrices with non-zero determinant, which can be seen as the set of linear automorphisms of an n-dimensional real vector space, is a Lie group with matrix multiplication. Similarly, the group $GL(n, \mathbb{C})$ of $n \times n$ non-singular complex matrices is a Lie group.

Texto revisado

This groups are the *general linear* groups. From them, we can get a variety of subgroups:

Example 1.19. The real *singular linear* group is the subgroup $SL(n, \mathbb{R})$ of $GL(n, \mathbb{R})$ of matrices with determinant 1. The complex *singular linear* group is obtained analogously.

These groups, however, are not compact. In $GL(n)$ the determinant is not bounded, and in $SL(n)$ one can consider the matrices of the form $\operatorname{diag}(\lambda, 1, \dots, 1/\lambda)$ and its trace (a continuous function) is not bounded.

In this text, we will be centred in compact groups. The compact matrix groups are:

Example 1.20. The *orthogonal group* is the group $O(n) = \{A \in GL(n, \mathbb{R}) : A^t A = Id\}$. Similarly, the *unitary group* is the group $U(n) = \{A \in GL(n, \mathbb{C}) : A^* A = Id\}$, where A^* denotes the complex transpose of A. The group $O(n)$ has two different connected components, split by the value of the determinant ± 1. The connected component of the identity $SO(n) = \{A \in O(n) : \det(A) = 1\}$ is called the *special orthogonal group*, which is a connected compact group. Similarly, one can define the *special unitary group* as $SU(n) = \{A \in U(n) : \det(A) = 1\}$. However, in this case it is not the connected component of the identity as the group $U(n)$ is itself connected.

Definition 1.21. The **twisted product** of two G-spaces X and Y, where G acts on the right on X and on the left on Y, is the space $X \times_G Y = (X \times Y)/G$, where G acts on $X \times Y$ by $g(x, y) = (x \cdot g^{-1}, g \cdot y)$.

Definition 1.22. For a Lie group G, we define the **maximal torus** T as the maximal compact abelian subgroup of G. The **rank** of a Lie group is defined as the dimension of its maximal torus. We define the **Weyl group** of G as the quotient $W(G) = N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G. The Weyl group is a finite group, and it acts on the maximal torus by conjugation.

Proposition 1.23. *The maximal torus of a compact Lie group is unique up to conjugation. In particular, the rank of a compact Lie group is well defined, and the Weyl group is well defined.*

Example 1.24. The maximal torus of $SO(n)$ is the group $T^n = \{A \in SO(n) : A^t A = Id\}$, which is the group of diagonal matrices with ± 1 in the diagonal. The maximal torus of $SU(n)$ is the group $T^n = \{A \in SU(n) : A^* A = Id\}$, which is the group of diagonal matrices with $e^{i\theta}$ in the diagonal.

1.4.2 Lie algebras

Before continuing with the content in Lie groups, it is useful to first define the concept of Lie algebras.

Definition 1.25. A **Lie algebra** $\mathfrak{L}$ is a vector space, bundled with a skew-symmetric bilinear map, the Lie bracket (or commutator), written as $[\cdot, \cdot] : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ which satisfies the Jacobi identity:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \forall x, y, z \in \mathfrak{L}$$

Some examples of Lie algebras include:

Example 1.26. Any vector space V is a Lie algebra with bracket $[u, v] = 0$. This algebra is called an abelian algebra.

Example 1.27. The classical matrix algebras $\mathfrak{gl}_n(\mathbb{R})$ of $n \times n$ matrices, $\mathfrak{sl}_n(\mathbb{R})$ of those matrices with determinant 1, are some of the matrix Lie algebras, with commutator $[X, Y] = XY - YX$.

The general example will be the Lie algebra of a Lie group, which we will define shortly.

For every element $x \in \mathfrak{L}$, we can define the **adjoint representation** of $\mathfrak{L}$ as the linear map $\mathrm{ad}_x : \mathfrak{L} \longrightarrow \mathfrak{L}$ defined as

$$\mathrm{ad}_x(y) = [x, y].$$

The adjoint representation is a Lie algebra homomorphism, and therefore it satisfies that

$$\mathrm{ad}_{[x,y]} = \mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x.$$

Definition 1.28. The Killing form of a Lie algebra $\mathfrak{L}$ is the bilinear form defined as

$$K(x, y) = \mathrm{Tr}(\mathrm{ad}_x \circ \mathrm{ad}_y).$$

Definition 1.29. A Lie algebra $\mathfrak{L}$ is called simple if it is non-abelian and has no non-trivial ideals, where an ideal $\mathfrak{I}$ of a Lie algebra $\mathfrak{L}$ is defined as subalgebra such that $[\mathfrak{I}, \mathfrak{L}] \subseteq \mathfrak{I}$.

A Lie algebra $\mathfrak{L}$ is called semisimple if it is a direct sum of simple Lie algebras, and $\mathfrak{L}$ is called reductive if it is a direct sum of semisimple and abelian Lie algebras.

One can see that a Lie algebra $\mathfrak{L}$ is semisimple if and only if its Killing form is non-degenerate. This is a consequence of the fact that the adjoint representation is a homomorphism, and therefore it preserves the structure of the algebra. Moreover, the Killing form gives as a way to relate the dual space of the algebra with the algebra itself, as it gives a bilinear form on $\mathfrak{L}^* \times \mathfrak{L}^*$, which can be used to identify $\mathfrak{L}$ with its dual space. This is important in the study of representations of Lie algebras, as it allows us to define a dual representation.

Definition 1.30. For a real Lie algebra $\mathfrak{L}$ (a vector space over the real numbers), we define the **complexification** of $\mathfrak{L}$ as the complex vector space $\mathfrak{L}_{\mathbb{C}} = \mathfrak{L} \otimes_{\mathbb{R}} \mathbb{C}$ with the Lie bracket defined as

$$[x \otimes z, y \otimes w] = [x, y] \otimes zw.$$

The complexification of a Lie algebra is a complex Lie algebra. It is a common practice to work with complex Lie algebras, as they are easier to classify, as we will see in 1.50. The process of complexification cannot be reversed, as different real Lie algebras can have the same complexification. However, for all complex algebras, there exists a unique compact form, which will be the Lie algebra of a compact Lie group.

Definition 1.31. A **Cartan subalgebra** $\mathfrak{t}$ of a Lie algebra $\mathfrak{L}$ is a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{L}$ such that the adjoint representation ad_x is diagonalizable for all $x \in \mathfrak{t}$. A Cartan subalgebra is called **regular** if it is self-normalizing, that is, if $[\mathfrak{t}, \mathfrak{t}] = \mathfrak{t}$.

The Cartan subalgebra is the equivalent concept of a maximal torus in a Lie group, and as such, all Cartan subalgebras are conjugate to each other.

1.4.3 Lie groups and Lie algebras

Left invariant vector fields

For a Lie group G, we can consider its action on itself by left multiplication. For an element $g \in G$, we define it as

$$L_g : G \longrightarrow G \quad a \longmapsto g \cdot a.$$

Definition 1.32. A vector field X is called **left-invariant** if $(L_g)_*X = X$

The We can see that left-invariant vector fields are defined by their value at the identity element of G, as one must have that $(L_g)_*X_{g^{-1}} = X_e$, so we have a map from the tangent space at the identity to the set of left-invariant vector fields. This map allows us to define a Lie bracket on T_eG by taking $[x, y] = \mathcal{L}_X Y$, where X and Y are the left-invariant vector fields generated by $x,\ y$.

Definition 1.33. The Lie algebra $\mathfrak{g}$ of a Lie group G is the tangent space at the identity T_eG with the induced Lie given on the left-invariant vector fields.

Moreover, the left action induces a natural bundle isomorphism between the tangent space of G and the product $G \times \mathfrak{g}$, which is called the **left trivialisation** of the tangent bundle. It can also be applied to the cotangent bundle, $T^*G \cong G \times \mathfrak{g}^*$.

For each element $g \in G$, we define the function $\Phi : G \longrightarrow \mathrm{Aut}(G)$ as $\Phi_g(a) = gag^{-1}$.

Definition 1.34. The adjoint representation of a Lie group G is the homomorphism $\mathrm{Ad} : G \longrightarrow GL(\mathfrak{g})$ defined as

$$\mathrm{Ad}_g(X) = (d\Phi(g))_e$$

The dual of the adjoint representation is called the coadjoint representation, which quite an important object. In particular, the study of coadjoint orbits is a important topic, mainly due to the following theorem by Kostant-Souriau (see [3, §2.5], [8, §1.4]):

Theorem 1.35 (Konstant-Souriau)**.** *Let G be a Lie group with $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$. Then, the coadjoint orbits of G are symplectic manifolds. And there is a one to one correspondence between G-orbits in $\mathfrak{g}^*$ and symplectic manifolds with a transitive G-action.*

This theorem can be used to identify the spaces $T^*(G/H)$ with the coadjoint orbits of G, where H is a closed subgroup of G. andrew bad

Sentence after Thm 1.34 doesn't look right–the coadjoint orbits of G will be compact if G is, but the cotangent bundle $T^*(G/H)$ is noncompact

Exponential Map

A **one parameter group** of a Lie group G is a homomorphism of Lie groups $\varphi : \mathbb{R} \longrightarrow G$ such that $\varphi(0) = e$. The correspondence $\varphi \longmapsto \dot{\varphi}(0)$ gives a bijection between the set of one-parameter groups and the Lie algebra $\mathfrak{g}$ of G due to the existence of solutions of ODEs, which allows us to define the exponential map:

Definition 1.36. The **exponential map** $\exp : \mathfrak{g} \longrightarrow G$ is the map

$$\exp : \mathfrak{g} \longrightarrow G \qquad X \longmapsto \varphi_X(1),$$

where φ_X is the unique one-parameter group such that $\dot{\varphi}_X(0) = X$. The map is differentiable, and its differential at 0 is the identity map.

As the differential at 0 is the identity map, we can see that the exponential map is a local diffeomorphism at the identity. However, it is not generally a global diffeomorphism. Nevertheless, for compact Lie groups it is surjective over the connected component of the identity.

Example 1.37. For matrix Lie groups, the exponential map is given by the usual matrix exponential, which is defined as the series

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}.$$

This series converges for all matrices, and it is a smooth map. The exponential map is a local diffeomorphism at the identity.

Correspondence of Lie algebras and Lie groups

As we have seen, all Lie groups have an associated Lie algebra, taken as the tangent space at the identity. There is also a correspondence in the other direction, as we can construct a Lie group from a Lie algebra, although this process is not unique.

Theorem 1.38 (Lie's third theorem)**.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{R}$. Then, there exists a unique simply connected Lie group G such that $\mathfrak{g}$ is isomorphic to the Lie algebra of G.*

For any connected Lie group G, one can consider its universal covering group $\tilde{G}$, which is a simply connected Lie group. The Lie algebra of G is naturally isomorphic to the Lie algebra if $\tilde{G}$, as the covering map is a local diffeomorphism which induces an isomorphism between the tangent spaces at the identity. Then, due to Lie's third theorem, any two connected Lie groups with the same Lie algebra have the same universal covering.

Another important fact in the correspondence between Lie groups and Lie algebras is that subgroups correspond to subalgebras, and vice versa.

Theorem 1.39. *If G is a Lie group with Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then there exists a unique connected Lie subgroup H of G such that $\mathfrak{h}$ is the Lie algebra of H.*

Although we have subgroup H in G, the group might not be embedded. Take, for example, the lines of irrational slope in a torus. In general, the group H will be generated by the $\exp(\mathfrak{h})$.

1.4.4 Root systems, Weyl groups and regular subalgebras

Definition 1.40. Let $\mathfrak{k}$ be a compact form of $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$, a complex semisimple Lie algebra. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Then, the complex Lie algebra can be decomposed as

$$\mathfrak{g} = \mathfrak{t} \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \quad \forall T \in \mathfrak{t}\},$$

and

$$\Phi = \{\alpha \in \mathfrak{t}^* \backslash \{0\} : \mathfrak{g}_\alpha \{0\}\}.$$

Each element $\alpha \in \Phi \subset \mathfrak{t}^*$ is called a **root**, and the corresponding space $\mathfrak{g}_\alpha$ is called a **root space**. The set of roots Φ is called the **root system** of $\mathfrak{g}$ with respect to $\mathfrak{t}$, or equivalently, of $\mathfrak{k}$ with respect to $\mathfrak{t} \cap \mathfrak{k}$.

Proposition 1.41. *A root system Φ satisfies that:*

- *If $\alpha \in \Phi$, then $\lambda\alpha \in \Phi$ only for $\lambda = \pm 1$,*
- *If $\alpha, \beta \in \Phi$, then $\langle \alpha, \beta \rangle := \frac{2[\alpha,\beta]}{[\alpha,\alpha]} \in \mathbb{Z}$*
- *The set Φ is closed under reflections under the hyperplane normal to α, $\alpha \in \Phi$.*

Definition 1.42. The Weyl group of $\mathfrak{g}$ with respect to $\mathfrak{t}$ is the group $W(\mathfrak{g}, \mathfrak{t})$ generated by reflections on the normal hyperplanes to the roots.

Definition 1.43. A **root subsystem** Ψ of a root system Φ is a subset $\Psi \subseteq \Phi$ that satisfies that for any $\alpha, \beta \in \Psi, \alpha + \beta \in \Phi$, $-\alpha \in \Psi$ and $\alpha + \beta \in \Psi$. Equivalently, Ψ is a root subsystem if $(\mathrm{span}_{\mathbb{Z}}\Psi) \cap \Phi = \Psi$.

A root subsystem is always a root system itself. We use the notation $\Psi \leq \Phi$ to denote that Ψ is a root subsystem of Φ, which gives a partial order on the set of root subsystems.

Root subsystems are closely related to regular subalgebras.

Definition 1.44. A subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called regular if there exist a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ such that $[\mathfrak{t}, \mathfrak{h}] \subseteq \mathfrak{h}$.

This notion was studied by Dynkin [13], who introduced the terminology. We denote the set of conjugacy classes of regular semisimple subalgebras of $\mathfrak{g}$ by $\mathcal{C}_{\mathfrak{g}}$. As all Cartan subalgebras are conjugate, all elements of $\mathcal{C}_{\mathfrak{g}}$ have a representative regular with respect to a fixed $\mathfrak{t}$.

Proposition 1.45. *The set of semisimple subalgebras of $\mathfrak{g}$ that are regular with respect to $\mathfrak{t}$ are in a one-to-one correspondence with the set of roots subsystems of Φ. The correspondence, for $\Psi \leq \Phi$ is*

$$\mathfrak{g}_\Psi = \mathfrak{t}_\Psi \bigoplus_{\alpha \in \Psi} \mathfrak{g}_\alpha \tag{1.1}$$

where $\mathfrak{t}_\Psi$ is the span of the coroots h_α for $\alpha \in \Psi$.

In particular, this equivalence also gives an equivalence between semisimple subalgebras of $\mathfrak{k}$ that are regular with respect to $\mathfrak{t}_{\mathfrak{k}} = \mathfrak{t} \cap \mathfrak{k}$, yielding $\mathfrak{k}_\Psi = \mathfrak{g}_\Psi \cap \mathfrak{k}$.

Definition 1.46. A bilinear form in $\mathfrak{g}$ is called **admissible** if it is a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ which is positive definite on the real span of the coroots. Equivalently, a bilinear form is admissible if its restriction to all compact real forms are negative definite. For example, the Killing form is admissible.

Proposition 1.47. *Any admissible bilinear form on $\mathfrak{g}$ ($\mathfrak{k}$) remains admissible on $\mathfrak{g}_\Psi$ ($\mathfrak{k}_\Psi$). Also, $\mathfrak{t}_\Psi$ ($\mathfrak{t}_\Psi \cap \mathfrak{k}$) is a Cartan subalgebra of $\mathfrak{g}_\Psi$ ($\mathfrak{k}_\Psi$), and 1.1 is the corresponding Cartan decomposition.*

The Weyl group W_Φ acts on the set of root subsystems by $w \cdot \Psi := \{w \cdot \alpha : \alpha \in \Psi\}$

Proposition 1.48. *Let Ψ_1, Ψ_2 be two root subsystems. Then, $\mathfrak{g}_{\Psi_1} = \mathfrak{g}_{\Psi_2} \iff \exists w \in W_\Phi$ such that $w \cdot \Psi_1 = \Psi_2$. In particular, the map $\Psi \longmapsto \mathfrak{g}_\Psi$ descends to a poset isomorphism*

$$\{\Psi \leq \Phi\}/W_\Phi \longleftrightarrow \mathcal{C}_{\mathfrak{g}}.$$

This bijection also occurs for the compact form, as the compact forms of $\mathcal{C}_{\mathfrak{g}}$ are in bijection with $\mathcal{C}_{\mathfrak{g}}$.

The order in $\mathcal{C}_{\mathfrak{g}}$ comes from inclusion too, $[\mathfrak{h}_1] \leq [\mathfrak{h}_2]$ if there exist representatives $\mathfrak{h}_1$, $\mathfrak{h}_2$ of the classes such that $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$

For a regular semisimple algebra $\mathfrak{h}$ of $\mathfrak{g}$, let $\mathfrak{g}_\Psi$ be a representative of $[\mathfrak{h}]$. Then, the Weyl group $W_{\mathfrak{h}}$ can be seen as a subgroup of $W_{\mathfrak{g}}$ as the group generated by the roots Ψ, and therefore the index $|W_{\mathfrak{g}} : W_{\mathfrak{h}}|$ is well defined

Definition 1.49. We call the **embedding number** of $\mathfrak{h}$ in $\mathfrak{g}$ is defined as

$$m_{\mathfrak{g}}(\mathfrak{h}) := |W_{\mathfrak{g}} : W_{\mathfrak{h}}||\{w \cdot \Psi : w \in W_{\mathfrak{g}}\}|$$

In particular, note that $m_{\mathfrak{g}}(\mathfrak{g}) = 1$ and that $m_{\mathfrak{g}}(0) = |W_{\mathfrak{g}}|$.

1.4.5 Classification of Semisimple Lie algebras

Theorem 1.50 (Classification of complex simple Lie algebras)**.** *A simple Lie algebras over the complex numbers belongs to one of the following families of classical algebras*

$$A_n := \mathfrak{sl}_{n+1}\mathbb{C}, \quad B_n := \mathfrak{so}_{2n+1}\mathbb{C}, \quad C_n := \mathfrak{sp}_{2n}\mathbb{C}, \quad D_n := \mathfrak{so}_{2n}\mathbb{C},$$

or is one of the five exceptional Lie algebras $\mathfrak{g}_2,\ \mathfrak{f}_4,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8$.

This theorem can be generalized to any closed field with characteristic 0, as it is based on the construction of *Dynkin diagrams*, and can be applied to any such field.

Moreover, it can be generalized to real algebras. However, the classification is quite more complex, but as we have seen, there is a one to one correspondence of complex simple algebras and compact real forms, we can classify them by the same families. More specifically,

Theorem 1.51 (Classification of compact Lie algebras)**.** *A compact Lie algebra belongs to one of the following families of classical algebras*

$$A_n := \mathfrak{su}_{n+1}, \quad B_n := \mathfrak{so}_{2n+1}, \quad C_n := \mathfrak{sp}_n, \quad D_n := \mathfrak{so}_{2n},$$

or is one of the five exceptional compact Lie algebras $\mathfrak{g}_2,\ \mathfrak{f}_4,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8$.

1.5 Hamiltonian Actions

In this section we will describe group actions on symplectic manifolds, which represent symmetries in those spaces, and which will be the main focus of this text. We will mainly follow [2, 5, 12, 23, 26, 33, 42].

Definition 1.52. Let G be a Lie group, acting symplectically on a manifold (M, ω). We say that the $G-$action is Hamiltonian if there exists a $G-$equivariant map $\mu : M \longrightarrow \mathfrak{g}^*$, called the **moment map**, such that

$$d\langle \mu, \xi \rangle = \iota_{\xi^M} \omega$$

$\forall \xi \in \mathfrak{g}$, and ξ^M is the generating vector field of ξ in M.

T

One can see that for any G-Hamiltonian space (M, G, μ), for a Lie subgroup $H \subseteq G$ the induced $H-$action is Hamiltonian with moment map $\mu_H = \iota^* \circ \mu$, where ι^* is the dual of the inclusion function of H in G.

Example 1.53. Let the circle S^1 act on the sphere S^2, by the action generated by $\frac{\partial}{\partial \theta}$, where we use the standard coordinates $\{\theta, h\}$ on the sphere with symplectic form $\omega = d\theta \wedge dh$. Then, we can see that

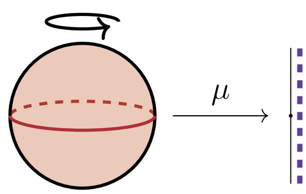


FIGURE 1.1: Moment map of the S^1-action by rotation on a symplectic S^2 [35]

$$\iota_{\frac{\partial}{\partial \theta}} \omega = dh$$

Therefore, the moment map is just the height function of the sphere.

Example 1.54. Let G act smoothly on a smooth manifold M. Then, the $G-$action can be lifted to a Hamiltonian $G-$action on the cotangent bundle T^*M.

Example 1.55. A **toric manifold** M^{2n} is a symplectic manifold together with a Hamiltonian toric action of maximal rank, T^n. These manifolds are completely classified by their images, which are a special class of polytopes in $\mathbb{R}^n$ called **Delzant**

polytopes. To be more precise, **Delzant theorem** establishes a bijection between toric manifolds and Delzant polytopes

An important theorem in Hamiltonian spaces is the slice theorem. There are a multitude of slice theorems, but the one we will introduced is the Hamiltonian slice theorem, first formulated by Guillemin-Sternberg in [21], and independently by Marle in [31].

Theorem 1.56 (Hamiltonian slice theorem). *Let (M, G, μ) be a Hamiltonian space. Let p be a point in M such that $\mathcal{O}_p$ is contained in the zero level set of the moment map. Denote H the stabilizer of p, K the stabilizer of $\mu(p)$, $\mathcal{O}_p$ the orbit of p, $\mathfrak{h}$ the Lie algebra of H, $\mathfrak{k}$ the Lie algebra of K, and $V_p = T_pM/T_p\mathcal{O}_p$ the symplectic slice, there exists neighbourhood of the orbit $\mathcal{O}_p$ which is equivariantly diffeomorphic to a neighbourhood of the orbit $G \cdot [e, 0, 0]$ in*

$$Y := G \times_H ((\mathfrak{h}^\circ \cap \mathfrak{k}^*) \times V_p).$$

In this local model, the moment map $\mu : M \longrightarrow \mathfrak{g}^*$ can be written as

$$\mu([g, \gamma, v]) = Ad_g^*(\mu(p) + \gamma + \phi(v)),$$

where $\phi : V_p \longrightarrow \mathfrak{h}^*$ is the moment map for the symplectic slice. The symplectic form on the space Y is called the MGS-symplectic form and will usually be noted as ω_{MGS}.

1.5.1 Marsden-Weinstein Reduction

For a Hamiltonian $G-$space, if the action of the group is free, the Marsden and Weinstein proved that the reduced space is a symplectic manifold

Theorem 1.57 (Marsden-Weinstein reduction). *Let a Lie Group G have a free Hamiltonian action on the symplectic manifold (M, ω), with proper moment map μ. Then, if 0 is a regular value of the moment map, the reduced space*

$$M//G := \mu^{-1}(0)/G$$

is a symplectic space, with the unique symplectic form ω_0 such that the pullback of the quotient $\pi^*\omega_0 = \omega|_{\mu^{-1}(0)}$ coincides with the restriction of the symplectic form on the 0 level set.

The proof of the theorem relies on the slice Theorem 1.56, which gives a normal form on the neighbourhood of an orbit, and the construction relies on the fact that the 0 level set is a coisotropic submanifold such that the leaves of the null foliation of $\omega|_Z$ are precisely the $G-$orbits.

The theorem is proven for the 0 value of the moment map. However, for a regular value ξ of the moment map, it is possible to see the reduction at value ξ as another reduction on the 0 level set. This is called the *shifting trick*, and is used in most constructions of reductions to allow the reduction to be focused at 0 value.

Proposition 1.58. *For a value $\xi \in \mathfrak{g}^*$, the reduced space at ξ is defined as*

$$M_\xi = \mu^{-1}(\mathcal{O}_\xi)/G.$$

This reduction can be seen as the reduction at level 0 of another $G-$Hamiltonian space.

Proof. Consider the symplectic manifold $M \times \mathcal{O}_{-\xi}$, which is a product of symplectic manifolds. The diagonal $G-$action on $M \times \mathcal{O}_{-\xi}$ is Hamiltonian with moment map $\mu_\xi(p, \zeta) = \mu(p) + \zeta$. Zero is a regular value of μ_ξ, and the reduced space at zero is

$$M_\xi = \mu_\xi^{-1}(0)/G = \mu^{-1}(\mathcal{O}_\xi)/G.$$

□

The Marsden Weinstein reduction has an easy generalization. If the action is locally free, instead of free, the reduced space is singular. However, the singularities are well-behaved, and the reduced space is an orbifold (see [27])cambiar cita.

Chapter 2

Singular reduction

In the case where G does not act freely, the reduction space will not be a symplectic manifold, and not even a smooth manifold. However, the resulting space has a rich structure which we call a stratified symplectic space. Loosely speaking, a stratified manifold consists of a union of symplectic manifolds, the symplectic strata, which fit together nicely. On this space one has a set of "smooth functions", which are smooth functions when restricted to the strata, and form a Poisson algebra whose bracket coincides with the natural brackets on the strata.

This generalization of the Marsden-Weinstein reduction was proven in 1991 by Sjamaar and Lerman in [42], and in this section we will cover the basic definitions and results exposed in it, as well as giving an outline of the involved proofs, following [42] and [30].

2.1 Stratified spaces

The main idea is that a stratification is a partition of a topological space in a disjoint union of manifolds which satisfy certain conditions. In particular, a manifold is trivially a stratified space. A more interesting example, and the local model for stratified manifold, is the the cone over a manifold:

Definition 2.1. The **open cone** $\mathring{C}M$ over a manifold (or topological space) M is the product $M \times [0, \infty)$ modulo $(x, 0) \sim (y, 0)$.

This cone $\mathring{C}M$ is the disjoint union of two manifolds, $M \times (0, \infty)$ and the vertex $*$ of the cone. Similarly one can consider the cone over the cone of a manifold $\mathring{C}(\mathring{C}M)$, which will be a stratified space and decomposes in three disjoint manifolds:

- the manifold $(M \times (0, \infty)) \times (0, \infty)$,
- the open half line $* \times (0, \infty)$ through the vertex of $\mathring{C}M$,
- the vertex $*$ of $\mathring{C}(\mathring{C}M)$.

In general, stratified spaces are locally a cone over a cone over a cone ....

Definition 2.2. A **decomposed space** is a Hausdorff and paracompact topological space X equipped with a locally finite partition $\mathcal{P} = \{S_i\}$ of disjoint locally closed manifolds $S_i \subset X$, called pieces. If $\mathcal{I}$ is a poset and the pieces satisfy

$$S_i \cap \bar{S}_j \neq 0 \Longleftrightarrow S_i \subset \bar{S}_j \Longleftrightarrow i \leq j,$$

which is called the **frontier condition**, we call the space X an $\mathcal{I}$-decomposed space. In particular, the condition $S_i \subset \bar{S}_j$ imposes a partial order on the strata, and all $\mathcal{I}$-decomposed spaces can be seen as $\mathcal{P}-$decomposed spaces, where $\mathcal{P}$ is the set of the strata with this partial order. By abuse of notation, we sometimes might omit $\mathcal{I}$ and write that X is a decomposed space even if it satisfies the frontier condition, and the poset will then be assumed then to be the order on the strata.
The **dimension** of a decomposed space X is defined as $\dim X = \sup_{i \in \mathcal{I}} \dim S_i$.

Proposition 2.3. *Let X be an $\mathcal{I}$-decomposed space. Then, the closure of a strata S_i can be written as*

$$\overline{S_i} = \bigsqcup_{j \leq i} S_j = \bigsqcup_{S \leq S_i} S$$

Proof. As $\mathcal{P} = \{S_j\}$ is a partition, we have

$$\overline{S_i} = \overline{S_i} \cap X = \overline{S_i} \cap \bigcup_{S \in \mathcal{P}} S = \bigcup_{S \in \mathcal{P}} \overline{S_i} \cap S = \bigcup_{j \leq i} \overline{S_i} \cap S_j = \bigcup_{j \leq i} S_j,$$

as $\overline{S_i} \cap S_j \neq \emptyset \Rightarrow j \leq i$ and $\overline{S_i} \cap S_j = S_j$. The union is disjoint as $\mathcal{P}$ is a partition. $\square$

Example 2.4. Consider the subset of $\mathbb{R}^2$

$$X = \left\{(x, y) \in \mathbb{R}^2 : y = 0\right\} \cup \left\{(x, y) \in \mathbb{R}^2 : x^2 - y \leq 0\right\}. \tag{2.1}$$

The space X can be broken as a union of manifolds as

$$X = \{x^2 - y < 0\} \cup \{x^2 - y = 0\} \cup \{x < 0, y = 0\} \cup \{x > 0, y = 0\} \tag{2.2}$$

or as

$$X = \{x^2 - y < 0\} \cup \{x^2 - y = 0, x < 0\} \cup \{x^2 - y = 0, x > 0\} \cup \tag{2.3}$$
$$\{x < 0, y = 0\} \cup \{x > 0, y = 0\} \cup \{(0,0)\}. \tag{2.4}$$

In both cases, as expected, its dimension as a decomposed space is $\dim Y = 2$.

Example 2.5. The product of two decomposed spaces $X = \bigsqcup S_i$ and $Y = \bigsqcup P_j$ is a decomposed space

$$X \times Y = \bigsqcup_{i,j} S_i \times P_j.$$

Its dimension $\dim X \times Y$ will be the product of the base dimensions $\dim X \cdot \dim Y$. We will consider finite-dimensional spaces exclusively.

Definition 2.6. For an $\mathcal{I}$-decomposition $\mathcal{P} = \{S_i\}$ of a space X, we define the **depth** of a piece as the integer

$$\mathrm{depth}_X S = \sup_n \{\exists\ S^j \in \{S_i\},\ 0 \leq j \leq n : S = S^0 < S^1 < \ldots < S^n\}.$$

It is clear that the depth of a piece is bounded by its codimension. For a decomposed space X we define its **depth** as

$$\mathrm{depth} X = \sup_{i \in \mathcal{I}} \mathrm{depth}_X S_i.$$

We will use the definition of depth to define a stratification by recursion. If a decomposition of X has depth 0, then X is a manifold (it only has one piece) and is automatically a stratification.

Definition 2.7. A decomposed space X is called a **stratified space** if the pieces of X have **conical slices**, i.e., they satisfy that:
For all points x in a piece S there exists an open neighbourhood $U \subset X$ of x, an open ball $B \subset S$ around x, a compact stratified space L, called the link of x, and a homeomorphism

$$\varphi : B \times \mathring{C}L \longrightarrow U \tag{2.5}$$

that preserves the decompositions, i.e., it maps pieces into pieces. In the case of a stratification, the pieces are called strata.

Remark. One can prove that a stratified space is an $\mathcal{I}$-decomposed space, as it satisfies the frontier condition.

Example 2.8. The decomposition 2.2 does not fulfil the frontier condition, therefore it is not an stratification. For the decomposition 2.3 the frontier condition is fulfilled, and one can prove that it is an stratification.

Example 2.9. For a compact Lie group G acting smoothly, but not freely, on a manifold M, the orbit space M/G is a stratified space. For each closed subgroup $H \subseteq G$, let (H) denote the conjugation class of H in G. We say that a point $p \in M$ has orbit type (H) if its stabilizer $\mathrm{Stab}_G(p)$ is conjugate to H, and we denote the the set of points of orbit type (H) as

$$M_{(H)} := \{p \in M : \mathrm{Stab}_G(p) \in (H)\}.$$

The orbit type decomposition $\mathcal{P} = \{M_{(\mathrm{Stab}_G(p))}/K\}$ is a stratification in the sense of Definition 2.7, for the fact that the strata might not be a proper manifold, they might have connected components of different dimensions. However, it is possible to refine the partition, taking the connected components of the strata, and obtain a genuine stratification.

Now we want to give this stratification a differential structure, which we will do by defining which set of functions $C^0(X)$ are smooth. This will be a subalgebra $C^\infty(X)$ of $C^0(X)$, which will be called a smooth structure on X, and must be compatible with the stratification:

Definition 2.10. A **smooth structure** $C^\infty(X)$ on a stratified space X whose strata are smooth manifolds is a subalgebra of the continuous functions $C^0(X)$ such that for any $f \in C^\infty(X)$ the restriction to a stratum is smooth, $f|_S \in C^\infty(S)$.

Example 2.11. For a stratified space X which is a subspace of a smooth manifold M, its *Whitney smooth functions* are:

$$C^\infty(X) = \{f : X \longrightarrow \mathbb{R} : \exists \bar{F} \in C^\infty(M), f = \bar{F}|_X\}$$

Definition 2.12. Given two spaces X, Y with smooth structures $C^\infty(X)$, $C^\infty(Y)$, a continuous map $\varphi : X \longrightarrow Y$ is **smooth** if for all $f \in C^\infty(Y)$, $\varphi^* f \in C^\infty(X)$. In particular, the inclusion of the strata into the space is smooth.

Definition 2.13. A **stratified symplectic space** X is a stratified space with a smooth structure $C^\infty(X)$ such that:

- the strata are symplectic manifolds,
- $C^\infty(X)$ is a Poisson algebra,
- the embeddings $S \hookrightarrow X$ of the strata are Poisson

The third condition briefly means that the Poisson bracket on the restriction of functions to a stratum S and the restriction of the Poisson bracket of those functions to the stratum coincides, i.e., $\forall\ f, g \in C^\infty(X)\ \{f|_S, g|_S\}_S = \{f, g\}\big|_S$

Example 2.14. The prototypical example is the singular symplectic reduction 2.16. For a symplectic manifold (M, ω) with Hamiltonian $G-$action and proper map μ, the reduced space $M_0 = \mu^{-1}(0)$ is a stratified symplectic space.

We will lastly introduce the concept of equivalences between stratified symplectic spaces:

Definition 2.15. An homeomorphism between stratified symplectic spaces is called a stratified symplectomorphism if it preserves the stratification and the restriction on the strata are symplectomorphism. In particular, one can prove (see [30, 2.14], [42, §3],) that a continuous map is a stratified symplectomorphism if and only if its pullback is a isomorphism of the Poisson algebras.

2.2 Sjamaar-Lerman Reduction

We will now cover the main results of Sjamaar-Lerman [42]. This lengthy paper was published in 1991, and it extended the theory developed by Marsden and Weinstein in [33]. The main result of the paper is

Theorem 2.16. *Let (M, G, μ) be a Hamiltonian space with proper moment map μ. The reduced space M_0 is a stratified symplectic space with a decomposition*

$$M_0 = \bigsqcup_{(H)\leq(G)} (M_0)_{(H)}, \tag{2.6}$$

where $(M_0)_{(H)} = \left(M_{(H)} \cap \mu^{-1}(0)\right)/G$ is a symplectic manifold with natural symplectic form $(\omega_0)_{(H)}$, which is determined by the fact that its pullback to $M_{(H)} \cap Z$ coincides with the restriction of the symplectic form on M.

This is a generalization of the MW symplectic reduction to non-free actions and singular values of the moment map, but where one obtains a stratified symplectic space instead of a symplectic manifold due to the less strict hypothesis.

The lengthy proof of this theorem is the subject of the 57 page-long paper [42], and in this section we will sketch the main ideas of the proof, and cover the main results found along the paper. Therefore, the structure of the section will be similar to [42], to which we will incorporate results of [30].

2.2.1 Decomposition of the reduced phase space

Theorem 2.17. *Let (M, G, μ) a Hamiltonian space with proper moment map μ. The intersection of the manifold $M_{(H)}$ of orbit type (H) with the zero level set $Z = \mu^{-1}(0)$ is a smooth manifold, and the orbit space*

$$(M_0)_{(H)} = \left(M_{(H)} \cap Z\right)/G$$

has a natural symplectic structure $(\omega_0)_{(H)}$, determined by the fact that its pullback to $Z_{(H)} := M_{(H)} \cap Z$ coincides with the restriction of the symplectic form on M. Therefore, the stratification of M by orbit types induces a decomposition of the reduced space $M_0 = Z/G$ into a disjoint union of symplectic manifolds,

$$M_0 = \bigsqcup_{(H)\leq(G)} (M_0)_{(H)}$$

The first step on the proof of Theorem 2.17 is to first prove the decomposition of the reduced space into the different $(M_0)_{(H)}$, the proof of which utilizes a generalization of the isotropic embedding theorem.

Theorem 2.18 (Constant Rank Embedding)**.** *Let A be a manifold with a closed two-form with constant rank τ. Then there exists a bijection between symplectic vector bundles over B and embeddings i of B into higher dimensional symplectic manifolds (A, σ) such that $i^*\sigma = \tau$.*

In particular, we can apply this theorem to two special cases:

- If the form τ on the manifold B is 0, the theorem gives a one to one correspondence between the symplectic vector bundles over B and the isotropic embeddings of B.

- If the τ has maximal rank, the manifold B is a symplectic manifold, and the theorem is the symplectic embedding theorem.

Using this theorem, and the Hamiltonian slice Theorem 1.56, one can prove the decomposition of the reduced space into the different $(M_0)_{(H)}$, applying it to the points with different isotropy groups. Similarly to the orbit type decomposition 2.9, the space $(M_0)_{(H)}$ might not be strictly a manifold, as it might contain components with different dimensions. However, we can refine the decomposition into the different connected components of the strata.

2.2.2 Dynamics on the reduced space

One can stablish Hamiltonian dynamics on the reduced space M_0, for which we will need to define a Poisson algebra of smooth functions on the reduced space.

Definition 2.19. We say a function $f \in \mathbb{C}^0(M_0)$ is **smooth** if there exists a function $F \in C^\infty(M)^G$ such that $F|_{M_0} = \pi^* f$, where π is the orbit map. We denote the space of smooth functions on M_0 as $C^\infty(M_0)$. To be more precise, we can show that this space is isomorphic to $C^\infty(M)^G/I^G$, where I^G is the set of G-invariant functions vanishing on $\mu^{-1}(0)$, and that it inherits a Poisson structure from the Poisson algebra of smooth functions on M (see [1]).

Proposition 2.20. *The bracket of two smooth functions $f, g \in C^\infty(M_0)$ is a function in $C^\infty(M_0)$.*

The bracket structure on the reduced space is therefore well behaved, and by coincides with the bracket of the functions on the strata. The reduced space is therefore a Poisson algebra, and we can define Hamiltonian dynamics on it, with the typical Hamiltonian relations in Poisson algebras. In particular, we have that the Hamiltonian flows of functions $f \in C^\infty(M_0)$ on the reduced space preserve the symplectic pieces of the symplectic pieces of M_0, as the restriction of the Hamiltonian flow to a stratum is a Hamiltonian flow of the restricted function on the stratum. This is a common trait of stratified symplectic spaces, and is why an homeomorphism whose pullback is a Poisson isomorphism is a stratified symplectomorphism.

The dynamics on the reduced space, allow us to observe that the intersection of the manifold $M_H = \{p \in M : \mathrm{Stab}_G(p) = H\}$ and the zero level set of the moment map, is a manifold which fibres over the piece $(M_0)_{(H)}$ with fibre space $N_G(H)/H$. The fibre projection can be seen to coincide with the orbit map of the induced $N_G(H)/H$-action on M'_H, the union of components of M_H which intersects the zero level set Z

non-trivially. If we call the induced moment map of the $mu_H : M'_H \longrightarrow \mathfrak{l}^*$, we obtain that:

Theorem 2.21. *Zero is a regular value of the moment map μ_H. The piece $(M_0)_{(H)}$ is a symplectic manifold obtained from the regular Marsden-Weinstein reduction $M'_H//(N_G(H)/H)$.*

2.2.3 Reduction in stages

We have already seen that the reduced space M_0 is a decomposition, that it satisfies the frontier condition, and that we have Hamiltonian mechanics on the which agree with the Hamiltonian dynamics on the strata. Therefore, the only condition left to prove is the existence of conical slices, which will be the hardest to prove. But before we jump onto it, lets look at another previous result.

An important result in the paper is the ability to proceed by stages in the reduction. This is formulated as the following.

Theorem 2.22 (SL Reduction by stages)**.** *Let G_1 and G_1 act with Hamiltonian actions and moment maps μ_1, μ_2 on the symplectic space (M, ω). If the actions commute, and we have an action of the product space $G_1 \times G_2$, then the action is Hamiltonian with moment map $\mu = (\mu_1, \mu_2)$. If the map μ_1 is not G_2 invariant, we can construct this moment map by averaging over the group G_2, and analogously for the μ_2 map.*

Then the reduction

$$M//(G_1 \times G_2) = (M//G_1)//G_2 = (M//G_2)//G_1.$$

To be more precise, $M//G_1$ is a G_2 Hamiltonian space, and the reduction by G_2 is equal to $M//(G_1 \times G_2)$. It is analogous for the other group.

2.2.4 A normal form and a Whitney embedding

To finalize we will look at the final steps of the prove. They consist on both a normal form for the reduction, and, finally, the proof of the existence of conical slices.

Reduction in stages allows us to construct a simple local model for the neighbourhood of a point in the reduced space.

Theorem 2.23. *Let x be a point of the reduced space $M//G$, p a point on the 0 level set such that its image under the reduction is x. Let H be the stabilizer of p, $V = (T_p(\mathcal{O}_p))^\omega / T_p(\mathcal{O}_p)$ be the fibre at p of the normal symplectic bundle of the orbit* *through p, and ω_V its symplectic form. Let $\bar{0}$ be the image of the origin in the reduced space $\Phi^{-1}(0)/H$, where Φ is the moment map of the $H-$action on V.*

Then, there exists a neighbourhood U of x in M_0 that is isomorphic to a neighbourhood U_2 of $\bar{0}$ in $\Phi^{-1}(0)/H$. More precisely, there exists an homeomorphism $\varphi : U_1 \longrightarrow U_2$ that induces an isomorphism

$$\varphi^* : C^\infty(U_2) \longrightarrow C^\infty(U_1)$$

of Poisson algebras, a stratified symplectomorphism.

This normal form has a two-fold importance. First, it allows us to describe the top strata of the stratification. Secondly, it allows us to construct the link of a point.

Theorem 2.24. *If the moment map μ is proper, there exists a unique piece $(M_0)_{(H)}$ which is open in the reduced space. It is also connected and dense.*

As the strata is dense, $\overline{(M_0)_{(H)}} = M_0$ and therefore, all strata S are contained in its closure $S \subseteq \overline{(M_0)_{(H)}}$, so the piece $(M_0)_{(H)}$ is maximal, $S \leq \overline{(M_0)_{(H)}}$ for all strata S, an the strata is the top strata. In particular, the existence of this strata also proves that the Poisson algebra $C^\infty(M_0)$ is non degenerate, which means that its centre is formed only by locally constant functions.

The normal form might allow us to construct a link. However, it is not immediate that it is a link, and to prove it we will construct a Whitney embedding of the reduced space, whose image is a Whitney stratification.

Definition 2.25. Let X be a subspace of $\mathbb{R}^n$ a decomposition of X is called a **Whitney stratification** if the pieces of X are smooth submanifolds of $\mathbb{R}^n$, and for pieces P, Q with $P \leq Q$, the following condition, called Whitney's condition B holds.

For an arbitrary point $p \in P$, let $\{p_i\}$ and $\{q_i\}$ be sequences in P and Q, respectively, such that both of them converge to p. Assume that the lines l_i joining p_i and q_i converge in the projective space $\mathbb{R}P^{n-1}$ to a line l, and that the tangent planes $T_{q_i}Q$ converge in the Grassmanian of $(\dim Q)-$planes in $\mathbb{R}^n$ to a plane τ. Then l is contained in τ.

Whitney stratified spaces are stratified spaces in the sense of the Definition 2.7, and the local form 2.23 will allows us to obtain an embedding into $\mathbb{R}^n$ such that the image is a Whitney stratification, which will finally prove the existence of the conical slices (see [42, §6]).

2.2.5 Examples of singular reduction

Example 2.26. This example is taken from [30], where the complete details can be found. We can consider the space $\mathbb{R}^2$ with the standard action of the circle group $SO(2)$. If we lift the action to the cotangent bundle $T^*\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2$, in standard coordinates the action will be

$$\begin{pmatrix} p_1 \\ p_2 \\ q^1 \\ q^2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \\ p_1 \\ p_2 \end{pmatrix}.$$

The cotangent bundle will have with the canonical symplectic form $\omega = dq^1 \wedge dp_1 + dq^2 \wedge dp_2$ and the $SO(2)$ action has moment map $\mu(q,p) = q^1 p_2 - q^2 p_1$. The zero level set is then the union of a point 0 and the hypersurface

$$Z = \{dq^1 \wedge dp_1 + dq^2 \wedge dp_2 = 0, (q,p) \neq (0,0)\}.$$

The hypersurface is, as one would expect, a $SO(2)-$invariant coisotropic submanifold of $T^*\mathbb{R}^2$. On it $SO(2)$ has a free action, and the null directions of the restriction of the symplectic form ω to Z are the orbital directions. The reduced space $(T^*\mathbb{R}^2)_0$ is therefore a disjoint union $C_0 \sqcup C_1$, where $C_0 = \{0\}/SO(2) \cong \{0\}$ and $C_1 = Z/SO(2)$, which can be seen to be symplectomorphic to the standard $\mathbb{R}^2/\{0\}$, and that the whole space is homeomorphic to the upper half of the standard cone in $\mathbb{R}^3$, $T^*\mathbb{R}^2 \simeq \{x_1^2 = x_2^2 + x_3^2, x_1 \geq 0\}$, a symplectic surface with a point singularity at the origin.

Example 2.27. Let D be a discrete group that acts by symplectomorphisms on a symplectic manifold (M, ω). Then, the trivial map $\mu = 0$ is a moment map, and the space (M, D, μ) is Hamiltonian. The reduced space $M/\!/D = M/D$ is a symplectic orbifold (see [27]). In particular, the dimension of the reduction will remain constant.

2.2.6 Some useful lemmas

In this section we will include some lemmas that are be useful for the calculation of reductions:

Lemma 2.28. *Let (M, K, μ) and (N, L, ν) be Hamiltonian spaces, $f : K \longrightarrow L$ a Lie group morphism and $F : M \longrightarrow N$ an $f-$equivariant symplectic map which sends $\mu^{-1}(0)$ to $\nu^{-1}(0)$. If K and L act freely, then F descends to a symplectic map $\bar{F}$:*

$M/\!/K \longrightarrow N/\!/L$. And more generally, if $F(M_A) \subseteq N_B$ for some $A \subseteq K, B \subseteq L$, then the restriction $\bar{F} : (M/\!/K)_{(A)} \longrightarrow (N/\!/L)_{(B)}$ is a symplectic map.

Proof. Lets first focus on the case K and L act freely. The symplectic form on $M/\!/K$ is characterized by its pullback on $\mu^{-1}(0)$ being the restriction of the symplectic form on M, as is the case in in $N/\!/L$, and therefore it is immediate that $\bar{F}$ preserves the symplectic form.

Now suppose that $F(M_A) \subseteq N_B$ for some $A \subseteq K, B \subseteq L$. It is well known (see [14, Proposition 27.5]) that M_A is a symplectic submanifold of M, and that $N_G(A)/A$ acts freely on it, and $(M_A, N_G(A)/A, \mu|_{M_A})$ is a Hamiltonian system. The symplectic form in $(M/\!/K)_{(A)} = \mu^{-1}(0)_(A)/K$ comes from its identification with the space $M_A/\!/(N_G(A)/A)$, and therefore we can use the previous argument replacing $M,\ N$ with $M_A,\ N_B$ respectively. $\square$

Lemma 2.29. *Let (M, G, μ) and (N, G, ν) be Hamiltonian spaces, $F : M \longrightarrow N$ a $G-$equivariant injective smooth (symplectic) map which sends $\mu^{-1}(0)$ into $\nu^{-1}(0)$. Then, the map F descends to a stratified (symplectic) homeomorphism between $M/\!/G$ and its image $\bar{F}(M/\!/G) \subseteq N/\!/G$.*

Proof. For a point $m \in M$, we have that $Z_G(m) = Z_G(n)$, where $n = F(m) \in N$. Let $g \in Z_G(m)$. Then

$$n = F(m) = F(gm) = gF(m) = gn \Rightarrow g \in Z_G(n) \Rightarrow Z_G(m) \subseteq Z_G(n).$$

On the other hand, let $g \in Z_G(n)$. Then

$$F(m) = n = gn = gF(m) = F(gm) \Rightarrow m = gm \Rightarrow g \in Z_G(m).$$

Therefore, the isotropy groups are preserved by the function F, and the function $\bar{F}$ preserves the stratification. The map $\bar{F}$ is open, as it is the composition of open functions, the projection onto the quotient (see [29, Proposition 4.28]), and the diffeomorphism of M onto its image. And as all maps, it is injective on its image. Therefore, the map is an homeomorphism, which preserves the stratification.

Moreover, if the map F is symplectic, by Lemma 2.28 it is symplectic over the strata that it preserves, all of them. $\square$

Lemma 2.30. *Let G, H be a connected Lie groups and (M, G, μ_G),(M, H, μ_H) be Hamiltonian spaces such that their effective actions are the same, i.e., the images* *of the two actions in* $\mathrm{Diff}(M)$ *coincide. Then the reduced spaces* $M/\!/G$ *and* $M/\!/H$ *are canonically isomorphic.*

Proof. Let $K \subset \mathrm{Diff}(M)$ be the common image of both actions. The maps

$$\phi_G \colon G \twoheadrightarrow K, \quad \phi_H \colon H \twoheadrightarrow K$$

are surjective Lie-group homomorphisms with discrete kernels acting trivially on M, from which we can obtain the exact sequences

$$1 \longrightarrow \ker \phi_G \longrightarrow G \xrightarrow{\phi_G} K, \quad 1 \longrightarrow \ker \phi_H \longrightarrow H \xrightarrow{\phi_H} K.$$

By equivariance, there exists a single moment map

$$\mu_K \colon M \longrightarrow \mathfrak{k}^* \quad \text{such that} \quad \mu_G = \phi_G^* \circ \mu_K, \qquad \mu_H = \phi_H^* \circ \mu_K.$$

Since ϕ_G and ϕ_H are surjective on the Lie algebras, $\ker(\phi_G^*) = \{0\}$ and $\ker(\phi_H^*) = \{0\}$, which gives us

$$\mu_G^{-1}(0) = \{\, x \in M : \mu_K(x) \in \ker \phi_G^* \,\} = \mu_K^{-1}(0) = \mu_H^{-1}(0).$$

Denote this common zero-level by $Z = \mu_K^{-1}(0)$. The pullback $\iota^*\omega$ of the ambient symplectic form to Z is basic for the K–action, and hence descends to the reduced form ω_{red}^K on Z/K. Likewise it descends to ω_{red}^G on Z/G and ω_{red}^H on Z/H.

On Z, both $\ker \phi_G$ and $\ker \phi_H$ act trivially and preserve $\iota^*\omega$. Therefore the natural identifications

$$Z/G \cong (Z/\ker \phi_G) \,/\, \big(G/\ker \phi_G\big) \cong Z/K, \quad Z/H \cong (Z/\ker \phi_H) \,/\, \big(H/\ker \phi_H\big) \cong Z/K$$

are in fact symplectomorphisms (they carry ω_{red}^G and ω_{red}^H to the same ω_{red}^K on Z/K).

Composing these gives a canonical symplectomorphism

$$M/\!/G = (Z, \iota^*\omega)/G \;\cong\; (Z, \iota^*\omega)/K \;\cong\; (Z, \iota^*\omega)/H = M/\!/H,$$

as stratified symplectic spaces, completing the proof.

$\square$

Chapter 3

Singular Reduction of the Cotangent Bundle of a Lie Group

This chapter will cover two examples of singular reduction, covering the reduction of the tangent bundle of a Lie group G by subgroups of the natural product action of $G \times G$. We will first cover a construction analogous to Mayrand's work [36] but adapted to the real, compact case, and later we will cover another different construction based on similar ideas.

Let G be a *compact connected semisimple* Lie group. G acts on itself by the left and right action, which can be lifted to the cotangent bundle T^*G. Using the trivialization of the cotangent bundle, $G \times \mathfrak{g}^*$, the action of $(g, h) \in G \times G$ on an element $(x, \xi) \in G \times \mathfrak{g}^*$ is $(gxh^{-1}, \mathrm{Ad}_h^* \xi)$. The action is Hamiltonian with a moment map

$$\mu_0 : G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^* \times \mathfrak{g}^* \quad \mu_0(x, \xi) \longmapsto (\mathrm{Ad}_x^* \xi, -\xi).$$

From this moment map we will derive the moment map for the various subgroups that we will consider.

For the duration of this chapter, let $\mathfrak{g}$ be the Lie algebra of G, let T be a maximal torus of G with corresponding Cartan subalgebra $\mathfrak{t} = \mathrm{Lie}(T)$, and let Φ be the corresponding root system. In addition, let $\tilde{G}$ be the universal covering of G, and suppose that $G = \tilde{G}/\Gamma$, with $\Gamma \subseteq Z(\tilde{G})$.

3.1 Reduction by $T \times T$

In this section we will look at the reduction of T^*G by the subgroup $T \times T$, which we will noted as $\mathcal{D}(G) := T^*G/\!/T \times T$. The action is Hamiltonian, and has a moment map that consists of the moment map μ_0 projected onto the dual Lie algebra of the subgroup $\mathfrak{t}^* \times \mathfrak{t}^*$, $\mu_T(x,\xi) = (\mathrm{Ad}_x^*\xi|_{\mathfrak{t}}^*, \xi|_{\mathfrak{t}}^*)$.

The main results in this section include the independence of the reduction on the specific group G, just on its Lie Algebra $\mathfrak{g}$; that the stratification poset can be seen as the root subsystems of Φ; and we will give an iterative description of the strata.

3.1.1 The stratification only depends on the Lie algebra $\mathfrak{g}$

To start, we will prove that the reduction depends only on the Lie algebra $\mathfrak{g}$, which will justify our later notation of $\mathcal{D}(\mathfrak{g})$. We will prove this by giving a model for the reduction, which will not depend on the Lie group, which will enable us to proof the following theorem:

Theorem 3.1. *Let $\tilde{G}$ be the universal covering space of G. There exists a stratified symplectomorphism between $\mathcal{D}(G)$ and $\mathcal{D}(\tilde{G})$.*

Therefore, for a compact semisimple Lie algebra $\mathfrak{h}$ we will define $\mathcal{D}(\mathfrak{h}) := \mathcal{D}(G_{\mathfrak{h}})$, where $G_{\mathfrak{h}}$ is a compact semisimple Lie Group (see Theorem 1.39).

To obtain the model for the reduction, we will use the reduction by stages procedure described in [42], from which we have that $\mathcal{D}(G) = T^*G/\!/T \times T = (T^*G/\!/1 \times T)/\!/T \times 1$.

For the right torus action, the moment map will be the second component of the μ_T moment map, $\mu_R(x,\xi) = \xi|_{\mathfrak{t}}^*$, and as the action is free, the reduced space is a symplectic manifold. Therefore, we can identify the 0 level of the moment map with $G \times \mathfrak{t}^\circ$, and quotiented by the right T-action we obtain the space $G \times_T \mathfrak{t}^\circ$. This manifold can be identified with the tangent bundle of a regular coadjoint orbit. More specifically, let $\tau \in \mathfrak{t}^{reg}$ be a regular element, and $\mathcal{O} = G \cdot \tau$ its regular orbit, then $G \times_T \mathfrak{t}^\circ$ is symplectomorphic to the cotangent bundle $T^*\mathcal{O}$ via the G-equivariant map $(g,\xi) \to (\mathrm{Ad}_g\tau, \mathrm{Ad}_{g^*}\xi)$, where G acts on $T^*\mathcal{O}$ by the coadjoint action (See [10, Lemma 1.4.9] or [40, Theorem 6.6.1]).

For the left torus action, the action is Hamiltonian with a moment map that is the first component of μ_T, $\mu_L(x,\xi) = \mathrm{Ad}_x^*\xi|_{\mathfrak{t}}^*$, with 0 level set $\mu_2^{-1}(0) = \{(g,\xi) \in G \times_T \mathfrak{t}^\circ : Ad_g^*\xi \in \mathfrak{t}^\circ\}$. By the previous isomorphism one can identify it with $T^*\mathcal{O} \cap \mathfrak{g} \times \mathfrak{t}^\circ$.

Therefore, it descends to a stratified symplectomorphism on the reduced spaces by T, and we obtain

Theorem 3.2. *The reduction $\mathcal{D}(G)$ is isomorphic to the reduced space $\left(T^*\mathcal{O} \cap \mathfrak{g} \times \mathfrak{t}^\circ\right)/T$. Moreover, the $T \times T$ stratification in $\mathcal{D}(G)$ coincides with the T stratification in $\left(T^*\mathcal{O} \cap \mathfrak{g} \times \mathfrak{t}^\circ\right)/T$.*

Therefore, we can now prove the independence on the Lie Group:

Proof. [Theorem 3.1] The reduction for the group G is $\mathcal{D}(G) = \left(T^*\mathcal{O} \cap \mathfrak{g} \times \mathfrak{t}^\circ\right)/T$, and for its universal covering is $\mathcal{D}(\tilde{G}) = \left(T^*\mathcal{O} \cap \mathfrak{g} \times \mathfrak{t}^\circ\right)/\tilde{T}$, where $\tilde{T}$ is the universal covering of the maximal torus $T = \tilde{T}/\Gamma$. As T and $\tilde{T}$ act by the coadjoint action, which is trivial over central elements, they act by the same effective action on $T^*\mathcal{O} \cap \mathfrak{g} \times \mathfrak{t}^\circ$ and therefore, by Lemma 2.30 their reductions are equal . □

3.1.2 The stratification poset

We will now give a calculation of the stratification poset for this reduction. We will find a poset equivalence between the stabilizers and root subsystems $\Psi \leq \Phi$ of the Lie algebra $\mathfrak{g}$. As the reduction is stratified by the conjugacy class of the stabilizers, but the $T \times T$-action is abelian, the reduction is stratified by just the stabilizers.

From Theorem 3.2 we know that $\mathcal{D}(\mathfrak{g}) = (T^*\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\circ)/T$. For our calculations it will be convenient to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Killing form, which gives us the identification of $\mathfrak{t}^\circ$ with $\mathfrak{t}^\perp$. Then, we have that $\mathcal{D}(\mathfrak{g}) = (T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp)/T$, where T acts by the adjoint action on both factors.

The first step will be to determine the stabilizer of a general point.

Lemma 3.3. *The stabilizer of the point $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ under the adjoint T action is $Z_{\Phi(X,Y)}$, where*

$$\Phi(X, Y) = \Phi \cap span_{\mathbb{Z}}\{\alpha \in \Phi : (X_\alpha, Y_\alpha) \neq (0, 0)\}.$$

$\Phi(X, Y)$ is a root subsystem of Φ.

Proof. As the elements of the stabilizer of $(X, Y) \in \mathfrak{g} \times \mathfrak{g}$ must both fix X and Y, using [8, §V.2] we get that $t \in T$ fixes (X, Y) if and only if $t \in Z_{\Phi(X)} \cap Z_{\Phi(Y)} = Z_{\Phi'(X,Y)}$, where $\Phi'(X, Y) = \Phi(X) \cup \Phi(Y) = \{\alpha \in \Phi : (X_\alpha, Y_\alpha) \neq (0, 0)\}$. However, we don't necessarily have that $\Phi'(X, Y)$ is a root subsystem, but we can take the minimal root subsystem that contains $\Phi'(X, Y)$, which is $\Phi(X, Y)$, and prove that $Z_{\Phi'(X,Y)} = Z_{\Phi(X,Y)}$.

Trivially, as $\Phi'(X,Y) \subseteq \Phi(X,Y) \Rightarrow Z_{\Phi(X,Y)} \subseteq Z_{\Phi'(X,Y)}$. Conversely, if we take $\gamma \in \Phi(X,Y)$, it can be written as an integer combination $\gamma = \sum n_i\alpha_i$, with $\alpha_i \in \Phi'(X,Y)$. Therefore, for an element $t \in Z_{\Phi'(X,Y)}, \gamma(t) = \prod \alpha_i(t)^{n_i} = 1$, which gives us $Z_{\Phi(X,Y)} \subseteq Z_{\Phi'(X,Y)}$. □

Therefore, we obtain that orbit types can be identified with root subsystems via the map $Z_\Psi \longmapsto \Psi$. If we define $\mathcal{D}(\mathfrak{g})_\Psi := \mathcal{D}(\mathfrak{g})_{Z_\Psi}$ for a root subsystem Ψ, the previous map can be seen as a map from the strata to the root subsystems $\mathcal{D}(\mathfrak{g})_\Psi \longmapsto \Psi$. We now set to prove that this identification is a poset isomorphism.

Lemma 3.4. *$\mathcal{D}(\mathfrak{g})_\Psi \neq \emptyset$ for all root subsystems $\Psi \leq \Phi$.*

Proof. For a root subsystem Ψ, take $0 \neq X_\alpha \in \mathfrak{g}_{\mathbb{C}\alpha}$ for all $\alpha \in \Psi^+$. Then $\bar{X}_\alpha \in \mathfrak{g}_{\mathbb{C},-\alpha}$, and we can define $X = \sum_{\alpha\in\Psi^+}\left(X_\alpha + \bar{X}_\alpha\right) \in \mathfrak{g}$ an the point $(\tau, X) \in T\mathcal{O} \cap (\mathfrak{g} \times \mathfrak{t}^\perp)$. Moreover, as $\tau \in \mathfrak{t}_{reg} \Rightarrow \tau_\alpha = 0\ \forall\alpha \in \Psi$, the root subsystem associated to the point (τ, X) will be $\Phi(\tau, X) = \Phi(X) = \Psi$ and therefore $(\tau, X) \in \mathcal{D}(\mathfrak{g})_\Psi$. □

With this we have proven that the identification is a surjective. We now prove that preserves the order over the elements. For this, the following lemma will be useful.

Lemma 3.5. *If Ψ is a root subsystem, and $\alpha \in \Phi$ is such that $\alpha(t) = 1$ for all $t \in Z_\Psi$, then $\alpha \in \Psi$.*

Proof. We will see α as an element of $\mathfrak{t}^*$, so we want to prove that if $\alpha(H) \in 2\pi i\mathbb{Z}$ for all $H \in \mathfrak{t}$ such that the set $\{\beta(H) : \beta \in \Psi\} \subseteq 2\pi i\mathbb{Z}$, then $\alpha \in \Psi$. Let $\beta_i, \ldots, \beta_k \in \Psi$ be a set of simple roots and complete it to a basis $\beta_1, \ldots, \beta_n$ of $\mathfrak{t}^*$, and let $H_1, \ldots, H_n$ be its dual base of $\mathfrak{t}$, i.e. $\beta_i(H_j) = \delta_{ij}$. Our element $\alpha \in \mathfrak{t}^*$ can be written as $\alpha = \sum_i a_i\beta_i$. For an element $H_j\ j > k$ and an scalar θ we have that $\beta(\theta H_j) = \theta \cdot 0 = 0$ for all $\beta \in \Psi$, so we must have that $\alpha(\theta H_j) = a_j\theta \in 2\pi i\mathbb{Z}$ for all values of θ, which implies that $a_j = 0$ for all $j > k$.

For $H_j\ \ \leq k$ we have that $\beta(2\pi i H_j) \in 2\pi i\mathbb{Z}$ for all $\beta \in \Psi$. Therefore, we have that $\alpha(2\pi i H_j) = 2\pi i a_j \in 2\pi i\mathbb{Z}$, which gives us that $a_j \in \mathbb{Z}$, giving that α is a integer combination of elements in Ψ and therefore $\alpha \in \operatorname{span}_Z \Psi \cap \Phi = \Psi$. □

Theorem 3.6. *The map $\mathcal{D}(\mathfrak{g})_\Psi \longmapsto \Psi$ is a poset isomorphism. Therefore, the stratification of $\mathcal{D}(\mathfrak{g})$ is given by the root subsystems $\Psi \subseteq \Phi$.*

Proof. By definition, if $\Psi_1 \subseteq \Psi_2 \Rightarrow \mathcal{D}(\mathfrak{g})_{\Psi_1} \leq \mathcal{D}(\mathfrak{g})_{\Psi_2}$, as $Z_{\Psi_2} \subseteq Z_{\Psi_1}$. Suppose then that $\mathcal{D}(\mathfrak{g})_{\Psi_1} \leq \mathcal{D}(\mathfrak{g})_{\Psi_2} \iff Z_{\Psi_2} \subseteq Z_{\Psi_1}$. For $\alpha \in \Psi_1$, as $Z_{\Psi_2} \subseteq Z_{\Psi_1}$, we have that $\alpha(t) = 1\ \forall Z_{\Psi_2}$, and using Lemma 3.5 we obtain that $\mathcal{D}(\mathfrak{g})_{\Psi_1} \leq \mathcal{D}(\mathfrak{g})_{\Psi_2}$. Therefore, the map preserves the order, and is injective as it preserves the equality case of the order. As it is also surjective, we have that is a bijection that preserves the order, therefore it is a poset equivalence. □

3.1.3 Description of the strata

Now that we have identified the stratification of our reduced system, is time to describe the individual strata $\mathcal{D}(\mathfrak{g})_\Psi$, which we will be able to do in a recursive manner. For a particular root subsystem Ψ, let W_Ψ the Weyl group generated by reflections on the roots of Ψ, and $\left|W_\Phi : W_\Psi\right|$ the index of W_Ψ in W_Φ. We will prove that $\mathcal{D}(\mathfrak{g})_\Psi$ is a disjoint union of $\left|W_\Phi : W_\Psi\right|$ copies of $\mathcal{D}(\mathfrak{g}_\Psi)_{\text{top}}$.

Lemma 3.7. *For a root system Ψ, there exists a compact semisimple connected Lie subgroup $G_\Psi \subseteq G$ with Lie algebra $Lie(G_\Psi) = \mathfrak{g}_\Psi$.*

Proof. Semisimple subgroups of compact Lie groups are closed, so the connected subgroup $G_\Psi = \exp(\mathfrak{g}_\Psi)$ of G with Lie algebra $\mathfrak{g}_\Psi$ is closed, and therefore compact. □

Let $\mathfrak{z}_\Psi = \text{Lie}(Z_\Psi)$, so we have $\mathfrak{t} = \mathfrak{z}_\Psi \oplus \mathfrak{t}_\Psi$. Then, the reductive Lie algebra $\mathfrak{g}'_\Psi = \mathfrak{z}_{\Psi\mathbb{C}} \oplus \mathfrak{g}_{\Psi\mathbb{C}} = \mathfrak{t} \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ has semisimple factor $\mathfrak{g}_\Psi$.

Lemma 3.8. *The intersection $\mathcal{O} \cap \mathfrak{g}'_\Psi$ has $\left|W_\Phi : W_\Psi\right|$connected components of the form $\tau_0 + \mathcal{O}_\Psi$ for some $\tau_0 \in \mathfrak{z}_\Psi$, and where $\mathcal{O}_\Psi$ is a regular semisimple G_Ψ-orbit in $\mathfrak{g}_\Psi$.*

Proof. Recall τ the regular element with regular orbit $\mathcal{O}$. It decomposes as $\tau = \tau_\Psi + \tau_0$, with $\tau_0 \in \mathfrak{z}_\Psi$ and $\tau_\Psi \in \mathfrak{t}_\Psi$. Every element in $\mathcal{O}$ is $G-$conjugate to exactly one point $w \cdot \tau$, $w \in W$. Since τ_0 centralises G,

$$w \cdot \tau = w \cdot \tau_\Psi + \tau_0.$$

Therefore, $\mathcal{O} \cap (\mathfrak{z}_\Psi \oplus \mathfrak{g}_\Psi)$ consists of the elements $w \cdot \tau$ for which $w \cdot \tau_\Psi \in \mathfrak{t}_\Psi$. This happens if and only if $w \in N_G(T_\Psi)$, equivalent to say that the class of $[w]$ lies in W_Φ / W_Ψ. For a representative w, inside the subgroup G_Ψ acts transitively on the regular orbit $\mathcal{O}_\Psi = G_\Psi \cdot (w \cdot \tau_\Psi)$. Adding the central element, one obtains the intersection $\tau_0 + \mathcal{O}_\Psi$. Different classes give rise to disjoint sets, and each $\tau_0 + \mathcal{O}_\Psi$ is connected, as G_Ψ is connected. The number of components are $|W_\Phi / W_\Psi| = \left|W_\Phi : W_\Psi\right|$. □

Proposition 3.9. *For all $\Psi \subseteq \Phi$, the strata $\mathcal{D}(\mathfrak{g})_\Psi$ is isomorphic to a disjoint union of $\left|W_\Phi : W_\Psi\right|$copies of $\mathcal{D}(\mathfrak{g}_\Psi)_{top}$.*

$$D(\mathfrak{g})_\Psi := \Big(T^*\mathcal{O} \cap (\mathfrak{g} \times z_\Psi^\perp)\Big)^T_{Z_\Psi} / T$$

Proof. First, we define $M_\Psi = M \cap (\mathfrak{g}'_\Psi \times \mathfrak{g}'_\Psi)$. We will proof that $M_{Z_\Psi} \subseteq M_\Psi$.

If $(X, Y) \in M_{Z_\Psi}$ then by Lemma 3.3 we have that $\Phi(X, Y) = \Psi$. Therefore, $\{\alpha \in \Phi : (X_\alpha, Y_\alpha) \neq (0, 0)\} \subseteq \Psi$ and $(X, Y) \in M_\Psi$. Therefore, as $M_{Z_\Psi} \subseteq M_\Psi \subseteq M$, we have that $\mathcal{D}(\mathfrak{g})_\Psi = M_{Z_\Psi} = (M_\Psi)_{Z_\Psi}/T$.

By Lemma 3.8, the manifold

$$M_{Z_\Psi} = (M_\Psi)_{Z_\Psi} = \bigsqcup_{[w] \in W_\Phi / W_\Psi} \Big(T^*\big(\zeta_w + \mathcal{O}_\Psi\big) \cap (\mathfrak{g} \times \mathfrak{t}^\perp)\Big)_{Z_\Psi} \cong$$

$$\bigsqcup_{[w] \in W_\Phi / W_\Psi} (T^*\mathcal{O}_\Psi \cap (\mathfrak{g}_\Psi \times \mathfrak{t}_\Psi^\perp))_{Z_\Psi \cap T_\Psi} = \bigsqcup_{[w] \in W_\Phi / W_\Psi} (T^*\mathcal{O}_\Psi \cap (\mathfrak{g}_\Psi \times \mathfrak{t}_\Psi^\perp))_{Z_\Psi \cap T_\Psi},$$

Where the symplectomorphism $\Big(T^*\big(\zeta_w + \mathcal{O}_\Psi\big) \cap (\mathfrak{g}_\Psi \times \mathfrak{t}_\Psi^\perp)\Big)_{Z_\Psi} \cong (T^*\mathcal{O}_\Psi \cap (\mathfrak{g}_\Psi \times \mathfrak{t}_\Psi^\perp))_{Z_\Psi \cap T_\Psi}$ is given via the map $(X, Y) \longmapsto (\zeta_\omega + X, Y)$, which is a symplectic vector-space translation due to $\mathfrak{z}_\Phi$ being isotropic to the canonical form.

In particular, $(T^*\mathcal{O}_\Psi \cap (\mathfrak{g}_\Psi \times \mathfrak{t}_\Psi^\perp))_{Z_\Psi \cap T_\Psi}$ is the top strata of the reduction $\mathcal{D}(\mathfrak{g}_\Psi)_{\mathrm{top}}$, as the centre for the T_Ψ-action on $\mathfrak{g}_\Psi \times \mathfrak{g}_\Psi$ is the group $Z_\Psi \cap T_\Psi$, so the strata $\mathcal{D}(\mathfrak{g})_\Psi$ is a union of $\big|W_\Phi : W_\Psi\big|$ disjoint manifolds, which are symplectomorphic to $\mathcal{D}(\mathfrak{g}_\Psi)_{\mathrm{top}}$. ☐

Corollary 3.10. *The stratum $\mathcal{D}(\mathfrak{g})_{top}$ is a dense, open, connected set of dimension $2(\dim \mathfrak{g} - 2 rank\ \mathfrak{g})$. The stratum $\mathcal{D}(\mathfrak{g})_{bottom}$ is a finite set of $|W_\Phi|$ points.*

Proof. The description of the top strata comes from Theorem 2.24. The dimension is a simple calculation of the dimension of the manifold minus twice the dimension of the minimal stabilizer. The description of the bottom strata comes from the Theorem 3.9, with the root subsystem $\emptyset$. ☐

3.1.4 A coarser stratification

The objective for this section is to prove a coarser stratification for the $\mathcal{D}(\mathfrak{g})$ reduction, which will be useful for explicit calculations. Moreover, as stratifications can be refined arbitrarily, it is desirable to obtain the stratification as coarse as possible. The stratification is the following:

Theorem 3.11. *The partition $\mathcal{P} = \big\{\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]} : [\mathfrak{h}] \in \mathcal{C}_g\big\}$ is a stratification of $\mathcal{D}(\mathfrak{g})$. The map $[\mathfrak{h}] \longmapsto \mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ is a poset isomorphism. Moreover, the strata $\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ is a disjoint union of $m_\mathfrak{g}(\mathfrak{h})$ copies of $\mathcal{D}(\mathfrak{h})_{top}$.*

For a conjugacy class $[\mathfrak{h}]$, one has a representative $\mathfrak{g}_\Psi$ with $\Psi \subseteq \Phi$. By Proposition 1.48, we have that

$$\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]} = \bigcup_{w \in W_\Phi} \mathcal{D}(\mathfrak{g})_{w\cdot\Psi}.$$

Let n, $\{w_i\} \subseteq W$ be such that $\{w \cdot \Psi : w \in W\} = \{w_1 \cdot \Psi, \ldots, w_n \cdot \Psi\}$ with $w_i \cdot \Psi \neq w_j \cdot \Psi$ for $i \neq j$.

$$\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]} = \bigcup_{i=1}^{n} \mathcal{D}(\mathfrak{g})_{w_i\cdot\Psi}$$

Lemma 3.12. *The previous union is topologically disjoint.*

Proof. We only need to prove that if $u, v \in W_\Phi$, $\overline{\mathcal{D}(\mathfrak{g})_{u\cdot\Psi}} \cap \mathcal{D}(\mathfrak{g})_{v\cdot\Psi} \neq \emptyset$, we have that $u \cdot \Psi = v \cdot \Psi$. Using Proposition 2.3, we have that:

$$\overline{\mathcal{D}(\mathfrak{g})_{u\cdot\Psi}} \cap \mathcal{D}(\mathfrak{g})_{v\cdot\Psi} = \bigcup_{\chi \leq u\cdot\Psi} \mathcal{D}(\mathfrak{g})_\chi \cap \mathcal{D}(\mathfrak{g})_{v\cdot\Psi} \neq \emptyset.$$

Therefore, for certain $\chi \leq u \cdot \Psi$, we have that $\mathcal{D}(\mathfrak{g})_\chi \cap \mathcal{D}(\mathfrak{g})_{v\cdot\Psi} \neq \emptyset$, which implies that $v \cdot \Psi = \chi \subseteq u \cdot \Psi$, and therefore $v \cdot \Psi = u \cdot \Psi$. □

The Lemma 3.12 proves that each piece of our coarser stratification is a symplectic manifold and is locally closed, and that the partition has conical slices, which will be the same as the conical slices of the finer stratification. We can combine it with Proposition 3.9, we prove that each strata $\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ is a disjoint union of $m_\mathfrak{g}(\mathfrak{h}) = n \cdot \left|W_\Phi : W_\Psi\right|$ copies of $\mathcal{D}(\mathfrak{h})_{\mathrm{top}}$.

Lemma 3.13. *For a conjugacy class $[\mathfrak{h}] \in \mathcal{C}_\mathfrak{g}$, we have that*

$$\overline{\mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}} = \bigcup_{[\mathfrak{q}] \leq [\mathfrak{h}]} \mathcal{D}(\mathfrak{g})_{[\mathfrak{q}]}$$

Proof. Let $\mathfrak{g}_\Psi$ be a representative of $[\mathfrak{h}]$. If we consider the class $[\Psi]$ of Ψ in $\{\Psi \leq \Phi\}/W_\Phi$, and using Proposition 2.3, we obtain

$$\begin{aligned}\overline{\mathcal{D}(\mathfrak{g})_{[\mathfrak{g}_\Psi]}} &= \bigcup_{w \in W_\Phi} \overline{\mathcal{D}(\mathfrak{g})_{w\cdot\Psi}} = \bigcup_{w \in W_\Phi} \bigcup_{\chi \leq w\cdot\Psi} \mathcal{D}(\mathfrak{g})_\chi = \bigcup_{[\chi] \leq [\Psi]} \bigcup_{w \in W_\Phi} \mathcal{D}(\mathfrak{g})_{w\cdot\chi} \\ &= \bigcup_{[\chi] \leq [\Psi]} \mathcal{D}(\mathfrak{g})_{[\mathfrak{g}_\chi]} = \bigcup_{[\mathfrak{q}] \leq [\mathfrak{h}]} \mathcal{D}(\mathfrak{g})_{[\mathfrak{q}]}\end{aligned}$$

□

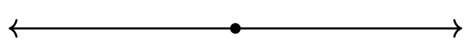
With this description, it is immediate to see that $[\mathfrak{q}] \leq [\mathfrak{h}] \iff \mathcal{D}(\mathfrak{g})_{[\mathfrak{q}]} \leq \mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$, which proves that the map $[\mathfrak{h}] \longmapsto \mathcal{D}(\mathfrak{g})_{[\mathfrak{h}]}$ is an isomorphism of posets. It also is immediate to see that the coarser partition satisfies the frontier condition, finalizing the proof of the Theorem 3.11.

3.1.5 Examples of $\mathcal{D}(\mathfrak{g})$

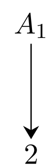
This section will cover some specific examples of the reduction $\mathcal{D}(\mathfrak{g})$. In particular, we will give the coarser stratification, drawing their Hasse diagram. For each node, we will write it as nL, where $n \in \mathbb{N}$ is the number of disjoint components of the strata, each of them isomorphic to the top strata of the reduction of the Lie algebra class L. We write the classes of Lie algebras multiplicatively, for example $A_2^2 B_3$ is $\mathfrak{su}_3 \oplus \mathfrak{su}_3 \oplus \mathfrak{so}_7$.

Example 3.14. The Lie algebra $A_2 : \mathfrak{su}_2$.

The root system of $\mathfrak{su}_2$ consist just of a pair of roots, and embeds in $\mathbb{R}^2$ as 3.1a. Therefore, the Hasse diagram 3.1b will just consist on the two main strata, and the bottom strata will consist of just $|W| = 2$ points. As the Lie group $\mathfrak{su}_2$ has dimension 3 and rank 1, the top stratum is a symplectic manifold of dimension 2(3-2·1)=2, and the reduced space will consist on a surface with two isolated singularities.



(A) A_2 root diagram



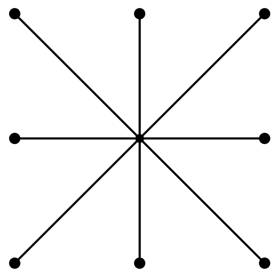
(B) A_2 hasse diagram

FIGURE 3.1: Stratification of $\mathfrak{su}_2$

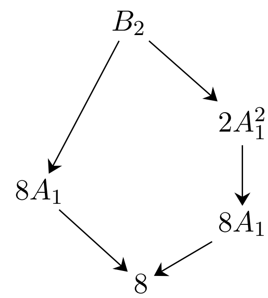
Example 3.15. The Lie algebra $B_2 : \mathfrak{so}_5$.

The root system of $\mathfrak{so}_5$ embeds into $\mathbb{R}^2$ as 3.2a. The four long roots 3.3a, one pair perpendicular to the other, form a unique A_1^2 system. It is fixed by the Weyl group, and its embedding number is 2.

Each pair of opposite long roots forms an A_1 system, with embedding number $2|W_{B_2} : W_{A_1}| = 8$. Similarly, each pair of opposite short roots forms an A_1 system with embedding number 8 as well. Lastly, we have to consider the empty set, a set of $|W_{B_2}|$ points, and the whole root system 3.2a, with embedding number one. Using all these strata, we can construct the corresponding Hasse diagram 3.2b.

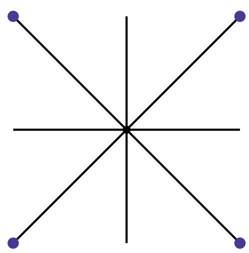


(A) B_2 root diagram.

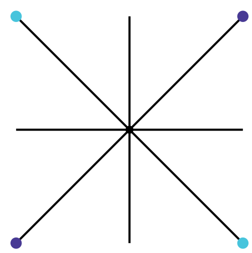


(B) B_2 Hasse diagram

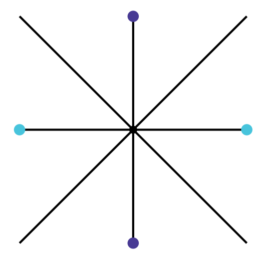
FIGURE 3.2: Stratification of $\mathfrak{so}_5$



(A) A_1^2 root subsystem.



(B) Long A_1 root subsystems.

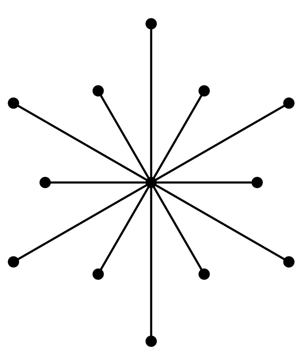


(C) Short A_1 root subsystems.

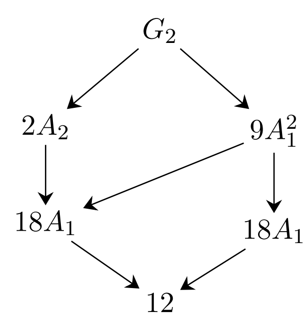
FIGURE 3.3: Root subsystems of $\mathfrak{so}_5$

Given that the dimension of $\mathfrak{so}_5$ is 10, and its rank is 2, the top strata of the reduction is a symplectic manifold of dimension 2(10-2·2)=12. For A_1, its top strata is of dimension 2, and for A_1^2 it has dimension 4. Therefore, the reduced space is a 12-dimensional symplectic manifold with 2 disjoint singularities of dimension 4 and 8 disjoint singularities of dimension 2.

Example 3.16. The exceptional Lie algebra $\mathfrak{g}_2$.



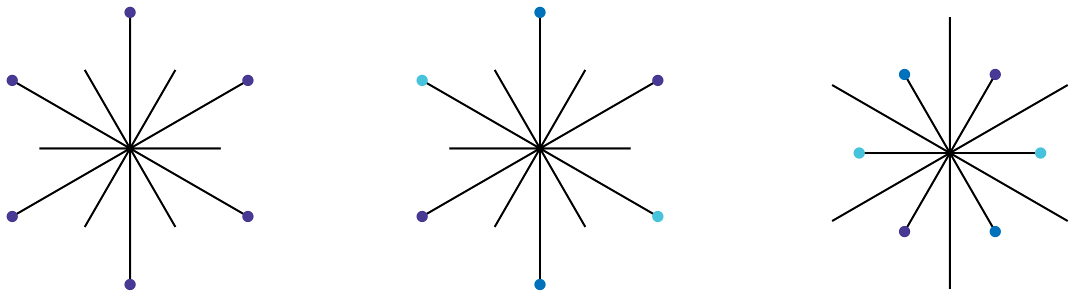
(A) G_2 root diagram.



(B) G_2 Hasse diagram

FIGURE 3.4: Stratification of $\mathfrak{g}_2$

The root system of $\mathfrak{g}_2$ embeds into $\mathbb{R}^2$ as 3.4a. The six long roots 3.5a form a unique A_2 system, which is fixed by the Weyl group. Its embedding number is $|W_{G_2} : W_{A_2}| = 2$.



(A) A_2 root subsystem.

(B) Long A_1 root subsystems.

(C) Short A_1 root subsystems.

FIGURE 3.5

There are three conjugate A_1^2 systems 3.6, generated each by a long root and the perpendicular short root (perpendicular A_1 systems). The index of its Weyl group is 3, so its embedding number is $3 \cdot 3 = 9$.

Each of the three pairs of opposite long roots 3.5b form a conjugate A_1 system, with index $|W_{G_2} : W_{A_1}| = 6$, and have an embedding number 18. Similarly, each of the three pairs of opposite short roots 3.5c form a conjugate A_1 system with embedding number 18 as well.

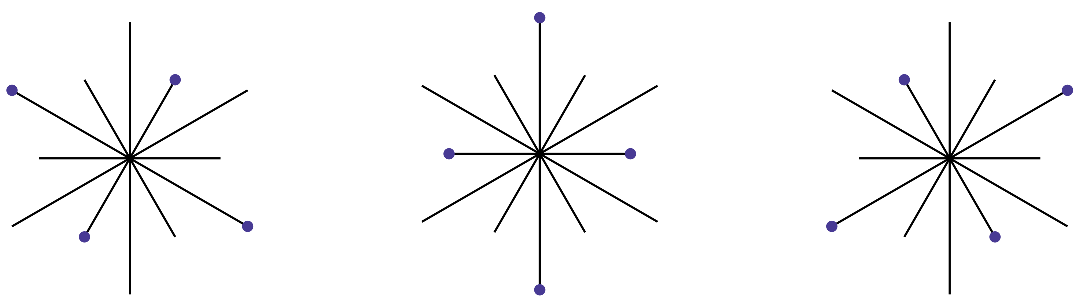


FIGURE 3.6: A_1^2 Root subsystems in G_2.

Lastly, we have the whole root system 3.4a, with embedding number 1, and the empty set formed by $|W_{G_2}|$ points. Putting all this information together, we can construct the Hasse diagram 3.4b.

As $\mathfrak{g}_2$ has dimension 14 and rank 2, the top stratum has dimension 20. Considering A_2 has dimension 8 and rank 2, its top stratum has dimension 8, we see that the reduced space $\mathcal{D}(\mathfrak{g}_2)$ is a symplectic manifold with 2 disjoint singularities of dimension 8 and 9 disjoint singularities of dimension 4.

An exhaustive list of the Hasse diagrams for simple Lie algebras of rank≤ 4 can be found in [36, §6].

3.2 Reduction by the Diagonal Group

In this section we will see the reduction of T^*G by the diagonal action $\Delta G \hookrightarrow G \times G,\ g \mapsto (g,g)$, which we will denote by $\mathcal{R}(G) = T^*G/\!/\Delta G$. The action is Hamiltonian, with moment map μ_G that consists of the moment map μ_0 projected onto the dual Lie group $\mathfrak{g}^*$, $\mu_G(x,\xi) = \mathrm{Ad}^*_G\xi - \xi$.

This reduction has a lot of similarities to the previous one, however there is a key difference: in this case the reduction does depend on the Lie group. The easiest way to see this is by comparing the points in the centre of the group, which will be the bottom strata.

Proposition 3.17. *The bottom strata consist of the points in the with trivial $G-$action, $T^*G_{(G)} = \{(z,0) : z \in Z(G)\}$ where $Z(G)$ is the centre of G. Therefore, the reduction $\mathcal{R}(G)$ depends on the Lie group G.*

Proof. All the points of the form $(x,0)$ are clearly in the 0 level set of the moment map. For a point (x,ξ) to be in the bottom strata, it needs to have stabilizer G, so we must have that $gxg^{-1}\ \forall g \in G \Rightarrow x \in Z(G)$. For the component $\xi \in \mathfrak{g}^*$, we have that $\mathrm{Ad}^*_g\xi = \xi\ \forall g \in G$. If we differentiate at the identity, we see that $\mathrm{ad}^*_X\xi = 0$ for all $X \in \mathfrak{g}$. Therefore for any $Y \in \mathfrak{g}$, $(\mathrm{ad}^*_X\xi)(Y) = -\xi([X,Y]) = 0$, so ξ vanishes on $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ as $\mathfrak{g}$ is semisimple. Therefore, $\xi(\mathfrak{g}) = 0 \Rightarrow \xi = 0$. □

However, we will find a relation between the reductions with the same Lie algebra. The main results we prove in this section will be a model for the reduction, and its stratification poset.

3.2.1 The model of the stratification

The first result we will prove is a form for the reduction. This identification is well known and appears in [4], [25] or [38].

Theorem 3.18. *The reduced space $\mathcal{R}(G)$is isomorphic to the reduced space $T \times \mathfrak{t}/W$.*

To be specific, let $(x,X) \in \mu_G^{-1} \subseteq G \times \mathfrak{g}$. The orbit of (x,X) under the adjoint $G-$action contains an element of $T \times \mathfrak{t}$. This element is unique under the Weyl action, and the application induces an homeomorphism on the quotient sets

$$\mathcal{R}(G) \simeq (T \times \mathfrak{t})/W$$

In particular, the homeomorphism preserve the strata and is symplectic over them.

Proof. For this proof we will use the correspondence of $\mathfrak{g}^*$ with $\mathfrak{g}$ to identify T^*G with $G \times \mathfrak{g}$. The 0 level set of the moment map $\mu_G(x, X) = \mathrm{Ad}_{x^{-1}} X - X$ are the pairs (x, X) that commute, $\mathrm{Ad}_{x^{-1}} X = X$.

First consider the Lie subalgebra $\mathfrak{h}_X \subset \mathfrak{g}$ generated by X. The Lie group $\exp \mathfrak{h}_X = \exp tX$ forms a torus in G, and since (x, X) commute, we have that

$$x \exp(tX) x^{-1} = \exp(\mathrm{Ad}_x tX) = \exp(tX)$$

so x commutes with the torus $\exp \mathfrak{h}_X$. In particular, this implies that there exists a maximal torus T' in G that contains x and $\exp \mathfrak{h}_X$, which is conjugate to T as $gT'g^{-1}$, so $gxg^{-1}, \exp(\mathrm{Ad}_g \mathfrak{h}_X) \in T$, and therefore $\mathfrak{h}_X \subseteq \mathfrak{t} \Rightarrow \mathrm{Ad}_g X \in \mathfrak{t}$ and $(gxg^{-1}, \mathrm{Ad}_g X) \in T \times \mathfrak{t}$.

We now define the map

$$\varphi : G \times_{N(T)} (T \times \mathfrak{t}) \longrightarrow \mu^{-1}(0) \quad [g, (x, X)] \longmapsto (gxg^{-1}, \mathrm{Ad}_g X).$$

This map is well defined. For $n \in N(T)$, we have that

$$\varphi(gn, (n^{-1}xn, \mathrm{Ad}_{n^{-1}} X)) = (gnn^{-1}xn^{-1}g^{-1}, \mathrm{Ad}_{gn} \mathrm{Ad}_{n-1} X) = (gxg^{-1}, \mathrm{Ad}_g X).$$

The map is by definition a G-equivariant continuous map. We show it is injective.

Lets suppose that

$$\varphi(g, (x, X)) = \varphi(h, (y, Y)) \Rightarrow (gxg^{-1}, \mathrm{Ad}_g X) = (hyh^{-1}, \mathrm{Ad}_h Y) \Rightarrow (y, Y) = \alpha(x, X)$$

where $\alpha = h^{-1}g$. We consider the centralizer $Z_G(x, X) = Z_G(x) \cap Z_G(X)$, with Lie algebra

$$Z_{\mathfrak{g}}(x, X) := \{H \in \mathfrak{g} : \mathrm{Ad}_x H = H \text{ and } [X, H] = 0\}$$

which contains $\mathfrak{t}$. It contains $\mathrm{Ad}_\alpha^{-1} \mathfrak{t}$ too, as for $H \in \mathfrak{t}$, we have

$$\mathrm{Ad}_x \mathrm{Ad}_{\alpha^{-1}} H = \mathrm{Ad}_{\alpha^{-1}} \mathrm{Ad}_{\alpha x \alpha^{-1}} H = \mathrm{Ad}_{\alpha^{-1}} H$$

as $\alpha x \alpha^{-1} = y \in T$. We also have that

$$[\mathrm{Ad}_{\alpha^{-1}} H, X] = \mathrm{Ad}_{\alpha^{-1}}[H, \mathrm{Ad}_\alpha X] = \mathrm{Ad}_{\alpha^{-1}} 0 = 0$$

as $\mathrm{Ad}_\alpha^{-1} X = Y \in \mathfrak{t}$. Therefore, $\mathfrak{t}$ and $\mathrm{Ad}_\alpha \mathfrak{t}$ are maximal abelian subalgebras of $Z_{\mathfrak{g}}(x, X)$, so they are conjugated by an element k in $Z_G(x, X)$ such that $\mathrm{Ad}_{k\alpha^{-1}} \mathfrak{t} = \mathfrak{t}$, implying that $k\alpha^{-1} \in N(T)$, which relates $k\alpha^{-1}(y, Y) = k\alpha^{-1}(gxg^{-1}, \mathrm{Ad}_g X) = k(x, X) = (x, X)$.

By Lemma 2.29 the map descends to the quotient, where it will be a stratified homeomorphism in its image. In particular, by the fact we determined that all orbits cut $T \times \mathfrak{t}$ in a point, the function is surjective, and an homeomorphism between the reduced spaces, which are on one side, $\mathcal{R}(G)$, and on the other, $(T \times \mathfrak{t})/N_G(T) \cong (T \times \mathfrak{t})/W$ if we take into account that they have the same effective action.

We still have to prove that the map is symplectic. However, we can take into account that the function $\bar{\varphi}$ in the quotient coincides with the inclusion of $T^*T \hookrightarrow T^*G$. The map send the 0 level set of the W-action (the group is discrete so the moment map is $\mu = 0$) to the 0 level set of T^*G, as the elements of $T \times \mathfrak{t}$ commute. It preserves the stratification the quotient, as $\bar{\varphi}$ preserves it, and is $G-$equivariant. And clearly, the map is symplectic, as the form in T^*T coincides with the restriction of the form in T^*G. Therefore, using Lemma 2.28, we obtain that the map in the quotient, which coincides with $\bar{\varphi}$, is symplectic.

Therefore, the spaces $\mathcal{R}(G) \cong T \times \mathfrak{t}/W$ are isomorphic as stratified symplectic spaces.

□

Proposition 3.19. *The universal covering map $\pi : \tilde{G} \longrightarrow G$ descends to a continuous map over the reduced spaces $\overline{\pi} : \mathcal{R}(\tilde{G}) \longrightarrow \mathcal{R}(G)$ that preserves the stratification and is a symplectic covering map over the strata.*

Proof. The projection can be lifted to the cotangent bundles, where it is a symplectic map. Moreover, using the trivialization of the cotangent bundle, that the lifted map will be

$$\tilde{\pi} : \tilde{G} \times \mathfrak{g} \longrightarrow G \times \mathfrak{g} \ \ \tilde{\pi}(g, X) = (\pi(g), X).$$

Its therefore immediate that the lifted map will also be π-equivariant.

The moment maps of the actions will satisfy that $\tilde{\mu} = \mu \circ \tilde{\pi}$, so the reduction sends the 0 level of the moment map $\tilde{\mu}$, and in particular, it is surjective $\tilde{\pi}(\tilde{\mu}^{-1}(0)) = \mu^{-1}(0)$.

As $G = \tilde{G}/\Gamma$, with Γ a central subgroup, Γ will be a subgroup of all stabilizers and the stabilizer of an element will be related to the one in its preimage as $Z_G(\tilde{\pi}(g, X)) =$

$Z_{\tilde{G}}(g, X)/\Gamma$, and the projection $\tilde{\pi}$ preserves the stratification. Therefore, by Lemma 2.28, it descends to the stratification as a stratified symplectic map. $\square$

3.2.2 The stratification poset

Proposition 3.20. *The stabilizer of a point $(t, H) \in T \times \mathfrak{t}$ is $W_{\Psi(x,H)}$, the subgroup of the Weyl group generated by the reflections $\{s_\alpha | \alpha \in \Psi(t, H)\}$ in the root subsystem*

$$\Psi(t, H) = \{\alpha \in \Psi | \alpha(t) = 0, \alpha(H) = 1\}$$

Proof. In this case, the set $\Psi(t, H)$ is a genuine root subsystem. In particular, if $\alpha, \beta \in \Psi$, we have that

- It is closed under the opposite sign: $\alpha(t) = 1,\ \alpha(H) = 0 \Rightarrow (-\alpha)(t) = -\alpha(t) = 1^{-1} = 1$ and $(-\alpha)(H) = -\alpha(H) = -0 = 0 \Rightarrow -\alpha \in \Psi(t, H)$
- It is closed under addition: if $\alpha + \beta \in \Phi$, then $(\alpha + \beta)(t) = \alpha(t)\beta(t) = 1 \cdot 1 = 1$ and $(\alpha + \beta)(H) = \alpha(H) + \beta(H) = 0 + 0 = 0$, and therefore $\alpha + \beta \in \Psi$

And therefore $\Psi(t, H)$ is a root subsystem. Trivially, the stabilizer of the point (t, H) under the Weyl group action

$\square$

Proposition 3.21. *The conjugacy classes of regular subalgebras of $\mathfrak{g}$, $\mathcal{C}_{\mathfrak{g}}$, stratifies the reduced space.*

Proof. The space is stratified by the conjugacy class of stabilizers. In particular, we have seen that the Stabilizers can be seen as Weyl groups generated by root subsystem. In particular, two of those Weyl groups W_{Ψ_1}, W_{Ψ_2} are conjugated if and only if there is an element $w \in W_\Phi$ such that $w \cdot \Psi_1 = \Psi_2$, as $ws_\alpha w^{-1} = s_{w(\alpha)}$. Then, the stratification poset can be seen as the root subsystems modulo the Weyl group, which by Proposition 1.48 is the conjugacy classes of regular subalgebras of $\mathfrak{g}$, $\mathcal{C}_{\mathfrak{g}}$. $\square$

Therefore, the stratification poset is the same for both reductions.

Although I do not have a general result for the description of the strata as in the previous case, we still can talk about the top and bottom strata.

Proposition 3.22. *The stratum $\mathcal{R}(G)_{top}$ is a dense, open, connected set of dimension $2\text{rank}\ G$. The stratum $\mathcal{R}(G)_{bottom}$ is a finite set of $|Z(G)|$ points.*

Proof. The nature of the top strata of $\mathcal{R}(G)$ is an immediate consequence of Theorem 2.24. The dimension comes from the fact that our reduction is that of a manifold of dimension 2rank G by a discrete group, so the dimension is conserved (see 2.27).

The nature of the bottom strata comes from Proposition 3.17. As there are $|Z(G)|$ fixed points, all contained in the 0 level set, which have orbit type (G), in the reduced space the (G) strata will still be a union of $|Z(G)|$ points. $\square$

Chapter 4

b^m-Symplectic Geometry

In this section, we will explore the world of b-symplectic geometry, covering the work I have done under the supervision of Eva Miranda during my stay at Barcelona, and some of the ideas Eva and I intend to develop in the near future.

b-Symplectic geometry began as a generalization of the work on calculus on manifolds with *boundary* by Melrose in [37], which gave the origin to the letter "b". These manifolds were first introduced by Melrose in his book [37], in which he proved the Atiyah-Patodi-Singer theorem replicating the proof of the Atiyah-Singer theorem for manifolds with boundary. This framework was later developed in [17], associating Poisson structures to b-forms of degree 2 as bivector fields that drop rank along the critical hypersurface. The results on this paper will cover the generalization of b-forms, called b^m-forms, which are obtained by imposing more general transversality conditions. In particular, b-symplectic forms can be seen as a particular case of b^m-symplectic forms for $m = 1$.

The first section of this chapter will be a review of the definitions and concepts in b^m-symplectic geometry, while the second section will be focused on the b^m-symplectic reduction.

4.1 Singular Symplectic Structures

We will now define the notion of b-symplectic geometry. b-Symplectic forms were defined and extensively studied in the works [16, 17, 37, 39]. The section will mainly follow [6], [17], [28], and [35], and we will cover the definitions and main results in b-symplectic geometry and its generalization, b^m-symplectic geometry, as well as other singular symplectic structures.

4.1.1 b^m-Symplectic manifolds

Definition 4.1. A b-**manifold** is an oriented manifold M together with an oriented hypersurface Z, (M,Z).

Definition 4.2. A b-**map** is a map $f:(M_1,Z_1)\longrightarrow(M_2,Z_2)$ between b-manifolds such that f is transverse to Z_2 and $f^{-1}(Z_2)=Z_1$.

Definition 4.3. A b^m-vector is a vector field on (M,Z) which is tangent to Z on order m.

For a point $p\in Z$ with a neighbourhood U, assume that Z is given locally as the 0 set of the **defining function** f. Then, the vector field $f^m\frac{\partial}{\partial f}$ is tangent to order m to Z. If we take a coordinate chart on U such that the coordinates are of the form $(f,x_2,\ldots,x_n)$, the b^m-vector fields form a free $C^\infty-$module generated by the basis $\left\{f^m\frac{\partial}{\partial f},\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right\}$.

Definition 4.4. The b^m-**tangent bundle** of (M,Z) is the unique vector bundle that has as sections b^m-vector fields. It is denoted as ${}^{b^m}TM$.

The existence and uniqueness of the b^m-tangent bundle is derived from the Serre-Swan Theorem [43]. From this object we can define the b^m-cotangent bundle.

Definition 4.5. The b^m-**cotangent bundle** of a b-manifold is defined as the dual of the tangent bundle, ${}^{b^m}T^*M=({}^{b^m}TM)^*$.

Definition 4.6. A b^m-**form of degree** k is a smooth section of $\bigwedge^k({}^{b^m}T^*M)$.

We can use the dual base of the local one we obtained for b^m-vector fields, obtaining then the base $\left\{\frac{dx_1}{{x_1}^m}f,dx_1,\ldots,dx_n\right\}$.

The definition of the b^m-forms allows us to introduce ${}^{b^m}\Omega^k(M)$ as $\bigwedge^k({}^{b^m}T^*M)$, and we obtain the associated b^m-cohomology ${}^{b^m}H^*(M)$.Using the local coordinates for the b^m-vector fields, which is related to de Rham cohomology by the theorem

Theorem 4.7 (The b^m-Mazzeo-Melrose [37])**.**

$$ {}^{b^m}H^*(M)\cong H^*(M)\oplus(H^{*-1}(Z))^m. $$

Among the b^m-forms, we can focus on a special type of forms of degree 2 that mimic the standard symplectic forms.

Definition 4.8. For a b-manifold (M^{2n}, Z), we say that a b^m-form of rank two $\omega \in {}^{b^m}\Omega^2(M)$ is a b^m**-symplectic form** if ω is closed and ω_p is an element of maximal rank of $\bigwedge^2({}^{b^m}T_p^*M)$ for all points $p \in M$. We say that the triple (M, Z, ω) is a b^m-**symplectic manifold**.

We can describe a b^m-symplectic form in a neighbourhood of the critical set Z. To be precise, in a neighbourhood $U = Z \times (\varepsilon, \varepsilon)$, the b^m-symplectic form can be written as

$$\omega = \sum_{j=1}^{m} \frac{df}{f^j} \wedge \pi^*(\alpha_j) + \beta, \tag{4.1}$$

where β is a closed 2-form on U, the α_j are closed one forms on Z, $\pi : U \longrightarrow Z$ is the projection of U onto the critical set, and f is the defining function for the critical set Z. The non-degeneracy of the form ω makes α_m nowhere vanishing, and that $\beta|_Z$ is of maximal rank. The form α_m defines the symplectic foliation of the Poisson structure associated with ω, and β gives the symplectic form on the leaves of the foliation.

Similarly to the case of symplectic manifolds, there is a analogue of the Darboux theorem which lustrate that the only local invariant for a b^m-symplectic manifold is the dimension.

Theorem 4.9 (b^m-Darboux)**.** *Let (M^{2n}, Z, ω) be a b^m-symplectic manifold, and $p \in Z$ a point in the critical set. Then, there exist a coordinate chart centred at p such that the hypersurface Z is locally defined by $y_1 = 0$, and*

$$\omega = dx_1 \wedge \frac{dy_1}{y_1^m} + \sum_{i=2}^{n} dx_i \wedge dy_i. \tag{4.2}$$

We also have a b^m-equivalent of the Moser theorem for symplectic manifolds, proven in [17], which is useful to analyse other invariants.

Theorem 4.10 (Equivariant b^m-Moser Theorem)**.** *Let ω_1, ω_2 be b^m-symplectic forms on a b-manifold (M, Z) in the same cohomology class $[\omega_1] = [\omega_2]$ for a closed manifold M^{2n}. Then, there exists a path ω_t connecting the b^m-symplectic form, and a b^m-symplectomorphism*

$$\varphi : (M^{2n}, Z) \longrightarrow (M^{2n}, Z)$$

such that $\varphi^*(\omega_2) = \omega_1$. If the b-manifold admits an action of a compact Lie group G that preserves the path ω_t, then φ can be chosen to be $G-$equivariant

As b^m-symplectic manifolds are a generalization of symplectic manifolds, one can define a generalization of Poisson manifolds, b^m-Poisson manifold, which are dual to

b^m-symplectic manifolds. This duality allows us to introduce elements from Poisson geometry, such as *modular vector fields*.

Definition 4.11. On a b^m-symplectic manifold (M, Z, ω), fix a volume form Ω. The **modular vector field** v_{mod}^{Ω} on M, written as v_{mod} if the volume form is clear from the context, is the vector field defined by the derivation

$$f \longmapsto \frac{\mathcal{L}_{u_f}\Omega}{\Omega}$$

where u_f is a Hamiltonian vector field of the smooth function f.

Aclarar notacion cociente forma

Although the definition of the modular vector field depends on the volume form, different choices of Ω result in modular vector fields that differ by Hamiltonian vector fields.

As we already seen, the only local invariant in a b^m-symplectic manifold is the dimension. However, the geometry of the Poisson structure of the critical set Z allows us to define new semilocal invariants. Moreover, we can see that the structure induced by the b^m-symplectic form on the critical set Z is *cosymplectic*.

Definition 4.12. A **cosymplectic manifold** is a manifold of odd dimension M^{2n+1} with a closed one-form ν and a closed two-form ω such that

$$\nu \wedge \omega^n$$

is a volume form.

One can see that, using the flow of the modular vector field, the critical set Z can be shown to be a mapping torus (see [16]), as the modular vector fields are tangent to the critical hypersurface Z and preserves its symplectic foliation.

Proposition 4.13. *Let (M, Z, ω) be a b^m-symplectic manifold, and suppose that the critical set Z is compact and connected, and that its symplectic foliation as a Poisson manifold has a compact leaf $\mathcal{L}$. The critical set Z is then a mapping torus*

$$Z \cong \frac{[0, c] \times \mathcal{L}}{(0, x) \sim (c, \phi(x))}$$

where ϕ is the time c-flow of a modular vector field v_{mod}, and the time $t-$flow of the v_{mod} vector field corresponds to the translation by t in the first coordinate.

*The number $c > 0$ is called the **modular period** of Z and is independent of the choice of modular vector field v_{mod}.*

4.1.2 Folded symplectic manifolds

By definition, a symplectic form ω on a manifold M^{2n} induces a volume form, sometimes called the Liouville volume, ω^n. However, one might consider a form ω such that ω^n is degenerate at some points, but with a generally *good behaviour*. In this line, we can define folded symplectic manifolds [18].

Definition 4.14. Let (M^{2n}, ω) be a manifold with a closed 2-form ω such that the map

$$p \in M \longmapsto (\omega(p))^n \in \Lambda^{2n}(T^*M)$$

is transverse to the zero section of $\Lambda^{2n}(T^*M)$, then the set $Z = \{p \in M | (\omega(p))^n = 0\}$ is a hypersurface, and we say that ω defines a **folded symplectic structure** on the manifold (M, Z) if the restriction of the ω form to Z is of maximal rank. Then, the hypersurface X is called a **folding hypersurface** and the pair (M, Z) is called a **folded symplectic manifold**.

Once again, we have a normal form analogous to the Darboux theorem, proved by Martinet in [34].

Theorem 4.15 (Folded Darboux)**.** *Let ω be a folded symplectic form on (M^{2n}, Z), and let $p \in Z$. Then, there exists a local chart centred in p with coordinates $\{x_i, y_i\}$ where the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = y_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

4.1.3 Relations between singular symplectic manifolds

One can relate the three structures we have seen: symplectic manifolds, b^m-symplectic manifolds, and folded symplectic manifolds. This relation is given by a process called **desingularization**, first formulated by Guillemin-Miranda-Weitsman in [19].

Theorem 4.16 (Desingularization theorem)**.** *let ω be a b^m-symplectic form on a compact b-manifold (M, Z).*

- *If the degree of the singularity $m = 2k$ is **even**, there exist a family of symplectic forms that coincide with the b^m-symplectic form outside of an ε-neighbourhood of the critical set Z, and where the family of bivector fields $(\omega_\varepsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\varepsilon \to 0$.*

- *If the degree of the singularity $m = 2k$ is **odd**, there exists a family of **folded symplectic** forms ω_ε which coincide with the b^m-symplectic form ω outside an ε neighbourhood of the critical set Z.*

This theorem has as an immediate consequence that a b^{2k}-symplectic manifold must admit a symplectic form.

4.1.4 b^m-Symplectic group actions

As with the theory, one can generalize the theory of Hamiltonian group actions on b^m-symplectic manifolds, and the theory of moment maps. In particular, we will follow the results found mainly in [6] and [18].

Let G be a Lie group, with corresponding Lie algebra $\mathfrak{g}$. We say that a group G acts in a transverse way on a b^m-manifold if the group action acts transversally to the fibres of the mapping torus in Z (see [6]). Then, for transverse actions, one has the following characterization of the group G.

Theorem 4.17 (Bradell, Keisenhofer, Miranda)**.** *Let G be a compact group acting on a b^m-symplectic manifold in a transverse way. Then, the group G decomposes as $S^1 \times H$ or $S^1 \times H/\Gamma$, where $\Gamma = \mathbb{Z}_l \times \mathbb{Z}_k$ and the group $\mathbb{Z}_k$ is a non trivial subgroup of H.*

The idea behind this theorem is that the action must preserve the critical surface Z, and as it is a mapping torus, if the action is transverse we have an S^1-action that rotates the fibres of the mapping torus.

Definition 4.18. A $G-$action on a b^m-symplectic manifold (M, Z, ω) is called b^m-Hamiltonian if there exists a **moment map** $\mu \in {}^{b^m}C^\infty(M) \otimes \mathfrak{g}^*$ such that

$$\iota_{\xi^M}\omega = \langle d\mu, \xi\rangle$$

where ξ^M is the fundamental vector field generated by ξ and the **set of b^m-functions** is

$${}^{b^m}C^\infty(M) = \left(\bigoplus_{i=1}^{m-1} t^{-i}C^\infty(t)\right) \oplus {}^{b}C^\infty(M)$$

and

$$^{b}C^{\infty}(M) = \{a \log|t| + g, g \in C^{\infty}(M)\}$$

Which is to say that an action is b^m-Hamiltonian if it preserves the b^m-symplectic form and the contraction $\iota_{\xi^M}\omega$ is exact.

The image of a moment map outside the critical surface is just in $\mathfrak{g}$. However, on the critical surface, the log component explodes. However, there is a construction which allows one to see the image of the moment map as a smooth object (see [18]).

For example, we can consider the b-line, constructed by gluing copies of the real line $\mathbb{R}$ with points at infinity, glued together at the infinity points in a zig-zag pattern.

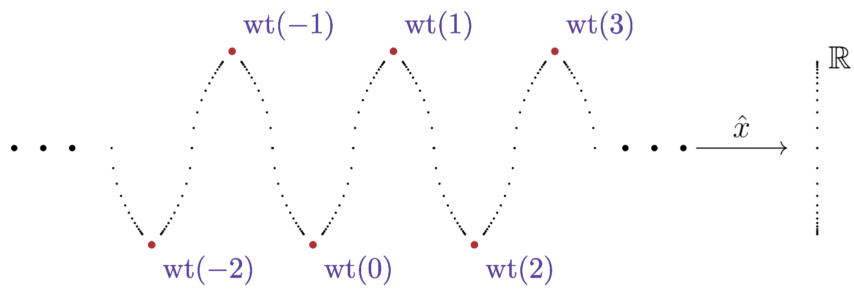


FIGURE 4.1: The b-line [35]

This can be generalized with the notion of a b-group.

Definition 4.19. A b-manifold (G,H) is called a b-**Lie Group** if G is a Lie group and $H \subset$ is a closed co-dimension one subgroup.

An example of such is the b-line, with $H = \mathbb{Z}$, and for more details see [18]. However, we will be considering standard b^m-symplectic actions.

As per Theorem 4.17, if the $G-$action is transverse to Z, G decomposes as $S^1 \times H/\Gamma$. We can consider the restriction of the action ρ to the S^1-component $\rho|_{S^1}$, we obtain a S^1 action on the b^m-symplectic manifold. We will now define the modular weights of the G-action, which will be important in the context of the b^m-symplectic reduction. We will follow [20] and [22], where the notion of modular weight of a torus was defined.

For an ε-neighbourhood $U = Z \times (-\varepsilon, \varepsilon)$ of the critical set, we recover the Expression 4.1 for the b^m-symplectic form.

$$\omega = \sum_{j=1}^{m} \frac{df}{f^j} \wedge \pi^*(\alpha_j) + \beta$$

If we assume that the action $\rho|_{S^1}$ is Hamiltonian, with a moment map $\mu \in {}^{b^m}C^\infty(M) \otimes \mathfrak{t}$, we define the modular weights:

Definition 4.20. The **modular weights** of $a_1, \dots, a_m \in \mathfrak{t}^*$ are given, in each connected component of Z, by

$$a_j(\xi) = \alpha_j(\xi^M)$$

It can be shown (see [22]) that these weights are constants.

Definition 4.21. The modular weights of the $G-$action ρ are defined as the modular weights of the $\rho|_{S^1}$ S^1-action.

By construction, b^m-Hamiltonian actions with non-vanishing highest modular weight are transverse actions.

Example 4.22. The b^m-symplectic sphere.

Let S^2 be the sphere with standard coordinates $\{h, \theta\}$, and consider it as a b^m-manifold with the critical set $Z = \{h = 0\}$, and b^m-symplectic form

$$\omega = \frac{dh}{h^m} \wedge d\theta$$

.

Consider the circle S^1 action on the axis given by the flow of $\frac{\partial}{\partial \theta}$. We can see that the action is Hamiltonian, and compute the moment map. Lets consider two cases:

- If $m = 1$, we have that $\iota_{\frac{\partial}{\partial\theta}}\omega = -\frac{dh}{h} = -\log(|h|)$, so the action is Hamiltonian with moment map $\mu(h, \theta) = log(|h|)$
- If $m > 1$, we have that $\iota_{\frac{\partial}{\partial\theta}}\omega = -\frac{dh}{h^m} = -d(-\frac{1}{(m-1)h^{m-1}})$, and the action is Hamiltonian with moment map $M \backslash Z$ is $\mu(h, \theta) = -\frac{1}{(m-1)h^{m-1}}$.

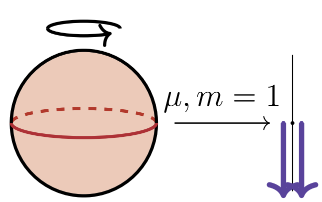


FIGURE 4.2: Moment map of the S^1-action by rotation on a b-symplectic S^2 [35]

We can see in Figure 4.2 the image of the moment map, which will be the negative half of the real numbers, and each point will have two preimages.

Although it might not be smooth over the standard real line, the moment map can be understood as a smooth section of ${}^{b^m}C^\infty(M)$ including the points at infinity.

Example 4.23. The b^m-symplectic Torus.

In distinction of the symplectic case, the b^m torus is a b^m-toric manifold. In particular, consider the torus T^2 as a b^m-symplectic manifold as

$$\left(T^2, Z = \{\theta_1 \in \{0,\pi\}\}, \omega = \frac{d\theta_1}{\sin^m \theta_1} \wedge d\theta_2\right)$$

Consider the the action of S_1 on the b^m-torus generated by the flow of the vector field $\frac{\partial}{\partial\theta}_2$. We can find that this action is b^m-Hamiltonian.

$$\iota_{\frac{\partial}{\partial\theta_2}}\omega = -\frac{d\theta_1}{\sin^m \theta_1} = d\left(\frac{|\cos\theta_1|}{\cos\theta_1}\frac{{}_2F_1\left(\frac{1}{2},\frac{1-m}{2};\frac{3-m}{2};\sin^2(\theta_1)\right)}{(1-m)\sin^{m-1}\theta_1}\right)$$

And therefore, it has a moment map

$$-\frac{|\cos\theta_1|}{\cos\theta_1}\frac{{}_2F_1\left(\frac{1}{2},\frac{1-m}{2};\frac{3-m}{2};\sin^2(\theta_1)\right)}{(1-m)\sin^{m-1}\theta_1},$$

where ${}_2F_1$ is the hypergeometric function.

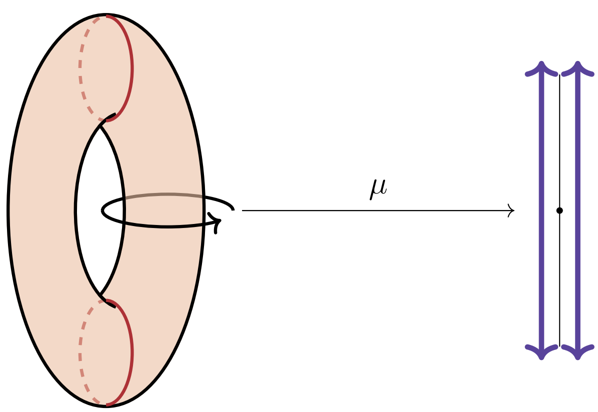


FIGURE 4.3: An S^1-action on a b-symplectic T^2 and its moment map [35]

In both of this examples, if we take a level set of the moment map, a circle, when quotiented by the group action we obtain a point, with the trivial symplectic form. This is an example of b^m-symplectic reduction, in both cases trivial, and we can already see what will become an important result: the b^m-symplectic reduction, for appropriate actions, will be a symplectic manifold.

4.2 b^m-Symplectic Reduction

We are now ready to cover the main ideas and results found in Matveeva-Miranda [35], where the symplectic reduction was generalized to the b^m-symplectic case. This generalization is well-behaved in the case of free actions and regular values, as in the Marsden-Weinstein case [33], but has not yet been adapted for the singular case.

Moreover, for an action G with non-vanishing modular weight, the S^1 action will "erase" the singularity from the manifold, and the reduction will therefore be a standard symplectic manifold.

4.2.1 b^m-Cotangent lift

Before we give the b^m-slice theorem, it is useful to consider the generalization of the cotangent lift to the b^m-symplectic case. The idea is the same as the cotangent Lift. Given a G-action ρ on a b-manifold, one can lift the action to a b^m-Hamiltonian G-action $\hat{\rho}$ on the b^m-cotangent bundle ${}^{b^m}T^*M$. However, we have two distinct procedures to do this cotangent lift. The standard lift and the twisted lift.

In the standard case the lifted action is given by $\hat{\rho} := \rho_{g^{-1}}$, and we obtain the following commuting diagram.

$$\begin{tikzcd} {}^{b^m}T^*M \arrow[r, "\hat{\rho}_g"] \arrow[d, "\pi"] & {}^{b^m}T^*M \arrow[d, "\pi"] \\ M \arrow[r, "\rho_g"] & M \end{tikzcd}$$

For the twisted lift, we have to consider the cotangent bundle of a circle T^*S^1. With standard coordinates, we have a logarithmic Liouville one-form

$$\lambda_{tw,c} = \left(c_1 \log|t| + \sum_{i=1}^{m-1} c_{i+1} \frac{t^{-i}}{i} \right) d\theta$$

for $t \neq 0$. This induces a b^m-symplectic structure on the space T^*S^1. The action of S^1 lifts to T^*S^1 in a b^m-Hamiltonian way, with moment map

$$\mu_S = c_1 \log|a| + \sum_{i=1}^{m-1} c_{i+1} \frac{a^{-i}}{i}$$

and with a twisted form

$$\tilde{\omega}_S = \sum_1^m \frac{\tilde{c}_i}{t_1^i} d\theta \wedge dt.$$

Then, we can combine this twisted action with the standard cotangent-lifted action of H to the b^m-cotangent bundle. In particular, the lifted action is Hamiltonian.

4.2.2 A b^m-symplectic slice theorem

Similarly to the proofs of 1.57 and 2.16, the proof will mainly lean on the existence of a symplectic slice. This theorem, first introduced in the b-symplectic case by Braddell-Kiesenhofer-Miranda in [6], was generalized by Matveeva-Miranda in [35] to prove the b^m-symplectic reduction theorem.

Theorem 4.24 (b^m-symplectic slice theorem [35])**.** *Let G be a compact group acting by b^m-symplectomorphisms, with* ***non-vanishing highest modular weight*** *, and suppose that $G = S^1 \times H/\Gamma$ the decomposition of the Lie Group G given by Theorem 4.17. Let $\mathcal{O}_z = Gz$, $z \in Z$ be an orbit contained in the critical set of M. Then there is a neighbourhood of $\mathcal{O}_z$ isomorphic to a neighbourhood of the zero section of ${}^{b^m}T^*G \times_\Gamma V_z$ equipped with the b^m-symplectic model*

$$\omega = \sum_{i=1}^m c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H), \tag{4.3}$$

where t is a defining function for Z, π is the projection $\pi : (T^*S^1 \times T^*H) \times_\Gamma V_z \to T^*H \times_{H_z} V_z$ and ω_H is the symplectic form on $T^*H \times_{H_z} V_z$ given by the symplectic slice Theorem 1.56. G acts on its ${}^{b^m}-$cotangent bundle ${}^{b^m}T^*G$ via the twisted b^m-cotangent lift, and the moment map for this action is given by

$$\mu = c_1 \log|t| + \sum_{i=1}^{m-1} c_{i+1} \frac{t^{-i}}{i} + \mu_0(x, y). \tag{4.4}$$

4.2.3 The b^m-symplectic reduction

In this section, we will cover the generalization of the Marsden-Weinstein reduction for the case of the b^m-Hamiltonian group actions. There will be two different cases, depending on the *modular weights*. If the highest modular weight is non vanishing, then the reduction will be a *symplectic manifold*. In the other case we will cover, when the modular weights are zero, the reduced space will be a b^m-*symplectic manifold*.

Note that we are considering the reduction at points $\mu(p)$ that are contained in the image of the critical set, $p \in Z$, as away of the critical set the manifold is locally symplectic and standard MW or SL reduction can be applied.

4.2.3.1 The case of non-vanishing highest modular weight

We will denote the images of points at infinity as $\mathbf{0}$, for which we mean the point $\mathbf{0} = (p_\infty, 0)$, considering the splitting of the moment map 4.4 given by the slice Theorem 4.24. Moreover, when we refer to $\mu^{-1}(\mathbf{0})$ we refer to the intersection of the space $\mu_0^{-1}(0)$ with the set $t = 0$.

For our action $G = S^1 \times H/\Gamma$, which can be seen as an action of $S^1 \times H$ on the universal cover of M, we will assume that:

- the action of H is locally free,
- the action of S^1 on the covering model associated with the finite group Γ is free,
- and 0 is a regular value of the moment map μ_0.

The last condition will be, by abuse of notation, usually denoted by saying that $\mathbf{0}$ is a regular point of μ.

Theorem 4.25 (The b^m-Marsden-Weinstein reduction)**.** *Given a b^m-Hamiltonian (locally) free action of a Lie group G on a b^m-symplectic manifold (M^{2n}, Z, ω). Assume that the highest modular weight is non-vanishing. Then, the image of a regular point $\mu^{-1}(\mathbf{0})$ is a b^m-presymplectic manifold with an induced action of G. The space of orbits of this induced action $M//G$ is a symplectic manifold (orbifold). This reduced space is symplectomorphic to the standard symplectic reduction of a symplectic leaf $\mathcal{L}$ on Z by a Lie subgroup of G.*

Proof. We will give a sketch of the proof, which is divided into three main steps.

Step 1: The reduced space, when the action is free (locally free) can be endowed a smooth (orbifold) structure (see [15]).

Step 2: We can use the slice Theorem 4.24 to describe the induced geometrical structure on the quotient. If the action is free, we have that a neighbourhood of the orbit will be diffeomorphic to a product of the orbit with a symplectic slice. If the action is not free, one can argue on a covering and reduce it to the product case.

Due to the b^m-symplectic slice Theorem 4.24, the tubular neighbourhood of the orbit $\mathcal{O}_x$ is equipped with the symplectic model:

$$\omega = \sum_{i=1}^{m} c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H)$$

and the moment map

$$\mu = c_1 \log|t| + \sum_{i=1}^{m-1} c_{i+1} \frac{t^{-i}}{i} + \mu_0(x, y).$$

Then, consider first the reduction with respect only to the S^1 component. The reduced space is symplectic, which we denote $M//S^1$, and with a Hamiltonian G-action corresponding to the moment map $\mu_0(x, y)$, which is a standard Hamiltonian map.

Step 3: The $H-$action on the cover can be seen as the usual Hamiltonian action on the symplectic slice, so we can use the Marsden-Weinstein reduction can be applied directly to the second component, and the reduction $\mu^{-1}(0)/G$ is a symplectic manifold (orbifold) which is symplectically equivalent to $\mathcal{L}//H$, where $\mathcal{L}$ is any symplectic leaf on Z. □

4.2.3.2 The case of vanishing modular weights

In the case of vanishing modular weight, the action is properly Hamiltonian, and the reduction in this case will still result in a b^m-symplectic manifold.

In this case, we can use the symplectic Lie algebroid reduction, proved by Marrero, Padrón and Rodriguez-Olmos in [32]. Applying directly their theorem to the Lie algebroid given by the b^m-cotangent bundle, we obtain the following.

Theorem 4.26 (The b^m-Marsden-Weinstein reduction with zero modular weight)**.** *Given a b^m-Hamiltonian (locally) free action of a Lie group G on a b^m-symplectic manifold (M^{2n}, Z, ω). Assume that the modular weights vanish. Then, the pre-image of a regular point $\mu^{-1}(\mathbf{0})$ is a b^m-presymplectic manifold with an induced action of G. The space of orbits of this induced action $M//G$ is a b^m-symplectic manifold (orbifold).*

4.2.3.3 Reduction commutes with desingularization

As we emphasized before, the desingularization procedure is an important result to relate b^m-symplectic structures with symplectic structures. In particular, one can prove the following result:

Theorem 4.27. *The desingularization procedure commutes with the b^m-Hamiltonian reduction.*

Which, equivalently, makes the following diagram commute

$$\begin{tikzcd}
(M, Z, \omega, G) \arrow[r, "b^m-\text{Ham. red.}"] \arrow[d, "\text{desing.}"'] & (M//G, \omega|_{M//G}) \\
(M, \omega_\varepsilon, G) \arrow[ur, "\text{Ham. red.}"'] &
\end{tikzcd} .$$

From this result, one can prove the following corollary.

Corollary 4.28. *The b^m-Hamiltonian reduction admits a reduction by stages procedure.*

Proof. As the Marsden-Weinstein reduction commutes with the desingularization, and the Marsden-Weinstein reduction by a Hamiltonian $G_1 \times G_2$-action admits a reduction by stages procedure, we can deduce that the b^m-Hamiltonian reduction admits a reduction by stages procedure. □

4.3 The singular b^m-symplectic reduction

As I have said, the SL reduction has not yet been extended to the b^m case. However, Miranda and I have proposed the following conjectures

Conjecture 4.29 (Singular b^m-symplectic reduction)**.** *Given a b^m-Hamiltonian* ***action*** *of a compact Lie group G on a b^m-symplectic manifold M^{2n} with non-vanishing highest modular weight, the space of orbits of the preimage of a point $M//G = \mu^{-1}(\boldsymbol{0})/G$ is a* ***stratified symplectic space****. This stratified symplectic manifold is symplectomorphic to the standard singular symplectic reduction of a symplectic leaf on Z by a Lie subgroup of G.*

Conjecture 4.30. *The singular b^m-symplectic reduction admits a procedure by stages.*

Although these are conjectures, we have a clear vision on how to develop the proofs.

The main idea is just to consider the reduction by stages procedure. As we know from Theorem 4.17 that the action decomposes as a product $S^1 \times H/\Gamma$, we can just deal with the reduction by the S^1-action, which is enough to put us back into the symplectic case, and then we will be able to just use the SL reduction to prove the result. A key point in the formulation is to use the b^m-cotangent lift of a non-free action following the recipe in [28]. For free S^1-actions the theorem is almost proven, and the main difficulty will just be the non-free S^1-actions. However, S^1 will locally act via the twisted b^m-cotangent lift of a non-free action of S^1, which are completely classified, as they are just rotations with different integer velocities.

Chapter 5

Methodology and planning

To develop the work shown in this thesis, the methodology employed and how the project was planned are described in this chapter.

Before starting my mobility period, i already started preparing for the project. To start, I attended Eva's Smooth Manifolds master course at the end of the previous year. In addition I attended two conferences during the summer. A workshop organized by the Fields Institute in Toronto regarding Hamiltonian Geometry and Quantization, and another organized by the CRM in Barcelona, Fluid Dynamics, Geometry, and Computer Science in Interaction.

Regarding the main body of the work, it has been developed in two different phases. The first phase took place in Oxford under the supervision of Andrew Dancer, spanning five and a half months from September to mid-March, and the second phase that took place under the supervision of Eva Miranda, spanning two months from mid-March until mid-May.

Under Andrew's supervision we met once per week, on Monday, and reviewed the literature i had read during the week and the work I had done, and establishing what should I continue reading about for the next meeting. I also attended the geometry seminar at the Mathematical Institute, alongside some courses imparted in the context of the Master in Mathematical Sciences, such as Lie Groups, Infinite Groups, and Analytic Number Theory. Under Eva's supervision we established meetings as we saw fit and depending on the topics we were covering, reviewing the topics I had looked. I also attended relevant events at SYMCREA, the lab where I worked during my stay in Barcelona. During the first phase, the project was not focused on a determined problem, as it was conceived as literature exploration, covering symplectic geometry, group actions, Lie groups, representation theory. . . . We later decided to cover some

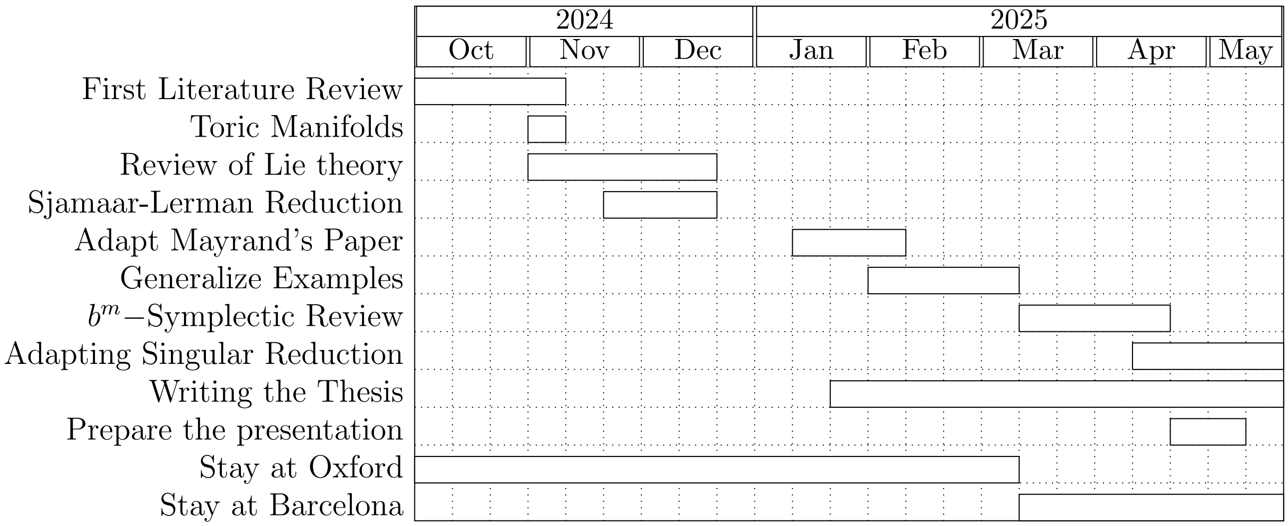


FIGURE 5.1: Gantt Chart for the tasks performed preparing the thesis.

examples of singular redaction, as it had been one of the topics I was the most interested in, and we decided to adapt Mayrand's paper [36] to the compact case, and this enabled us to develop another similar example.

During the second phase, we decided to adapt the singular reduction production to the b^m-symplectic case, an area of expertise of Eva, so I started an introduction to b^m-symplectic geometry and b^m-symplectic reduction. Finally we sketched a proof for the singular b^m- symplectic reduction, which we have not been able to finalize to include in this document, but are planning to finish could be ready during the summer.

Chapter 6

Sustainability Report

The final chapter of the body of the work is a sustainability analysis of the thesis, which is a mandatory section in the final degree projects at UPC. We will asses the environmental, economic and social impacts of this work, and asses the possible ethical implications that come from it.

We will follow the sustainability matrix, which divides three aspects of the project —project development, project execution, and risk and limitations— against thee perspectives —environmental, economic, and social—. We will analyse each of this intersections, alongside the ethical implications and the relationship to the UN Sustainable Developement Goals.

6.1 Environmental impact

Development

The environmental footprint during the development phase has been modest. The work was primarily theoretical, developed using digital tools. Approximately 400 sheets of paper were used for printing and note-taking purposes, along with two A5 physical notebooks. As a very rough estimate, this corresponds to a carbon footprint of around $4 \cdot 400 = 1600$ grams of CO_2, mainly associated with paper production and printing. This calculation is quite rough, as it is an estimation error compared to the main pollutant during the production of this thesis: the emissions related to the 4 flights taken, two Madrid-London and two Barcelona-London flights. The estimations of the CO_2 emissions for these flights vary widely, between 300 and 600 kg of CO_2, with an approximate mean value of 500 kg per flight, gives an impact of about 2 tonnes of CO_2.

The other main impact has been computer related. I use my personal laptop, which has an estimated power consumption of 65 W. I have used it around 7 hours per day, 6 days per week, and during an approximate 24 weeks, gives a consumption of 65 kWh. The UK has an approximate carbon intensity of 0.2 $kg\ CO_2/kWh$, gives a total of 13 kg of carbon emissions derived from computer usage.

Execution

The research has no direct physical implementation, and thus no impact from future deployment. Potential applications of algorithms built on this theory would be limited to theoretical physics, particularly celestial mechanics (e.g. orbit prediction), where the tools may assist in symbolic derivations or modelling. In any case, this would not imply material resource use or energy consumption beyond what is standard in academic computation.

Risks and Limitations

Environmental risk is negligible, as the work is purely theoretic. If such methods were used in high-fidelity numerical simulations, energy costs could arise, but this would depend mainly on the application methodology.

A limitation in environmental analysis is the difficulty of quantifying the energy impact of cloud and web infrastructure have, alongside the CO_2 usage approximations from the diverse tools, including the carbon intensity of the grid and the impact of the flights.

6.2 Economic impact

Development

The analysis of the economic impact includes material and labour costs. Regarding the direct cost, the flights have had a total cost of 300€, and the personal laptop, which I bough for this project, had a cost of 1200€. Other material costs for office material have totalled around 40€. During my stay at Oxford, my labour costs were covered by a grant of CFIS, and an Erasmus grant, which totalled the monthly limit established by CFIS of 1800€ per month. In the contination of the work in Barcelona, the labour costs have been financed by a scholarship given by the SYMCREA lab, which has totaled 875€ per month.

Execution

As the project is purely academic, it has no commercial deployment plan. In a professional context, similar work could support advanced research or symbolic computation software, but no financial costs can be associated with it.

Risks and Limitations

Theoretical research inherently carries economic risk in terms of uncertain applicability or return. This work does not aim to have a transfer to industry, and its viability is purely academic. The economic risk that has been undertaken has been by the university in the form of salaries, and no other risk are forseen.

6.3 Social impact

Development

The thesis has not led to direct ethical reflections, but it has highlighted a feature of the field of symplectic geometry: its strong representation of women in both research and teaching roles, which has also caused a reflection about the role of women in Mathematics.

Execution

As a review of research work, and in particular, the inclusion and developments of generalized examples, the potential social benefit lies primarily in knowledge generation and long-term support of science. In theoretical physics, for instance, techniques from singular symplectic reduction could eventually aid in modelling complex systems. There are no foreseen negative social impacts or dependencies.

Risks and Limitations

No social risks have been identified. Given the abstract and foundational nature of the work, it is unlikely to affect vulnerable groups or introduce bias. One limitation is the restricted audience for such advanced mathematical knowledge, but this is a general trait of specialised theoretical research.

6.4 Ethical implications

The project responds to a need for better understanding of singular spaces in geometry, which can have long-term relevance in mathematical physics. While it is not tied to a professional code of conduct or any immediate societal need, it adheres to the principles of academic integrity and open research.

6.5 Relationship with the sustainability development goals

While the project is not made with the idea of contributing directly to any of the 17 Sustainable Development Goals (SDGs) approved by the UN, the thesis contributes indirectly to different goals, related to education, research and innovations:

- **SDG 4: Quality Education** – The project supports the development of advanced mathematical knowledge and contributes to the academic ecosystem of higher education, improving the quality of research and the instruction of better mathematicians, and in such, teachers and professors.

- **SDG 8: Decent work and economic growth** – Investment in research can lead to economic growth in the long run. Moreover, academic positions have usually very good working conditions. Therefore, promoting research can not only lead to economic growth but to the creation of better jobs.

- **SDG 9: Industry, Innovation and Infrastructure** – Although not this thesis is not created with industry in mind, research is the basis of innovation, and new techniques in physics could be developed after the information treated here.

Conclusion

The sustainability analysis of this TFG reveals a low environmental impact, a modest economic footprint, and a socially neutral to positive contribution. The work, situated in a field with inclusive participation, supports long-term knowledge generation aligned with responsible research practices.

Chapter 7

Conclusion and future work

In this thesis, we have explored the theory of symplectic reduction in the presence of singularities, tracing an arc that starts with classical constructions and finalizes in a modern generalisation to the b^m-symplectic setting.

We began by laying the foundational material: smooth and symplectic manifolds, Lie group actions, and the classical Marsden–Weinstein reduction. Building on them, looked at the theory of stratified symplectic spaces as introduced by Sjamaar and Lerman, and their generalization of the Marsden–Weinstein reduction, seeing how the singular structure of the quotient, far from being pathological, is a well-behaved space partitioned into symplectic manifolds with a coherent structure between them.

Following this, we focused on a case study inspired by the work of one of Andrew Dancer students, Maxence Mayrand, adapting his analysis of the cotangent bundle T^*G of a complex reductive Lie group G to the compact, real setting. We established that the quotient $T^*G/\!/(T \times T)$ is a stratified symplectic space whose stratification depends only on the Lie algebra $\mathfrak{g}$ of G, and can be parametrised by the root subsystems of the corresponding root system Φ. The upper stratum is open and dense, while the lower dimensional strata are encoded in the structure of conjugacy classes of regular semisimple subalgebras. We constructed explicit Hasse diagrams to describe these posets in selected examples. We also proved analogous results for the reduction by the diagonal G-action, although not as complete due to the more complex nature of the example.

The final phase of the thesis has been an exploration into the realm of b^m-symplectic geometry, initiated under the guidance of Eva Miranda. These manifolds, which allow for symplectic forms with prescribed singularities along hypersurfaces, generalise the classical setting and open up new pathways for studying Hamiltonian dynamics on singular phase spaces. We reviewed the local models of b^m-symplectic structures and their desingularization, and the b^m-symplectic reduction. We then presented a sketch of a generalized reduction procedure in the b^m-symplectic category, motivated by the slice theorem and the results of Matveeva and Miranda [35].

Beyond the pure results obtained, this thesis has been an intellectual and personal journey into the depths of the academic world, and brand new intersecting branches of mathematics: symplectic geometry, representation theory, Lie theory, and Poisson geometry. The mobility period in Oxford and the stay in Barcelona have enriched not only the mathematical content but also the pedagogical and collaborative dimensions of the work.

Future work includes the finalization of the proof of the singular b^m-symplectic reduction theorem, extending the case studies explored in this thesis to the b^m-framework, as well as looking at the reduction procedure in other mathematical settings.

Bibliography

[1] Judith M. Arms, Richard H. Cushman, and Mark J. Gotay, *A universal reduction procedure for hamiltonian group actions*, The Geometry of Hamiltonian Systems (New York, NY) (Tudor Ratiu, ed.), Springer US, 1991, pp. 33–51.

[2] M. F. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), no. 1, 1–28. MR 721448

[3] Rolf Berndt and Michael Klucznik, *An introduction to symplectic geometry*, Graduate studies in mathematics, volume 26, AMS, Providence, Rhode Island, 2001 - 2001 (eng).

[4] Jord Boeijink, Klaas Landsman, and Walter van Suijlekom, *Quantization commutes with singular reduction: cotangent bundles of compact lie groups*, 2018.

[5] Raoul Bott, Susan Tolman, and Jonathan Weitsman, *Surjectivity for Hamiltonian loop group spaces*, Invent. Math. **155** (2004), no. 2, 225–251. MR 2031427

[6] Roisin Braddell, Anna Kiesenhofer, and Eva Miranda, *A b-symplectic slice theorem*, Bulletin of the London Mathematical Society **55** (2023), no. 1, 90–112.

[7] Glen E. Bredon (ed.), *Introduction to compact transformation groups*, Pure and Applied Mathematics, vol. 46, Elsevier, 1972.

[8] T. Bröcker and T. Dieck, *Representations of compact lie groups*, Graduate Texts in Mathematics, Springer Berlin Heidelberg, 2003.

[9] Henrique Bursztyn, Gil R. Cavalcanti, and Marco Gualtieri, *Reduction of courant algebroids and generalized complex structures*, Advances in Mathematics **211** (2007), no. 2, 726–765.

[10] Neil Chriss and Victor Ginzburg, *Representation theory and complex geometry*, Modern Birkhäuser Classics, Birkhäuser, Boston [etc, 1997 (eng).

[11] Peter Crooks and Maxence Mayrand, *Symplectic reduction along a submanifold*, 2021.

[12] Ana Cannas da Silva, *Lectures on symplectic geometry*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2008.

[13] E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Trans. Am. Math. Soc. Ser. 2 **6** (1957), 111–244.

[14] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, 1990.

[15] Victor Guillemin, Viktor Ginzburg, and Yael Karshon, *Moment maps, cobordisms, and Hamiltonian group actions*, Mathematical Surveys and Monographs, vol. 98, American Mathematical Society, Providence, RI, 2002, Appendix J by Maxim Braverman. MR 1929136

[16] Victor Guillemin, Eva Miranda, and Ana Rita Pires, *Codimension one symplectic foliations and regular Poisson structures*, Bull. Braz. Math. Soc. (N.S.) **42** (2011), no. 4, 607–623. MR 2861781

[17] Victor Guillemin, Eva Miranda, and Ana Rita Pires, *Symplectic and poisson geometry on b-manifolds*, Advances in Mathematics **264** (2014), 864–896.

[18] Victor Guillemin, Eva Miranda, Ana Rita Pires, and Geoffrey Scott, *Toric actions onb-symplectic manifolds*, International Mathematics Research Notices **2015** (2014), no. 14, 5818–5848.

[19] Victor Guillemin, Eva Miranda, and Jonathan Weitsman, *Desingularizing $\boldsymbol{b^m}$-symplectic structures*, International Mathematics Research Notices **2019** (2017), no. 10, 2981–2998.

[20] ———, *On geometric quantization of b-symplectic manifolds*, Advances in Mathematics **331** (2018), 941–951.

[21] Victor Guillemin and Shlomo Sternberg, *A normal form for the moment map*, Differential geometric methods in mathematical physics (Jerusalem, 1982), Math. Phys. Stud., vol. 6, Reidel, Dordrecht, 1984, pp. 161–175. MR 767835

[22] Victor W. Guillemin, Eva Miranda, and Jonathan Weitsman, *Convexity of the moment map image for torus actions on b^m-symplectic manifolds*, Philos. Trans. Roy. Soc. A **376** (2018), no. 2131, 20170420, 6. MR 3868423

[23] Victor W. Guillemin and Shlomo Sternberg, *Supersymmetry and equivariant de Rham theory*, Mathematics Past and Present, Springer-Verlag, Berlin, 1999. MR 1689252

[24] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyperkähler metrics and supersymmetry*, Communications in Mathematical Physics **108** (1987), no. 4, 535–589.

[25] J. Huebschmann, G. Rudolph, and M. Schmidt, *A gauge model for quantum mechanics on a stratified space*, Communications in Mathematical Physics **286** (2008), no. 2, 459–494.

[26] Lisa C. Jeffrey, *Extended moduli spaces of flat connections on Riemann surfaces*, Math. Ann. **298** (1994), no. 4, 667–692. MR 1268599

[27] Francisco C. Caramello Jr, *Introduction to orbifolds*, 2022.

[28] Anna Kiesenhofer and Eva Miranda, *Cotangent models for integrable systems*, Communications in Mathematical Physics **350** (2016), no. 3, 1123–1145.

[29] John M. Lee, *Introduction to smooth manifolds*, Springer New York, New York, NY, 2012.

[30] Eugene Lerman, Richard Montgomery, and Reyer Sjamaar, *Examples of singular reduction*, London Mathematical Society Lecture Note Series, p. 127–156, Cambridge University Press, 1994.

[31] Charles-Michel Marle, *Modèle d'action hamiltonienne d'un groupe de Lie sur une variété symplectique*, Rend. Sem. Mat. Univ. Politec. Torino **43** (1985), no. 2, 227–251 (1986). MR 859857

[32] Juan Carlos Marrero, Edith Padrón, and Miguel Rodríguez-Olmos, *Reduction of a symplectic-like lie algebroid with momentum map and its application to fiberwise linear poisson structures*, Journal of Physics A: Mathematical and Theoretical **45** (2012), no. 16, 165201.

[33] Jerrold Marsden and Alan Weinstein, *Reduction of symplectic manifolds with symmetry*, Reports on Mathematical Physics **5** (1974), no. 1, 121–130.

[34] Jean Martinet, *Sur les singularités des formes différentielles*, Ann. Inst. Fourier (Grenoble) **20** (1970), no. fasc. 1, 95–178. MR 286119

[35] Anastasia Matveeva and Eva Miranda, *Reduction theory for singular symplectic manifolds and singular forms on moduli spaces*, Advances in Mathematics **428** (2023), 109161.

[36] Maxence Mayrand, *Stratified hyperkäler spaces from semisimple lie algebras*, Transformation Groups **25** (2018), no. 1, 191–215.

[37] Richard B. Melrose, *The Atiyah-Patodi-Singer index theorem*, Research Notes in Mathematics, vol. 4, A K Peters, Ltd., Wellesley, MA, 1993. MR 1348401

[38] Ryszard Mrugala, *Lie groups beyond an introduction, 2nd edition, vol. 140, anthony w. knapp, in: Birkhäuser series: Progress in mathematics. birkhäuser, boston basel berlin (2002), xviii+812 pp., chf 138.-/eur 88 (hardcover)., isbn: 0-8176-4259-5*, Reports on Mathematical Physics **55** (2005), 150–151.

[39] Ryszard Nest and Boris Tsygan, *Formal deformations of symplectic manifolds with boundary*, J. Reine Angew. Math. **481** (1996), 27–54. MR 1421945

[40] Juan-Pablo Ortega and Tudor S. Ratiu, *Regular symplectic reduction theory*, pp. 205–269, Birkhäuser Boston, Boston, MA, 2004.

[41] Mark R. Sepanski (ed.), *Compact lie groups*, Springer New York, New York, NY, 2007.

[42] Reyer Sjamaar and Eugene Lerman, *Stratified symplectic spaces and reduction*, Annals of Mathematics **134** (1991), no. 2, 375–422.

[43] Richard G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962), 264–277. MR 143225

[44] L.W. Tu, *An introduction to manifolds*, Universitext, Springer New York, 2010.

[45] Izu Vaisman, *Introduction*, pp. 1–3, Birkhäuser Basel, Basel, 1994.

[46] Alan Weinstein, *The local structure of Poisson manifolds*, Journal of Differential Geometry **18** (1983), no. 3, 523 – 557.