Addressing the Principal-Agent Problem in Public Private Partnerships via Mixed-Integer Bilevel Linear Programming

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***Abstract*—Public Private Partnerships (PPPs) are associations between a government and a private party with the objective of delivering public assets and/or services. The private party provides financial leverage and technical expertise to develop large-scale projects, responding to the requirements and conditions specified by the public party, which, in turn, must oversee the adequate development of the project. A key challenge arises due to the Principal-Agent problem (PA problem), namely: a conflict of interests in which the public may not be able to ascertain whether the private party's actions respond to its own interests rather than the project's. The interactions in a PPP can thus be viewed as a Stackelberg Game and consequently as a bilevel optimization problem, in which a leader (the government) and a follower (a private contractor) partake in coupled decision processes, where the follower's decisions depend on previous leader decisions and cause an impact on the leader's objective. We propose a Mixed Integer Bilevel Linear Programming (MIBLP) framework to model PPPs in the context of a road maintenance as a typical example of infrastructure operation projects. The problem under consideration involves integer decision variables, which complicates the solution strategy for bilevel optimization problems. Thus, we incorporate a recent Branch & Cut approach found in the operations research literature as a means to solve the bilevel framework for the PPP in mention. The use of MIBLPs allows to model interactions in PPPs (e.g., PA problem), while considering potentially complex combinatorial problems resulting from the project's characteristics (e.g., infrastructure maintenance).**

CONTENTS

Keywords: Public Private Partnerships, Infrastructure Maintenance, PA problem, Stackelberg Games, Bilevel Programming, Branch & Bound, Branch & Cut, Intersection Cuts.

I. Introduction

A. Infrastructure maintenance importance and PPPs advantages/disadvantages

Infrastructure maintenance is a challenging task for a government since public assets are exposed to deterioration [13] due to factors such as increasing demands, deterioration by aging or vulnerability to environmental factors, among others [7]. Governments often establish long-term associations with private companies to build and/or operate physical infrastructure, seeking to exploit the technical expertise and financial leverage that such companies can provide. This associations are known as PPPs, where the public party establishes the requirements for an infrastructure project and monitors their fulfillment, while the private party is responsible for the construction, maintenance and/or administration of the underlying public asset.

B. Conflict of interests and possible solutions

The divergence in objective functions may result in poor infrastructure performance, as the public party faces limited capacity to inspect and enforce the compliance of contractual terms [17]. The government is thus interested in applying policies that grant the best incentives (e.g., rewards, penalties, or bonds) for the private party in order to maximize the public infrastructure's performance.

C. PA problem as an example of conflicts of interests

The Principal-Agent problem (PA problem) arises when a person/entity (the principal) hires a person/entity (the agent) to perform a task, and the agent may act in detriment of the task by pursuing its own interests [12]. For instance, when a car technician recommends an engine replacement, the car owner may question whether the technician is acting in his own interest, taking advantage of information asymmetry. In the Colombian context, scandals related to large-scale projects such as La Ruta del Sol, La Via al Llano or the Chirajara bridge motivate the need for analysis tools that help elucidate potential bad practices or conflicts of interests in PPPs.

D. Relation between PA-Stackelberg Games-MIBLPs

A way to address the PA problem is by means of incentive mechanisms based on inspection (i.e., associating rewards or penalties depending on the system's performance). This situation can be viewed as a Stackelberg Game, where we find a leader (the principal) that makes decisions pursuing a certain objective and then, sequentially, a follower (the agent) counterpart makes decisions reacting to those of the leader [18]. Bilevel optimization allows to model such situations by incorporating the follower's constraints that depend on leader's decisions, and an influence of the follower's decisions on the leader's objective. Bilevel optimization is a suitable framework for modeling PPPs since it allows to have both 'players' responding to a shared set of rules, but recognizing that each can act in pursuit of their own objectives. Potential PA problems can, thus, be modeled by establishing divergent objective functions for the principal and the agent (in this case, the government and a private contractor).

E. Context of application

We focus on the maintenance of a road as one example of a PPP project. In this problem, the agent must decide whether to perform a maintenance action at any period of a planning horizon. Because of this, and potentially other binary decisions related to contractual terms, binary and integer variables may arise. While bilevel optimization problems are relatively well solved for the case of continuous variables, the case of integer problems is much more challenging due to the non-convex, combinatorial nature of the problem. We adopt a state-of-the-art technique to solve Mixed-Integer Bilevel Linear Programming problems (MIBLPs) based on a Branch & Cut approach. The contribution of this research is the implementation of state-of-the-art algorithms to solve MIBLPs to address the PA problem in PPPs in the context of infrastructure maintenance.

F. Structure

This document is organized as follows. Section **II** presents the literature review related to the problem's context; we provide an in-depth definition of the PPP context under study, as well as its components; also, we review Bilevel Linear Programming problems (BLPs), MIBLPs, and the challenges they pose. Section **III** explains the solution approach we will implement to solve our MIBLP; a Branch & Cut solution approach. Section **IV** presents the mathematical model for the PPP problem. Section **V** presents the results for a variety of testbed instances from the literature that we considered in our computational study as well as an illustrative example and results analysis for the PPP problem. Section **VI** synthesizes the main contents of our research, challenges, and solution approaches, while presenting a brief analysis for the overall results of our instances testbed, as well as it poses the limitations, assumptions, and future work regarding our implementation. Finally, the appendix presents the mathematical formulation for our small instance testbed.

II. Literature Review

A. Public Private Partnerships (this goes in the paper)

B. Problem statement (this one does not go in the paper)

We consider the situation presented in [10] where a government hires a private contractor for performing maintenance actions to a road through a PPP. The road's physical conditions or reliability are quantified using any performance metric. The road is prone to decrease its performance through time; therefore, this performance decrease is represented as a deterioration on the physical conditions with a degradation function. Although physical deterioration is represented as a continuous process, the performance of the road at any given time is mapped into a discrete set of categories. Gómez et al. (2020) classify the road's performance into discrete service levels, which are associated with different levels of social benefit.

C. PA problem in the PPP

The private party's task is to define a maintenance plan for the road given its deterioration process. The government's objective is to guarantee a minimum social benefit derived from the road's performance and service levels throughout a planning horizon, whereas the private party pursues the cost-effectiveness of the maintenance plan. Such divergence may lead to significantly different maintenance plans depending on whose objective is being considered, possibly leading to a PA problem. PPP contracts can try to address this issue by imposing pre-contractual conditions (e.g., establishing minimum performance levels). The monitoring and enforcement of such conditions may have technical, budgetary, and organizational limitations in practice. Alternatively, PPP contracts may include performance dependent payments to the private party as rewards or incentives for maintaining high service levels, even in the absence of monitoring. The specific design of such conditions and incentives is challenging as their success depends on understanding the integrated effect of physical and financial processes, observed through the rationality of the decision-making process of the involved parties.

D. Benefits of implementing a computational model

Computational tools can significantly contribute to design and understand mechanisms that allows trial-and-error experimentation via simulation rather than actual real-life high-stakes projects. We propose a MIBLP model that captures both government's and private party's interests in a comprehensive yet tractable way. The central decisions in the model are the definition of inspection and maintenance plans for the road, performed by the government and the private contractor, respectively. The private party bases its maintenance decisions on its own information (e.g, the fixed and variable costs of maintenance actions, a quantification of the economic benefit obtained from the system at each service level and the magnitudes of payments and possible penalties/rewards from the government at the moment of inspection), whereas the government's decisions are subject to guarantee a minimum social benefit derived from the road's service levels, in which case inspection plays an important role.

E. Implementation details

Our model keeps track of the road's performance at any time as a function of maintenance actions, given a deterioration function, which can be non-linear (as it is modeled as a parameter rather than a variable) and may be stochastic; we focus on the deterministic case for simplicity in exposition. The service level and social benefit produced by the road's performance, as well as the associated cash flows for the government and the private party, can be derived from this performance, inspection, and maintenance actions over time.

F. Mixed-Integer Bilevel Linear Programming (this goes in the paper)

G. MIBLP review

BLPs are a type of optimization problems that contain a nested inner optimization problem in the constraints of an outer optimization problem [20] and they are classified as NP-Complete problems [15]. The outer problem is often called as the *upper level* or leader problem while the nested problem is called the *lower level* or follower problem. A MIBLP is a BLP that has some (if not all) integer constrained decision variables. An example of a MIBLP structure, as shown in [5], is presented as follows:

$$\min_{x,y} c_x^T x + c_y^T y \tag{1}$$

$$G_x x + G_y y \leqslant q \tag{2}$$

$$x_j \text{ integer}, \ \forall\ j \in J_x \tag{3}$$

$$x^- \leqslant x \leqslant x^+ \tag{4}$$

$$y^- \leqslant y \leqslant y^+ \tag{5}$$

$$\begin{aligned} y \in \arg\min_{y \in \mathbb{R}^{n_2}} \{ d^T y : Ax + By \leqslant b,\ l \leqslant y \leqslant u, \\ y_j \text{ integer}, \ \forall\ j \in J_y \} \end{aligned} \tag{6}$$

$$J_x \subseteq N_x \coloneqq \{1, \ldots, n_1\},\ J_y \subseteq N_y \coloneqq \{1, \ldots, n_2\} \tag{7}$$

Where $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, while c_x, c_y, G_x, G_y, q, d, A, B, b, l, u, x^-, x^+, y^- and y^+ are given matrices/vectors of appropriate size. Also, sets J_x and J_y represent the indexes for the integer-constrained x (leader) and y (follower) variables, respectively. Expressions (1)–(5) correspond to the leader problem while expression (6) corresponds to the follower problem. Expressions (4) & (5) are often included in the constraints of expression (2) (if any lower and/or upper bounds for the respective variables are given).

1) Importance of MIBLPs in the scientific community: MIBLPs have been granted special attention in recent years due to their challenging nature, depending on which level of the MIBLP (leader or follower) presents the discrete conditions for the corresponding decision variables. Different approaches have been proposed for solving BLPs and MIBLPs to optimality or to retrieve a solution close to it. Some of these include single reduction techniques, which consists on reformulating the two-level problem into a global single-level one by exploiting its Karush–Kuhn–Tucker (KKT) conditions [21]; single reduction applying Branch & Bound methods [3]; descent methods, which consists on decreasing the leader's objective function value while keeping the new point follower-optimal [14]; among others. Our research assumes an optimistic position for MIBLPs, meaning that given multiple follower optimal solutions, the leader expects the follower to choose the solution that leads to the best objective function value for the leader. The reader can refer to [20] for an extended in-depth review on BLPs, their history, solution approaches, and the pessimistic position.

III. Methodology

A. Introducing Fischetti's work to the reader

Fischetti et al. (2016 & 2017) proposed new approaches for solving MIBLPs to optimality using Branch and Bound [6] and Branch & Cut [5] methods, the latter incorporating the ideas of the corner polyhedron explained in [4], firstly introduced by [11] and the associated Intersection Cuts (ICs) presented by [1]. They framed their Branch & Cut methodology following the "disjunctive interpretation" ([8] & [9]) for generating numerically reliable ICs. Before introducing the Branch & Cut approach, we descrive an alternative way to formulate the MIBLP and describe some necessary concepts.

First the MIBLP has to be restated to its *value function formulation* [19] as follows:

$$\min_{x,y} c_x^T x + c_y^T y \tag{8}$$

$$G_x x + G_y y \leqslant q \tag{9}$$

$$A_x x + B_y y \leqslant b \tag{10}$$

$$l \leqslant y \leqslant u \tag{11}$$

$$x_j \text{ integer}, \ \forall\ j \in J_x \tag{12}$$

$$y_j \text{ integer}, \ \forall\ j \in J_y \tag{13}$$

$$d^T y \leqslant \Phi(x) \tag{14}$$

Where $\Phi(x)$ in expression (14) denotes the *follower value function* for a given $x^* \in \mathbb{R}^{n_1}$ by computing the follower MILP:

$$\begin{aligned} \Phi(x^*) \coloneqq \min_{y \in \mathbb{R}^{n_2}} \{ d^T y : By \leqslant\ b - Ax^*,\ l \leqslant y \leqslant u, \\ y_j \text{ integer}, \ \forall\ j \in J_y \} \end{aligned} \tag{15}$$

By dropping expression (14) from the model (8)-(14), the above formulation results in a MILP called the *high-point-relaxation* (HPR), whose Linear Programming problem (LP) relaxation is denoted by $\overline{\text{HPR}}$. The following notation:

$$J_F \coloneqq \{ j \in N_x : A_j \neq 0 \} \tag{16}$$

is also proposed in [5] to denote the index set of leader variables x_j that appear in the follower problem. In expression (16), A_j represents the j-th column of matrix A.

Bilevel feasibility is violated for a point (x, y) if it violates expression (14), whereas if it satisfies conditions (8)–(14), bilevel feasibility is met.

B. goal of the BnC

For the Branch & Cut approach, as stated in [6] and [5], the goal is to derive valid inequalities that are violated by a solution point. In this case, after retrieving a bilevel infeasible $\overline{\text{HPR}}$ solution point, say (x^*, y^*), we would want to enforce a cutting plane, named an IC, that cuts off this point, while keeping the rest of *bilevel feasible* solutions intact.

C. Why the Hipercube?

Fischetti et al. (2017) propose 3 different methodologies for retrieving a valid convex set S that contains a *bilevel infeasible* vertex that violates a generated IC derived from S. In this research project, we focus on the third method to generate such a set and ICs, named as the Hypercube. The reader is refered to [5] for a deeper explanation on the other two sets and ICs generation methodologies. However, these other two methodologies require the incorporation of a strong assumption, stating that $Ax + By - b$ is integer for every solution point (x, y). This assumption does not need to be satisfied for the use of an Hypercube *bilevel-free* set, reason why we decided to use this methodology to generate valid ICs, since it is likely to enounter that this assumption does not hold for our instances: both in the testbed ones as in the PPP.

To generate the ICs from a Hypercube convex set we assume $J_F \subseteq J_x$. Starting from a given HPR solution point (x^*, y^*), we define $(\hat{x}, \hat{y})$ as the best bilevel feasible solution point satisfying $\hat{x}_j = x_j^* \;\forall\; j \in J_F$.

1) We solve the follower MILP from expression (15) for $x = x^*$ and compute $\Phi(x^*)$.
2) We add the constraints (17) (18), to the HPR.
$$x_j = x_j^* \tag{17}$$
$$d^T y \leqslant \Phi(x^*) \tag{18}$$
3) We solve the restricted HPR and set $(\hat{x}, \hat{y})$ as the best bilevel solution given x^*.
4) Then, the Hypercube:
$$\text{HC}^+(x^*) = \left\{(x, y) \in \mathbb{R}^n : x_j^* - 1 \leqslant x_j \leqslant x_j^* + 1,\; \forall\; j \in J_F\right\} \tag{19}$$
results in the convex set S of interest.

Some definitions to generate ICs derived from the set S, as mentioned in Fischetti et al. (2017), are presented as follows.

1) Let ξ represent the whole variable vector $(x, y) \in \mathbb{R}^n$.
2) Let the $\overline{\text{HPR}}$ at the given Branch & Bound node to be formulated in its standard form:
$$\min\left\{\hat{c}^T\xi : \hat{A}\xi = \hat{b},\; \xi \geqslant 0\right\} \tag{20}$$
3) Let ξ^* be an optimal vertex for the above $\overline{\text{HPR}}$, associated to an optimal basis, say $\hat{B}$.
4) Let the convex set S of interest be defined as:
$$S = \left\{\xi : g_i^T\xi \leqslant g_{0i},\; i = 1, \ldots, k\right\} \tag{21}$$
5) To derive a valid IC violated by ξ^*, satisfying that all other feasible ξ must not belong to the inside of S, expression (20) can be restated to its k-term disjunction (see [8] & [9]):
$$\bigvee_{i=1}^{k}(g_i^T\xi \geqslant g_{0i}) \tag{22}$$
It is important to note that in expression (22), the $\geqslant$ permits the feasible point ξ to belong to any facet of S (in our case, the set denoted by expression (19)).
6) Solve the above $\overline{\text{HPR}}$ to retrieve an optimal vertex ξ^* along with its associated basis $\hat{B}$.

D. ICs derivation description

In expression (21), g_i^T represents the coefficient vector for all variables in the $i - th$ row of the convex set S. By construction, our convex set S being the Hypercube, only spans the x (leader) variable space. This is, for every leader variable $x_j \;\forall\; j \;\in\; J_F$ there is going to be two associated rows (one for the condition $x_j^* - 1$ and one for the condition $x_j^* + 1$), in which its coefficients will be 1 for that specific variable and 0 otherwise. Similarly, g_{0i} represents the right hand side of the $i - th$ row of S. In the Hypercube, for every $x_j \;\forall\; j \;\in\; J_F$ and for every couple of rows associated to x_j, the right hand side will be $x_j^* - 1$ and $x_j^* + 1$, in the corresponding row. Moreover, in the Branch & Cut algorithm presented up next to generate ICs, the expression $(g_i)_{\hat{B}}^T$ in (23) represents the coefficient vector for the $i - th$ row of the associated basis for the vertex ξ^* in S. Analogously to g_i^T, the coefficient $(g_{i,j})_{\hat{B}}$ is going to be 1 for the leader variables $x_j \;\forall\; j \;\in\; J_F$ and 0 otherwise, if and only if the variable $x_j \;\forall\; j \;\in\; J_F$ is part of the basis at the current solution vertex ξ^*.

Algorithm 1 Branch & Cut Algorithm [5]

Input: An optimal $\overline{\text{HPR}}$ vertex ξ^* and the associated convex *bilevel-free* set S.
Output: An IC violated by ξ^*.
Require: Branch as usual on fractional integer-constrained variables.

for $i := 1$ to k **do**

$$(\bar{g}^T, \bar{g}_{i0}) := (g_i^T, g_{i0}) - (g_i)^T_{\hat{B}} \hat{B}^{-1}(\hat{A}, \hat{B}) \tag{23}$$

end for
for $j := 1$ to n **do**

$$\gamma_j := \max\{\bar{g}_{ij}/\bar{g}_{i0} : \ i \in \{1, \dots, k\}\} \tag{24}$$

end for
if $\gamma \geqslant 0$ and ξ is integer constrained **then**
  for for $j := 1$ to n **do**

$$\gamma_j := \min\{\gamma_j, 1\} \tag{25}$$

  end for
end if
Return the violated intersection cut $\gamma^T \xi \geqslant 1$

E. Implementation details

As for the implementation details for the Branch & Cut approach, we had to build from zero our own Branch & Bound exploration algorithm. The reason for this is, although we used **Gurobi Optimizer 9.0.1 in Sublime-Python IDE**, we were not able to retrieve information about the inherited cuts from "father" nodes for every node in the Branch & Bound tree. We ran our experiment over the $\overline{\text{HPR}}$ in its standard form. After finding any integer (x^*, y^*), we implemented a bilevel feasibility check and, if this point did not satisfy bilevel feasibility, we computed $\Phi(x^*)$ to capture the best bilevel feasible $(x^*, \hat{y})$ solution for the current node.

IV. PPP Instance

A. Intro to PPP problem

Our research consists on the retrieval and analysis of optimal policies to address the PA problem in PPPs, based on establishing planning actions for a road that suffers performance deterioration over time, modeling divergent objectives using a MIBLP and retrieving solutions using the Q-MIBLP algorithm.

B. Problem components: Parameters

This section formally introduces the mathematical model for our PPP problem. Let $T = \{1, \dots, |T|\}$ be the set of time periods making up the planing horizon. Let $L = \{1.0, \dots, 5.0\}$ be the set of discrete performance levels for the road, where 1.0 means the road is at optimal conditions while 5.0 means the road is at the worst conditions. Let γ_t be the performance obtained after t periods without restoration. Let $c_t^{(f)}$ be the fixed cost for restoration and $c_t^{(v)}$ be the unit cost to restore a performance unit at period $t \in T$. Let a be the fix income from the government to the private party. Let ϵ be the target return rate for the private party. Let $\check{\xi}_l$ be the lower bound and $\hat{\xi}_l$ be the upper bound for service level $l \in L$. Let k_l be the penalty, reward or bond given to the private party by the government if the road's performance is at service level $l \in L$. Let g_l be the social benefit perceived by the government for the operation of the road for a performance level $l \in L$ and let g^* be the social benefit target. Finally, let $c^{(i)}$ be the cost incurred by the government regarding inspection actions at any given period.

C. Problem components: Variables

Variables for both parties are as follows. Let q_t be a binary variable to denote if an inspection action performed at period $t \in T$. Let x_t be a binary variable to denote whether a maintenance action is applied at period $t \in T$. Let y_t be an integer variable to denote the number of periods elapsed after last restoration at period $t \in T$. Let $b_{t,\tau}$ be a binary variable to denote Whether $y_t = \tau$. Let $z_{t,l}$ be a binary variable to denote whether the system is at service level $l \in L$ at period $t \in T$. Let v_t be a continuous variable to capture the road's performance at period $t \in T$. Let $p_t^{(+)}$ be a continuous variable to denote the private party's earnings at period $t \in T$. Let $p_t^{(-)}$ be a continuous variable to represent the private party's expenses at period $t \in T$. Let $p_t^{(\cdot)}$ be a continuous variable to capture the available cash of the private party at period $t \in T$. Let w_t be an integer variable to capture the linearization of the product $y_{t-1} \cdot x_t$ at period $t \in T$. Let u_t be a continuous variable to capture the linearization of the product $v_t \cdot x_t$ at period $t \in T$. Finally, let $aux_{t,l}$ be an integer variable to capture the linearization of $z_{t,l} \cdot q_t$ at period $t \in T$ for service level $l \in L$.

TABLE I
SUMMARY OF SETS, PARAMETERS AND VARIABLES IN THE MATHEMATICAL MODEL.

Sets	T:	set of periods in planning horizon
	L:	set of discrete service-levels
Parameters	γ_t:	performance obtained after t periods without restoration
	$c^{(f)}$:	fix cost for restoration action
	$c^{(u)}$:	unit cost to restore a performance unit
	a:	fixed income from the government to the private party
	ϵ:	target return rate for the private party
	$\breve{\xi}_l$:	lower bound for service level $l \in L$
	$\hat{\xi}_l$:	upper bound for service level $l \in L$
	k_l:	penalty/reward from the road being at service-level $l \in L$
	g_l:	economic benefit obtained from the system at service-level $l \in L$
	g^*:	target benefit
	$c^{(i)}$:	Cost of inspection
Variables	q_t:	whether an inspection action is applied at period $t \in T$
	x_t:	whether a maintenance action is applied at period $t \in T$
	y_t:	number of periods elapsed after last restoration
	$b_{t,\tau}$:	whether $y_t = \tau$ for $\tau \in T$
	$z_{t,l}$:	whether system is at service level $l \in L$ at period $t \in T$
	v_t:	performance at period $t \in T$
	$p_t^{(+)}$:	earnings at period $t \in T$
	$p_t^{(-)}$:	expenditures at period $t \in T$
	$p_t^{(\cdot)}$:	available budget at period $t \in T$
	w_t:	linearization of $y_{t-1}x_t$
	u_t:	linearization for $v_t x_t$
	$aux_{t,l}$:	linearization of $z_{t,l} \cdot q_t$

The PPP MIBLP formulation is presented as follows:

$$\min \sum_{t\in T} q_t \cdot c^{(i)} + \sum_{l\in L}\sum_{t\in T} k_l \cdot aux_{t,l} + a \cdot |T| - \sum_{l\in L}\sum_{t\in T} g_l \cdot z_{l,t} \tag{26}$$

s.t.

$$\textstyle\sum_{l\in L} g_l \cdot z_{t,l} \geqslant g^* \quad \forall\, t \in T \tag{27}$$

$$x_t,\, y_t,\, b_{t,\tau},\, z_{l,t},\, v_t,\, p_t^{(+)},\, p_t^{(-)},\, p_t^{(\cdot)},\, w_t,\, u_t,\, aux_{t,l} \in \arg\min\{\text{Private party MILP}\} \quad \forall\, t, \tau \in T, l \in L \tag{28}$$

Where the private party MILP:

$$\min \sum_{t\in T} p_t^{(-)} - p_t^{(+)} \tag{29}$$

s.t.

$$y_1 = 0 \tag{30}$$

$$w_1 = 0 \tag{31}$$

$$u_1 = 0 \tag{32}$$

$$p_1^{(\cdot)} = p_1^{(+)} - p_1^{(-)} \tag{33}$$

$$y_t = y_{t-1} + 1 - w_t - x_t \quad \forall\, t \in T \mid t > 1 \tag{34}$$

$$w_t \leqslant y_{t-1} \quad \forall\, t \in T \mid t > 1 \tag{35}$$

$$w_t \geqslant y_{t-1} - |T| \cdot (1 - x_t) \quad \forall\, t \in T \mid t > 1 \tag{36}$$

$$w_t \leqslant |T| \cdot x_t \quad \forall\, t \in T \mid t > 1 \tag{37}$$

$$u_t \leqslant v_{t-1} \quad \forall\, t \in T \mid t > 1 \tag{38}$$

$$u_t \geqslant v_t - (1 - x_t) \quad \forall\, t \in T \mid t > 1 \tag{39}$$

$$u_t \leqslant x_t \quad \forall\, t \in T \mid t > 1 \tag{40}$$

$$p_t^{(\cdot)} = p_{t-1}^{(\cdot)} + p_t^{(+)} - p_t^{(-)} \quad \forall\, t \in T \mid t > 1 \tag{41}$$

$$y_t = \textstyle\sum_{\tau \in T} \tau \cdot b_{t,\tau} \quad \forall\, t \in T \tag{42}$$

$$\textstyle\sum_{\tau \in T} b_{t,\tau} = 1 \quad \forall\, t \in T \tag{43}$$

$$v_t = \textstyle\sum_{\tau \in T} \gamma_\tau \cdot b_{t,\tau} \quad \forall\, t \in T \tag{44}$$

$$v_t \leqslant \textstyle\sum_{l \in L} \hat{\xi}_l \cdot z_{t,l} \quad \forall\, t \in T \tag{45}$$

$$v_t \geqslant \textstyle\sum_{l \in L} \breve{\xi}_l \cdot z_{t,l} \quad \forall\, t \in T \tag{46}$$

$$\textstyle\sum_{l \in L} z_{t,l} = 1 \quad \forall\, t \in T \tag{47}$$

$$p_t^{(-)} = (c_t^{(f)} + c_t^{(v)}) \cdot x_t - c_t^{(v)} \cdot u_t \quad \forall\, t \in T \tag{48}$$

$$p_t^{(-)} \leqslant p_t^{(\cdot)} \quad \forall\, t \in T \tag{49}$$

$$\textstyle\sum_{t \in T} p_t^{(+)} \geqslant (1 + \epsilon) \cdot \sum_{t \in T} p_t^{(-)} \tag{50}$$

$$aux_{t,l} \leqslant q_t \quad \forall\, t \in T,\, l \in L \tag{51}$$

$$aux_{t,l} \leqslant z_{t,l} \quad \forall\, t \in T,\, l \in L \tag{52}$$

$$aux_{t,l} \geqslant q_t + z_{t,l} - 1 \quad \forall\, t \in T,\, l \in L \tag{53}$$

$$p_t^{(+)} = a + \textstyle\sum_{l \in L} k_l \cdot aux_{t,l} \quad \forall\, t \in T \tag{54}$$

$$q_t,\, x_t \in \{0, 1\} \quad \forall\, t \in T \tag{55}$$

$$b_{t,\tau} \in \{0, 1\} \quad \forall\, t,\, \tau \in T \tag{56}$$

$$z_{t,l},\, aux_{t,l} \in \{0, 1\} \quad \forall\, t \in T,\, l \in L \tag{57}$$

$$y_t,\, w_t \in \mathbb{Z}^+ \cup \{0\} \quad \forall\, t \in T \tag{58}$$

$$v_t,\, p_t^{(+)},\, p_t^{(-)},\, p_t^{(\cdot)},\, u_t \geqslant 0 \quad \forall\, t \in T \tag{59}$$

D. Problem components: Constraints and objectives

Expressions (26)–(28) represent the government's (leader) problem, while expressions (29)–(54) denote the private party's (follower). Both parties objective functions have a minimize sense in order to follow the notation posed in section **III**. Expression (26) is composed of the costs of inspection at every period, the bond/penalty given to the private entity with respect to the road's service level at the moment of inspection, the fixed payment per period to the private party and, finally, the social benefit perceived from the road being at every service level through the planning horizon. Expression (27) seeks to satisfy the minimum social benefit the road produces at the corresponding service level at every period. Expression (28) denotes the inner optimization problem for the private party as a constraint for the government's outer optimization problem. Expression (29) is composed of the expenses and profits for the private party. Expressions (30)–(33) initializes the corresponding variables for the first period of the planning horizon. Expression (34) captures the periods after last restoration as an inventory. Expressions (35)–(37) captures the linearization for the performance. Expressions (38)–(40) captures the linearization for getting the private party's objective function appropriately. Expression (41) updates the available cash for the private party at every period. Expressions (42) and (43) serve to capture a "binarization" for the restoration inventory to retrieve the road's performance. Expression (44) captures the quantification of the road's performance. Expressions (45)–(46) captures the upper and lower bounds for the service level of the road, respectively, given the road's performance at every period. Expression (47) denotes that, for every period, the road can only have one service level. Expressions (48) and (49) denote the expenses and budget balance for the implementation of maintenance actions to the road, respectively, by the private party. Expression (50) denotes the monetary return for the private party as a factor of their expenses through time. Expressions (51)–(53) capture the linearization of whether an inspection action is carried away by the government at any period when the road is on a respective service level. Expression (54) captures the private party's profits for every period, adding the fixed income paid by the government and the bond/penalty given if inspection is performed and the road is at a correspondent service level. Ultimately, expressions (55)–(59) denote the nature of the variables.

V. Testbed

A. Small Instances

B. Intro to the testbed results

We present in **Table II** the computational results for our small instance runs for the Branch & Cut (B&C) approach. The reported times are given in seconds.

C. Testbed details

We tested 11 small instance problems from the literature ([2], [16]). A special consideration must be addressed related to instance 10, where we encountered a huge execution time. This situation may arise because instance 10 is purely continuous in both the leader and follower variables, leading to non-optimal but *bilevel-feasible* solutions using the Branch & Cut, resulting in a very interesting example for the practical use of ICs (maybe exclusively) on MILPs. As mentioned in [5], ICs effectiveness deteriorates when they are applied several times iteratively over the same LP (or MILP Branch & Bound node). The main difference of applying ICs over LP instances rather than MILPs arises from, on the LP's, Branch & Bound cuts are not an available tool for speeding up the feasible region exploration. Hence, ICs violation for a given vertex, say (x^*, y^*), is going to be weaker every time an intersection cut is applied to the same Branch & Bound node (the root node for the case of a purely continuous problem). The number of cuts generated before stopping the Branch & Cut approach over instance 10 after approximately a 10 hours run was 587 with a 6.37% gap from the optimal solution. It is important to note that we did not implement any stopping criteria for our Branch & Cut implementation over the testbed instances. Since all these problems were relatively small, we allowed the exploration to visit every possible node in the Branch & Bound tree, resulting in a variant for an exhaustive enumeration methodology.

D. PPP Instances

E. Intro to the PPP results

For our Branch & Cut implementation over the PPP instance, and relating to the ICs weakening issue when applied iteratively over the same LP, we defined a maximum number of ICs to be applied on a specific node in the Branch & Bound tree. Therefore, we settled this value at a maximum of 10 ICs per Branch & Bound node.Figures **1** and **2** can be interpreted as follows. The black-stripped line segment represents the road's performance, the red squares denote when an inspection action was performed by the government and the blue triangles represent the moments in which maintenance actions were performed. Figure **1** shows the inspection actions performed by the government that maximize the social benefit under the assumption that they are accountable for the maintenance actions as well.

Nonetheless, for this set up of inspection actions and the private party being the real accountant for the road's maintenance, its response to the government's decisions are to maximize its utility, as seen in **2**, by arranging the maintenance actions that maximize the amount of incentives while minimizing the penalties from the corresponding road's service levels at the times of inspection.

F. Results considerations

This private party's maintenance response is, however, infeasible for the government since constraint (27) is violated, meaning that the road's performance under such inspection-maintenance actions can not guarantee the minimum social benefit over the planning horizon. After enforcing appropriate ICs that violate such infeasible solution points, we found a bilevel feasible solution after exploring 269 nodes on the B&B tree. Given the size of the problem, let alone 2^30 combinations for inspection actions only, we considered a stopping criteria: since we did not have acces to valid lower bounds, we opted to use a *best know solution* (BKS) obtained while using the solution approach in **CITEMYSELF**. Given this (BKS), we stablished a threshold of 25% for the stopping criterion gap at any moment the Branch & Cut found a bilevel feasible solution. Hence, the solution presented up next, was the first solution found using the Branch & Cut, with a gap of 20% away from the BKS.

G. Solution description

Figure **3** represents the equilibrium (*bilevel feasible*) solution for the road's performance. Note that these maintenance actions match the inspection periods, taking the road's performance at its highest value by accommodating to the moments when inspections are carried away, deriving on greater rewards from the government given the assumption that the private party has early access to the inspection decisions information. It can be seen that the trade-off is achieved at the moment where the private party decides to increase the frequency of maintenance actions in order to accommodate to the inspections performed by the government. This accommodation is performed in a way that maximizes both the social benefit of the government alongside the profits perceived by the private party with respect to bonds/penalties given by the government. One would expect that, when the incentives/penalties get bigger, fewer inspection actions need to be performed in order to achieve the project's objectives. Same way around, when the bonds/penalties get smaller, more inspection actions need to be carried away since there is no extra incentive for the private party to align their interests to the one's of the project.

TABLE II
SMALL INSTANCE RESULTS TABLE

Instance	(x^*, y^*)	z_l^*	z_f^*	Time B&C
1	(4,4)	-12.00	4.00	0.45
2	(6,2)	12.00	-2.00	0.89
3	(2,2)	-22.00	2.00	1.41
4	(3,1)	5.00	-1.00	0.37
5	(2,0,1,0)	-3.50	-4.00	0.15
6	(1,9,0)	-19.00	-9.00	0.49
7	(0,1,0,1,0,75,21.67)	-1,1011.67	-4,673.33	0.10
8	(17,10)	-76.00	30.00	5.09
9	(1,75,21.67)	-961.67	-4,673.33	0.06
10	(12,3)	27.00; 28.72*	-3.00; -3.47*	36,050.17*
11	(3,4,2,0,3,0)	49.00	-27.00	22.17

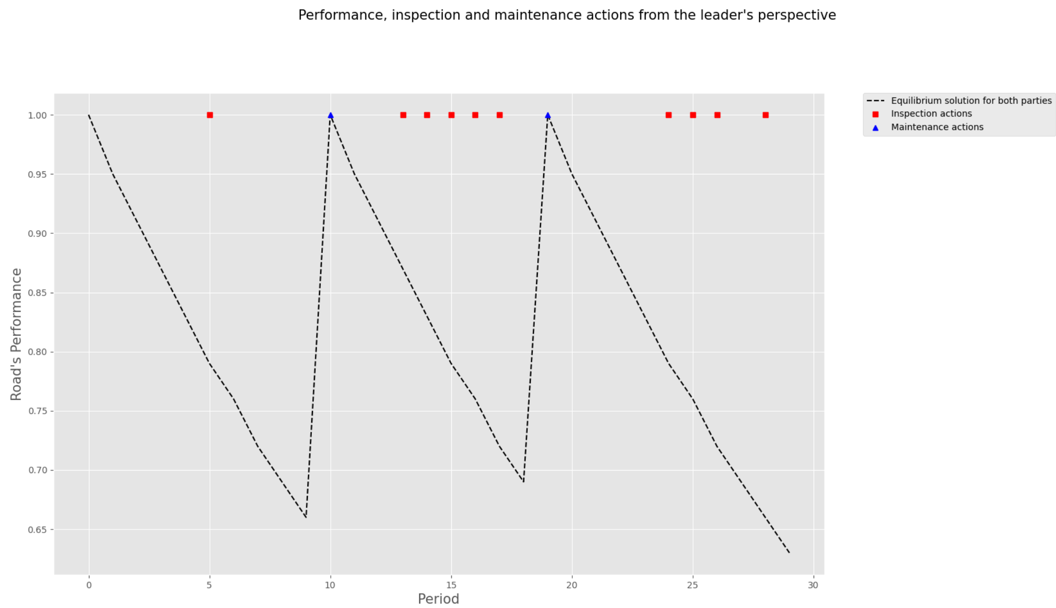

Fig. 1. Solution under the leader's perspective

H. Disclaimer

It is important to point out that the Branch & Cut algorithm only found solutions for a specific set of parameter configurations in a considerable low computation time. When the original parameters values were changed to a more tighten, specifically in the values of the incentives/penalties given from the government to the private party, the Branch & Cut approach requires a much greater amount of time to find bilevel solutions (i.e., there were no solutions found yet after exploring 10,000 nodes in the Branch & Bound tree). Consequently, this approach seems to be very sensitive with respect to changes in the model's parameters at the moment of returning solutions for this type of high-dimension combinatorial problems.

VI. CONCLUSIONS

A. Research project summary

We address projects of infrastructure operation via PPPs, particularly focusing on road maintenance, considering the well-known advantages PPPs have (e.g., financial leverage or technical expertise) but acknowledging their vulnerability to the PA problem (i.e., conflicting interest between the principal and the agent, that may compromise the project's performance). We address the PA problem by means of MIBLPs, which allows to model the coupled interactions of the principal and the agent while considering divergent objectives. A key challenge in this research is that the problem under consideration is integer, thus, com-

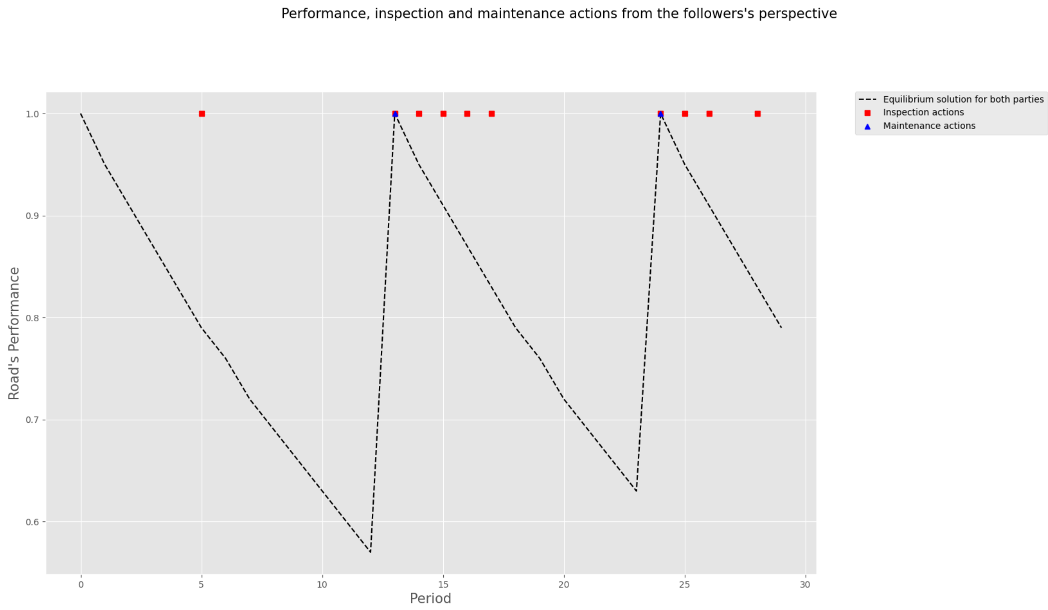

Fig. 2. Solution under the follower's perspective

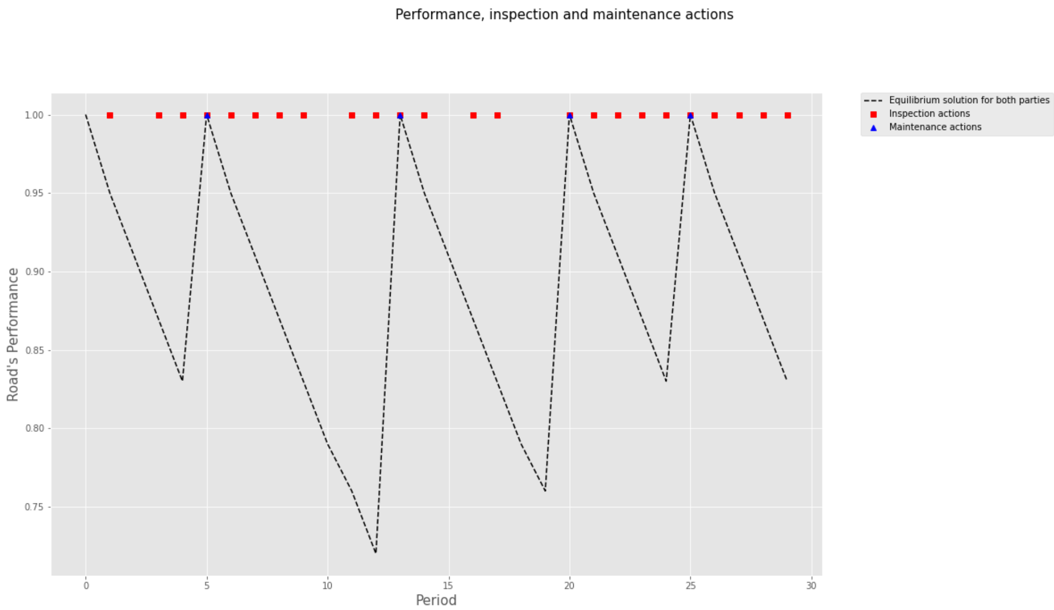

Fig. 3. Branch & Cut Solution

plicating the solution strategies for bi-level approaches. We exploited a state-of-the-art approach to solve MIBLPs: a Branch & Cut, and tested the capabilities of this approach under a set of small instances, as well as a case study for the PPP road maintenance example, which constitutes the main contribution of this research.

B. *Research Q/A*

The question addressed in this research project is:

1) how can operations research methods be articulated into a methodology to provide quantitative support for decisions in engineering projects in which conflicts of interest such as the PA problem are a matter of concern to guarantee acceptable outcomes?

Our analysis is useful to estimate an equilibrium solution for such conflict of interests that may arise between the public and private parties by incorporating conditions (e.g., inspections deriving in penalties or rewards given a determined service level) that aid their contractual or policy settlements.

C. Capabilities of MIBLPs as models for PPPs

Computational experiments over the PPP instance confirm the capabilities of MIBLPs to model the expected behavior of the involved decision-makers, both when analyzed independently, and within the coupled context. On the one hand, the government prioritizes maintaining higher performance levels (associated with the respective inspection actions) that derive on a greater social benefit, considering that for this party there is no need to satisfy any budgetary constraint over maintenance actions. On the other hand, given the government's inspection decisions, the private party reacts in a way that best fits their interests related to the project's cost-efficiency. We achieve a conciliation between both parties that can be seen as an equilibrium and a trade-off for each one of them; the government accommodates their inspection actions so the private party agrees to perform maintenance actions as a best response for maximizing social benefit, since the government provides rewards to the private party over high performance levels at the moment of inspection.

D. Disclaimers

It is important to acknowledge that our model's parameters only consider the deterministic case. However, natural phenomena and man-made activities are very likely to be stochastic. Because of this, we consider that a natural extension for our model would be to incorporate uncertainty over the deterioration functions, for example. Similarly, robust optimization approaches may provide a risk averse analysis for the changes over the system when the uncertainty is applied over the probability distributions for deterioration functions. Additionally, we rely on an assumption of perfect information between the parties, which is often not the case in practice.

E. Future Work

Future work is directed towards addressing limitations and relaxing assumptions of our model, as discussed below. On the MIBLP methodology background, we considered the optimistic position for this kind of problems (e.g., given multiple follower optimal solutions, the leader expects the follower to choose the solution that leads to the best objective function value for the leader). Also, it is important to highlight that the Branch & Cut approach are both variants for an exhaustive enumeration method over the feasible region, which can be computationally demanding, since we do not have any means for proving optimality other that letting the algorithm explore the Branch & Bound tree to its full extension and choosing the best solution. Thus, we are considering the implementation of a strategy to prune nodes in the Branch & Bound tree for both approaches by following methods like the *follower upper bound* proposed for finding valid lower bounds for the HPR. In addition, we consider the use of the *follower pre-processing* technique for speeding up the searching for an initial *bilevel feasible* solution. Finally, we can also speed up the IC generation process for the Branch & Cut approach if we opt to discard ICs whose violation by the vertex (x^*, y^*) is not strong enough. These last three considerations follow the respective proposals presented in [5].

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Instance problems:

1)
$$\min x - 4y$$
s.t.,
$$\min y$$
s.t.,
$$-x - y \leqslant -3$$
$$-2x + y \leqslant 1$$
$$2x + y \leqslant 12$$
$$3x - 2y \leqslant 4$$
$$10x + y \geqslant 20$$
$$x,\ y \in \mathbb{Z}^+ \cup \{0\}$$

2)
$$\min x + 3y$$
s.t.,
$$x \geqslant 0$$
$$x \leqslant 6$$
$$\min -y$$
s.t.,
$$x + y \leqslant 8$$
$$x + 4y \leqslant 8$$
$$x + 2y \geqslant 13$$
$$x,\ y \in \mathbb{Z}^+ \cup \{0\}$$

3)
$$\min -x - 10y$$
s.t.,
$$\min y$$
s.t.,
$$-25x + 20y \leqslant 30$$
$$x + 2y \leqslant 10$$
$$2x - y \leqslant 15$$
$$2x + 10y \geqslant 15$$
$$x,\ y \in \mathbb{Z}^+ \cup \{0\}$$

4)
$$\min x + 2y$$
s.t.,
$$\min -y$$

s.t.,

$$-x + 2.5y \leqslant 3.75$$
$$x + 2.5y \geqslant 3.75$$
$$2.5x + y \leqslant 8.75$$
$$x,\ y\ \in\ \mathbb{Z}^{+}\ \cup\ \{0\}$$

5)

$$\min -2x_1 + x_2 + 0.5y_1$$

s.t.,

$$x_1 + x_2 \leqslant 2$$
$$\min -4y_1 + y_2$$

s.t.,

$$2x_1 - y_1 + y_2 \geqslant 2.5$$
$$-x_1 + 3x_2 - y_2 \geqslant -2$$
$$x_1,\ x_2,\ y_1,\ y_2\ \in\ \mathbb{Z}^{+}\ \cup\ \{0\}$$

6)

$$\min -x - 2y_1 - 3y_2$$

s.t.,

$$\min -y_1 + y_2$$

s.t.,

$$x + y_1 + y_2 \leqslant 10$$
$$x,\ y_1,\ y_2\ \in\ \mathbb{Z}^{+}\ \cup\ \{0\}$$

7)

$$\min -20x_1 - 60x_2 - 30x_3 - 50x_4 - 15y_1 - 10y_2 - 7y_3$$

s.t.,

$$\min -20y_1 - 60y_2 - 8y_3$$

s.t.,

$$5x_1 + 10x_2 + 30x_3 + 5x_4 + 8y_1 + 2y_2 + 3y_3 \leqslant 230$$
$$20x_1 + 5x_2 + 10x_3 + 10x_4 + 4y_1 + 3y_2 \leqslant 240$$
$$5x_1 + 5x_2 + 10x_3 + 5x_4 + 2y_1 + y_3 \leqslant 90$$
$$x_1,\ x_2,\ x_3,\ x_4\ \in\ \{0, 1\}$$
$$y_1,\ y_2,\ y_3\ \geqslant\ 0$$

8)

$$\min 2x - 11y$$

s.t.,

$$\min x + 3y$$

s.t.,

$$x - 2y \leqslant 4$$
$$2x - y \leqslant 24$$
$$3x + 4y \leqslant 96$$
$$x + 7y \leqslant 126$$
$$-4x + 5y \leqslant 65$$
$$-x - 4y \leqslant -8$$
$$x,\ y\ \in\ \mathbb{Z}^{+}\ \cup\ \{0\}$$

9)

$$\min -60x - 10y_1 - 7y_2$$

s.t.,

$$\min -60y_1 - 8y_3$$

s.t.,

$$10x + 2y_1 + 3y_2 \leqslant 225$$
$$5x + 3y_1 \leqslant 230$$
$$5x + 3y_2 \leqslant 85$$
$$x\ \in\ \{0, 1\}$$
$$y_1,\ y_2\ \geqslant 0$$

10)

$$\min x + 5y$$

s.t.,

$$\min -y$$

s.t.,

$$-x - y \leqslant -8$$
$$-3x + 2y \leqslant 6$$
$$3x + 4y \leqslant 48$$
$$2x - 5y \leqslant 9$$
$$x,\ y\ \geqslant 0$$

11)

$$\min -4x_1 + 8x_2 + x_3 - x_4 + 9y_1 - 9y_2$$

s.t.,

$$-9x_1 + 3x_2 - 8x_3 + 3x_4 + 3y_1 \leqslant 1$$
$$4x_1 - 10x_2 + 3x_3 + 5x_4 + 8y_1 + 8y_2 \leqslant 25$$
$$4x_1 - 2x_2 - 2x_3 + 10x_4 - 5y_1 + 8y_2 \leqslant 21$$
$$9x_1 - 9x_2 + 4x_3 - 3x_4 - y_1 - 9y_2 \leqslant -1$$
$$-2x_1 - 2x_2 + 8x_3 - 5x_4 + 5y_1 + 8y_2 \leqslant 20$$
$$7x_1 + 2x_2 - 5x_3 + 4x_4 - 5y_1 \leqslant 11$$
$$\min -9y_1 + 9y_2$$

s.t.,

$$-6x_1 + x_2 + x_3 - 3x_4 - 9y_1 - 7y_2 \leqslant -15$$
$$4x_2 + 5x_3 + 10x_4 \leqslant 26$$
$$-9x_1 + 9x_2 - 9x_3 + 5x_4 - 5y_1 - 4y_2 \leqslant -5$$
$$5x_1 + 3x_2 + x_3 + 9x_4 + y_1 + 5y_2 \leqslant 32$$
$$x_1,\ x_2,\ x_3,\ x_4,\ y_1\ y_2\ \in\ \mathbb{Z}^{+}\ \cup\ \{0\}$$