A stochastic optimization model to support decisions in infrastructure operation projects

Natalia Torres Ramírez

Department of Industrial Engineering, Universidad de los Andes

Abstract

Public Private Partnerships (PPPs) are associations between a private party and a government agency for providing a public asset or service, most commonly infrastructure. Financing comes from an upfront investment from the private sector but requires funding from financial institutions, and payments from the public sector and/or users over the project's lifetime. The private partner's role in the partnership is the design, completion, implementation and funding of the project, while the public partner focuses on defining and monitoring compliance with the objectives. This research is focused on PPPs associated to maintenance of a road throughout a given horizon. We propose a stochastic optimization model that describes the life-cycle performance of the system (as a result of deterioration and maintenance processes), and the private partner's earnings and costs associated to maintenance decisions. A Mixed Integer Program (MIP) is formulated to model the situation, and the resulting problem is solved by the Sample Average Approximation (SAA) method using Monte Carlo simulation. In addition, increasing non-homogeneous Lévy processes are considered for both types of deterioration: progressive and shock-based. Specifically, we generate realizations of the deterioration process from a non-homogeneous gamma process (for progressive) and a compound Poisson process (for shock-based), and we consider multiple sources of degradation by the superposition of any of these models. We provide a decision-support tool for private entities involved in infrastructure projects operated via PPPs that can be used for the analysis of diverse scenarios by running the model under different conditions. We test our methodology with a synthetic case study based on available information for infrastructure projects executed under PPPs in Colombia.

Keywords: Public-Private Partnerships, infrastructure, deterioration, maintenance, stochastic optimization, Lévy process, Monte Carlo sampling, Sample Average Approximation.

1. Introduction

Infrastructure systems are an essential component of sustainable development and their adequate operation is central to socioeconomic vitality. Public Private Partnerships (PPPs) have become a basic instrument for public management of infrastructure projects, where large investments are required. PPPs are a tool that helps governments leverage the expertise and efficiency of the private sector, raise capital, and promote development; allocating the risk across the public and private sectors. In PPPs, the public party defines the requirements of an infrastructure project and monitors its fulfillment periodically, while the private party is responsible for the design, construction, operation and management of the public asset during the established time horizon. To achieve the adequate operation of the infrastructure, maintenance actions are needed periodically to restore the condition of the system, which continuously deteriorates due to multiple factors. In PPP projects the private party needs to select the most appropriate design for the infrastructure, based on estimated profits. Once a specific design is selected, the entity needs to decide the most appropriate times to perform maintenance actions to the system in order to satisfy public requirements, without certain knowledge about the deterioration process of the infrastructure. Comprehensive decision support tools are, thus, necessary to evaluate trade offs between investments at the design stage (e.g., robust designs) versus the operation stage (e.g., recurrent maintenance), accounting for the stochastic nature of deterioration.

Projects operated via PPPs have been studied from different perspectives, as they integrate concepts from a variety of domains. Early contributions considered game theory as a mean to model and understand the behavior of the parties involved in the project. Later, the principal-agent approach became a common tool to model the dynamic relation between the principal (i.e., the government) and the agent (i.e., the private entity), where each actor makes decisions in their own behalf. Regarding the infrastructure maintenance problem, life-cycle analysis of infrastructure systems are widely studied using probabilistic models, and optimization models have been formulated to include financial and economic evaluation of the projects considering maintenance and restoration actions. Recently, Gómez et al. (2020) proposed a methodological framework based on exact optimization models, in which the life-cycle analysis framework is adopted to describe technical, financial, and organizational aspects of PPPs in the context of infrastructure management projects, with a deterministic and known deterioration process. We extend the model of Gómez et al. (2020) by introducing the inherent uncertainty associated with the deterioration process of the system, in order to provide a quantitative tool capable of generating optimal maintenance policies for infrastructure systems considering multiple sources of degradation. The proposed approach addresses the maintenance problem, but can be also used as a second stage problem for decisions related to system design in order to estimate the eventual maintenance costs when a specific design for the infrastructure is selected.

This research proposes a methodological framework based on stochastic optimization models to integrate the uncertainty associated with the deterioration process of the system into the operation stage of infrastructure PPP projects. We extend the model proposed by Gómez et al. (2020) to its stochastic version, where the scenario set (i.e., realizations of the deterioration process) is an input parameter. We adopt state-of-the-art techniques to generate sample paths for progressive, shock-based and combined deterioration, based on the Lévy degradation framework defined by Riascos-Ochoa et al. (2016). We use the sum of a deterministic linear drift and a non-homogeneous Gamma process to model the progressive deterioration, a compound Poisson process with different distribution of shock sizes to model the shock-based deterioration process, and the sum of these two process to model combined deterioration. We calibrate the models in order to replicate the expected behavior of the infrastructure system. To deal with the computational effort required to solve the stochastic optimization problem for a large number of scenarios, we use the Sample Average Approximation (SAA) method by Kleywegt et al. (2002). This methodology approximates the expected objective function of the stochastic optimization problem by the sample average over a reduced scenario set generated via Monte Carlo simulation. The main contribution of this work is the incorporation of the scenario set, generated via Lévy processes, in the optimization-based methodology proposed by Gómez et al. (2020), and its implementation using sampling techniques.

The proposed methodology can help the private entity through different stages of the project, given a set of rules (e.g., a PPP contract), which specifies minimum performance thresholds and non-compliance penalties, etc. In the construction stage, the model could be used to inform design decisions by calculating the expected NPV of the project as a function of design decisions, and the expected maintenance actions that would be necessary to respond to possible deterioration realizations. In turn, this analysis can estimate the savings in maintenance operations from choosing a robust design over a basic design. In the operation stage, the model can find the best maintenance plan, considering various factors (internal and external) that affect the infrastructure system. Consequently, this research is also useful for the public party, as it allows decision makers to perform sensitivity analysis on some parameters in order to define the conditions for a PPP contract, or to anticipate the actions of its counterpart for a specific situation. With the appropriate data it can represent an instrument for public entities that allows it to make better informed decisions. For the proposed methodology, computational complexity depends on the number of discrete service levels and the periods in the planning horizon. In our example, these values were set to 7 levels and 30 years, which is representative of the size expected for a realistic case, and does not represent a challenge in computational terms.

This document is organized as follows. Section 2 gives a general review of the relevant literature on life-cycle analysis, deterioration process, PPPs and sampling techniques. Section 3 provides a detailed definition of the problem, and presents a methodological framework for analysis. Section 4 presents the formulation of the corresponding mathematical model. Section 5 demonstrates the use of the proposed methodology considering the maintenance of a road via PPPs. Finally, Section 6 provides a discussion about the proposed methodology and results, as well as posing the limitations, assumptions and future work.

2. Literature review

Infrastructure projects include the construction and improvement of roads, railways, pipelines, gas pipelines, power lines, waterways, ports, airports, dams, hydroelectric plants and others. These can be seen as the basic structures, systems and services required for the economic development of a country. This implies that the design and evaluation of maintenance policies, that support the decision-making process in infrastructure public assets management, require the combination of techniques from different disciplines. In this section we provide a brief account of the literature related to the operation of infrastructure projects. First, we provide a general overview about life-cycle analysis of infrastructure systems. Second, we present different methodologies to model the effect of multiple sources of degradation on deteriorating systems. Third, we give an overview of existing approaches to model PPP projects with operations research (OR) techniques. Finally, we conclude the section by discussing sampling techniques used in stochastic optimization problems.

2.1. Life-cycle of infrastructure systems in OR

Every project has a life-cycle which undergoes several processes. Each phase involves various parties and activities that are interrelated and overlapped. All the processes in a project life-cycle need coordination, decision making, technical capabilities, benchmarking and scheduling techniques. Over the years, life-cycle thinking, in the context of infrastructure systems, has gained increasing attention. Frangopol et al. (2004) offer a review of probabilistic models for maintaining and optimizing the life-cycle performance of deteriorating structures. Later, Frangopol & Liu (2007) review recent development of life-cycle maintenance and management planning for deteriorating civil infrastructure using optimization techniques and considering simultaneously multiple and often competing criteria in terms of condition, safety and life-cycle cost, while Frangopol (2011) highlight recent accomplishments in the life-cycle performance assessment, maintenance, monitoring, management and optimization of structural systems under uncertainty.

Academic and practitioners in the field of OR have increased their attention in infrastructure problems. Liu & Frangopol (2005) formulate a multiobjective optimization problem that treats the lifetime condition and safety levels as well as life-cycle maintenance cost as separate objective functions. Then, Santander & Sánchez-Silva (2008) present a strategy for optimizing the maintenance of technical facilities based on their life-cycle within which deterioration and sudden failure due to extreme events may occur, whereas Sánchez-Silva et al. (2012) integrates a life-cycle cost optimization model with a structural deterioration model that combines the action of progressive degradation (e.g. corrosion, fatigue) and sudden events (e.g. earthquakes).

2.2. Deterioration of infrastructure systems

Deterioration is the reduction in performance, reliability, and life span of systems. Most of systems deteriorate due to internal and external factors; the former occur due to the inner structure of systems (e.g. ageing and quality of materials) and the latter often occur due to the environmental conditions in which systems operate (e.g. vibrations, humidity, and pollution). Gorjian et al. (2010) offer a review of degradation models in reliability analysis and present merits, limitations, and applications of each model. They also provide potential applications of these degradation models.

According to Sánchez-Silva et al. (2011), degradation models are usually divided into: shock-based, progressive, and combined degradation. Shock-based degradation models consider the effect of sudden events, such as earthquakes or hurricanes, by decreasing the system's condition in finite amounts in the form of jumps or shocks at discrete points in time. In progressive degradation models, the system's condition decays continuously over time due to processes such as corrosion, fatigue or wear-out. Combined degradation models consider the system's condition decay due to multiple sources of degradation from both kinds: shock-based and progressive. In recent years, different methodologies have been used to model system's degradation. Kumar et al. (2015) propose a general stochastic framework to model the deterioration process as a combination of shock and gradual processes, they use a deterministic function of time to model gradual deterioration and a compound Poisson process to model the shock deterioration. Later, Wang et al. (2016) developed a degradation modeling approach in which degradation is represented by an independent increment process with linear mean and general quadratic variance functions. Riascos-Ochoa et al. (2016) present a framework to model the effect of multiple sources of degradation on deteriorating systems, by considering the system deterioration as a general increasing Lévy process (i.e., a subordinator).

2.3. Public Private Partnerships problem in OR

Public–private partnerships (PPP) are becoming an increasingly popular option of project delivery, and are capturing the attention of different fields. Over the years, diverse approaches have been proposed. First, researchers considered game theory as a means to model PPPs, Ho (2006) present a game-theory based model to provide foundations to policy makers for prescribing effective PPP procurement and management policies and for examining the quality of PPP policies. This study also offers a framework and a methodology to understand the behavioral dynamics of the parties in PPP. Later, the principal-agent framework became a common tool to model the dynamic relation between the principal (i.e., the government) and the agent (i.e., the private entity). Leruth (2009) proposes a principal-agent framework to help conceptualize the relations between the government and the private firm, discussing the economic and financial rationale, including risks, for partnerships between the public and the private sectors as a way to share the burden of developing infrastructure. Then, Páez-Pérez & Sánchez-Silva (2016) presents a novel approach to the problem of infrastructure development by integrating technical, economic and operational aspects, as well as the interactions between the entities.

Regarding the use of optimization, Ng et al. (2007) present a simulation model to assist the public partner to determine an optimal concession period. Song et al. (2018) propose a multi-objective programming model to find Pareto-optimal decisions and conduct a comparative statistic analysis using the parameters in the model to examine the impacts of exogenous variables on decision behaviors. Recently, some advances have been performed considering PPPs in infrastructure projects. Lozano & Sánchez-Silva (2019) presents a model to define a set of contract parameters that maximize the profit of both entities in the long term, and at the same time optimize the system performance, the model combines the advantages of game theory, simulation, optimization and agent-based modeling, and describes the complex interactions between the public, the private and nature. Lastly, Gómez et al. (2020) proposed an optimization model that describes infrastructure performance as a result of deterioration and maintenance processes, and computes the associated benefits and costs for the principal and the agent, combining game theoretical notions with life-cycle analysis. Their methodological framework integrates technical, financial, and organizational aspects of PPPs into exact optimization models that allow the analysis of the interactions and decision strategies of governments and companies.

2.4. Sampling techniques in stochastic optimization

Stochastic optimization refers to a collection of methods for minimizing or maximizing an objective function when uncertainty is included. Randomness can appear in the cost function or the constraint set. In these problems, the goal is to find some policy that is feasible for all (or almost all) the possible data instances and maximize or minimize the expectation of some function of the decisions and the random variables (Shapiro et al., 2009). More generally, such models are formulated, solved analytically or numerically, and analyzed in order to provide useful information to a decision-maker.

According to Ekblom & Blomvall (2020), most of decision problems that involve uncertainty are often combined with a sequential decision process and can not be solved exactly, so a commonly used strategy is to replace expectations with sample average approximations (e.g., by Monte Carlo sampling). Shapiro (2003) provides a comprehensive review of theoretical properties of sampling-based approaches for stochastic programming problems. In recent years, sampling-based methods have been successfully used in many different applications of stochastic optimization. Homem-de-Mello & Bayraksan (2014) offer an overview of some of that work, providing a practical guide to solve a stochastic optimization problem with sampling. Kleywegt et al. (2002) discuss a Monte Carlo simulation–based approach to stochastic discrete optimization problems, and Verweij et al. (2003) present a detailed computational study of the application of the SAA method to solve three classes of stochastic routing problems.

2.5. Research gap

The remainder of this document discusses the methodology used to address a problem of infrastructure management projects via PPPs, where key physical and financial aspects are described using the life-cycle analysis framework. We present a stochastic version of the model proposed by Gómez et al. (2020), scenarios for the deterioration process are generating via Lévy process following the framework defined by Riascos-Ochoa et al. (2016), and the problem is solved by means of sampling techniques described in Kleywegt et al. (2002) and Verweij et al. (2003). To the extend of our knowledge, this is a novel approach, since stochastic optimization models have not been used to address the problem of maintenance in the operational stage of a infrastructure project via PPP.

3. Methodology

The proposed methodology addresses the maintenance problem in infrastructure PPP projects by means of stochastic optimization considering a life-cycle performance framework and stochastic processes to model system's deterioration. The overall objective of this study is to find the best maintenance policy for a set of input parameters. We provide a detailed definition of the problem of interest in 3.1, describe the stochastic optimization approach in 3.2, present the stochastic processes used to generate realizations of the deterioration process in 3.3, and detail sampling methods used to solve the optimization model in 3.4.

3.1. Problem Definition

Public infrastructure assets are a foundation of a country's economic development. Therefore, maintaining such assets in good condition is critical as infrastructure wears-out with time and use. Infrastructure projects are services to support the public's basic needs of living and can be seen as complex systems designed and constructed to accomplish a specific function during a period of time. In these projects, outcomes depend on the mechanical behavior of physical structures, as well as on uncertain factors from the external environment (e.g., natural hazards, aging), and on the organizational context of the project (e.g., legal, economic, and socio-political conditions). To illustrate the situation, consider an infrastructure system for which stakeholders define a performance metric. The deterioration of the system (due to multiple sources) causes the performance measure to decrease over time. The performance measure is a continuous magnitude related to the condition of the system, however a discrete set of performance ranges related to service-levels is also considered.

As mentioned by Gómez et al. (2020) the core of the problem is in the benefits and costs derived from the system service-levels (hence, performance) over time, which depend on the maintenance policy and the structure of payments. This model considers fixed and variable costs of maintenance actions, and a system of payments and rewards/penalties from the operation of the the system at each service-level. In this context, the government can specify a minimum performance threshold, whereas the private contractor can define a minimum profit threshold. We considered a discount rate to assess the value of money throughout the planning horizon. Finally, we assume that maintenance actions take place immediately, and without loss of generality, that all maintenance actions make the system as good as new.

3.2. Stochastic optimization

We are addressing the problem defined in 3.1 through stochastic optimization for two main reasons. First, our main objective is to find the best maintenance policy in response to a set of possible realizations generated via stochastic processes. Second, the private contractor has to define a maintenance policy at the start of the planning horizon, before the realization of the system's deterioration process occurs.

Stochastic programs are mathematical programs that include data that is not known with certainty, but can be approximated by probability distributions. The goal here is to find some policy that is feasible for all (or almost all) the possible data instances and maximizes the expectation of some function of the decisions and the random variables. In these programs, the decision-maker takes some actions in the first-stage, after which a random event occurs affecting the outcome of the first-stage decision. Then, second-stage decisions can be made to compensates for any bad effects that might have been experienced as a result of the first-stage decision. A general stochastic programming problem is in the form:

$$z^* = \min_{x \in X} c^T x + \mathbb{E}_{\mathcal{P}}[Q(x, \xi(\omega))] \tag{1}$$

where x denotes the first-stage solution, X denotes the first-stage feasible set, $\omega \in \Omega$ denotes a *scenario* that is unknown when the first-stage decision x have to be made, Ω is the set of all scenarios, in this problems we assume that the probability distribution $\mathcal{P}$ on Ω is known in the first stage. The quantity $Q(x, \xi(\omega))$ represents the optimal value of the second-stage problem corresponding to first-stage solution x and the parameters $\xi(\omega)$. This formulation approaches the problem by optimizing the average value of the objective function. This is a valid approach, because if the process repeats for a given (fixed) x, then by the Law of Large Numbers the average of the objective function, over many repetitions, will converge (with probability one) to the expectation $\mathbb{E}[Q(x, \xi(\omega))]$, and, indeed, in that case the solution of problem will be optimal on average (Shapiro et al., 2009, Chapter 1).

If Ω contains a finite number of scenarios, say $\Omega = \{\omega_1, \omega_2, \ldots, \omega_{|\Omega|}\}$, each with probability p_k, $k = 1, 2, \ldots, |\Omega|$, it is possible to build and solve the deterministic equivalent (DE) problem by writing the expected value as a weighted sum:

$$\mathbb{E}[Q(x, \xi(\omega))] = \sum_{k=1}^{|\Omega|} p_k Q(x, \xi(\omega_k)) \tag{2}$$

For stochastic programs with a prohibitively large number of scenarios, a number of sampling based approaches have been proposed to estimate the expected second-stage values. According to Shapiro et al. (2009), such approaches are classified into two main groups: interior sampling and exterior sampling. In interior sampling methods, different samples are used at different steps of a particular optimization procedure. In exterior sampling methods, a sample $\omega_1, \omega_2, ..., \omega_N$ of N sample scenarios is generated from Ω according to probability distribution $\mathcal{P}$ and the expected value function $\mathbb{E}[Q(x, \xi(\omega))]$ is approximated by the sample average function. We will be focusing on the latter approach, using the detailed procedure presented in 3.4.

3.3. Deterioration process

We generate a set of realizations of the deterioration process for the infrastructure system, following the framework defined by Riascos-Ochoa et al. (2016). We provide a conceptual framework for characterizing system degradation over time, describe the basics of Lévy processes, and how they can be used for modeling degradation.

Consider a infrastructure system that is placed into operation at time $t = 0$ and whose condition decreases with time. The system condition at time t is denoted by V_t, modeled as a random variable that takes values in the set of positive real numbers. The value of V_0 is deterministic and represents the conditions when the system is new. The condition decreases over time due to degradation, and the accumulated deterioration until time t is defined by the random variable D_t, with $D_0 = 0$. We are interested in obtaining the condition of a system at time t under the assumption that there is no maintenance. Then, V_t is related to D_t by:

$$V_t = \max\{V_0 - D_t, 0\} \tag{3}$$

where the sets $\{V_t\}_{t\geq 0}$ and $\{D_t\}_{t\geq 0}$, formed by the random variables of condition and deterioration for all times $t \geq 0$, constitute stochastic processes. We will be modeling the accumulated deterioration as a non-homogeneous Lévy process. First we introduce Lévy processes, then its generalization to non-homogeneous Lévy processes.

3.3.1. Lévy process

Lévy processes are continuous-time stochastic processes with independent and stationary increments and with right continuous sample paths having left limits. Formally, a Lévy process is defined as follows (Riascos-Ochoa et al., 2016): given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, an adapted process $\{X_t\}_{t\geq 0}$ with $X_0 = 0$ almost surely is a *Lévy process* if:

1. $\{X_t\}_{t\geq 0}$ has increments independent of the past.
2. $\{X_t\}_{t\geq 0}$ has stationary independent of the past.
3. X_t is continuous in probability.

Lévy processes are a superposition of three independent processes:

$$X_t = X_t^{\{1\}} + X_t^{\{2\}} + X_t^{\{3\}} \tag{4}$$

The first term is a deterministic (linear) process $X_t^{\{1\}} = qt$. The second term is a Gaussian process $X_t^{\{2\}}$ with zero mean, covariance matrix A, and having positive and negative increments and continuous sample paths (e.g., a Brownian motion). The third part $X_t^{\{3\}}$ is a pure-jump process. We discuss now a framework for modeling degradation based on Lévy processes. Our goal here is to illustrate an application rather than provide a complete mathematical exposition, for that, we refer the reader to Bertoin (2001) and papers referenced therein.

3.3.2. Lévy degradation framework

The Lévy degradation framework is supported on the following assumptions (Riascos-Ochoa et al., 2016):

1. Deterioration is described by a 1-dimensional stochastic process $\{X_t\}_{t\geq 0}$ with $X_0 = 0$ almost surely.
2. The deterioration process $\{X_t\}_{t\geq 0}$ has independent increments.
3. The deterioration process $\{X_t\}_{t\geq 0}$ has stationary increments.
4. In absence of maintenance, deterioration is increasing (i.e., not decreasing) almost surely.
5. Multiple sources of degradation (i.e.,different shock and/or progressive processes) act independently.

The first three assumptions allow modeling deterioration as a 1D Lévy process, assumption 4 restricts the process to the class of subordinators, and assumption 5 allows the combination of different degradation mechanisms. According to Riascos-Ochoa et al. (2016), subordinators are processes that take values in $\mathbb{R}^+ := [0, \infty)$ with increasing sample paths, where the Gaussian (Brownian) component $X_t^{\{2\}}$ in (4) is zero. A subordinator X_t has the general form:

$$X_t = qt + X_t^{\{3\}} \tag{5}$$

The described framework can be used to describe the main degradation models: progressive, shock-based and combined degradation models as follows.

Progressive deterioration: Progressive degradation, also called graceful degradation, constitutes those processes that reduce (almost) continuously the system's condition V_t over time. In the Lévy degradation formalism, the subordinator for modeling progressive deterioration can be constructed as the sum of two independent processes: a linear deterministic drift (LD) $X_t^{\{1\}}$ with $q > 0$ and a jump-process $X_t^{\{3\}}$:

$$Z_t = qt + X_t^{\{3\}} \tag{6}$$

where $X_t^{\{3\}}$ describes a jump process with infinite number of small jumps in any finite time interval, In this work we relax the assumption of stationary increments, which defines a non-homogeneous Lévy process (Riascos-Ochoa, 2016). Specifically the progressive degradation is a non-stationary Gamma process. We propose to model progressive degradation as a non-stationary Gamma Process (GP) with shape function $\Lambda(t)$, scale parameter β and increments of Δt. Where increments are *iid* distributed gamma with shape $\Lambda(t) - \Lambda(t - \Delta t)$ and scale β. The algorithm to generate a sample path to a desired time T is:

Algorithm 1: Non-stationary Gamma Process

$t = 0$, $Z = 0$;
while $t < T$ **do**
  Generate $X \sim \text{Gamma}(\Lambda(t) - \Lambda(t - \Delta t), \beta)$;
  Set $t = t + \Delta t$ and $Z = Z + X$
end
Return the sample path Z.

Shock-based deterioration: Shock-based degradation describes the effect of sudden events decrease the system's condition V_t in finite amounts in the form of jumps or shocks at discrete points in time. In the Lévy degradation formalism, the subordinator for modeling shock-based deterioration is a jump-process $X_t^{\{3\}}$:

$$W_t = X_t^{\{3\}} = \sum_{i=1}^{N_t} Y_i \tag{7}$$

where N_t represents the number of shocks that have occurred by time t, with $\{N_t\}_{t \geq 0}$ being a counting process. These models can be described by two stochastic processes: the inter-arrival times $\{T_i\}_{i \geq 1}$ (with T_i the random variable of the time between the $(i-1)^{th}$ and i^{th} shock) and shock sizes $\{Y_i\}_{i \geq 1}$ which represent the damage of each shock to the system. We propose to model shock-based degradation as a stationary Compound Poisson Process (CPP), where inter-arrival times T_i are *iid* distributed exponential with rate λ, and shock sizes Y_i are *iid* with distribution $H(\cdot)$ supported in $\mathbb{R}^+$. The algorithm to generate a sample path to a desired time T is:

Algorithm 2: Stationary Compound Poisson Process

$t = 0$, $N = 0$, $W = 0$;
while $t < T$ **do**
  Generate $U \sim \text{Uniform}(0, 1)$;
  Set $t = t + [-\frac{1}{\lambda} \ln(U)]$;
  Generate $Y \sim H(\cdot)$;
  Set $N = N + 1$ and $W = W + Y$
end
Return the sample path W.

Combined deterioration: Combined degradation accounts of the effect of both types of deterioration; once the system is put in service, damage starts accumulating as a result of progressive degradation and sudden events (i.e., shocks). Combined deterioration can be seen in the form of continuous deterioration plus a cumulative shock model. In the Lévy degradation formalism, the subordinator for modeling combined deterioration can be constructed as the sum of two independent processes: shock-based W_t and progressive Z_t:

$$K_t = W_t + Z_t = \sum_{i=1}^{N_t} Y_i + (qt + X_t^{\{3\}}) \tag{8}$$

In this case, both, shock-based and progressive deterioration processes are increasing and act on the infrastructure independently. We propose to model combined degradation considering a stationary CPP with shock magnitudes distributed $H(\cdot)$, supported in $\mathbb{R}^+$, for shock-based, and a non-stationary GP for progressive deterioration.

The stochastic process described above let us generate as many scenarios as we need. According to literature, the number of constructed scenarios Ω should be relatively modest so that the obtained DE can be solved with reasonable computational effort. To deal with that problem, a common approach to reduce the scenario set to a manageable size is by using Monte Carlo simulation.

3.4. Sampling techniques

We solve the stochastic optimization problem considering a scenario set built via stochastic processes, following the framework defined by Kleywegt et al. (2002), where a Monte Carlo simulation–based approach is provided. The basic idea is to generate a random sample of Ω and use it to approximate the expected value function by the corresponding sample average function. The obtained sample average optimization problem is solved, and the procedure is repeated several times until the stopping criteria are satisfied. We provide a brief overview of sampling process, and present the algorithm used to achieve our goal.

The sample average approximation (SAA) method is an approach for solving stochastic optimization problems by using Monte Carlo simulation. This method assumes that problem (1) has relatively complete recourse. A two-stage stochastic program is said to have relatively complete recourse if for each feasible first-stage solution $x \in X$ and each scenario $\omega \in \Omega$, there exists a second-stage solution (Shapiro, 2003). This method is used to generate a finite approximation Ω_N to Ω, where the size of Ω_N is N and each element of ω_N has the same probability $p = \frac{1}{N}$.

After generating a sample of N replications of the random vector ω, where $\omega^1, \ldots, \omega^N$ are *iid* random samples of N realizations the sample average function is:

$$\frac{1}{N} \sum_{i=1}^{N} Q(x, \xi(\omega^i)) \tag{9}$$

For any $x \in X$, the sample average function can be viewed as a numerical value associated with the generated sample, and the associated optimization problem is:

$$z_N = \min_{x \in X} c^T x + \frac{1}{N} \sum_{i=1}^{N} Q(x, \xi(\omega^i)) \tag{10}$$

We will refer to (1) and (10) as the *true* (or expected value) and sample average approximation (SAA) problems, respectively. The SAA method works by solving problem (10) repeatedly, where the optimal value z_N and an optimal solution $\hat{x}$ to the SAA problem provides an estimation to their true counterparts in the stochastic problem (1). We discuss now an algorithm design for the sample average approximation approach to solve (1), following the methodology proposed by Verweij et al. (2003). Our goal here is to illustrate the results rather than provide detailed mathematical statements, for that, we refer the reader to the surveys in Shapiro (2003) and Shapiro et al. (2009, Chapter 5) and papers referenced therein.

3.4.1. Algorithm design

By generating M independent samples, each of size N, and solving the associated SAA problems, we obtain values $z_N^1, z_N^2, \ldots, z_N^M$ and a set of candidate solutions $\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^M$. Let:

$$\bar{z}_N = \frac{1}{M} \sum_{m=1}^{M} z_N^m \tag{11}$$

denote the average of the M optimal values of the SAA problems. It is well-known that $\mathbb{E}[\bar{z}_N] \leq z^*$, therefore, $\bar{z}_N$ provides a statistical estimate for a lower bound of the true problem. Also, for any feasible point $\hat{x} \in X$, the objective value $c^T\hat{x} + \mathbb{E}[Q(\hat{x}, \xi(\omega))]$ is an upper bound of the true problem z^*. The upper bound can be estimated by:

$$\hat{z}_{N'} = c^T\hat{x} + \frac{1}{N'} \sum_{n=1}^{N'} Q(\hat{x}, \xi(\omega^n)) \tag{12}$$

where $\{\omega^1, \omega^2, \ldots, \omega^{N'}\}$ is a sample of size N', typically chosen to be quite large, $N' > N$, and the sample of size N' is independent of the sample, if any, used to generate $\hat{x}$. Then, we have that $\hat{z}_{N'}(\hat{x})$ is an unbiased estimator of $c^T\hat{x} + \mathbb{E}[Q(\hat{x}, \xi(\omega))]$, and hence, for any feasible solution $\hat{x}$, we have that $\mathbb{E}[\hat{z}_{N'}] \geq z^*$. The variance of the estimators $\bar{z}_N$ and $\hat{z}_{N'}(\hat{x})$ can be estimated by:

$$\hat{\sigma}^2_{\bar{z}_N} = \frac{1}{(M-1)M} \sum_{m=1}^{M} (z_N^m - \bar{z}_N)^2 \tag{13}$$

and:

$$\hat{\sigma}^2_{\hat{z}_{N'}(\hat{x})} = \frac{1}{(N'-1)N'} \sum_{n=1}^{N'} \left(c^T\hat{x} + Q(\hat{x}, \xi(\omega^n)) - \hat{z}_{N'}(\hat{x})\right)^2 \tag{14}$$

The above procedure produces up to M different candidate solutions. It is logical to take $\hat{x}^*$ as one of the optimal solutions $\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^M$ of the M SAA problems which has the smallest estimated objective value. That is:

$$\hat{x}^* \in \arg\min \left\{\hat{z}_{N'}(\hat{x}) \mid \hat{x} \in \{\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^M\}\right\} \tag{15}$$

To evaluate the quality of solution $\hat{x}^*$, we compute the optimality gap estimate as:

$$\hat{z}_{N''}(\hat{x}^*) - \bar{z}_N \tag{16}$$

where $\hat{z}_{N''}(\hat{x}^*)$ is recomputed after performing the minimization in (15) with an independent sample of size N'' to obtain an unbiased estimate. The estimated variance of this gap estimator is:

$$\hat{\sigma}^2_{\hat{z}_{N''}(\hat{x}^*) - \bar{z}_N} = \hat{\sigma}^2_{\hat{z}_{N''}(\hat{x}^*)} + \hat{\sigma}^2_{\bar{z}_N} \tag{17}$$

and the accuracy of the optimality gap estimator can be taken into account by adding a multiple z_α of its estimated standard deviation to the gap estimator expressed in Equation (17). Here $z_\alpha = \Phi^{-1}(1-\alpha)$, where $\Phi^{-1}(z)$ is the cumulative distribution function of the standard normal distribution. The optimality gap estimator taking accuracy into account is:

$$\hat{z}_{N''}(\hat{x}) - \bar{z}_N + z_\alpha \sqrt{\hat{\sigma}^2_{\hat{z}_{N''}(\hat{x}) - \bar{z}_N}} \tag{18}$$

The above procedure for statistical evaluation of a candidate solution was suggested in Mak et al. (1999), convergence properties of the SAA method were studied in Kleywegt et al. (2002). We address some issues that need tackling in order to design a specific algorithm for solving stochastic optimization problems.

Selection of the sample size: To choose a sample size N, several trade-offs should be taken into account. With larger N, the objective function of the SAA problem tends to be a more accurate estimate of the true objective function, and an optimal solution of the SAA problem tends to be a better solution. However, the computational complexity of solving the SAA problem increases at least linearly, and often exponentially, in the sample size N. However, estimating the objective value of a feasible solution x by the sample average requires much less computational effort than solving the SAA problem (for the same sample size N). In literature, authors recommend to choose a relatively small sample size N for the SAA problem, and choose a larger sample size N' to obtain an accurate estimate of the objective value $\hat{z}_{N'}(\hat{x})$ of an optimal solution $\hat{x}$ of the SAA problem.

Selection of the number of replications: Due to the fact that computational complexity of solving the SAA problem increases faster than linearly in the sample size N, it is more efficient to choose a smaller sample size N and to generate and solve several SAA problems with *iid* samples. That is, to replicate generating and solving M SAA problems. In literature, authors recommend to define a stopping criterion based on the optimality gap estimate, or based on the probability that the $(M+1)^{th}$ SAA replication would produce a better solution than the best of the solutions produced by the M replications. Thus, if the stopping criterion is satisfied, then no more replications are performed.

Selection of the best solution: Once the algorithm reaches the stopping condition, it is necessary to select the candidate solution with the best performance produced during the replications. To accomplish that goal, there are many screening and selection methods for selecting subsets of solutions or a single solution, among a (reasonably small) finite set of solutions, using samples of the objective values of the solutions. In literature, authors recommend a combined procedure: in the first stage, a subset of the candidate solutions $\{\hat{x}^1, \ldots, \hat{x}^M\}$ is chosen for further evaluation, based on its sample average values $\hat{z}_{N'}(\hat{x})$. Then, in the second stage, a larger sample size N'' is generated, and the candidate solution $\hat{x}$ with the best value of $\hat{z}_{N''}(\hat{x})$ is selected as the chosen solution.

We propose an algorithm for the studied stochastic optimization problem by making some modifications to the algorithm presented by Kleywegt et al. (2002).

Algorithm 3: Sample average approximation

---

1 Choose initial sample sizes N, N' and N'', a decision rule for determining the number M of SAA replications (involving a maximum number M^+ of SAA replications with the same sample size, such that $\frac{1}{(M^+ + 1)}$ is sufficiently small) and a tolerance ε.
2 Generate a sample of size N' and a sample of size N''.
3 **for** $m = 1, \ldots, M$ **do**
4   Generate a sample of size N.
5   Solve the SAA problem with objective value $\hat{z}_N^m$ and optimal solution $\hat{x}$.
6   Estimate the sample average value $\hat{z}_{N'}(\hat{x}^m)$.
7 **end**
8 Create a subset of potential solutions for further evaluation $\{\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^K\}$, based on its sample average values $\hat{z}_{N'}(\hat{x}^m)$.
9 Select $\hat{x}^*$ performing the minimization $\hat{x}^* \in \arg\min \left\{ \hat{z}_{N''}(\hat{x}) \mid \hat{x} \in \{\hat{x}^1, \hat{x}^2, \ldots, \hat{x}^K\} \right\}$.
10 Compute the optimality gap $\hat{z}_{N''}(\hat{x}^*) - \bar{z}_N$ and the variance of the gap estimator $\hat{\sigma}^2_{\hat{z}_{N''}(\hat{x}^*) - \bar{z}_N}$.
11 **if** $\frac{\hat{z}_{N''}(\hat{x}^*) - \bar{z}_N}{\bar{z}_N} > \varepsilon$ **then**
12   Increase the sample size N.
13   Go to step 3.
14 **end**
15 Return the optimal solution $\hat{x}^*$.

The main difference between our algorithm an the one proposed by Kleywegt et al. (2002) lies in steps 9 and 10, where we select the best solution among all potential solutions based on their sample average values, and then compute the optimality gap comparing the sample average value $\hat{z}_{N''}(\hat{x}^*)$ (obtained with an independent sample of size N'') against the average of the optimal objective values $\bar{z}_N$. This differs to Kleywegt et al. (2002) where they compute the optimality gap after the execution of each replication of the SAA problem.

4. Mathematical Model

We propose a stochastic programming model for infrastructure operation, considering life-cycle performance framework. The private party decides on when to perform maintenance actions on the system. The entity seeks to maximize its profit, including the earnings from payments and eventual rewards, minus the cost of maintenance actions and eventual penalties. An important contribution of this work is the incorporation of a scenario set with realizations of the deterioration process of the system. This section presents the mathematical formulation for the optimization model which responds to the problem defined in 3.1. This model is the stochastic version of the model presented by Gómez et al. (2020).

Table 1 summarizes the sets, parameters and variables defined for the problem.

Table 1: Summary of sets, parameters and variables in the mathematical model

Sets	$\mathcal{T}$:	planning horizon
	$\mathcal{L}$:	set of discrete service-levels
	$\mathcal{S}$:	set of scenarios
Parameters	$\gamma_{\tau,s}$:	performance obtained after τ periods without restoration in scenario $s \in \mathcal{S}$
	$\bar{\gamma}$:	performance when the system is as good as new
	γ^*:	minimum performance threshold
	$c^{(f)}$:	fix cost for restoration action
	$c^{(v)}$:	unit cost to restore a performance unit
	α:	fixed income from the government to the private party
	f_t:	income from finance institution at time $t \in \mathcal{T}$
	d_l:	toll income under service-level $l \in \mathcal{L}$
	k_l:	penalty/reward under service-level $l \in \mathcal{L}$
	ϵ:	target return rate
	$\hat{\zeta}_l$:	lower bound for service-level $l \in \mathcal{L}$
	$\check{\zeta}_l$:	upper bound for service-level $l \in \mathcal{L}$
	ρ_s:	probability of scenario $s \in \mathcal{S}$
Variables	x_t:	whether a maintenance action is applied at $t \in \mathcal{T}$
	y_t:	number of periods elapsed after last restoration
	$b_{t,\tau}$:	whether $y_t = \tau$ for $\tau \in \mathcal{T}$
	$z_{t,l,s}$:	whether system is at service-level $l \in \mathcal{L}$ at $t \in \mathcal{T}$ in scenario $s \in \mathcal{S}$
	$v_{t,s}$:	performance at $t \in \mathcal{T}$ in scenario $s \in \mathcal{S}$
	$p^{(+)}_{t,s}$:	earnings at $t \in \mathcal{T}$ in scenario $s \in \mathcal{S}$
	$p^{(-)}_{t,s}$:	expenditures at $t \in \mathcal{T}$ in scenario $s \in \mathcal{S}$
	$p^{(\cdot)}_{t,s}$:	available budget at $t \in \mathcal{T}$ in scenario $s \in \mathcal{S}$
	w_t:	linearization of $y_{t-1}x_t$
	$u_{t,s}$:	linearization of $v_{t,s}x_t$

Sets
We considered a set of periods in the planning horizon ($\mathcal{T}$), a set of service levels ($\mathcal{L}$), associated with discrete categories of the continuous value representing the performance of the system, and a set of scenarios ($\mathcal{S}$), which represent realizations of the deterioration process.

Parameters
Parameters $\gamma_{\tau,s}$, $\bar{\gamma}$ and γ^* are related to the performance of the system: $\gamma_{\tau,s}$ represents the performance level that would be observed after τ periods without executing maintenance actions in scenario $s \in \mathcal{S}$.

This parameter represent the condition of the system, and is precomputed based on the stochastic processes described in 3.3. Parameter $\bar{\gamma}$ indicates the maximum performance level that can be observed and γ^* represents the minimum performance threshold allowed for the operation of the system.

Parameters $c^{(f)}$, $c^{(v)}$, α, f_t, d_l, k_l, and ϵ (detailed in Table 1) correspond to important economic magnitudes in the problem (costs, payments, benefits, penalties, target profits), which must be carefully estimated based on the particular project, as they determine the outcomes for the private entity. The remaining parameters in Table 1, $\hat{\zeta}_l$ and $\check{\zeta}_l$ are used to categorize the performance of the system into discrete service-levels, while ρ_s indicates the probability of occurrence of each scenario.

Variables
The decision variables of the model are x_t which models maintenance actions, x_t takes the value of 1 when there is a maintenance operation performed in time period $t \in \mathcal{T}$ and 0 otherwise; y_t and $b_{t,\tau}$ which keep track of maintenance actions, $b_{t,\tau}$ takes the value of 1 if the number of periods elapsed after last maintenance is $\tau \in \mathcal{T}$ in time period $t \in \mathcal{T}$ and 0 otherwise; and $v_{t,s}$ and $z_{t,l,s}$ which model the performance of the system in each scenario (in terms of the continuous metric and the discrete service-levels respectively), $z_{t,l,s}$ takes the value of 1 if the system is at service-level $l \in \mathcal{L}$ in time period $t \in \mathcal{T}$ in scenario $\in \mathcal{S}$ and 0 otherwise. Variables $p_{t,s}^{(+)}$, $p_{t,s}^{(-)}$ and $p_{t,s}^{(\cdot)}$ model the financial aspects of the problem, keeping track of earnings, expenditures and available budget respectively. We introduce auxiliary variables w_t and $v_{t,s}$ to linearize products between other variables in the model.

The proposed mathematical model is expressed in Expressions (19) through (37).

$$\max \sum_{s \in \mathcal{S}} \rho_s \left(\sum_{t \in \mathcal{T}} p_{t,s}^{(+)} - p_{t,s}^{(-)} \right) \tag{19}$$

Subject to:

$$
\begin{aligned}
y_1 &= 0 && && (20)\\
y_t &= (y_{t-1}+1)(1-x_t) && \forall t \in \mathcal{T} \mid t > 1 && (21)\\
y_t &= \sum_{\tau \in \mathcal{T}} \tau b_{t,\tau} && \forall t \in \mathcal{T} && (22)\\
\sum_{\tau \in \mathcal{T}} b_{t,\tau} &= 1 && \forall t \in \mathcal{T} && (23)\\
v_{t,s} &= \sum_{\tau \in \mathcal{T}} \gamma_{\tau,s} b_{t,\tau} && \forall t \in \mathcal{T}, s \in \mathcal{S} && (24)\\
\sum_{l \in \mathcal{L}} \check{\zeta}_l z_{t,l,s} \le v_{t,s} &\le \sum_{l \in \mathcal{L}} \hat{\zeta}_l z_{t,l,s} && \forall t \in \mathcal{T}, s \in \mathcal{S} && (25)\\
\sum_{l \in \mathcal{L}} z_{t,l,s} &= 1 && \forall t \in \mathcal{T}, s \in \mathcal{S} && (26)\\
p^{(-)}_{t,s} &= \left[c^{(f)} + c^{(v)}(\bar{\gamma} - v_{t,s})\right] x_t && \forall t \in \mathcal{T}, s \in \mathcal{S} && (27)\\
p^{(+)}_{t,s} &= \alpha + f_t + \sum_{l \in \mathcal{L}} (d_l + k_l) z_{t,l,s} && \forall t \in \mathcal{T}, s \in \mathcal{S} && (28)\\
p^{(\cdot)}_{1,s} &= p^{(+)}_{1,s} - p^{(-)}_{1,s} && \forall s \in \mathcal{S} && (29)\\
p^{(\cdot)}_{t,s} &= p^{(\cdot)}_{t-1,s} + p^{(+)}_{t,s} - p^{(-)}_{t,s} && \forall t \in \mathcal{T} \mid t > 1, s \in \mathcal{S} && (30)\\
p^{(-)}_{t,s} &\le p^{(\cdot)}_{t,s} && \forall t \in \mathcal{T}, s \in \mathcal{S} && (31)\\
\sum_{t \in \mathcal{T}} p^{(+)}_{t,s} &= (1+\epsilon) \sum_{t \in \mathcal{T}} p^{(-)}_{t,s} && \forall s \in \mathcal{S} && (32)\\
x_t &\in \{0,1\} && \forall t \in \mathcal{T} && (33)\\
b_{t,\tau} &\in \{0,1\} && \forall t \in \mathcal{T}, \tau \in \mathcal{T} && (34)\\
z_{t,l,s} &\in \{0,1\} && \forall t \in \mathcal{T}, l \in \mathcal{L}, s \in \mathcal{S} && (35)\\
y_t, w_t &\in \mathbb{Z} && \forall t \in \mathcal{T} && (36)\\
v_{t,s}, u_{t,s}, p^{(+)}_{t,s}, p^{(-)}_{t,s}, p^{(\cdot)}_{t,s} &\in \mathbb{R} && \forall t \in \mathcal{T}, s \in \mathcal{S} && (37)
\end{aligned}
$$

Expression (19) states the objective function as private entity profit. Constraints (20) and (21) keep track of the number of periods since the last maintenance action. Constraints (22) and (23) transform the integer number of periods since the last maintenance (y_t) into a binary representation ($b_{t,\tau}$) stating whether, at time t, τ periods have passed since last maintenance. Constraints (24) quantifies the performance level ($v_{t,s}$) as a function of the number of periods elapsed since the last maintenance by means of parameter $\gamma_{\tau,s}$. Constraints (25) and (26) compute which of the discrete service levels can be associated to the (continuous) performance level ($v_{t,s}$) at the current period (compute $z_{t,l,s}$); this will be useful to associate such service level to its corresponding benefit. Constraints (27) and (28) describe the costs and earnings associated to the operation of the infrastructure system, while Constraints (29) and (30) model the budget based on such cash-flow. Constraints (31) and (32) guarantee that the available budget and target profit are honored, respectively. Finally, Constraints (33) through (37) define the domain of the variables.

Constraints (21) and (27) have nonlinear terms. We replace Constraints (21) using the linealization $w_t = y_{t-1}x_t$ as follows:

$$
\begin{aligned}
w_1 &= 0 && && (38)\\
w_t &\le y_{t-1} && \forall t \in \mathcal{T} \mid t > 1 && (39)\\
w_t &\ge y_{t-1} - |\mathcal{T}|(1-x_t) && \forall t \in \mathcal{T} \mid t > 1 && (40)\\
w_t &\le |\mathcal{T}| x_t && \forall t \in \mathcal{T} \mid t > 1 && (41)
\end{aligned}
$$

where $|\mathcal{T}|$ works as an upper bound for y_{t-1}. Similarly, we replace Constraints (27), by using the linearization $u_{t,s} = v_{t,s}x_t$ as follows:

$$u_{1,s} = 0 \qquad \forall s \in \mathcal{S} \tag{42}$$

$$u_{t,s} \leq v_{t,s} \qquad \forall t \in \mathcal{T} \mid t > 1, s \in \mathcal{S} \tag{43}$$

$$u_{t,s} \geq v_{t,s} - \bar{\gamma}(1 - x_t) \qquad \forall t \in \mathcal{T} \mid t > 1, s \in \mathcal{S} \tag{44}$$

$$u_{t,s} \leq \bar{\gamma}x_t \qquad \forall t \in \mathcal{T} \mid t > 1, s \in \mathcal{S} \tag{45}$$

where $\bar{\gamma}$ works as an upper bound for $v_{t,s}$. The differences between our model and the one proposed by Gómez et al. (2020) lies in the objective function expressed in Equation (19), where we optimize the expected value of the utility among scenarios in consideration. Also, constraints from (24) to (32) and (42) to (45) are modified to include the scenario set.

In stochastic optimization models risk measures are used to enable decision-makers to seek decisions that are less likely to yield a highly undesirable outcome. For us, an undesirable outcome occurs when the condition falls bellow a predefined threshold γ^* (with $0 \leq \gamma^* \leq \bar{\gamma}$) that represents a serviceability or operation limit for the system (i.e. a safety threshold or limit state). In infrastructure projects this limit is usually defined by the government, and PPP contracts can address this issue by imposing a minimum performance constraint that guarantees high service levels.

We will solve the model for three different cases, assuming different risk profiles for the private party. If the contractor is *risk prone* or *risk seeking*, no performance restriction and the model is solved as-is. Alternatively, if the contractor is *risk neutral*, constraints (46) are added to the model, guaranteeing that in average, the performance of the system in each period is above the desirable threshold. Instead, if the contractor is *risk averse* or *risk avoiding*, constraints (47) are added to the model, ensuring that for every scenario, the performance of the system is always above the desirable threshold.

$$\sum_{s \in \mathcal{S}} \rho_s v_{t,s} \geq \gamma* \qquad \forall t \in \mathcal{T} \tag{46}$$

$$v_{t,s} \geq \gamma* \qquad \forall t \in \mathcal{T}, s \in \mathcal{S} \tag{47}$$

The proposed model relies on the following assumptions: maintenance actions take place immediately and make the system as good as new, monetary parameters are defined as time indexed arrays to reflect the effect of discount rates, and inspection actions from the government are not considered. The limitations of the model arise mainly from the exclusion of the effects of organizational interactions (i.e., the relationships between the parties involved in the project). In this case, focusing only in the private contractor limits the analysis and ignores trade offs of the involved parties. The accuracy of the model to describe real situations, however, depends completely on its input information; i.e., on the parameters that describe deterioration, costs, benefits, etc.

5. Illustrative example

We provide a simple yet general illustrative example of an infrastructure project under PPPs. The focus of this section is to demonstrate the potential of the proposed methodology to produce relevant analyses for projects and policy on PPPs. In addition, we are interested in evaluating how conditions surrounding PPPs influence the decisions of the private party, and affect the system's performance and associated economic outcomes. The proposed model has the potential to provide coherent numerical results, but their relevance depends on the accuracy and completeness of the input parameters.

Consider a PPP contract between the government and a private contractor to provide a road infrastructure with a planning horizon given by a set $\mathcal{T}$ of 30 years. The performance of the road is measure from 0.0 to 1.0, and the condition of the system is classified in a set $\mathcal{L}$ of 7 service-levels. The condition of the road is measured using the Pavement Condition Index (PCI). The PCI is a numerical indicator that rates the surface condition of the pavement (ASTM International, 2018). The PCI provides a measure of the present condition of the pavement based on the distress observed on the surface of the pavement, which also indicates the structural integrity and surface operational condition. The PCI is a numerical rating of the pavement condition that ranges from 0.0 to 1.0 with 0.0 being the worst possible condition and 1.0 being the best possible condition. The ASTM International (2018) divides the PCI into seven classes shown in Table 2.

Table 2: Pavement Condition Index (PCI) rating scale

Condition	PCI range
Good	$(0.85, 1.00]$
Satisfactory	$(0.70, 0.85]$
Fair	$(0.55, 0.70]$
Poor	$(0.40, 0.55]$
Very poor	$(0.25, 0.40]$
Serious	$(0.10, 0.25]$
Failed	$[0.00, 0.10]$

Economic parameters are derived from available documentation regarding large infrastructure projects via PPPs in Colombia, while condition of the system is generated using stochastic processes. For the private entity, the earnings come from both the fixed payment agreed with the government, and the usufruct of the road, calculated using toll rates and average expected traffic for the fourth generation (4G) concessions in Colombia. These payments will depend on the road service-level. Additionally, the cash-flow will depend on the acquired funding from financial institutions. The costs incurred by the private contractor due to maintenance actions are project-specific and not available in public data. In this case, these values were set as a ratio with respect to the income. The discount rate was chosen based on literature analysis.

In the remainder of this section we implement the model, methodologies and analysis from Section 3, we generate realizations of the deterioration process in 5.1, and present the results for the described illustrative example in 5.2.

5.1. Realizations of the deterioration process

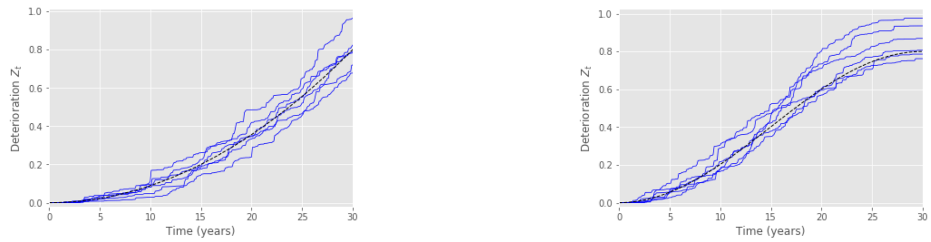
Roads are permanently exposed to the combined action of traffic loading and climatic effects. According to Paterson (1987) the main causes of pavement deterioration are traffic loading and thermal movement. Traffic causes repeated flexing of the pavement leading to fatigue, crazing and structural failure. Changes in temperature, between night and day, and seasonally, cause expansion and contraction of the carriageway.

For the deterioration process of the road, we included the following two sources. First, pavement deterioration, caused by three main processes: fatigue cracking in asphalt layers (caused by the repetition of traffic loads), permanent deformation or rutting in unbounded layers, and low temperature cracking in the asphalt course layer. And second, earthquake damage, generated when a civil infrastructure is subjected to a sudden acceleration which causes large inertial forces resulting in structural damage. These sources were modeled as progressive and shock-based deterioration processes, respectively.

We present the appropriate input parameters needed to replicate the behavior of the system, and show simulated sample paths of various Lévy deterioration models for each type of deterioration in consideration.

5.1.1. Progressive deterioration

Progressive deterioration is modeled using a non-stationary gamma process. The model was calibrated so the road (in the mean) reaches 80% of the total deterioration after 30 years, following the prediction of the pavement performance provided by Hong & Wang (2003). The parameters of the GP model were obtained matching the mean function of the degradation process. We generated sample paths from a quadratic and a sigmoid mean function. Sample paths generated are shown in Figure 1, where the mean of the deterioration process is indicated with a black dashed line.



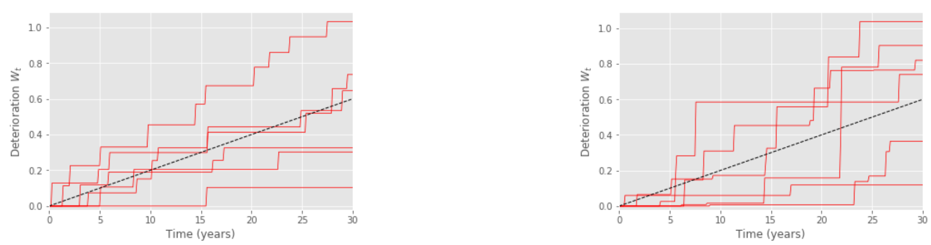
(a) GP with quadratic mean function

(b) GP with sigmoid mean function

Figure 1: Sample paths of progressive deterioration models described by gamma processes

5.1.2. Shock-based deterioration

Shock-based deterioration is modeled using a stationary compound Poisson process, with shock sizes distributed $H(\cdot)$ supported in $\mathbb{R}^+$. We generate sample paths with exponential and log-normal shock-sizes, the parameters of this distributions are calibrated to have the same mean. The Poisson rate for the CPP model is estimated from the expected number of earthquakes per year from Aktas et al. (2009). We show the generated sample paths in Figure 2, where the mean of the deterioration process is indicated with a black dashed line.



(a) CPP with log-normal distributed shock sizes

(b) CPP with exponential distributed shock sizes

Figure 2: Sample paths of shock-based deterioration models described by compound Poisson processes

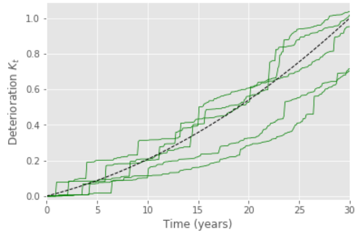
5.1.3. Combined deterioration

Combined deterioration is modeled using a non-stationary GP (progressive), plus a stationary CPP (shock-based). We generate sample paths combining the models mentioned above. We show the generated sample paths in Figure 3, where the mean of the deterioration process is indicated with a black dashed line.

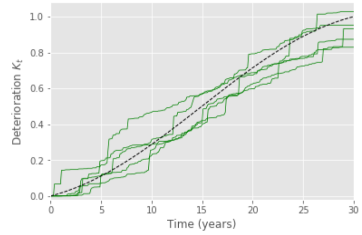
For this case study, we consider a road that is exposed to progressive or combined deterioration. Paths generated only via shock-based models were not used as its unrealistic to assume that a road will only suffer this type of degradation. In order to generate sample paths of the condition of the system V_t we use Equation (3), where $V_0 = 1.0$ represents the condition of the system when it is new, and D_t represents the total deterioration, which will be equal to Z_t in a progressive model, or to K_t in a combined model.

5.2. Results

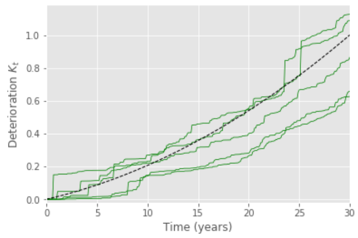
We performed numerical experiments in order to demonstrate the behavior of the model and the usefulness of the methodology to carry out diverse analyses. First, we verify that the model performs as expected. Then, we prove the impact of the size of the scenario set in the solution. Finally, we describe and present results for different instances, using the SAA method.



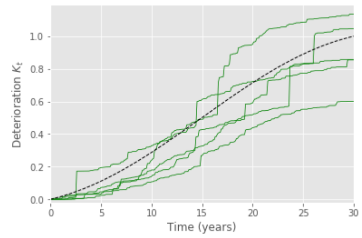
(a) GP with quadratic mean function plus a CPP with log-normal shock sizes



(b) GP with sigmoid mean function plus a CPP with log-normal shock sizes



(c) GP with quadratic mean function plus a CPP with exponential shock sizes

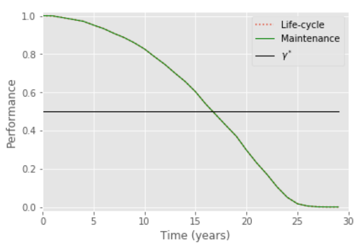


(d) GP with sigmoid mean function plus a CPP with exponential shock sizes

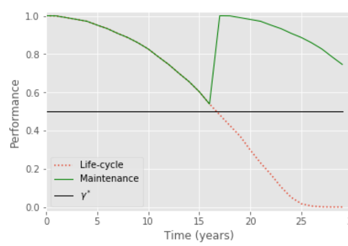
Figure 3: Sample paths of combined deterioration models described by a gamma process plus a compound Poisson process

5.2.1. Verifying the behavior of the model

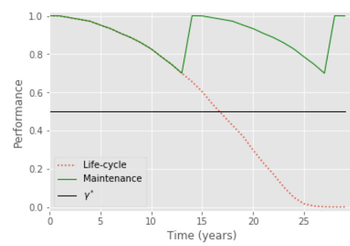
We test the model with small instances, to verify if it behaves as expected. We generate 10 scenarios for progressive and combined deterioration processes. We present our results in Figures 4 and 5, where the black line represents the government's minimum performance threshold for the road, the red dashed line represents the average life-cycle of the system without maintenance actions, and the green line represents the average condition of the system when the optimal maintenance policy is executed.



(a) Risk *prone* profile



(b) Risk *neutral* profile

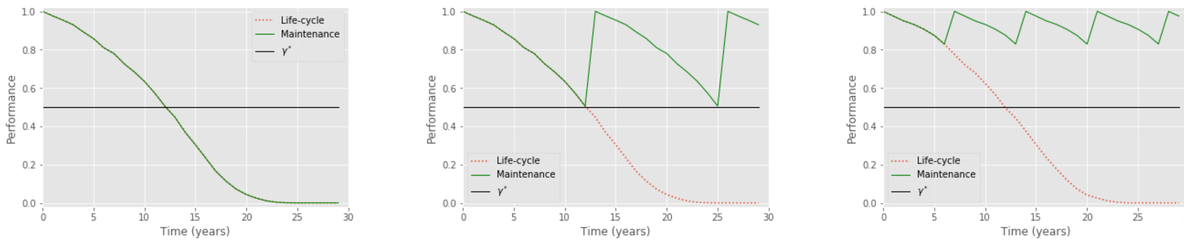


(c) Risk *averse* profile

Figure 4: Life-cycle of the road under different risk profiles considering progressive deterioration scenarios

Figure 4 shows the performance of the road under progressive deterioration for different risk profiles. In (a) the minimum performance constraint is not imposed, and the private contractor chooses not to apply any maintenance action. In (b) the minimum performance constraint is imposed in average, and the private contractor decides to apply one maintenance action to strictly satisfy the constraint. In (c) the minimum performance constraint is imposed for each scenario, and the private contractor performs two maintenance actions to satisfy the constraint even in the worst-case.

Figure 5 shows the performance of the road under combined deterioration for different risk profiles. The behavior of the private contractor is the same, however, more maintenance actions are required due to deterioration process being faster when shocks are considered.

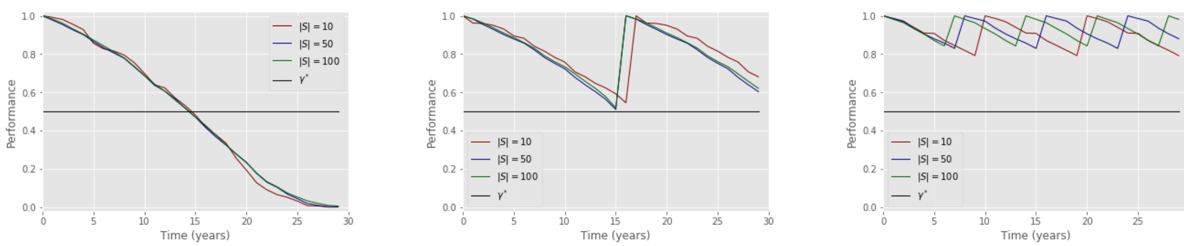


(a) Risk *prone* profile **(b)** Risk *neutral* profile **(c)** Risk *averse* profile

Figure 5: Life-cycle of the road under different risk profiles considering combined deterioration scenarios

5.2.2. Proving the impact of the size of the scenario set

We prove the impact of the number of scenarios in the optimal maintenance policy. To accomplish that goal, we run the DE assuming a combined deterioration process for different risk profiles. We increase the size of scenario set in order to asses the change in the optimal solution. We limit the scenario set at 100, due to the computational effort needed to solve each problem.



(a) Risk *prone* profile **(b)** Risk *neutral* profile **(c)** Risk *averse* profile

Figure 6: Life-cycle of the road under different risk profiles considering various sizes of the scenario set

As shown in Figure 6 different maintenance plans can be obtained depending on the size of the scenario set. In (a) the maintenance policy is the same. In (b) the mean between scenarios is similar, and the maintenance policy generated with 50 and 100 scenarios is the same, but different from the one generated with 10 scenarios. Instead, in (c) the worst-case scenario is different in each set, generating multiple maintenance policies. These results show the need to consider a large number of scenarios in order to obtain a maintenance policy that is feasible for most of the possible realizations of the deterioration process.

5.2.3. Including the SAA method to solve the model

As mentioned earlier, the computational effort needed to solve the model increases at least linearly, and often exponentially, in the scenario set size. With this in mind, we adopt the SAA method to obtain a maintenance policy in response to all (or almost all) realizations of the generated deterioration processes.

In this case study, the complete recourse assumption is violated because the maintenance plan obtained with a small scenario set, could be infeasible for a larger set where worst realizations of the deterioration process appear. To solve that issue, we include slack variables in the model used to estimate the sample average value, to capture the number of periods (in average or per scenario) where the obtained policy did not satisfy the constraint associated with minimum performance. These variables were included in the objective function with a negative cost representing a penalty, paid to the government, for exceeding the established limit.

For the implementation, we start with sample sizes $N = 10$, $N' = 500$ and $N'' = 1000$, and a number of replications $M = 10$. We also set the maximum number of SAA replications with the same sample size in $M^+ = 100$, where the probability that an additional SAA replication would produce a better solution is less than 1%. We set the maximum sample size in $N^+ = 100$, where each replication (run) takes about 30 seconds in a personal computer (Intel Core i7 @ 2.20 GHz; 16GB RAM), although runs differ in execution time.

We run the algorithm with M replications of sample size N, and if the optimality gap estimate was greater than a specified tolerance, we updated $M = M + 10$ until M^+ SAA replications had occurred. If $M = M^+$ and the optimality gap estimate was greater than a specified tolerance, then the sample size was updated $N = N + 10$ and the procedure was repeated until $N = N^+$. If both limits were reached, the algorithm stops and return the best policy found. We set a tolerance of 1% for the estimated relative optimality gap.

Table 3 presents solution values, optimality gaps (absolute and relative), and their standard deviations for all of the problems we considered, for both neutral and risk averse profiles. Instances 1 and 2 represent progressive deterioration with quadratic and sigmoid mean functions respectively. Instances 3 and 4 represent combined deterioration, with log-normal shock sizes for the shock-based model and quadratic and sigmoid mean functions for the progressive model respectively. Instances 5 and 6 represent combined deterioration, with exponential shock sizes for the shock-based model and quadratic and sigmoid mean functions for the progressive model respectively. The optimal solution found is denoted by $\hat{x}^*$ and generated policies are shown in Figure 7.

Table 3: Solution values, optimality gaps (absolute and relative) and estimated standard deviations for neutral and risk averse profiles

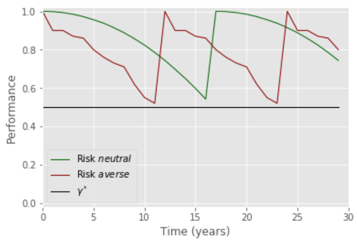
	Risk *neutral*				Risk *averse*			
Instance	$\hat{z}_{N''}(\hat{x}^*)$	Gap	Gap (%)	$\hat{\sigma}_{\text{Gap}}$	$\hat{z}_{N''}(\hat{x}^*)$	Gap	Gap (%)	$\hat{\sigma}_{\text{Gap}}$
1	445.45	0.87	0.19	1.78	412.13	2.02	0.49	2.18
2	419.80	0.60	0.14	1.14	372.97	1.78	0.47	5.73
3	427.71	2.95	0.69	7.15	278.27	3.99	1.41	11.82
4	403.23	1.23	1.04	3.34	276.68	3.06	1.09	8.20
5	424.44	3.53	0.83	14.82	271.10	4.28	1.56	26.53
6	394.80	4.83	1.21	14.48	255.42	5.77	2.13	23.25

From Table 3 it is clear that the objective function (in monetary units) decreases if the private contractor adopts an averse risk profile. Also, instances where only progressive deterioration is considered result in a greater profit, as less maintenance actions are required. All six runs reached the maximum number of replications with the larger sample size. In those cases where the performance of the best solution exceeded the tolerance, it was by a relatively small margin. This happens on instances were combined deterioration is considered, which is to be expected due to the incorporation of shocks with random sizes. Also, the estimated standard deviations of these estimated gaps are large for instances 5 and 6, where combined deterioration considered exponential shock sizes; this is explained by the variability of samples generated each replication.

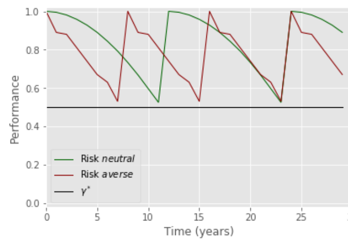
In Figure 7 we observe three lines created from 1,000 independent scenarios. The black line represents the government's minimum performance threshold for the road. The green line represents the average condition of the system (each period), among all scenarios, when risk neutral optimal maintenance policy is executed. Finally, the red line represents the minimum condition of the system (each period), among all scenarios, when risk averse optimal maintenance policy is executed. We can see that, in both cases, the private contractor decides to apply the necessary maintenance actions to fulfill the requirements imposed. As expected, instances 3 to 6, where combined deterioration is considered, had a faster deterioration process and required more interventions during the time horizon. Also, instances where progressive deterioration is modeled with a sigmoid mean function (instances 2, 4 and 6) required more maintenance actions that the ones modeled with a quadratic mean function (instances 1, 3 and 5).

6. Conclusions

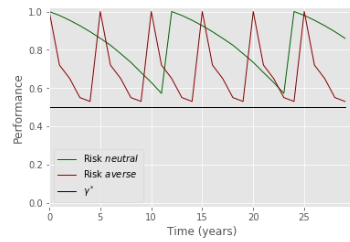
We proposed an optimization-based methodology to address the maintenance problem in infrastructure projects via PPPs, particularly focusing on road maintenance, adopting a life-cycle performance



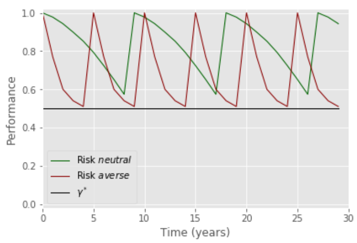
(a) Instance 1: GP with quadratic mean function



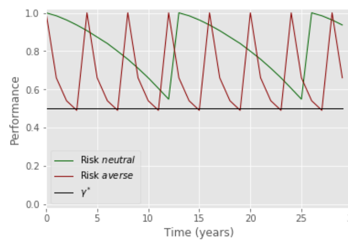
(b) Instance 2: GP with sigmoid mean function



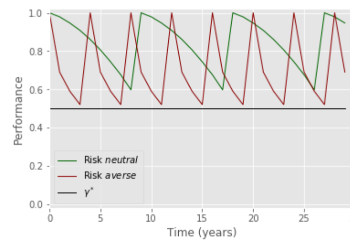
(c) Instance 3: GP with quadratic mean function plus a CPP with log-normal shock sizes



(d) Instance 4: GP with sigmoid mean function plus a CPP with log-normal shock sizes



(e) Instance 5: GP with quadratic mean function plus a CPP with exponential shock sizes



(f) Instance 6: GP with sigmoid mean function plus a CPP with exponential shock sizes

Figure 7: Life-cycle of the road obtained with the optimal maintenance policies generated for neutral and risk averse profiles

framework and modeling the deterioration process of the system as a subordinator (i.e., increasing Lévy process). To reduce the computational burden of our approach, we solve the stochastic optimization model using the SAA method by Monte Carlo simulation. We build a set of scenarios, realizations of the deterioration process, generating sample paths for progressive and combined degradation, where parameters were calibrated in order to replicate the expected behavior of the system. We integrated the generated scenario set into the stochastic optimization model. The proposed model describes the performance of the system as a function of the deterioration process, and the performed maintenance actions that restore the system's condition to a desirable service-level, seeking to maximize the profit of the private party under specific conditions fixed by the public agent. To solve the resulting problem, sampling techniques were required in order to reduce the scenario set into a manageable size. The main contribution of this work is the inclusion of the uncertainly associated with the deterioration of the system in the decision-making process, with the purpose of finding the optimal policy for a comprehensive set of possible realizations. An illustrative example was presented to demonstrate the usefulness of the proposed methodology and display key results.

This research attempts to provide a quantitative tool to support the decision-making process in infrastructure projects that operate via PPPs, where the system is subject to multiple sources of deterioration and stakeholders need to make decisions without a-priori knowledge of all information. The methodology is able to estimate the eventual maintenance costs given a specific initial design (and stochastic process that describes the deterioration of the system). The fundamental questions addressed with this work are: how does a maintenance plan look if the infrastructure system is subject to multiple sources of deterioration unknown at the beginning of the time horizon? How much does the private party need to pay in order to accomplish the specifications fixed by the public agency assuming different risk profiles? Computational experiments confirm the expected behavior of the system and emphasize the importance of sampling techniques to solve problems where the optimal decisions depends on the number of scenarios in consideration, and robust decisions are needed to fulfil the requirements imposed by the government under different reliability requirements (e.g., comply with metrics on average, on all cases, etc.). This methodology offers the possibility of performing further analysis in response to multiple questions. Through sensitivity analysis the public agency could estimate the limit up to which a contract is attractive for the private party, or assess how much it can demand from the counterpart in order to achieve economical and societal benefits. Risk profiles can be approached from different perspectives depending on parameters in consideration, to evaluate the viability of the project.

Future work should be directed towards addressing the following limitations. In our methodology we assume that the private contractor needs to define a maintenance plan at the beginning of the project, which is not completely realistic, as the contractor may decide at the start of each period whether or not carry out maintenance with knowledge of the information available up to that time and the expectations about the future. This issue could be addressed by reformulating the problem into a multi-stage stochastic program, and applying novel approaches to solve it, such as the Stochastic Dual Dynamic Programming (SDDP) method where the recourse cost function is progressively approximated by using cutting planes; or adopting the Markov decision process (MDP) framework that has found success in informing optimal decision-making under uncertainty when decisions are made sequentially over time. Also, in our model we are not considering the decisions made by the public agent, so it is not possible to capture the interaction between the involved parties, where each one pursues their own benefit. This problem can be managed by reformulating the proposed model into a bi-level optimization program. Under this framework, the Principal-Agent problem is easily addressed by means of incentive mechanisms, and the optimization model can simultaneously consider the coupled decisions of both parties. This approach allows both actors to respond to a shared set of rules, but recognizes that each can act in pursuit of their own objectives.

References

Aktas, S., Konsuk, H., & Yîgîter, A. (2009). Estimation of change point and compound poisson process parameters for the earthquake data in turkey. *Environmetrics*, *20*, 416–427. URL: `https://doi.org/10.1002/env.937`. doi:`10.1002/env.937`.

ASTM International (2018). Standard practice for roads and parking lots pavement condition index surveys. URL: `https://doi.org/10.1520/d6433-18`. doi:`10.1520/d6433-18`.

Bertoin, J. (2001). Some elements on lévy processes. In *Handbook of Statistics* (pp. 117–144). Elsevier. URL: `https://doi.org/10.1016/s0169-7161(01)19007-3`. doi:`10.1016/s0169-7161(01)19007-3`.

Ekblom, J., & Blomvall, J. (2020). Importance sampling in stochastic optimization: An application to intertemporal portfolio choice. *European Journal of Operational Research*, *285*, 106–119. URL: `https://doi.org/10.1016/j.ejor.2019.01.013`. doi:`10.1016/j.ejor.2019.01.013`.

Frangopol, D. M. (2011). Life-cycle performance, management, and optimisation of structural systems under uncertainty: accomplishments and challenges. *Structure and Infrastructure Engineering*, *7*, 389–413. URL: `https://doi.org/10.1080/15732471003594427`. doi:`10.1080/15732471003594427`.

Frangopol, D. M., Kallen, M.-J., & Noortwijk, J. M. v. (2004). Probabilistic models for life-cycle performance of deteriorating structures: review and future directions. *Progress in Structural Engineering and Materials*, *6*, 197–212. URL: `https://onlinelibrary.wiley.com/doi/abs/10.1002/pse.180`. doi:`10.1002/pse.180`.

Frangopol, D. M., & Liu, M. (2007). Maintenance and management of civil infrastructure based on condition, safety, optimization, and life-cycle cost. *Structure and Infrastructure Engineering*, *3*, 29–41. URL: `https://doi.org/10.1080/15732470500253164`. doi:`10.1080/15732470500253164`.

Gorjian, N., Ma, L., Mittinty, M., Yarlagadda, P., & Sun, Y. (2010). A review on degradation models in reliability analysis. In *Engineering Asset Lifecycle Management* (pp. 369–384). Springer London.

Gómez, C., Castiblanco, D., Lozano, J. M., & Daza, C. (2020). An exact optimization approach to the principal-agent problem in infrastructure projects via PPPs. *International Journal of Construction Management*, *0*, 1–11. URL: `https://doi.org/10.1080/15623599.2020.1757572`. doi:`10.1080/15623599.2020.1757572`.

Ho, S. P. (2006). Model for financial renegotiation in public-private partnership projects and its policy implications: Game theoretic view. *Journal of Construction Engineering and Management*, *132*, 678–688. URL: https://doi.org/10.1061/(asce)0733-9364(2006)132:7(678). doi:10.1061/(asce)0733-9364(2006)132:7(678).

Homem-de-Mello, T., & Bayraksan, G. (2014). Monte carlo sampling-based methods for stochastic optimization. *Surveys in Operations Research and Management Science*, *19*, 56 – 85. URL: http://www.sciencedirect.com/science/article/pii/S1876735414000038. doi:https://doi.org/10.1016/j.sorms.2014.05.001.

Hong, H., & Wang, S. (2003). Stochastic modeling of pavement performance. *International Journal of Pavement Engineering*, *4*, 235–243. URL: https://doi.org/10.1080/10298430410001672246. doi:10.1080/10298430410001672246.

Kleywegt, A. J., Shapiro, A., & Homem-de-Mello, T. (2002). The sample average approximation method for stochastic discrete optimization. *SIAM Journal on Optimization*, *12*, 479–502. URL: https://doi.org/10.1137/s1052623499363220. doi:10.1137/s1052623499363220.

Kumar, R., Cline, D. B., & Gardoni, P. (2015). A stochastic framework to model deterioration in engineering systems. *Structural Safety*, *53*, 36 – 43. URL: http://www.sciencedirect.com/science/article/pii/S016747301400112X. doi:https://doi.org/10.1016/j.strusafe.2014.12.001.

Leruth, L. E. (2009). Public-private cooperation in infrastructure development: A principal-agent story of contingent liabilities, fiscal risks, and other (un)pleasant surprises. *Networks and Spatial Economics*, *12*, 223–237. URL: https://doi.org/10.1007/s11067-009-9112-0. doi:10.1007/s11067-009-9112-0.

Liu, M., & Frangopol, D. M. (2005). Multiobjective maintenance planning optimization for deteriorating bridges considering condition, safety, and life-cycle cost. *Journal of Structural Engineering*, *131*, 833–842. URL: https://doi.org/10.1061/(asce)0733-9445(2005)131:5(833). doi:10.1061/(asce)0733-9445(2005)131:5(833).

Lozano, J.-M., & Sánchez-Silva, M. (2019). Improving decision-making in maintenance policies and contract specifications for infrastructure projects. *Structure and Infrastructure Engineering*, *15*, 1087–1102. URL: https://doi.org/10.1080/15732479.2019.1581818. doi:10.1080/15732479.2019.1581818.

Mak, W.-K., Morton, D. P., & Wood, R. (1999). Monte carlo bounding techniques for determining solution quality in stochastic programs. *Operations Research Letters*, *24*, 47 – 56. URL: http://www.sciencedirect.com/science/article/pii/S0167637798000546. doi:https://doi.org/10.1016/S0167-6377(98)00054-6.

Ng, S. T., Xie, J., Cheung, Y. K., & Jefferies, M. (2007). A simulation model for optimizing the concession period of public–private partnerships schemes. *International Journal of Project Management*, *25*, 791–798. URL: https://doi.org/10.1016/j.ijproman.2007.05.004. doi:10.1016/j.ijproman.2007.05.004.

Páez-Pérez, D., & Sánchez-Silva, M. (2016). A dynamic principal-agent framework for modeling the performance of infrastructure. *European Journal of Operational Research*, *254*, 576–594. URL: https://doi.org/10.1016/j.ejor.2016.03.027. doi:10.1016/j.ejor.2016.03.027.

Paterson, W. D. (1987). Road deterioration and maintenance effects: models for planning and management. http://documents.worldbank.org/curated/en/222951468765265396/Road-deterioration-and-maintenance-effects-models-for-planning-and-management.

Riascos-Ochoa, J. (2016). *The Lévy-based framework for deterioration modeling, reliability estimation and maintenance of engineered systems*. Ph.D. thesis Universidad de los Andes.

Riascos-Ochoa, J., Sánchez-Silva, M., & Klutke, G.-A. (2016). Modeling and reliability analysis of systems subject to multiple sources of degradation based on lévy processes. *Probabilistic Engineering Mechanics*, *45*, 164 – 176. URL: http://www.sciencedirect.com/science/article/pii/S0266892016300704. doi:https://doi.org/10.1016/j.probengmech.2016.05.002.

Santander, C. F., & Sánchez-Silva, M. (2008). Design and maintenance programme optimization for large infrastructure systems. *Structure and Infrastructure Engineering*, *4*, 297–309. URL: https://doi.org/10.1080/15732470600819104. doi:10.1080/15732470600819104.

Shapiro, A. (2003). Monte carlo sampling methods. In *Handbooks in Operations Research and Management Science* (pp. 353–425). Elsevier. URL: https://doi.org/10.1016/s0927-0507(03)10006-0. doi:10.1016/s0927-0507(03)10006-0.

Shapiro, A., Dentcheva, D., & Ruszczyński, A. (2009). *Lectures on Stochastic Programming: Modeling and theory*. Society for Industrial and Applied Mathematics. URL: https://doi.org/10.1137/1.9780898718751. doi:10.1137/1.9780898718751.

Song, J., Zhao, Y., Jin, L., & Sun, Y. (2018). Pareto optimization of public-private partnership toll road contracts with government guarantees. *Transportation Research Part A: Policy and Practice*, *117*, 158–175. URL: https://doi.org/10.1016/j.tra.2018.08.019. doi:10.1016/j.tra.2018.08.019.

Sánchez-Silva, M., Klutke, G.-A., & Rosowsky, D. V. (2011). Life-cycle performance of structures subject to multiple deterioration mechanisms. *Structural Safety*, *33*, 206 – 217. URL: http://www.sciencedirect.com/science/article/pii/S0167473011000105. doi:https://doi.org/10.1016/j.strusafe.2011.03.003.

Sánchez-Silva, M., Klutke, G.-A., & Rosowsky, D. V. (2012). Optimisation of the design of infrastructure components subject to progressive deterioration and extreme loads. *Structure and Infrastructure Engineering*, *8*, 655–667. URL: https://doi.org/10.1080/15732471003686157. doi:10.1080/15732471003686157.

Verweij, B., Ahmed, S., Kleywegt, A. J., Nemhauser, G., & Shapiro, A. (2003). The sample average approximation method applied to stochastic routing problems: A computational study. *Computational Optimization and Applications*, *24*, 289–333. URL: https://doi.org/10.1023/a:1021814225969. doi:10.1023/a:1021814225969.

Wang, Z., Zhang, Y., Wu, Q., Fu, H., Liu, C., & Krishnaswamy, S. (2016). Degradation reliability modeling based on an independent increment process with quadratic variance. *Mechanical Systems and Signal Processing*, *70-71*, 467 – 483. URL: http://www.sciencedirect.com/science/article/pii/S0888327015003830. doi:https://doi.org/10.1016/j.ymssp.2015.08.021.