Robust Model Reference Adaptive Control

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Abstract—**We propose a new model reference adaptive control algorithm and show that it provides the robust stability of the resulting closed-loop adaptive control system with respect to unmodeled plant uncertainties. The robustness is achieved by using a *relative error signal* in combination with a dead zone and a projection in the adaptive law. The extra *a priori* information needed to design the adaptive law, are bounds on the plant parameters and an *exponential bound on the impulse response of the inverse plant transfer function*.**

I. Introduction

ADAPTIVE control algorithms are designed with the assumption that the plant dynamics are exactly those of one member of a specified class of models. It is then natural to ask how the adaptive control system will behave when, as is inevitable in practice, the true plant is not perfectly described by any model in the given class. If the stability of the adaptive control system is guaranteed, provided only that the modeling error is sufficiently small in some sense, then we say that the adaptive control algorithm is robust, and we speak of robust stability. It is clear that robust stability is very important for the practical applicability of adaptive control algorithms.

Unfortunately, a stable adaptive control algorithm is not necessarily robustly stable (e.g., [1]). The reason is that the modeling error signal appears as a disturbance in the adaptive law and may cause the divergence of the adaptive process. The fact that the disturbance is correlated with the plant input and output signals and, in addition, is of the same order of magnitude, is part of the complexity of the robustness problem.

As a first step towards robustness results, the stability of adaptive control systems in the presence of *bounded external* disturbances has been investigated by several authors [2]–[8]. These investigations were prompted by observations (e.g., [3]) showing that a bounded external disturbance, even an asymptotically vanishing one, can cause the divergence of the adaptive process, and thereby instability. To prevent the latter, four main approaches have been made.

In the first approach (e.g., [2], [4], [6]–[8]) a dead zone is used in the adaptive law so that adaptation takes place only when the identification error exceeds a certain threshold. If the disturbance is bounded below this threshold, then it can be shown that the adaptation is always in the "right" direction and closed-loop system stability is achieved. In order to choose the size of the dead zone appropriately, a bound on the disturbance must be known. In the second approach (e.g., [2], [5]), a modification of the adaptive law is used, which comes into operation only when the norm of the estimated controller parameters exceeds a certain value and has the effect that the parameter estimates remain bounded for all time. Closed-loop system stability is thus obtained in the presence of bounded disturbances of arbitrary, unknown size. In this case a bound on the norm of the desired (unknown) controller parameters must be known. In the third approach, a σ-modification, i.e., an adaptive law of the form $\dot{\theta} = -\sigma\theta - \cdots$, is suggested and analyzed [10]. Again, if the disturbance is known to be bounded, closed-loop system stability is obtained. For plants of relative degree greater than two, this also requires the knowledge of a bound on the norm of the (unknown) desired controller parameters. In the fourth approach (e.g., [2], [11]–[13]), the idea is to produce persistency of excitation in order to make the adaptive control system exponentially stable, and to obtain stability in the presence of a bounded disturbance as a consequence of the exponential stability. However, since the persistency of excitation is to be produced by an external probing signal, and since the disturbance can counteract the excitation, the disturbance must in some sense be bounded below the probing signal [9]. The question of how the probing signal can be chosen to ensure the persistency of excitation in the presence of the disturbance has not been completely resolved as yet. In summary, we point out that in all four approaches the proof of stability depends crucially on the *a priori* boundedness of the external disturbance.

In the robustness problem, the disturbance is internally generated and thus depends on the actual plant input and output signals. In particular, if the adaptive control system were unstable and the plant input and output signals were to grow without bound, then the disturbance, caused by the model-plant mismatch, would also grow without bound. In other words, the robust stability problem becomes the problem of an internal, signal-dependent, and thus *potentially unbounded* disturbance. Therefore, when proving robust stability, boundedness of the disturbance cannot be assumed *a priori*, and, consequently, the aforementioned approaches for bounded disturbances do not necessarily solve the robustness problem.

In spite of this intrinsic difficulty, a number of robustness results have been obtained in the literature. For example, [9] shows the robust local stability for an indirect adaptive regulator. In [10] it is shown that the σ-modification discussed above yields the local stability of a model reference adaptive control scheme when the plant is a linear system of relative degree one and has unmodeled parasitics. Furthermore, robustness as a consequence of persistence of excitation through an external probing signal is shown in [11]–[15]. The results which have been proved in this approach are also local in nature because, as discussed above, a sufficiently large disturbance could invalidate the persistency of excitation assumption. A proof or a technique to ensure persistent excitation regardless of the size of the disturbance is still an unresolved problem. Another interesting approach, made for an indirect adaptive control scheme in [16], is to use a modified signal normalization, which suitably bounds the modeling error signal, and a projection, which keeps the parameter estimates bounded, in the adaptive law. Robust (global) stability is then shown subject to the assumptions that bounds on the unknown plant parameters are known and the estimated plant is uniformly controllable and observable. In [17] a robust indirect adaptive control approach is made, which uses a similar, modified normalizing signal in combination with a dead zone (relative dead zone) and an asymptotic projection in the adaptive law. Robust (global) stability is then demonstrated subject to the assumption that the unknown plant parameters lie in a known convex set of the parameter space, throughout which no unstable pole-zero cancellation occurs.

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The present paper extends the ideas of [17] to the model reference adaptive control problem. A direct model reference adaptive controller is given which ensures the robust (global) stability of the adaptive control system, assuming that bounds on the unknown plant and controller parameters and on the plant zeros are known. To achieve the robust stability a *relative* dead zone and a projection are used in the adaptive law. Relative dead zone means that the dead zone acts on a suitably normalized, relative identification error, in contrast to the absolute errors, which were used in case of bounded disturbances. A signal normalization as in [16], [17] is used to define the relative identification error. It is shown that if the model-plant mismatch is sufficiently small, then the relative modeling error signal is within the dead zone, and the adaptive law causes the parameter error to decrease monotonically as in the bounded disturbance case. However, the adaptive law can now only guarantee that the *relative* identification error becomes smaller than the dead zone eventually, and hence, if the closed-loop system were unstable, the absolute identification error would still grow without bound. A crucial step in the proof of stability is to carry this potentially unbounded identification error in such a way "backwards through the model" that the stability can be concluded and an expression for the admissible size of the dead zone is obtained. To achieve this, the above-mentioned *a priori* knowledge of bounds on the unknown plant and controller parameters and on the plant zeros appears to be necessary. In principle, these bounds can be extended arbitrarily to approach the assumptions made in the ideal nonrobust case, but it is also true that their values affect the admissible size of the dead zone, and hence the degree of robustness which can be guaranteed.

The relative dead zone approach, as compared to the approach taken in [16], gives rise to quite different properties of the overall adaptive system. In [16] the potential of parameter convergence to the true values is retained in the absence of modeling errors. However, in the presence of modeling errors, even when they are arbitrarily small, the parameter estimates can drift all over the specified region and the possibility of "burst" phenomena exists. In contrast to that, in the relative dead zone approach the parameter estimates will not converge to their true values in general, even when no modeling errors are present. However, regardless of the presence of modeling errors, the parameter estimates will converge to a certain neighborhood of the true parameter values, and no drift phenomena are encountered.

In summary, the main contributions of this paper are the introduction of the relative dead zone concept as a tool for dealing with modeling errors in model reference adaptive control schemes, the exploration of the admissible size of the dead zone and the extra *a priori* information needed, and the proof of robust stability of the closed-loop adaptive control system.

II. The Plant and the Control Objective

A. Definition of the Modeling Error

Consider a discrete-time plant with measurable input $u(t)$ and output $y(t)$. In an attempt to model the input–output behavior of the plant as that of a linear, time-invariant, nth order system, define polynomials

$$A(z^{-1}) = 1 + a_1 z^{-1} + \cdots + a_n z^{-n} \tag{1a}$$

$$\begin{aligned} B(z^{-1}) &= b_0 z^{-n^*}(1 + b_1 z^{-1} + \cdots + b_{n-n^*} z^{n^*-n}) \\ &= b_0 z^{-n^*} \tilde{B}(z^{-1}) \end{aligned} \tag{1b}$$

where z^{-1} denotes the unit delay operator and $n \geqslant n^* > 0$. Associated with these polynomials define the modeling error $\eta(t)$ as

$$\eta(t) = A(z^{-1})y(t) - B(z^{-1})u(t). \tag{2}$$

Further choose $0 < \sigma_0 < 1$, and define

$$m(t) = \sigma_0 m(t-1) + |u(t-1)| + |y(t-1)|. \tag{3}$$

The modeling error is said to be relatively bounded if there exist finite $\mu \geqslant 0$ and $m(0) \geqslant 0$ such that

$$\eta(t) \leqslant \mu m(t). \tag{4}$$

Note that a relatively bounded modeling error is not guaranteed to be bounded unless the plant input and output sequences are bounded.

The modeling error is said to be relatively small if it is relatively bounded and μ is small in (4).

Example: Suppose the true plant satisfies the equation

$$y(t) = f(\underline{\xi}(t), t) \tag{5a}$$

where

$$\underline{\xi}(t) = [y(t-1), \cdots, y(t-\bar{n}), u(t-1), \cdots, u(t-\bar{n})]^T \tag{5b}$$

and $\bar{n} \geqslant n$.

To evaluate the modeling error in a general form, let

$$C(z^{-1}) = 1 + c_1 z^{-1} + \cdots + c_{\bar{n}-n} z^{n-\bar{n}} \tag{6a}$$

be an arbitrary polynomial with

$$1/C(z^{-1}) = \sum_{i=0}^{\infty} \gamma_i z^{-i} \tag{6b}$$

and the property that

$$\sum_{i=0}^{\infty} |\gamma_i| \sigma_0^{-i} = \bar{\gamma} < \infty, \tag{6c}$$

i.e., $C(z_i^{-1}) = 0$ implies $|z_i| < \sigma_0$. Combining (2) and (5a) we obtain

$$\begin{aligned} C(z^{-1})\eta(t) &= C(z^{-1})[A(z^{-1})y(t) - B(z^{-1})u(t)] \\ &= y(t) - \underline{p}^T \underline{\xi}(t) \\ &= f(\underline{\xi}(t), t) - \underline{p}^T \underline{\xi}(t) \end{aligned} \tag{7}$$

where $\underline{p}$ is a constant vector containing the coefficients of the polynomials $C(z^{-1})A(z^{-1}) - 1$ and $C(z^{-1})B(z^{-1})$. Now suppose that for all $\underline{\xi}$ and $t \geqslant 0$

$$|f(\underline{\xi}, t) - \underline{p}^T \underline{\xi}| \leqslant \epsilon_f \|\underline{\xi}\|_{\sigma_0} \tag{8}$$

where $\|\underline{\xi}\|_\sigma = \sum_{i=1}^{\bar{n}} \sigma^{i-1}(|\xi_i| + |\xi_{i+\bar{n}}|)$ and $\sigma \epsilon (0, 1]$. (In the definition of $\|\underline{\xi}\|_\sigma$, the weight σ^{i-1} has been introduced to include certain infinite-dimensional plants. For finite $\bar{n}$ the weight is of minor significance, since $\sigma^{\bar{n}-1} \|\underline{\xi}\|_1 \leqslant \|\underline{\xi}\|_\sigma \leqslant \|\underline{\xi}\|_1$.)

Further assuming $m(0) \geqslant \|\underline{\xi}(0)\|_{\sigma_0}$, we have $m(t) \geqslant \|\underline{\xi}(t)\|_{\sigma_0}$ for all $t \geqslant 0$, and it follows that

$$\begin{aligned} |\eta(t)| &= |C^{-1}(z^{-1})[f(\underline{\xi}(t), t) - \underline{p}^T \underline{\xi}(t)]| \\ &= \left| \sum_{i=0}^{\infty} \gamma_i [f(\underline{\xi}(t-i), t-i) - \underline{p}^T \underline{\xi}(t-i)] \right| \\ &\leqslant \epsilon_f \sum_{i=0}^{\infty} |\gamma_i| \cdot \|\underline{\xi}(t-i)\|_{\sigma_0} \\ &\leqslant \epsilon_f \sum_{i=0}^{\infty} |\gamma_i| \cdot m(t-i) \\ &\leqslant \epsilon_f \sum_{i=0}^{\infty} |\gamma_i| \cdot \sigma_0^{-i} m(t) \\ &\leqslant \epsilon_f \bar{\gamma} m(t) = \mu m(t). \end{aligned} \tag{9}$$

Hence, a relatively small modeling error results, if (8) holds with a sufficiently small value of ϵ_f.

To illustrate the meaning of (8), let $\bar{\underline{p}}^T\underline{\xi}(t)$ be a linear approximation of $f(\underline{\xi}(t), t)$, and rewrite the quantity $y(t)-\bar{\underline{p}}^T\underline{\xi}(t)$ as $\bar{A}(z^{-1})y(t) - \bar{B}(z^{-1})u(t)$. Then we obtain

$$\begin{aligned} f-\underline{p}^T\underline{\xi} &= [f-\bar{\underline{p}}^T\underline{\xi}]+[\bar{\underline{p}}-\underline{p}]^T\underline{\xi} \\ &= [f-\bar{\underline{p}}^T\underline{\xi}]+[(CA-\bar{A})y-(CB-\bar{B})u]. \end{aligned} \tag{10}$$

The first term on the right-hand side of (10) is due to the deviation of f from linearity, whereas the second term is due to approximating the linearization $\bar{A}$, $\bar{B}$, which is of full order $\bar{n}$, by means of a reduced, nth order linearization A, B. Obviously, the modeling error is relatively small, if the plant is almost linear and the coefficients of the polynomials $CA - \bar{A}$ and $CB - \bar{B}$ are small.

The significance of the polynomial C is that it approximates those portions to be neglected in the reduced-order linearization. The zeros of $C(z^{-1})$ are either close to the origin in the z-plane, or they indicate an approximate pole-zero cancellation of $\bar{B}(z^{-1})/\bar{A}(z^{-1})$ in $|z| < \sigma_0$.

B. Model of the Plant

Motivated by the above discussion, let us assume that the plant, which we want to control, can be represented by the equations

$$A(z^{-1})y(t)=B(z^{-1})u(t)+\eta(t) \tag{11a}$$

$$m(t)=\sigma_0 m(t-1)+|u(t-1)|+|y(t-1)| \tag{11b}$$

$$\eta(t) \leqslant \mu[m(t)+\bar{m}_0\sigma_0^t]. \tag{11c}$$

where $m(0) \geq 0$, and $\bar{m}_0$ is a sufficiently large positive constant depending only on $m(0)$ and the initial state of the plant.

The dimension n and the delay n^* of the plant model in (11a), as well as the constant $\sigma_0 \in (0, 1)$ are chosen by the designer. Such a choice is appropriate if there exist parameter vectors $\underline{a}^T = [a_1, \cdots, a_n]$, $\underline{b}^T = b_0[0, \cdots, 1, b_1, \cdots, b_{n-n*}]$ such that:

A1: $A(z^{-1})$ and $B(z^{-1})$ are relatively prime,

A2: $A(z^{-1})$, $B(z^{-1})$ result in a sufficiently small measure μ of the modeling error. (How small "sufficiently small" is, depends on $A(z^{-1})$, $B(z^{-1})$, and the control objective and will be made more precise later.)

It is further assumed known that:

A3: $\tilde{B}(z^{-1}) \neq 0$ for all $|z| > \sigma_B$, where $\sigma_B \in [0, 1)$ is a known constant,

A4: $b_0 \in [b_{0,\min}, b_{0,\max}]$, where $b_{0,\min}$, $b_{0,\max}$ are known positive constants,

A5: a bound on $\|\underline{a}\|$ is known. Note that a bound on $\|\underline{b}\|$ follows from A3 and A4.

C. Control Objective

A reference model is chosen as

$$A_M(z^{-1})y_M(t)=z^{-n*}b_M r(t) \tag{12}$$

where $b_M > 0$,

$$A_M(z^{-1})=1+a_{M,1}z^{-1}+\cdots+a_{M,n*}z^{-n*} \tag{13}$$

is a strictly stable polynomial and the reference input $r(t)$ is bounded in modulus by a known constant.

The problem is to devise an adaptive controller for the plant (11) so that, in spite of the modeling error $\eta(t)$, the adaptive control system is globally stable, and the output of the plant follows the output of the reference model closely in some sense.

III. The Model Reference Adaptive Controller

A. Structure of the Controller

Defining

$$\underline{q}^T(z^{-1})=[z^{-1}, z^{-2}, \cdots, z^{-n}] \tag{14}$$

and

$$\begin{aligned} \underline{v}_1(t)&=\underline{q}(z^{-1})u(t) \\ \underline{v}_2(t)&=\underline{q}(z^{-1})y(t), \end{aligned} \tag{15}$$

and using the notations

$$\underline{v}^T(t)=[r(t), \underline{v}_1^T(t), \underline{v}_2^T(t)] \tag{16}$$

$$\underline{\theta}^T(t)=[\theta_0(t), \underline{\theta}_1^T(t), \underline{\theta}_2^T(t)]$$

we choose the control law

$$u(t)=\underline{v}^T(t)\underline{\theta}(t). \tag{17}$$

Lemma 1: Let θ_0^*, $\underline{\theta}_1^*$, and $\underline{\theta}_2^*$ be such that

$$\theta_0^*=b_M/b_0 \tag{18a}$$

$$[1-\underline{q}^T(z^{-1})\underline{\theta}_1^*]A(z^{-1})-B(z^{-1})\underline{q}^T(z^{-1})\underline{\theta}_2^*=A_M(z^{-1})\tilde{B}(z^{-1}) \tag{18b}$$

and let $\underline{\phi}(t) = \underline{\theta}(t) - \underline{\theta}^*$. Then there exists a polynomial $P(z^{-1})$ of degree n^* such that the controller defined in (15)–(17) causes

$$A_M(z^{-1})y(t)=z^{-n*}\{b_M r(t)+b_0[\underline{v}^T(t)\underline{\phi}(t)+z^{n*}P(z^{-1})\eta(t)]\} \tag{19}$$

and the model reference control objective is satisfied if $\underline{\phi}(t) \equiv \underline{0}$ and $\eta(t) \equiv 0$, i.e., if $\underline{\theta}(t) \equiv \underline{\theta}^*$ and $\mu = 0$.

Proof: From (17) and the definition of $\underline{\phi}(t)$, we have $u(t) = \underline{v}^T(t)\underline{\theta}^* + \underline{v}^T(t)\underline{\phi}(t)$. This and (15), (16) yield

$$[1-\underline{q}^T(z^{-1})\underline{\theta}_1^*]u(t)=\theta_0^*r(t)+\underline{q}^T(z^{-1})\underline{\theta}_2^*y(t)+\underline{v}^T(t)\underline{\phi}(t), \tag{20}$$

which together with (11a) gives

$$\begin{aligned} &\{[1-\underline{q}^T(z^{-1})\underline{\theta}_1^*]A(z^{-1})-B(z^{-1})\underline{q}^T(z^{-1})\underline{\theta}_2^*\}y(t) \\ &\quad=B(z^{-1})[\theta_0^*r(t)+\underline{v}^T(t)\underline{\phi}(t)]+[1-\underline{q}^T(z^{-1})\underline{\theta}_1^*]\eta(t). \end{aligned} \tag{21}$$

Since $\tilde{B}(z^{-1})$ divides $B(z^{-1})$, it follows from (18b) that it must also divide $1 - \underline{q}^T(z^{-1})\underline{\theta}_1^*$, and we must have for some polynomial $P(z^{-1})$

$$1-\underline{q}^T(z^{-1})\underline{\theta}_1^*=b_0P(z^{-1})\tilde{B}(z^{-1}). \tag{22}$$

Substituting (22) and (18a), (18b) in (21), and recalling that $B(z^{-1}) = b_0 z^{-n*}\tilde{B}(z^{-1})$, we finally obtain (19). Q.E.D.

It is assumed as known that

A6: $\underline{\theta}^* \in [\underline{\theta}_{\min}, \underline{\theta}_{\max}]$, where $\underline{\theta}_{\min}$, $\underline{\theta}_{\max}$ are known constant vectors, in particular, $\theta_{0,\min} = b_M/b_{0,\max}$, $\theta_{0,\max} = b_M/b_{0,\min}$.

Assumption A6 essentially means that, in addition to A1, some measure of the coprimeness of $A(z^{-1})$, $B(z^{-1})$ is known. This may be seen from (18b).

B. Adaptive Law with a Relative Dead Zone

From (12) and (19) we obtain

$$(1/b_0)A_M(z^{-1})[y(t)-y_M(t)]=z^{-n*}[\underline{v}^T(t)\underline{\phi}(t)]+P(z^{-1})\eta(t). \tag{23}$$

Defining

$$\underline{w}^T(t)=[(1/b_M)z^{n*}A_M(z^{-1})y(t), \underline{v}_1^T(t), \underline{v}_2^T(t)] \tag{24}$$

$$\eta_1(t)=P(z^{-1})\eta(t) \tag{25}$$

it can be verified that the left-hand side of (23) may be written as $z^{-n*}[\underline{w}(t) - \underline{v}(t)]^T\underline{\theta}^*$. Thereby, (23) results in

$$z^{-n*}[\underline{w}^T(t)\underline{\theta}^*-\underline{v}^T(t)\underline{\theta}(t)]-\eta_1(t)=0. \tag{26}$$

Based on (26), an identification error (constructable from measured data) is defined as

$$E(t)=\underline{w}^T(t-n^*)\underline{\theta}(t)-\underline{v}^T(t-n^*)\underline{\theta}(t-n^*) \tag{27}$$

which by virtue of (26) is equivalent to

$$E(t)=\underline{w}^T(t-n^*)\underline{\phi}(t)+\eta_1(t). \tag{28}$$

Further, define a normalizing factor $N(t)$ and a relative error signal $E_1(t)$ as

$$N(t)=\gamma_0+m(t), \qquad \gamma_0>0 \tag{29}$$

$$E_1(t)=E(t)/N(t) \tag{30}$$

and, related to (27), (30),

$$E_2(t)=[\underline{v}^T(t-n^*)\underline{\phi}(t-n^*)+\eta_1(t)]\theta_0(t)/[\theta_0^*N(t)] \tag{31a}$$

$$E_3(t)=[\underline{v}^T(t-n^*)\underline{\phi}(t-n^*)]/\{N(t)[1+\theta_0^*/\theta_0(t)]\}. \tag{31b}$$

The adaptive law is then chosen as the following two subsequent steps. In step 1 set

$$\underline{\tilde{\theta}}(t+1)=\underline{\theta}(t)-\frac{\underline{w}(t-n^*)N(t)D(E_1(t))}{\gamma_1+\underline{w}^T(t-n^*)\underline{w}(t-n^*)} \tag{32a}$$

where $D(E_1)$ is the continuous function defined by (see also Fig. 1)

$$D(E_1)=\begin{Bmatrix} 0 & \text{if } |E_1| \leqslant d_0 \\ E_1-d_0 & \text{if } E_1>d_0 \\ E_1+d_0 & \text{if } E_1<d_0 \end{Bmatrix} \tag{32b}$$

and in step 2 set

$$\theta_i(t+1)=\begin{Bmatrix} \tilde{\theta}_i(t+1) & \text{if } \theta_{i,min} \leqslant \tilde{\theta}_i(t+1) \leqslant \theta_{i,\max} \\ \theta_{i,\min} & \text{if } \quad \tilde{\theta}_i(t+1)<\theta_{i,\min} \\ \theta_{i,\max} & \text{if } \quad \tilde{\theta}_i(t+1)>\theta_{i,\max} \end{Bmatrix}. \tag{32c}$$

Equation (32a) is a gradient projection type algorithm, but with the error replaced by $N\cdot D(E_1)$, where $D(E_1)$ is the dead zone function of the relative identification error. The size d_0 of the dead zone will be specified later. Equation (32c) simply takes the result of the first step and maps it into the closest point in the set $[\underline{\theta}_{\min}, \underline{\theta}_{\max}]$.

The closed-loop adaptive control system thus established comprises the plant (11), the controller (15)–(17), and the adaptive law (32).

Lemma 2: For arbitrary $d_0>0$ there exists a $\mu_0>0$ (which depends on d_0) such that for all $0 \leqslant \mu \leqslant \mu_0$ and arbitrary initial conditions, the closed-loop adaptive control system generates parameter estimates $\underline{\theta}(t)$ with the properties

a) $\|\underline{\phi}(t)\| \leqslant \text{const}.$

b) $\lim_{t\to\infty} D(E_1(t))=0$

c) $\lim_{t\to\infty} [\underline{\phi}(t+1)-\underline{\phi}(t)]=0$

d) $\lim_{t\to\infty} D(E_2(t))=0$

e) $\lim_{t\to\infty} D(E_3(t))=0.$

Proof: Part a) of the lemma is obvious from step 2 of the adaptive law.

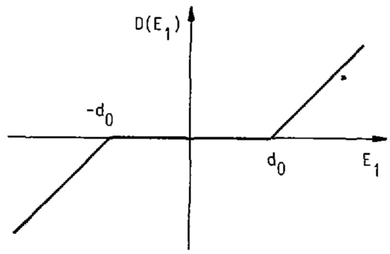

Fig. 1. The dead zone function $D(\cdot)$ of the relative identification error E_1.

To show part b) of the lemma, we have from (25) and (11c)

$$\begin{aligned}|\eta_1(t)| &= \left|\sum_{i=0}^{n^*} p_i\eta(t-i)\right| \\ &\leqslant \mu \sum_{i=0}^{n^*} |p_i| \cdot [m(t-i)+\bar{m}_0\sigma_0^{t-i}] \\ &\leqslant \mu\left(\sum_{i=0}^{n^*} |p_i| \cdot \sigma_0^{-i}\right)[m(t)+\bar{m}_0\sigma_0^{t-n^*}] \end{aligned} \tag{33}$$

where $\bar{m}_0$ is a sufficiently large positive number, which depends only on $m(0)$ and the initial state of the plant.

$$\mu_0=d_0\Big/\sum_{i=0}^{n^*}|p_i|\sigma_0^{-i} \tag{34}$$

it follows for all $0 \leqslant \mu \leqslant \mu_0$ that

$$\begin{aligned}|\eta_1(t)| &\leqslant d_0[m(t)+\bar{m}_0\sigma_0^{t-n^*}] \\ &< d_0N(t), \qquad (t \geqslant t_1)\end{aligned} \tag{35}$$

where t_1 is sufficiently large and finite. This, together with (28), (30), implies that we may write

$$D(E_1)=\alpha(t)\underline{w}^T(t-n^*)\underline{\phi}(t)/N(t) \tag{36}$$

where $0 \leqslant \alpha(t) \leqslant 1$. Therefore, (32a) reduces to

$$\underline{\tilde{\phi}}(t+1)=\left\{\underline{I}-\alpha(t)\,\frac{\underline{w}(t-n^*)\underline{w}^T(t-n^*)}{\gamma_1+\underline{w}^T(t-n^*)\underline{w}(t-n^*)}\right\}\underline{\phi}(t) \tag{37}$$

where $\underline{\tilde{\phi}}(t)=\underline{\tilde{\theta}}(t)-\underline{\theta}^*$. Letting $V(t)=\underline{\phi}^T(t)\underline{\phi}(t)$, and noting that, due to (32c) we have $\underline{\phi}^T(t+1)\underline{\phi}(t+1) \leqslant \underline{\tilde{\phi}}^T(t+1)\underline{\tilde{\phi}}(t+1)$, yields

$$\begin{aligned}V(t+1)-V(t) &\leqslant \underline{\tilde{\phi}}^T(t+1)\underline{\tilde{\phi}}(t+1)-\underline{\phi}^T(t)\underline{\phi}(t) \\ &\leqslant -\alpha^2(t)[\underline{w}^T(t-n^*)\underline{\tilde{\phi}}(t)]^2 \\ &\quad /[\gamma_1+\underline{w}^T(t-n^*)\underline{w}(t-n^*)] \\ &\leqslant -D^2(E_1)\cdot N^2(t)/[\gamma_1+\underline{w}^T(t-n^*)\underline{w}(t-n^*)] \\ &\leqslant -D^2(E_1)\cdot \text{const} \qquad (t \geqslant t_1).\end{aligned} \tag{38}$$

The constant in the last inequality of (38) follows from (29), (24), (15), and (11b). This implies part b) of the lemma.

The second inequality of (38) also implies $\alpha(t)\underline{w}^T(t-n^*)\underline{\phi}(t)/[\gamma_1+\underline{w}^T(t-n^*)\underline{w}(t-n^*)]^{1/2} \to 0$ as $t\to\infty$. Hence, $\underline{\tilde{\phi}}(t+1)-\underline{\phi}(t) \to 0$ as $t\to\infty$ and, by the continuity of (32c), part c) of the lemma follows.

To show part d) of the lemma, rewrite (28) as

$$\begin{aligned}E(t)=\underline{v}^T(t-n^*)\underline{\phi}(t-n^*)+\eta_1(t)+\underline{v}^T(t-n^*)[\underline{\phi}(t)-\underline{\phi}(t-n^*)] \\ +[\underline{w}(t-n^*)-\underline{v}(t-n^*)]^T\underline{\phi}(t).\end{aligned} \tag{39}$$

Recalling that only the first component of $w - v$ is nonzero and that $[w(t-n^*) - v(t-n^*)]^T\theta^*$ equals the left-hand side of (23), we get

$$[w(t-n^*)-v(t-n^*)]^T\phi(t)$$
$$=[w(t-n^*)-v(t-n^*)]^T\theta^*\phi_0(t)/\theta_0^*$$
$$=[v^T(t-n^*)\phi(t-n^*)+\eta_1(t)]\phi_0(t)/\theta_0^* \quad (40)$$

which can be substituted in (39) to obtain

$$E(t)=[v^T(t-n^*)\phi(t-n^*)+\eta_1(t)]\theta_0(t)/\theta_0^*$$
$$+v^T(t-n^*)[\phi(t)-\phi(t-n^*)]$$
$$=N(t)E_2(t)+v^T(t-n^*)[\phi(t)-\phi(t-n^*)]. \quad (41)$$

Dividing both sides of (41) by $N(t)$, noting that $v(t-n^*)/N(t)$ is bounded in norm by a constant, and using parts b) and c) and the continuity of $D(\cdot)$ implies part d) of the lemma.

To show part e) of the lemma, we obtain from (31a)

$$|v^T(t-n^*)\phi(t-n^*)|\cdot\frac{\theta_0(t)}{\theta_0^* N(t)} \leqslant |E_2(t)|+|\eta_1(t)|\cdot\frac{\theta_0(t)}{\theta_0^* N(t)}.$$

Multiplying this inequality by $\theta_0^*/(\theta_0^* + \theta_0(t))$ and using (35) gives

$$|E_3(t)| \leqslant |E_2(t)|\cdot\theta_0^*/(\theta_0^*+\theta_0(t))+d_0\theta_0(t)/(\theta_0^*+\theta_0(t)).$$

Since $E_2(t)$ converges to the set $[-d_0, d_0]$ by part d) of the lemma, part e) follows. Q.E.D.

IV. Proof of Robust Stability

In order to establish the robust stability of the closed-loop adaptive control system, we derive a bound on $N(t+1)/N(t)$ first. From (11b) and (29) we have

$$N(t+1)=\sigma_0 N(t)+\gamma_0(1-\sigma_0)+|y(t)|+|u(t)|. \quad (42)$$

Equation (11a) can be written in the form

$$y(t)=b^T v_1(t)-a^T v_2(t)+\eta(t), \quad (43)$$

and from (17)

$$u(t)=\theta_0(t)r(t)+\theta_1^T(t)v_1(t)+\theta_2^T(t)v_2(t). \quad (44)$$

Since (15), (11b), (29) imply

$$\|v_1(t)\|+\|v_2(t)\| \leqslant n\sigma_0^{-n}N(t), \quad (45)$$

for some finite $t_2 \geqslant t_1$, it follows that for $t \geqslant t_2$

$$N(t+1)=[\sigma_0+\sigma_0^{-n}k_1+\mu]N(t)+\gamma_0(1-\sigma_0)+|\theta_0(t)r(t)|$$
$$\leqslant [1+\sigma_0^{-n}k_1+\mu]N(t)+|\theta_0(t)r(t)|$$
$$\leqslant [1+\sigma_0^{-n}k_1+d_0k_0+k_2]N(t)=\Lambda(d_0)N(t) \quad (46)$$

where (34) was used, and

$$k_0 \geqslant 1\Big/\sum_{i=0}^{n^*}|p_i|\sigma_0^{-i} \quad (47a)$$

$$k_1 \geqslant \sup_{t\geqslant t_2}\{\|\theta_1(t)\|+\|b\|,\ \|\theta_2(t)\|+\|a\|\} \quad (47b)$$

$$k_2 \geqslant \sup_{t\geqslant t_2}\{|\theta_0(t)r(t)|/N(t)\}. \quad (47c)$$

Suitable constants k_0, k_1, k_2 can be obtained using the available prior information and the fact that $N(t) \geqslant \gamma_0$.

Second, we rewrite the input and output of the closed-loop system in a form permitting us to apply Lemma 2 and to derive simplified equations for analyzing the stability of the closed-loop system in terms of $m(t)$.

Combining (19), (25), and (31a) gives

$$y(t)=[b_M/A_M(z^{-1})]\{r(t-n^*)+E_2(t)N(t)/\theta_0(t)\}. \quad (48)$$

On the other hand, multiplying (11a) by $q^T(z^{-1})\theta_2^*$ and using (20), (18b), we get

$$A_M(z^{-1})\tilde{B}(z^{-1})u(t)=A(z^{-1})\{\theta_0^*r(t)+v^T(t)\phi(t)\}$$
$$+q^T(z^{-1})\theta_2^*\eta(t) \quad (49)$$

which together with (31b) gives

$$u(t)=\frac{A(z^{-1})}{B(z^{-1})}\cdot\frac{b_M z^{-n^*}}{A_M(z^{-1})}$$
$$\cdot\left\{r(t)+E_3(t+n^*)N(t+n^*)\left[\frac{1}{\theta_0^*}+\frac{1}{\theta_0(t+n^*)}\right]\right\}$$
$$+\frac{q^T(z^{-1})\theta_2^*}{A_M(z^{-1})\tilde{B}(z^{-1})}\eta(t). \quad (50)$$

Let the impulse responses of b_M/A_M, $b_M z^{-n^*}A/(A_M B)$, and $q^T\theta_2^*/(A_M\tilde{B})$ all be bounded in modulus by $K_c\sigma_c^t$, where $0 \leqslant \sigma_c < 1$. Suitable constants K_c, σ_c can be obtained using the available prior information.

Note that $r(t)$ is bounded by a constant, that for arbitrary $d_1 > d_0$ there exists $T \geqslant t_2$ such that $E_2(t) \leqslant d_1$ and $E_3(t+n^*) \leqslant d_1$ whenever $t \geqslant T$, and that $N(t)$, $(t \leqslant T)$ is bounded. Hence, for $t \geqslant T$ we obtain from (48)

$$|y(t)| \leqslant \text{const}+\sum_{i=T}^{t}K_c\sigma_c^{t-i}d_1N(i)/\theta_0(i)$$
$$\leqslant \text{const}+(1/\theta_{0,\min})K_cd_1\sum_{i=T}^{t}\sigma_c^{t-i}m(i) \quad (51)$$

and from (50)

$$|u(t)| \leqslant \text{const}+\sum_{i=T}^{t}K_c\sigma_c^{t-i}$$
$$\cdot\{d_1N(i+n^*)[1/\theta_0^*+1/\theta_0(i+n^*)]+\mu N(i)\}$$
$$\leqslant \text{const}+K_c[2d_1\Lambda^{n^*}(d_0)/\theta_{0,\min}+\mu]\sum_{i=T}^{t}\sigma_c^{t-i}N(i)$$
$$\leqslant \text{const}+K_c[2d_1\Lambda^{n^*}(d_0)/\theta_{0,\min}+d_0k_0]\sum_{i=T}^{t}\sigma_c^{t-i}m(i) \quad (52)$$

where we used the fact that $N(t+n^*) \leqslant \Lambda^{n^*}(d_0)N(t)$ and $\mu \leqslant \mu_0 \leqslant d_0k_0$. Further, (11b), (51), (52) yield

$$m(t+1) \leqslant \sigma_0 m(t)+K_0+K_m(d_1)\sum_{i=T}^{t}\sigma_c^{t-i}m(i) \quad (53)$$

where K_0 is a corresponding constant and

$$K_m(d_1)=d_1K_c\{[1+2\Lambda^{n^*}(d_1)]/\theta_{0,\min}+k_0\}. \quad (54)$$

Defining

$$\bar{m}(t+1)=\sigma_0\bar{m}(t)+K_0+K_m(d_1)\sum_{i=T}^{t}\sigma_c^{t-i}\bar{m}(i), \quad (t \geqslant T) \quad (55)$$

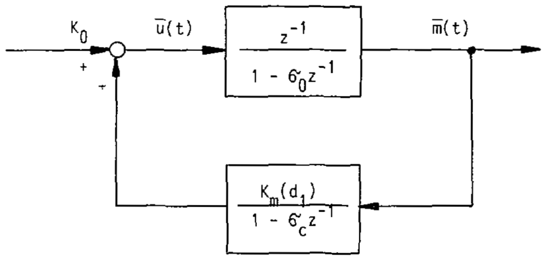

Fig. 2. Feedback configuration for the robust stability investigation.

it follows, by comparison to (53), that if $\bar{m}(T) \geqslant m(T)$ then $\bar{m}(t) \geqslant m(t)$ for all $t \geqslant T$, i.e., $\bar{m}(t)$ is an upper bound of $m(t)$. Equation (55) can also be written as

$$(z-\sigma_0)\bar{m}(t)=K_0+\{K_m(d_1)/(1-\sigma_c z^{-1})\}\bar{m}(t) \tag{56}$$

or, equivalently, as the feedback system

$$\bar{u}(t)=K_0+\{K_m(d_1)/(1-\sigma_c z^{-1})\}\bar{m}(t) \tag{57a}$$

$$\bar{m}(t)=\{z^{-1}/(1-\sigma_0 z^{-1})\}\bar{u}(t) \tag{57b}$$

which is depicted in Fig. 2.

Finally, we can investigate the stability of the feedback configuration (57). Its closed-loop characteristic equation is

$$1-[\sigma_0+\sigma_c+K_m(d_1)]z^{-1}+\sigma_0\sigma_c z^{-2}=0. \tag{58}$$

Noting that σ_0, σ_c lie in (0, 1) and $K_m(d_1)$ is positive, it is not hard to see that stability of the feedback configuration (57) is obtained if

$$K_m(d_1)<(1-\sigma_0)(1-\sigma_c). \tag{58}$$

This gives rise to the following conclusion.

If the dead zone is chosen so that $d_0 > 0$ and

$$K_m(d_0)<(1-\sigma_0)(1-\sigma_c) \tag{59}$$

then, since $K_m(\cdot)$ is a continuous function, there always exists $d_1 > d_0$ satisfying (58), which implies the stability of (57). Since $\bar{m}(t)$ is an upper bound of $m(t)$, the boundedness of $m(t)$, and in turn the boundedness of all signals of the adaptive control system follows. The result thus proved is summarized in the following theorem.

Theorem (Robust Global Stability): Consider the closed-loop adaptive control system, consisting of the plant (11), the controller (15)–(17), and the adaptive law (32) with $d_0 > 0$ satisfying (59). Then for arbitrary bounded initial conditions and arbitrary plant uncertainties $0 \leqslant \mu \leqslant \mu_0$, where μ_0 satisfies (34), all signals in the adaptive control system remain bounded.

It is seen from (12) and (48) that the error between the output of the plant and the output of the model satisfies the equation

$$y(t)-y_M(t)=[b_M/A_M(z^{-1})]\{E_2(t)N(t)/\theta_0(t)\} \tag{60}$$

where $E_2(t)$ is known to converge to the interval $[-d_0, d_0]$. Hence, the magnitude of the relative output error $[y(t) - y_M(t)]/N(t)$ will roughly be of an order which is proportional to the size d_0 of the dead zone.

V. Conclusions

A new model reference adaptive control algorithm is proposed, which ensures the robust stability of the resulting closed-loop adaptive control system. The robustness is achieved by using a relative error signal in combination with a dead zone (relative dead zone) and a projection in the adaptive law. To design the adaptive law, some extra *a priori* information about the plant is needed, which comprises bounds on the parameters of the plant (or the controller) as well as an exponential bound on the impulse response of the inverse plant transfer function.

Robustness is achieved in the sense that if the actual plant is "almost like the assumed linear, time invariant, nth order system" (in some well-defined sense), then the closed-loop will be stable, and the relative model following error will roughly be of an order which is proportional to the size of the dead zone. The plant uncertainties, which can be tolerated, include parasitic poles and zeros, approximate pole-zero cancellations in $|z| < \sigma_o$, slight parameter uncertainties [which may change the actual delay of the plant from n^* to $\bar{n}^*$, $(1 \leqslant \bar{n}^* \leqslant n^*)$], as well as slight nonlinearities and slight time variation effects.

The result thus presented can be used in two different ways. One would be as stated in the Introduction, namely to achieve robustness against unmodeled plant uncertainty. The other one would be to design reduced order (or otherwise simplified) model reference adaptive controllers, treating the purposely introduced mismatch like an uncertainty.

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