DTU

Theoretical development and FVM numerical solution

Bulk-flow equations for annular seals

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Mass and linear momentum

From BSL [Bird et al., 2002, pp.340-341]

$$\frac{\partial \rho}{\partial t} + \left(\nabla \cdot \rho \vec{v}\right) = 0 \tag{1}$$

$$\frac{\partial \rho \vec{v}}{\partial t} + \left(\nabla \cdot \rho \vec{v}\vec{v}\right) = -\nabla p + \nabla \cdot \bar{\bar{\tau}} + \rho \vec{g} \tag{2}$$

Assumption 1 : Newtonian fluid.

$$\bar{\bar{\tau}} = \mu \left(\nabla \vec{v} + \left(\nabla \vec{v}\right)^T\right) + \underbrace{\left(\kappa - 2/3\mu\right)}_{\lambda} \left(\nabla \cdot \vec{v}\right) \bar{\bar{I}} \tag{3}$$

Energy

$$\frac{\partial \rho \hat{U}}{\partial t} = -\left(\nabla \cdot \rho \hat{U} \vec{v}\right) - \nabla \cdot \vec{q} - p\left(\nabla \cdot \vec{v}\right) + \overline{\overline{\tau}} : \nabla \vec{v} \tag{4}$$

where the irreversible viscous dissipation is given by [Bird et al., 2002, p.82] (note that : denotes the double inner product)

$$\overline{\overline{\tau}} : \nabla \vec{v} = 1/2\mu \sum_i \sum_j \left[\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - 2/3\left(\nabla \cdot \vec{v}\right)\delta_{ij}\right]^2 + \kappa\left(\nabla \cdot \vec{v}\right)^2 \tag{5}$$

$$= \mu \Phi + \kappa \Psi \tag{6}$$

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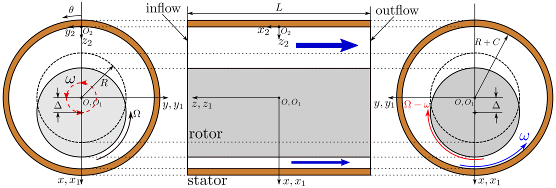
Disparity of length scales
Bearings and seals

$$H_0/L \approx 10^{-3} \tag{7}$$

Assumption 2 : Neglect curvature effects. Can then utilize Cartesian coordinates to describe the "unwrapped" bearing/seal film



Example : Cartesian x-component of the linear momentum equation

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho g_x \tag{8}$$

Non-dimensionalize equations

Non-dimensionalize equations and exploit difference in length scales

$$x = \tilde{x}L \quad y = \tilde{y}L \quad z = \tilde{z}H_0$$

$$v_x = \tilde{v}_x U_0 \quad v_y = \tilde{v}_y U_0 \quad v_z = \tilde{v}_z U_0 \frac{H_0}{L}$$

$$t = \tilde{t}t_0$$

$$\rho = \tilde{\rho}\rho_0 \quad \mu = \tilde{\mu}\mu_0 \quad \lambda = \tilde{\lambda}\lambda_0$$

$$k = \tilde{k}k_0 \quad c_p = \tilde{c}_p c_{p0} \quad T = \tilde{T}T_0$$

Dimensionless x-momentum

Introduce into conservation of mass and linear momentum.
For example, the x-momentum equation becomes

$$
\begin{aligned}
& St\, Re_{H_0} \left(\frac{H_0}{L_0}\right) \frac{\partial(\tilde{\rho}\tilde{v}_x)}{\partial \tilde{t}} + Re_{H_0} \left(\frac{H_0}{L_0}\right) \frac{\partial(\tilde{\rho}\tilde{v}_x\tilde{v}_x)}{\partial \tilde{x}} + Re_{H_0} \left(\frac{H_0}{L_0}\right) \frac{\partial(\tilde{\rho}\tilde{v}_x\tilde{v}_y)}{\partial \tilde{y}} \\
& + Re_{H_0} \left(\frac{H_0}{L_0}\right) \frac{\partial(\tilde{\rho}\tilde{v}_x\tilde{v}_z)}{\partial \tilde{z}} = -\frac{H_0^2}{U_0\mu_0}\frac{\partial p}{\partial x} + \frac{H_0^2}{L^2}\frac{\partial}{\partial \tilde{x}}\left(2\tilde{\mu}\frac{\partial \tilde{v}_x}{\partial \tilde{x}}\right) + \frac{\lambda_0 H_0^2}{\mu_0 L^2}\frac{\partial}{\partial \tilde{x}}\left[\tilde{\lambda}\left(\nabla\cdot\tilde{\vec{v}}\right)\right] \\
& + \frac{H_0^2}{L^2}\frac{\partial}{\partial \tilde{y}}\left(\tilde{\mu}\left[\frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right]\right) + \frac{H_0^2}{L^2}\frac{\partial}{\partial \tilde{z}}\left(\tilde{\mu}\frac{\partial \tilde{v}_z}{\partial \tilde{x}}\right) + \frac{\partial}{\partial \tilde{z}}\left(\tilde{\mu}\frac{\partial \tilde{v}_x}{\partial \tilde{z}}\right) + \frac{\rho_0 H_0^2}{U_0\mu_0}\tilde{\rho} g_x
\end{aligned}
$$

where $Re = \dfrac{\rho_0 U_0 H_0}{\mu_0}$ is a Reynolds number and $St = \dfrac{L}{U_0 t_0}$ is a Strouhal numbers

Simplified continuity and momentum

Within each linear momentum equation, terms with coefficients $\frac{H_0^2}{L^2}$ or smaller can be neglected.

Also applying **Assumption 3 : Neglect body force.** one obtains

$$St\frac{\partial\tilde{\rho}}{\partial t} + \left(\nabla\cdot\tilde{\rho}\tilde{\vec{v}}\right) = 0$$

$$St\,Re_{H_0}\frac{\partial(\tilde{\rho}\tilde{v}_x)}{\partial\tilde{t}} + Re_{H_0}\frac{\partial(\tilde{\rho}\tilde{v}_x\tilde{v}_x)}{\partial\tilde{x}} + Re_{H_0}\frac{\partial(\tilde{\rho}\tilde{v}_x\tilde{v}_y)}{\partial\tilde{y}} + Re_{H_0}\frac{\partial(\tilde{\rho}\tilde{v}_x\tilde{v}_z)}{\partial\tilde{z}} = -\frac{\partial\tilde{p}}{\partial\tilde{x}} + \frac{\partial\tilde{\tau}_{xz}}{\partial\tilde{z}}$$

$$St\,Re_{H_0}\frac{\partial(\tilde{\rho}\tilde{v}_y)}{\partial\tilde{t}} + Re_{H_0}\frac{\partial(\tilde{\rho}\tilde{v}_y\tilde{v}_x)}{\partial\tilde{x}} + Re_{H_0}\frac{\partial(\tilde{\rho}\tilde{v}_y\tilde{v}_y)}{\partial\tilde{y}} + Re_{H_0}\frac{\partial(\tilde{\rho}\tilde{v}_y\tilde{v}_z)}{\partial\tilde{z}} = -\frac{\partial\tilde{p}}{\partial\tilde{y}} + \frac{\partial\tilde{\tau}_{yz}}{\partial\tilde{z}}$$

$$0 = -\frac{\partial\tilde{p}}{\partial\tilde{z}}$$

Simplified energy

Applying the same procedure to the energy equation one obtains

$$Pe_{H_0}\left(\frac{H_0}{L}\right)\tilde{\rho}\tilde{c}_p\frac{D\tilde{T}}{D\tilde{t}} = \frac{\partial}{\partial\tilde{z}}\left(\tilde{k}\frac{\partial\tilde{T}}{\partial\tilde{z}}\right) + Br\left[\tilde{\tau}_{xz}\left(\frac{\partial\tilde{v}_x}{\partial\tilde{z}}\right) + \tilde{\tau}_{yz}\left(\frac{\partial\tilde{v}_y}{\partial\tilde{z}}\right)\right]$$
$$- Br\left(\frac{H_0}{L}\right)\left(\frac{\partial\log\tilde{\rho}}{\partial\log\tilde{T}}\right)_{\tilde{p}}\left[\frac{D\tilde{p}}{D\tilde{t}}\right] + \left(\frac{H_0^2}{T_0 k_0}\right)\dot{s}$$

where $Pe_{H_0} = Re_{H_0}Pr_0 = \dfrac{U_0 H_0}{\alpha_0}$ is the Péclet number, $\alpha_0 = \dfrac{k_0}{\rho c_{p0}}$ is the thermal diffusivity, $Br = \dfrac{\mu_0 U_0^2}{k_0 T_0^2}$ is the Brinkman number, and $p = \tilde{p}\dfrac{\mu_0 U_0}{H_0}$ (same non-dim. as linear momentum equation)

Summary of dimensional forms

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \vec{v}) = 0$$

$$\frac{\partial(\rho v_x)}{\partial t} + \frac{\partial(\rho v_x v_x)}{\partial x} + \frac{\partial(\rho v_x v_y)}{\partial y} + \frac{\partial(\rho v_x v_z)}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xz}}{\partial z}$$

$$\frac{\partial(\rho v_y)}{\partial t} + \frac{\partial(\rho v_y v_x)}{\partial x} + \frac{\partial(\rho v_y v_y)}{\partial y} + \frac{\partial(\rho v_y v_z)}{\partial z} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}$$

$$0 = \frac{\partial p}{\partial z}$$

$$\rho c_p \frac{DT}{Dt} = \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \left[\tau_{xz}\left(\frac{\partial v_x}{\partial z}\right) + \tau_{yz}\left(\frac{\partial v_y}{\partial z}\right)\right] - \left(\frac{\partial \log \rho}{\partial \log T}\right)_p \left[\frac{Dp}{Dt}\right] + \dot{s}$$

Recap

Assumptions thus far

- Newtonian fluid
- $H_0 << L$
  - Neglect curvature effects, i.e. unwrapped film
  - Order-of-magnitude analysis eliminates many terms in equations
- Neglect body forces

These equations are the natural starting point for both low and high Reynolds number thin-film flows.

- Low-*Re* –> Reynolds equation
- High-*Re* –> Bulk-flow equations

Low-*Re* limit, Reynolds equation

Make additional assumptions

- **neglect inertia**
- **laminar flow**

Then integrate across the thin film. Momentum equations inserted into continuity equation **eliminating velocities as a dependent variables**

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho \vec{v}) = 0$$

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial z}\left(\mu \frac{\partial v_x}{\partial z}\right)$$

$$0 = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial z}\left(\mu \frac{\partial v_y}{\partial z}\right)$$

$$\rho c_p \frac{DT}{Dt} = \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) + \mu\left[\left(\frac{\partial v_x}{\partial z}\right)^2 + \left(\frac{\partial v_y}{\partial z}\right)^2\right] - \left(\frac{\partial \log \rho}{\partial \log T}\right)_p \left[\frac{Dp}{Dt}\right] + \dot{s}$$

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Overview and integrate conservation of mass

Integrate across the thin film before making any additional assumptions. Will show for conservation of mass, but the same procedure is applied for the momentum and energy equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} = 0$$

$$\int_{H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial \rho}{\partial t}\, dz + \int_{H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial(\rho v_x)}{\partial x}\, dz + \int_{H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial(\rho v_y)}{\partial y}\, dz + \int_{H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial(\rho v_z)}{\partial z}\, dz = 0$$

Apply Leibniz to flux terms

$$\int_{H_1(x,y,t)}^{H_2(x,y,t)} \frac{\partial \rho}{\partial t}\, dz + \frac{\partial}{\partial x}\int_{H_1(x,y,t)}^{H_2(x,y,t)} \rho v_x\, dz - \rho_2 v_{x,2}\frac{\partial H_2}{\partial x} + \rho_1 v_{x,1}\frac{\partial H_1}{\partial x}$$
$$+ \frac{\partial}{\partial y}\int_{H_1(x,y,t)}^{H_2(x,y,t)} \rho v_y\, dz - \rho_2 v_{y,2}\frac{\partial H_2}{\partial y} + \rho_1 v_{y,1}\frac{\partial H_1}{\partial y} + \rho_2 v_{z,2} - \rho_1 v_{z,1} = 0$$

Surface velocity approximations

The x-velocity of the upper surface can be approximated by the rotational velocity of the journal.

$$v_{x,2} = v_{x,t,2} + v_{x,r,2} \approx v_{x,r,2}$$

In the axial direction, there may be some translational sliding velocity.

$$v_{y,2} = v_{y,t,2} + v_{y,r,2} \approx v_{y,t,2}$$

The velocity across the film (z-direction) has both translational and rotational components. The translational component is related to the local (temporal) change in the film height and the rotational component is related to the rotational and axial sliding velocity of the journal (convective change).

$$v_{z,2} = v_{z,t,2} + v_{z,r,2} \approx \frac{\partial H_2}{\partial t} + v_{x,2}\frac{\partial H_2}{\partial x} + v_{y,2}\frac{\partial H_2}{\partial y}$$

Bulk-flow conservation of mass

1. Insert surface velocity approximations
2. Apply Leibniz to transient term
3. Expand terms and simplify

$$\frac{\partial\left(\rho H\right)}{\partial t}+\frac{\partial\left(\rho H\overline{v}_x\right)}{\partial x}+\frac{\partial\left(\rho H\overline{v}_y\right)}{\partial y}=0$$

where

$$H=\int_{H_1(x,y,t)}^{H_2(x,y,t)} dz = H_2 - H_1$$

$$\overline{v}_x=\frac{1}{H}\int_{H_1(x,y,t)}^{H_2(x,y,t)} v_x\, dz$$

$$\overline{v}_y=\frac{1}{H}\int_{H_1(x,y,t)}^{H_2(x,y,t)} v_y\, dz$$

Bulk-flow linear momentum

Applying the same procedure to the linear momentum equations

$$\frac{\partial\left(\rho H \overline{v}_x\right)}{\partial t} + \frac{\partial\left(\rho H I_{xx}\right)}{\partial x} + \frac{\partial\left(\rho H I_{xy}\right)}{\partial y} = -\frac{\partial p}{\partial x} H + \tau_{xz,2} - \tau_{xz,1}$$

$$\frac{\partial\left(\rho H \overline{v}_y\right)}{\partial t} + \frac{\partial\left(\rho H I_{yx}\right)}{\partial x} + \frac{\partial\left(\rho H I_{yy}\right)}{\partial y} = -\frac{\partial p}{\partial y} H + \tau_{yz,2} - \tau_{yz,1}$$

$$I_{xx} = \frac{1}{H} \int_{H_1(x,y,t)}^{H_2(x,y,t)} v_x v_x \, dz$$

$$I_{xy} = I_{yx} = \frac{1}{H} \int_{H_1(x,y,t)}^{H_2(x,y,t)} v_x v_y \, dz$$

$$I_{yy} = \frac{1}{H} \int_{H_1(x,y,t)}^{H_2(x,y,t)} v_y v_y \, dz$$

Bulk-flow energy

$$c_p\left(\frac{\partial\left(\rho H\overline{T}\right)}{\partial t}+\frac{\partial\left(\rho H\overline{v}_x\overline{T}\right)}{\partial x}+\frac{\partial\left(\rho H\overline{v}_y\overline{T}\right)}{\partial y}\right)$$

$$=-\left(q_z\right)_{H_1}^{H_2}-\left(\tau_{zx}\overline{v}_x+\tau_{zy}\overline{v}_y\right)_{H_1}^{H_2}++\alpha_{\hat{v}}\overline{T}H\left(\frac{\partial p}{\partial t}+\overline{v}_x\frac{\partial p}{\partial x}+\overline{v}_y\frac{\partial p}{\partial y}\right)$$

$$\overline{T}=\frac{1}{H}\int_{H_1(x,y,t)}^{H_2(x,y,t)}T\,dz$$

$$\alpha_{\hat{v}}=\frac{1}{\hat{v}}\left(\frac{\partial\hat{v}}{\partial T}\right)_p \tag{9}$$

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Simplify notation and summarize

Drop overbars on film-averaged quantities and consider only conservation of mass and linear momentum

$$\frac{\partial\left(\rho h\right)}{\partial t} + \frac{\partial\left(\rho h v_x\right)}{\partial x} + \frac{\partial\left(\rho h v_y\right)}{\partial y} = 0$$

$$\frac{\partial\left(\rho h v_x\right)}{\partial t} + \frac{\partial\left(\rho h i_{xx}\right)}{\partial x} + \frac{\partial\left(\rho h i_{xy}\right)}{\partial y} = -\frac{\partial p}{\partial x} h + \tau_{xz,2} - \tau_{xz,1}$$

$$\frac{\partial\left(\rho h v_y\right)}{\partial t} + \frac{\partial\left(\rho h i_{yx}\right)}{\partial x} + \frac{\partial\left(\rho h i_{yy}\right)}{\partial y} = -\frac{\partial p}{\partial y} h + \tau_{yz,2} - \tau_{yz,1}$$

Approximation of inertia terms

Assume profiles are nearly flat for fully-developed turbulent flow.

$$i_{xx} = \frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_x v_x \, dz \approx v_x^2$$

$$i_{xy} = i_{yx} = \frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_x v_y \, dz \approx v_x v_y$$

$$i_{yy} = \frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_y v_y \, dz \approx v_y^2$$

Alternative approximation methods have been applied to turbulent hydrodynamic bearings [Constantinescu and Galetuse, 1982]

Friction factors and relative velocities

Shear stresses expressed in terms of friction factors

$$\tau_r = 0.5\rho f_r v_r^2$$

$$\tau_s = 0.5\rho f_s v_s^2$$

$$v_s = \sqrt{\left(v_x^2 + v_y^2\right)}$$

$$v_r = \sqrt{\left(v_x^2 + (v_y - R\Omega)^2\right)}$$

Shear stress components

$$\tau_{xz,r} = \tau_r \frac{v_x}{U_r} = 0.5\rho f_r v_r^2 \frac{v_x}{v_r} = 0.5\rho f_r v_r v_x$$

$$\tau_{xz,s} = \tau_s \frac{v_x}{v_s} = 0.5\rho f_s U_s^2 \frac{v_x}{U_s} = 0.5\rho f_s U_s v_x$$

$$\tau_{yz,r} = \tau_r \frac{v_y - R\Omega}{v_r} = 0.5\rho f_r v_r^2 \frac{v_y - R\Omega}{v_r} = 0.5\rho f_r v_r \left(v_y - R\Omega\right)$$

$$\tau_{yz,s} = \tau_s \frac{v_y}{v_s} = 0.5\rho f_s v_s^2 \frac{v_y}{v_s} = 0.5\rho f_s v_s v_y$$

Shear stress summary

$$\tau_{xz,2} - \tau_{xz,1} = \tau_{xz,s} - \tau_{xz,r} = -0.5\rho\left(f_r v_r + f_s v_s\right) v_x$$

$$\tau_{yz,2} - \tau_{yz,1} = \tau_{yz,s} - \tau_{yz,r} = -0.5\rho\left(f_r v_r + f_s v_s\right) v_y + 0.5\rho f_r v_r R\Omega$$

Non-dimensionalize

Desirable for multiphysics models to prevent large disparity in length and/or time scales

$$\tilde{h} = \frac{h}{C} \quad \tilde{x} = \frac{x}{R} \quad \tilde{y} = \frac{y}{R}$$

$$\tilde{t} = \omega t \quad \tilde{\rho} = \frac{\rho}{\rho_*} \quad \tilde{\mu} = \frac{\mu}{\mu_*}$$

$$\tilde{v}_x = \frac{v_x}{u_*} \quad \tilde{v}_y = \frac{v_y}{u_*}$$

$$\tilde{p} = \frac{p}{\rho_* u_*^2} \quad \sigma == \frac{\omega R}{u_*}$$

Note that $y \in 0, 2\pi R$, so following the non-dimensionalization above, $\tilde{y} = \theta \in 0, 2\pi$

Dimensionless bulk-flow summary

$$\sigma \frac{\partial \left(\tilde{\rho}\tilde{h}\right)}{\partial \tilde{t}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x\right)}{\partial \tilde{x}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_y\right)}{\partial \tilde{y}} = 0$$

$$\sigma \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x\right)}{\partial \tilde{t}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x^2\right)}{\partial \tilde{x}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x\tilde{v}_y\right)}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}\tilde{h} - 0.5\frac{R}{C}\tilde{\rho}\left(f_r\tilde{v}_r + f_s\tilde{v}_s\right)\tilde{v}_x$$

$$\begin{aligned}\sigma \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_y\right)}{\partial \tilde{t}} &+ \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x\tilde{v}_y\right)}{\partial \tilde{x}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_y^2\right)}{\partial \tilde{y}} \\ &= -\frac{\partial \tilde{p}}{\partial \tilde{y}}\tilde{h} - 0.5\frac{R}{C}\tilde{\rho}\left(f_r\tilde{v}_r + f_s\tilde{v}_s\right)\tilde{v}_y + 0.5\frac{R}{C}\tilde{\rho}f_r\tilde{v}_r\frac{\Omega R}{u_*}\end{aligned}$$

Dimensionless bulk-flow summary, simplifying notation

$$\sigma \frac{\partial \left(\rho h\right)}{\partial t} + \frac{\partial \left(\rho h v_x\right)}{\partial x} + \frac{\partial \left(\rho h v_y\right)}{\partial y} = 0$$

$$\sigma \frac{\partial \left(\rho h v_x\right)}{\partial t} + \frac{\partial \left(\rho h v_x^2\right)}{\partial x} + \frac{\partial \left(\rho h v_x v_y\right)}{\partial y} = -\frac{\partial p}{\partial x} h - 0.5 \frac{R}{C} \rho \left(f_r v_r + f_s v_s\right) v_x$$

$$\sigma \frac{\partial \left(\rho h v_y\right)}{\partial t} + \frac{\partial \left(\rho h v_x v_y\right)}{\partial x} + \frac{\partial \left(\rho h v_y^2\right)}{\partial y} = -\frac{\partial p}{\partial y} h - 0.5 \frac{R}{C} \rho \left(f_r v_r + f_s v_s\right) v_y + 0.5 \frac{R^2 \Omega}{C u_*} \rho f_r v_r$$

$$v_s = \sqrt{v_x^2 + v_y^2}$$

$$v_r = \sqrt{v_x^2 + \left(v_y - v_t\right)^2}$$

where $v_t = R\Omega / u_*$ is the dimensionless rotor surface velocity.

Bulk-flow equations summarizing remarks

- Nonlinear inertia and friction factor terms
- Velocities retained as dependent variables in addition to pressure (unlike Reynolds equation)
- Coupled system of equations
- Retains mathematical character of full Navier-Stokes
- Very general

CFD-based solution approach

- Finite volume discretization
- Predictor-corrector method
- Capture linearized dynamics by perturbing equations
- Solve steady form of equations

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Perturb variables

$$h = h_0 + \epsilon_\psi h_\psi e^{i\omega t} = h_0 + \epsilon_\psi h_\psi e^{i\bar{t}}$$

Overbar dropped from the non-dimensional form of the equations below.

$$h = h_0 + \epsilon_\psi h_\psi e^{it}$$

$$v_x = v_{x0} + \epsilon_\psi v_{x,\psi} e^{it}$$

$$v_y = v_{y0} + \epsilon_\psi v_{y,\psi} e^{it}$$

$$p = p_0 + \epsilon_\psi p_\psi e^{it}$$

$$v_r = v_{r0} + \epsilon_\psi v_{r,\psi} e^{it}$$

$$v_s = v_{s0} + \epsilon_\psi v_{s,\psi} e^{it}$$

$$f_r = f_{r0} + \epsilon_\psi f_{r,\psi} e^{it}$$

$$f_s = f_{s0} + \epsilon_\psi f_{s,\psi} e^{it}$$

Generalized film thickness perturbation

$$h = h_0 + \epsilon_\psi h_\psi e^{i\omega t} = h_0 + \left(\epsilon_X \frac{\partial h_0}{\partial \varepsilon_{X0}} + \epsilon_Y \frac{\partial h_0}{\partial \varepsilon_{Y0}} \right) e^{i\omega t}$$

The appropriate forms for the X- and Y-directions are derived by replacing the partial derivatives

$$h_X = \frac{\partial h_0}{\partial \varepsilon_{X0}} = \cos y$$

$$h_Y = \frac{\partial h_0}{\partial \varepsilon_{Y0}} = \sin y$$

remember y denotes the circumferential direction

Zeroth-order problem

$$\frac{\partial\left(\rho h v_x\right)}{\partial x}+\frac{\partial\left(\rho h v_y\right)}{\partial y}=0$$

$$\frac{\partial\left(\rho h v_x^2\right)}{\partial x}+\frac{\partial\left(\rho h v_x v_y\right)}{\partial y}+0.5\frac{R}{C}\rho\left(f_r v_r+f_s v_s\right)v_x=-\frac{\partial p}{\partial x}h$$

$$\frac{\partial\left(\rho h v_x v_y\right)}{\partial x}+\frac{\partial\left(\rho h v_y^2\right)}{\partial y}+0.5\frac{R}{C}\rho\left(f_r v_r+f_s v_s\right)v_y=-\frac{\partial p}{\partial y}h+0.5\frac{R^2\Omega}{Cu_*}\rho f_r v_r$$

$$v_s=\sqrt{v_x^2+v_y^2}$$

$$v_r=\sqrt{v_x^2+\left(v_y-v_t\right)^2}$$

where $v_t=R\Omega/u_*$ is the dimensionless rotor surface velocity.

1st-order, continuity

$$\frac{\partial\left(\rho_0 h_0 v_{x,\psi}\right)}{\partial x}+\frac{\partial\left(\rho_0 h_0 v_{y,\psi}\right)}{\partial y}=-\frac{\partial\left(\rho_0 h_\psi v_{x0}\right)}{\partial x}-\frac{\partial\left(\rho_0 h_\psi v_{y0}\right)}{\partial y}-i\sigma\rho_0 h_\psi \quad (10)$$

1st-order, axial momentum, arbitrary friction factor

$$\frac{\partial\left(\rho_0 h_0 v_{x0} v_{x,\psi}\right)}{\partial x} + \frac{\partial\left(\rho_0 h_0 v_{y0} v_{x,\psi}\right)}{\partial y} + 0.5\frac{R}{C}\rho_0\left(f_{r0} v_{r0} + f_{s0} v_{s0}\right) v_{x,\psi} = S_{v_{x,\phi}}$$

$$\begin{aligned} S_{v_{x,\phi}} = & -\frac{\partial p_\psi}{\partial x} h_0 - \rho_0 h_0 v_{x,\psi}\frac{\partial v_{x0}}{\partial x} - \rho_0 h_0 v_{y,\psi}\frac{\partial v_{x0}}{\partial y} \\ & - \rho_0 h_\psi v_{x0}\frac{\partial v_{x0}}{\partial x} - \rho_0 h_\psi v_{y0}\frac{\partial v_{x0}}{\partial y} - \frac{\partial p_0}{\partial x} h_\psi \\ & - 0.5\frac{R}{C}\rho_0 v_{x0}\left(f_{r0} v_{r,\psi} + f_{s0} v_{s,\psi} + f_{s,\psi} v_{s0} + f_{r,\psi} v_{r0}\right) - i\sigma\rho_0 h_0 v_{x,\psi} \end{aligned}$$

1st-order, axial momentum, Blasius friction factor

$$\frac{\partial\left(\rho_0 h_0 v_{x0} v_{x,\psi}\right)}{\partial x}+\frac{\partial\left(\rho_0 h_0 v_{y0} v_{x,\psi}\right)}{\partial y}+0.5\frac{R}{C}\rho_0\left(f_{r0} v_{r0}+f_{s0} v_{s0}\right) v_{x,\psi}=S_{v_{x,\phi}}$$

$$\begin{aligned}
S_{v_{x,\phi}} = & -\frac{\partial p_\psi}{\partial x} h_0 - \rho_0 h_0 v_{x,\psi}\frac{\partial v_{x0}}{\partial x} - \rho_0 h_0 v_{y,\psi}\frac{\partial v_{x0}}{\partial y} \\
& - \rho_0 h_\psi v_{x0}\frac{\partial v_{x0}}{\partial x} - \rho_0 h_\psi v_{y0}\frac{\partial v_{x0}}{\partial y} - \frac{\partial p_0}{\partial x} h_\psi \\
& - 0.5\frac{R}{C}\rho_0 v_{x0}\left\{-\frac{\left(1+m_r\right) f_{r0}}{v_{r0}}\left[v_{x0} v_{x,\psi}+\left(v_{y0}-v_t\right) v_{y,\psi}\right]\right. \\
& \quad \left. -\frac{\left(1+m_s\right) f_{s0}}{v_{s0}}\left(v_{x0} v_{x,\psi}+v_{y0} v_{y,\psi}\right)+\frac{v_{s0} m_s}{h_0} f_{s0} h_\psi+\frac{v_{r0} m_r}{h_0} f_{r0} h_\psi\right\} \\
& - i\sigma\rho_0 h_0 v_{x,\psi}
\end{aligned}$$

1st-order, circumferential momentum, Blasisus friction factor

$$\frac{\partial\left(\rho_0 h_0 v_{x0} v_{y,\psi}\right)}{\partial x}+\frac{\partial\left(\rho_0 h_0 v_{y0} v_{y,\psi}\right)}{\partial y}+0.5\frac{R}{C}\rho_0\left(f_{r0}v_{r0}+f_{s0}v_{s0}\right)v_{y,\psi}=S_{v_{y,\phi}}$$

where [1,2]

$$\begin{aligned}S_{v_{y,\phi}}=&-\frac{\partial p_\psi}{\partial y}h_0-\rho_0 h_0 v_{x,\psi}\frac{\partial v_{y0}}{\partial x}-\rho_0 h_0 v_{y,\psi}\frac{\partial v_{y0}}{\partial y}-\rho_0 h_\psi v_{x0}\frac{\partial v_{y0}}{\partial x}-\rho_0 h_\psi v_{y0}\frac{\partial v_{y0}}{\partial y}\\&-\frac{\partial p_0}{\partial y}h_\psi-0.5\frac{R}{C}\rho_0 v_{y0}\left\{-\frac{\left(1+m_r\right)f_{r0}}{v_{r0}}\left[v_{x0}v_{x,\psi}+\left(v_{y0}-v_t\right)v_{y,\psi}\right]\right.\\&\qquad\left.-\frac{\left(1+m_s\right)f_{s0}}{v_{s0}}\left(v_{x0}v_{x,\psi}+v_{y0}v_{y,\psi}\right)+\frac{v_{s0}m_s}{h_0}f_{s0}h_\psi+\frac{v_{r0}m_r}{h_0}f_{r0}h_\psi\right\}\\&+0.5\frac{R^2\Omega}{Cu_*}\rho\left\{-\frac{\left(1+m_r\right)}{v_{r0}}f_{r0}\left[v_{x0}v_{x,\psi}+\left(v_{y0}-v_t\right)v_{y,\psi}\right]+\frac{m_r v_{r0}}{h_0}f_{r0}h_\psi\right\}\\&-i\sigma\rho_0 h_0 v_{y,\psi}\end{aligned}$$

[1] [Wilcox, 2006]

[2] Constantinescu

Table of Contents

FVM of zeroth-order problem

The same solution framework developed for the zeroth-order problem can be applied to the 1st-order problem

$$\frac{\partial\left(\rho h v_x\right)}{\partial x}+\frac{\partial\left(\rho h v_y\right)}{\partial y}=0$$

$$\frac{\partial\left(\rho h v_x^2\right)}{\partial x}+\frac{\partial\left(\rho h v_x v_y\right)}{\partial y}+0.5\frac{R}{C}\rho\left(f_r v_r+f_s v_s\right)v_x=-\frac{\partial p}{\partial x}h$$

$$\frac{\partial\left(\rho h v_x v_y\right)}{\partial x}+\frac{\partial\left(\rho h v_y^2\right)}{\partial y}+0.5\frac{R}{C}\rho\left(f_r v_r+f_s v_s\right)v_y=-\frac{\partial p}{\partial y}h+0.5\frac{R^2\Omega}{C u_*}\rho f_r v_r$$

Integrate axial momentum over CV

$$\frac{\partial\left(\rho h v_x^2\right)}{\partial x}+\frac{\partial\left(\rho h v_x v_y\right)}{\partial y}+0.5\frac{R}{C}\rho\left(f_r v_r+f_s v_s\right)v_x=-\frac{\partial p}{\partial x}h$$

$$\int\frac{\partial\left(\rho h v_x^2\right)}{\partial x}\,dV+\int\frac{\partial\left(\rho h v_x v_y\right)}{\partial y}\,dV+\int 0.5\frac{R}{C}\rho\left(f_r v_r+f_s v_s\right)v_x\,dV$$
$$=-\int\frac{\partial p}{\partial x}h\,dV$$

Apply divergence theorem to convective terms

$$\int\left(\rho h v_x\right)\vec{v}\cdot d\vec{S}+\int 0.5\frac{R}{C}\rho\left(f_r v_r+f_s v_s\right)v_x\,dV=-\int\frac{\partial p}{\partial x}h\,dV$$

Axial momentum : convective term

$$\int (\rho h v_x)\, \vec{v} \cdot d\vec{S} \approx \sum_{i=1}^{nf} (\rho h v_x \vec{v})_i \cdot \vec{S}_i = \sum_{i=1}^{nf} \dot{m}_i\, (v_x)_i$$

Consider evaluation of the mass flux at the faces of the cell. This requires the velocities and density at the faces of the cells. Apply linear interpolation and approximate fluxes at the face center (midpoint rule).

Geometric face interpolation factor [Jasak, 1996, pp.81-82]

$$g_f = \frac{|\vec{r}_{f-nb}|}{|\vec{r}_{P-nb}|} = \frac{\sqrt{(x_f - x_{nb})^2 + (y_f - y_{nb})^2}}{\sqrt{(x_P - x_{nb})^2 + (y_P - y_n b)^2}}$$

_compute_int() method of mesh base class

```
for i in range(self.Nfint):
  fxc = 0.5 * (points[faces[i, 0], 0] + points[faces[i, 1], 0])  # face centroid
  fyc = 0.5 * (points[faces[i, 0], 1] + points[faces[i, 1], 1])
  xnb = cell[neighbor[i], 0]  # neighbor centroid
  ynb = cell[neighbor[i], 1]
  xp = cell[owner[i], 0]  # owner centroid
  yp = cell[owner[i], 1]
  rnb = np.sqrt((fxc - xnb) ** 2. + (fyc - ynb) ** 2)
  rp = np.sqrt((xp - xnb) ** 2. + (yp - ynb) ** 2)
  self.cn[i, 0] = fxc - xp
  self.cn[i, 1] = fyc - yp
  self.cf[i, 0] = fxc
  self.cf[i, 1] = fyc
  self.gf[i] = rnb / rp
  self.t[i] = rp
  self.fa[i] = np.sqrt(np.abs(sf[i, 0]) ** 2 + np.abs(sf[i, 1]) ** 2)
```

Face mass fluxes

Then the value of an arbitrary variable ϕ (e.g. density, velocity) at the face can be linear interpolated as

$$\phi_f = g_f \phi_P + (1 - g_f)\, \phi_{nb} \tag{11}$$

$$\dot{m}_f = \rho_f h_f \left(v_{x,f} S_f^x + v_{y,f} S_f^y \right) \tag{12}$$

_init_zeroth_massflux() method in seal class

```
for i in range(self.Nfint):
  # linear interpolation of density and velocity to face
  rhof = gf[i] * rho[owner[i]] + (1. - gf[i]) * rho[neighbor[i]]
  uf = gf[i] * u[owner[i]] + (1. - gf[i]) * u[neighbor[i]]
  vf = gf[i] * v[owner[i]] + (1. - gf[i]) * v[neighbor[i]]
  self.phi[i] = hf[i] * rhof * (uf * sf[i, 0] + vf * sf[i, 1])
```

Linear and upwind blending

$$\dot{m}_f v_{x,f} \approx \dot{m}_f \left[(1-\gamma)\, v_{x,f}^{UDS} + \gamma v_{x,f}^{LDS} \right]$$

$$v_{x,f}^{LDS} = f_f v_{x,P} + (1 - f_f)\, v_{x,nb}$$

The first-order upwind difference approximation is made based upon the sign of mass flux at the cell face

$$v_{x,f}^{UDS} = 0.5\,(\mathrm{sgn}\,(\dot{m}_f) + 1)\, v_{x,P} - 0.5\,(\mathrm{sgn}\,(\dot{m}_f) - 1)\, v_{x,nb}$$

$$\begin{aligned}\dot{m}_f v_{x,f} \approx\ & \dot{m}_f \left[(1-\gamma)\, 0.5\,(\mathrm{sgn}\,(\dot{m}_f) + 1) + \gamma g_f\right] v_{x,P} \\ & + \dot{m}_f \left[(-1+\gamma)\, 0.5\,(\mathrm{sgn}\,(\dot{m}_f) - 1) + \gamma\,(1 - g_f)\, v_{x,nb}\right]\end{aligned}$$

```
fluxp = phi[i] * ((1. - self.gamma) * 0.5 * (np.sign(phi[i]) + 1.) + self.gamma * gf[i])
fluxnb = phi[i] * ((-1. + self.gamma) * 0.5 * (np.sign(phi[i]) - 1.) + self.gamma * (1. - gf[
    i]))
# owner
self.A[p, p] += fluxp
self.A[p, nb] += fluxnb
```

Axial momentum : pressure gradient

Green-Gauss cell-center gradient (and all other gradients)

$$-\int \frac{\partial p}{\partial x} h \, dV \approx \left(\frac{\partial p}{\partial x} h \right)_P V$$

_cc_grad() method in seal class

```
for i in range(self.Nfint):
  # linear interpolation of density and velocity to face
  varf = gf[i] * var[owner[i]] + (1. - gf[i]) * var[neighbor[i]]
  grad_var[owner[i], 0] += varf * sf[i, 0]
  grad_var[owner[i], 1] += varf * sf[i, 1]
  grad_var[neighbor[i], 0] += - varf * sf[i, 0]
  grad_var[neighbor[i], 1] += - varf * sf[i, 1]
```

_setup_zeroth_uv() method in seal class

```
for i in range(self.Nc):
  self.bu[i] += - hc[i] * self.grad_p[i, 0] * self.cell[i, 2]
  self.bv[i] += - hc[i] * self.grad_p[i, 1] * self.cell[i, 2]
```

Axial momentum : shear stress term

$$\int 0.5\frac{R}{C}\rho\left(f_r v_r + f_s v_s\right) v_x \, dV$$

_setup_zeroth_uv() method in seal class

```
for i in range(self.Nc):
  # friction factor formulation
  flux = 0.5 * (self.R / self.C) * self.rho[i] * ( U_s[i] * f_s[i] + U_r[i] * f_r[i] ) * self.
      cell[i, 2]
  self.A[i, i] += self.uv_src_blend * flux  # implicit portion
  self.bu[i] += - (1. - self.uv_src_blend) * flux * self.u[i]  # explicit portion portion
```

In practice, get best results with fully implicit formulation, i.e. uv_src_blend = 1.0

Circumferential momentum

The coefficient matrix is exactly the same as the axial momentum equation. The only differences is the presence of an additional shear stress term which is treated explicitly.

$$\int 0.5 \frac{R^2 \Omega}{C u_*} \rho f_r v_r \, dV$$

_setup_zeroth_uv() method in seal class

```
for i in range(self.Nc):
  self.bv[i] += 0.5 * (self.R / self.C) * self.rho[i] * (U_r[i] * f_r[i]) * self.v_r * self.
      cell[i, 2]
```

Coupling pressure to velocity and continuity

A SIMPLE-like algorithm is applied to get a pressure-correction equation from the conservation of mass.

The discretized momentum equations are solved using a guessed pressure field [Ferziger and Perić, 2002, p. 174]

$$v_{x,P}^{m*} = \frac{-\sum_{nb} A_{nb} v_{x,nb}^{m*} + \overline{Q}_{v_x}^{m-1}}{A_p} - \frac{1}{A_P}\left(h\frac{\partial p}{\partial x}\right)_P^{m-1} V$$

where $\overline{Q}_{v_x}^{m-1}$ is the source term at the previous outer iteration $m-1$ less the pressure term. $v_{x,P}^{m}$ represents the current estimate of the solution.

The true solution (satisfies momentum and continuity) is denoted

$$v_{x,P}^{m} = \frac{-\sum_{nb} A_{nb} v_{x,nb}^{m} + \overline{Q}_{v_x}^{m}}{A_p} - \frac{1}{A_P}\left(h\frac{\partial p}{\partial x}\right)_P^{m} V$$

Velocity and pressure corrections

Add small corrections to the intermediate values of the velocity and pressure

$$v_x^m = v_x^{m*} + v_x^{'}$$

$$p^m = p^{m-1} + p^{'}$$

Combine the expressions to get the velocity corrections

$$v_{x,P}^{'} \approx \frac{-\sum_{nb} A_{nb} v_{x,nb}^{'}}{A_p} - \frac{1}{A_P} \left(h \frac{\partial p'}{\partial x} \right)_P V$$

SIMPLE approximation to velocity correction

$$v'_{x,P} \approx -\frac{h_P V_P}{A_P} \left(\frac{\partial p'}{\partial x} \right)_P$$

$$v'_{x,P} \approx -D_p \left(\frac{\partial p'}{\partial x} \right)_P$$

solve_zeroth() method in seal class

```
self.Dp = self.cell[:,2] * self.hc / self.apu
self.u = self.u_star - self.Dp * self.grad_p_corr[:,0]
self.v = self.v_star - self.Dp * self.grad_p_corr[:,1]
```

Rough approximation to velocity correction requires relaxation of pressure correction

Substitute corrections into continuity

Discrete continuity

$$\sum_{i=1}^{nf} \dot{m}_i = 0$$

$$\dot{m} = \dot{m}^* + \dot{m}'$$

$$\sum_{i=1}^{nf} \dot{m}_i' = -\sum_{i=1}^{nf} \dot{m}_i^*$$

$$\sum_{i=1}^{nf} \rho_f h_f \left(\frac{h_P V_P}{A_P}\right)_f \left(\frac{p_{nb}' - p_P}{x_{nb} - x_P} * S_f^x + \frac{p_{nb}' - p_P}{y_{nb} - y_P} * S_f^y\right) = \sum_{i=1}^{nf} \dot{m}_i^*$$

Pressure-correction equation

$$\sum_{i=1}^{nf} \rho_f h_f D_f \left(\frac{p'_{nb} - p_P}{x_{nb} - x_P} * S_f^x + \frac{p'_{nb} - p_P}{y_{nb} - y_P} * S_f^y \right) = \sum_{i=1}^{nf} \dot{m}_i^*$$

_setup_zeroth_p method in seal class

```
for i in range(self.Nfint):
  p = owner[i]
  nb = neighbor[i]
  fluxp = rhof[i] * Df[i] * hf[i] * (div(sf[i, 0], (cell[nb, 0] - cell[p, 0])) +
                       div(sf[i, 1], (cell[nb, 1] - cell[p, 1])))
  fluxnb = -fluxp
  # owner
  self.Ap[p, p] += fluxp
  self.Ap[p, nb] += fluxnb
  # neighbor
  self.Ap[nb, p] += -fluxp
  self.Ap[nb, nb] += -fluxnb
  self.bp[p] += - phi[i]
  self.bp[nb] += phi[i]
```

Rhie and Chow Correction

Ensures strong pressure-velocity coupling on cell-centered arrangement and prevents checkerboarding [Ferziger and Perić, 2002]

$$v_{x,f} = \overline{v}_{x,f} - D_f \left[\left(\frac{\partial p}{\partial x} \right)_f - \overline{\left(\frac{\partial p}{\partial x} \right)_f} \right]$$

```python
for i in range(self.Nfint):
  # linear interpolation of density and velocity to face
  idxp = owner[i]
  idxnb = neighbor[i]
  rhof = gf[i] * rho[owner[i]] + (1. - gf[i]) * rho[idxnb]
  uf = gf[i] * u[owner[i]] + (1. - gf[i]) * u[neighbor[i]]
  vf = gf[i] * v[owner[i]] + (1. - gf[i]) * v[neighbor[i]]
  gradpf_x = gf[i] * gradp[idxp, 0] + (1. - gf[i]) * gradp[idxnb, 0]
  gradpf_y = gf[i] * gradp[idxp, 1] + (1. - gf[i]) * gradp[idxnb, 1]
  coeff = Df[i]
  rx = cell[idxnb, 0] - cell[idxp, 0]
  ry = cell[idxnb, 1] - cell[idxp, 1]
  uadj = coeff * (div((press[idxnb] - press[idxp]), (cell[idxnb, 0] - cell[idxp, 0])) -
      gradpf_x)
  vadj = coeff * (div((press[idxnb] - press[idxp]), (cell[idxnb, 1] - cell[idxp, 1])) -
      gradpf_y)
  #
  phi[i] = hf[i] * rhof * (uf * sf[i, 0] + vf * sf[i, 1]) - hf[i] * rhof * (uadj * sf[i, 0] +
      vadj * sf[i, 1])
```

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