1 Bulk-flow equations

This document contains a summary of the bulk-flow equations for the analysis of high-Reynolds number thin films. The conservation equations for mass, linear momentum, and energy are included. Also included are the perturbed forms of the bulk-flow equations considering incompressible flow, transverse rotor perturbations (only), and friction factors of Blasius form.

- x : axial direction
- y : circumferential direction

1.1 Dimensional forms

$$\frac{\partial\left(\rho h\right)}{\partial t}+\frac{\partial\left(\rho h v_x\right)}{\partial x}+\frac{\partial\left(\rho h v_y\right)}{\partial y}=0 \tag{1}$$

$$\frac{\partial\left(\rho h v_x\right)}{\partial t}+\frac{\partial\left(\rho h i_{xx}\right)}{\partial x}+\frac{\partial\left(\rho h i_{xy}\right)}{\partial y}=-\frac{\partial p}{\partial x}h+\tau_{xz,2}-\tau_{xz,1} \tag{2}$$

$$\frac{\partial\left(\rho h v_y\right)}{\partial t}+\frac{\partial\left(\rho h i_{yx}\right)}{\partial x}+\frac{\partial\left(\rho h i_{yy}\right)}{\partial y}=-\frac{\partial p}{\partial y}h+\tau_{yz,2}-\tau_{yz,1} \tag{3}$$

$$\begin{aligned} c_p\left(\frac{\partial\left(\rho h T\right)}{\partial t}+\frac{\partial\left(\rho h v_x T\right)}{\partial x}+\frac{\partial\left(\rho h v_y T\right)}{\partial y}\right) &= -\left(q_z\right)_{h_1}^{h_2}-\left(\tau_{zx}v_x+\tau_{zy}v_y\right)_{h_1}^{h_2} \\ &\quad +\alpha_{\hat{v}}Th\left(\frac{\partial p}{\partial t}+v_x\frac{\partial p}{\partial x}+v_y\frac{\partial p}{\partial y}\right) \end{aligned} \tag{4}$$

where

$$h=\int_{h_1(x,y,t)}^{h_2(x,y,t)} dz=h_2-h_1 \tag{5}$$

$$v_x=\frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_x\,dz \tag{6}$$

$$v_y=\frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_y\,dz \tag{7}$$

$\alpha_{\hat{v}}$ is a thermal volumetric expansion coefficient.

$$\alpha_{\hat{v}}=\frac{1}{\hat{v}}\left(\frac{\partial \hat{v}}{\partial T}\right)_p \tag{8}$$

1.2 Convective integrals

The convective inertia integrals may be approximated in different ways depending on the nature of the flow. In the case of fully-developed, turbulent flow, the velocity profiles are nearly flat and the convective inertia integrals can be approximated as

$$i_{xx} = \frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_x v_x \, dz \approx v_x^2 \tag{9}$$

$$i_{xy} = \frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_x v_y \, dz \approx v_x v_y \tag{10}$$

$$i_{yx} = I_{xy} = \frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_y v_x \, dz \approx v_x v_y \tag{11}$$

$$i_{yy} = \frac{1}{h}\int_{h_1(x,y,t)}^{h_2(x,y,t)} v_y v_y \, dz \approx v_y^2 \tag{12}$$

1.3 Shear stresses

dimensional shear stress of the form

$$\tau_{xz,2} - \tau_{xz,1} = \tau_{xz}\big|_1^2 = \tau_{xz}\big|_0^h = -0.5\rho\left(f_r v_r + f_s v_s\right) v_x \tag{13}$$

where

$$v_s = \sqrt{v_x^2 + v_y^2} \tag{14}$$

$$v_r = \sqrt{v_x^2 + \left(v_y - R\Omega\right)^2} \tag{15}$$

$$\tau_{yz,2} - \tau_{yz,1} = \tau_{yz}\big|_1^2 = \tau_{yz}\big|_0^h = -0.5\rho\left(f_r v_r + f_s v_s\right) v_y + 0.5\rho f_r v_r R\Omega \tag{16}$$

1.4 Non-dimensionalization

Non-dimensional forms of the bulk-flow equations assuming the convective integral approximations provided in the previous section.

$$\tilde{h} = \frac{h}{C}$$
$$\tilde{x} = \frac{x}{R}$$
$$\tilde{y} = \frac{y}{R}$$
$$\tilde{t} = \omega t$$
$$\tilde{\rho} = \frac{\rho}{\rho_*}$$
$$\tilde{\mu} = \frac{\mu}{\mu_*}$$
$$\tilde{v}_x = \frac{v_x}{u_*}$$
$$\tilde{v}_y = \frac{v_y}{u_*}$$
$$\tilde{p} = \frac{p}{\rho_* u_*^2}$$
$$\sigma = = \frac{\omega R}{u_*}$$

Note that $y \in 0, 2\pi R$, so following the non-dimensionalization above, $\tilde{y} = \theta \in 0, 2\pi$

1.5 Dimensionless form

$$\sigma \frac{\partial \left(\tilde{\rho}\tilde{h}\right)}{\partial \tilde{t}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x\right)}{\partial \tilde{x}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_y\right)}{\partial \tilde{y}} = 0$$

$$\sigma \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x\right)}{\partial \tilde{t}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x^2\right)}{\partial \tilde{x}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x\tilde{v}_y\right)}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{x}}\tilde{h} - 0.5\frac{R}{C}\tilde{\rho}\left(f_r\tilde{v}_r + f_s\tilde{v}_s\right)\tilde{v}_x$$

$$\sigma \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_y\right)}{\partial \tilde{t}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_x\tilde{v}_y\right)}{\partial \tilde{x}} + \frac{\partial \left(\tilde{\rho}\tilde{h}\tilde{v}_y^2\right)}{\partial \tilde{y}} = -\frac{\partial \tilde{p}}{\partial \tilde{y}}\tilde{h} - 0.5\frac{R}{C}\tilde{\rho}\left(f_r\tilde{v}_r + f_s\tilde{v}_s\right)\tilde{v}_y + 0.5\frac{R}{C}\tilde{\rho}f_r\tilde{v}_r\frac{\Omega R}{u_*}$$

Dropping the overbars for simplicity...

$$\sigma \frac{\partial \left(\rho h\right)}{\partial t} + \frac{\partial \left(\rho h v_x\right)}{\partial x} + \frac{\partial \left(\rho h v_y\right)}{\partial y} = 0 \tag{17}$$

$$\sigma \frac{\partial \left(\rho h v_x\right)}{\partial t} + \frac{\partial \left(\rho h v_x^2\right)}{\partial x} + \frac{\partial \left(\rho h v_x v_y\right)}{\partial y} = -\frac{\partial p}{\partial x}h - 0.5\frac{R}{C}\rho\left(f_r v_r + f_s v_s\right)v_x \tag{18}$$

$$\sigma\frac{\partial\left(\rho h v_y\right)}{\partial t}+\frac{\partial\left(\rho h v_x v_y\right)}{\partial x}+\frac{\partial\left(\rho h v_y^2\right)}{\partial y}=-\frac{\partial p}{\partial y}h-0.5\frac{R}{C}\rho\left(f_r v_r+f_s v_s\right)v_y+0.5\frac{R^2\Omega}{Cu_*}\rho f_r v_r \tag{19}$$

Non-dimensionlize the rotor and stator velocities as well

$$v_s=\sqrt{v_x^2+v_y^2} \tag{20}$$

$$v_r=\sqrt{v_x^2+\left(v_y-v_t\right)^2} \tag{21}$$

where $v_t=R\Omega/u_*$ is the dimensionless rotor surface velocity.

2 Perturbed equations

2.1 Perturbations

The perturbed form of the bulk-flow equations are derived by introducing the following first-order Taylor-series perturbations into the film thickness and the dependent variables (shown here for incompressible flow):

$$h=h_0+\epsilon_\psi h_\psi e^{i\omega t}=h_0+\epsilon_\psi h_\psi e^{i\bar{t}}$$

But recall that the overline $\bar{*}$ was dropped from the non-dimensional form of the equations for ease of notation. And so, it is understood that the perturbations below of time t actually represent dimensionless time ωt.

$$\begin{aligned}
h&=h_0+\epsilon_\psi h_\psi e^{it}\\
v_x&=v_{x0}+\epsilon_\psi v_{x,\psi}e^{it}\\
v_y&=v_{y0}+\epsilon_\psi v_{y,\psi}e^{it}\\
p&=p_0+\epsilon_\psi p_\psi e^{it}\\
v_r&=v_{r0}+\epsilon_\psi v_{r,\psi}e^{it}\\
v_s&=v_{s0}+\epsilon_\psi v_{s,\psi}e^{it}\\
f_r&=f_{r0}+\epsilon_\psi f_{r,\psi}e^{it}\\
f_s&=f_{s0}+\epsilon_\psi f_{s,\psi}e^{it}
\end{aligned}$$

where $\psi\in X,Y$. Important: The capital Cartesian coordinates X,Y correspond to the global coordinate system describing the rotor motions while x,y describe the unwrapped film thickness. Note that separately perturbing the rotor and stator velocities and friction factors at this stage results in the flexible set of the perturbation equations. Different expressions for f_{i0} and its derivative $f_{i\psi}$ (where $i=r,s$) can be inserted for the friction factor or friction factor derivative of interest.

Dimensionless film thickness function at static equilibrium postion with no misalignment

$$h_0(y) = 1.0 + \varepsilon_{X0}\cos y + \varepsilon_{Y0}\sin y \tag{22}$$

Film thickness function including perturbation from static equilibrium position

$$h = h_0 + \epsilon_\psi h_\psi e^{i\omega t} = h_0 + \left(\epsilon_X \frac{\partial h_0}{\partial \varepsilon_{X0}} + \epsilon_Y \frac{\partial h_0}{\partial \varepsilon_{Y0}}\right) e^{i\omega t} \tag{23}$$

In the perturbed equations that follow, the appropriate forms for the X- and Y-directions are derived by replacing the partial derivatives with $h_X = \cos y$ or $h_Y = \sin y$ which follow from taking the partial derivatives of the film thickness function in Equation 22.

2.2 Rotor and stator velocity derivatives

$$v_{i,\psi} = \left.\frac{\partial v_i}{\partial v_x}\right|_0 v_{x,\psi} + \left.\frac{\partial v_i}{\partial v_y}\right|_0 v_{y,\psi} \tag{24}$$

where $i = r, s$

2.2.1 Stator

$$v_{s,\psi} = \left.\frac{\partial v_s}{\partial v_x}\right|_0 v_{x,\psi} + \left.\frac{\partial v_s}{\partial v_y}\right|_0 v_{y,\psi} \tag{25}$$

Evaluating the partial derivatives considering the stator velocity given by Equation 20

$$v_{s,\psi} = \frac{1}{v_{s0}}\left(v_{x0}v_{x,\psi} + v_{y0}v_{y,\psi}\right) \tag{26}$$

2.2.2 Rotor

$$v_{r,\psi} = \left.\frac{\partial v_r}{\partial v_x}\right|_0 v_{x,\psi} + \left.\frac{\partial v_r}{\partial v_y}\right|_0 v_{y,\psi} \tag{27}$$

Evaluating the partial derivatives considering the rotor velocity given by Equation 21

$$v_{r,\psi} = \frac{1}{v_{r0}}\left[v_{x0}v_{x,\psi} + \left(v_{y0} - R\Omega\right)v_{y,\psi}\right] \tag{28}$$

2.3 Friction factor derivatives

$$f_{i,\psi} = \left.\frac{\partial f_i}{\partial v_i}\right|_0 v_{i,\psi} + \left.\frac{\partial f_i}{\partial h}\right|_0 h_\psi \tag{29}$$

where $i = r, s$

2.3.1 Blasius

$$f_i = n_i Re_i^{m_i} = n_i \left(\frac{\rho_* v_i 2h}{\mu_*} \right)^{m_i} \tag{30}$$

$$f_{i,\psi} = \frac{m_i}{v_{i0}} f_{i0} v_{i,\psi} + \frac{m_i}{h_0} f_{i0} h_\psi \tag{31}$$

Stator:

$$f_{s,\psi} = \frac{m_s}{v_{s0}^2} f_{s0} \left(v_{x0} v_{x,\psi} + v_{y0} v_{y,\psi} \right) + \frac{m_s}{h_0} f_{s0} h_\psi \tag{32}$$

Rotor:

$$f_{r,\psi} = \frac{m_r}{v_{r0}^2} f_{r0} \left[v_{x0} v_{x,\psi} + \left(v_{y0} - R\Omega \right) v_{y,\psi} \right] + \frac{m_r}{h_0} f_{r0} h_\psi \tag{33}$$

2.4 Zeroth-order

2.4.1 Continuity

$$\frac{\partial \left(\rho_0 h_0 v_{x0} \right)}{\partial x} + \frac{\partial \left(\rho_0 h_0 v_{y0} \right)}{\partial y} = 0 \tag{34}$$

2.4.2 Axial momentum

$$\frac{\partial \left(\rho_0 h_0 v_{x0}^2 \right)}{\partial x} + \frac{\partial \left(\rho_0 h_0 v_{x0} v_{y0} \right)}{\partial y} = -\frac{\partial p_0}{\partial x} h_0 - 0.5 \frac{R}{C} \rho_0 \left(f_{r0} v_{r0} + f_{s0} v_{s0} \right) v_{x0} \tag{35}$$

2.4.3 Circumferential momentum

$$\frac{\partial \left(\rho h_0 v_{x0} v_{y0} \right)}{\partial x} + \frac{\partial \left(\rho h_0 v_{y0}^2 \right)}{\partial y} = -\frac{\partial p_0}{\partial y} h_0 - 0.5 \frac{R}{C} \rho \left(f_{r0} v_{r0} + f_{s0} v_{s0} \right) v_{y0} + 0.5 \frac{R^2 \Omega}{C u_*} \rho f_{r0} v_{r0} \tag{36}$$

2.5 First-order

2.5.1 Continuity

$$\frac{\partial \left(\rho_0 h_0 v_{x,\psi} \right)}{\partial x} + \frac{\partial \left(\rho_0 h_0 v_{y,\psi} \right)}{\partial y} = -\frac{\partial \left(\rho_0 h_\psi v_{x0} \right)}{\partial x} - \frac{\partial \left(\rho_0 h_\psi v_{y0} \right)}{\partial y} - i\sigma \rho_0 h_\psi \tag{37}$$

$h_\psi \neq f(x)$ so...

$$\frac{\partial \left(\rho_0 h_0 v_{x,X} \right)}{\partial x} + \frac{\partial \left(\rho_0 h_0 v_{y,X} \right)}{\partial y} = -h_\psi \frac{\partial \left(\rho_0 v_{x0} \right)}{\partial x} - h_\psi \frac{\partial \left(\rho_0 v_{y0} \right)}{\partial y} - \rho_0 v_{y0} \frac{\partial \left(h_\psi \right)}{\partial y} - i\sigma \rho_0 h_\psi \tag{38}$$

2.5.2 Axial momentum

$$
\begin{aligned}
&\frac{\partial\left(\rho_0 h_0 v_{x0} v_{x,\psi}\right)}{\partial x}+\frac{\partial\left(\rho_0 h_0 v_{y0} v_{x,\psi}\right)}{\partial y}+\left[\frac{\partial\left(\rho_0 h_0 v_{x0} v_{x,\psi}\right)}{\partial x}+\frac{\partial\left(\rho_0 h_0 v_{y,\psi} v_{x0}\right)}{\partial y}\right] \\
&\quad=-\frac{\partial p_\psi}{\partial x} h_0-0.5 \frac{R}{C} \rho_0\left(f_{r0} v_{r0}+f_{s0} v_{s0}\right) v_{x,\psi}-\left[\frac{\partial\left(\rho_0 h_\psi v_{x0}^2\right)}{\partial x}+\frac{\partial\left(\rho_0 h_\psi v_{x0} v_{y0}\right)}{\partial y}\right]-\frac{\partial p_0}{\partial x} h_\psi \\
&\qquad-0.5 \frac{R}{C} \rho_0 v_{x0}\left(f_{r0} v_{r,\psi}+f_{s0} v_{s,\psi}+f_{s,\psi} v_{s0}+f_{r,\psi} v_{r0}\right)-i \sigma \rho_0 h_0 v_{x,\psi}-\left[i \sigma \rho_0 h_\psi v_{x0}\right]
\end{aligned}
$$

Make use of first-order continuity equation to simplify the terms in brackets. Notably, derivatives of the dependent variables can be eliminated.

$$
\begin{aligned}
&\frac{\partial\left(\rho_0 h_0 v_{x0} v_{x,\psi}\right)}{\partial x}+\frac{\partial\left(\rho_0 h_0 v_{y0} v_{x,\psi}\right)}{\partial y}+\rho_0 h_0 v_{x,\psi} \frac{\partial v_{x0}}{\partial x}+\rho_0 h_0 v_{y,\psi} \frac{\partial v_{x0}}{\partial y} \\
&\quad=-\frac{\partial p_\psi}{\partial x} h_0-0.5 \frac{R}{C} \rho_0\left(f_{r0} v_{r0}+f_{s0} v_{s0}\right) v_{x,\psi}-\rho_0 h_\psi v_{x0} \frac{\partial v_{x0}}{\partial x}-\rho_0 h_\psi v_{y0} \frac{\partial v_{x0}}{\partial y} \\
&\qquad-\frac{\partial p_0}{\partial x} h_\psi-0.5 \frac{R}{C} \rho_0 v_{x0}\left(f_{r0} v_{r,\psi}+f_{s0} v_{s,\psi}+f_{s,\psi} v_{s0}+f_{r,\psi} v_{r0}\right)-i \sigma \rho_0 h_0 v_{x,\psi}
\end{aligned}
$$

If the magenta and cyan terms are treated explicitly, then the coefficient matrix from the zeroth-order problem can be re-used directly. Note that it is assumed that that the friction factor terms are treated implicitly within the zeroth-order problem. Additionally, implicit treatment of the latter magenta term would require coupled solution of the first-order momentum equations

$$
\frac{\partial\left(\rho_0 h_0 v_{x0} v_{x,\psi}\right)}{\partial x}+\frac{\partial\left(\rho_0 h_0 v_{y0} v_{x,\psi}\right)}{\partial y}+0.5 \frac{R}{C} \rho_0\left(f_{r0} v_{r0}+f_{s0} v_{s0}\right) v_{x,\psi}=S_{v_{x,\phi}}
$$

where

$$
\begin{aligned}
S_{v_{x,\phi}}=&-\frac{\partial p_\psi}{\partial x} h_0-\rho_0 h_0 v_{x,\psi} \frac{\partial v_{x0}}{\partial x}-\rho_0 h_0 v_{y,\psi} \frac{\partial v_{x0}}{\partial y}-\rho_0 h_\psi v_{x0} \frac{\partial v_{x0}}{\partial x}-\rho_0 h_\psi v_{y0} \frac{\partial v_{x0}}{\partial y} \\
&-\frac{\partial p_0}{\partial x} h_\psi-0.5 \frac{R}{C} \rho_0 v_{x0}\left(f_{r0} v_{r,\psi}+f_{s0} v_{s,\psi}+f_{s,\psi} v_{s0}+f_{r,\psi} v_{r0}\right)-i \sigma \rho_0 h_0 v_{x,\psi}
\end{aligned}
$$

Substitute expressions for partial derivatives of rotor and stator velocities as well as partial derivatives of rotor and stator friction factors into the source term. Also substitute friction factors assuming Blasius form.

$$
\begin{aligned}
S_{v_{x,\phi}}=&-\frac{\partial p_\psi}{\partial x} h_0-\rho_0 h_0 v_{x,\psi} \frac{\partial v_{x0}}{\partial x}-\rho_0 h_0 v_{y,\psi} \frac{\partial v_{x0}}{\partial y}-\rho_0 h_\psi v_{x0} \frac{\partial v_{x0}}{\partial x}-\rho_0 h_\psi v_{y0} \frac{\partial v_{x0}}{\partial y} \\
&-\frac{\partial p_0}{\partial x} h_\psi-0.5 \frac{R}{C} \rho_0 v_{x0}\left\{\frac{f_{r0}}{v_{r0}}\left[v_{x0} v_{x,\psi}+\left(v_{y0}-v_t\right) v_{y,\psi}\right]+\frac{f_{s0}}{v_{s0}}\left(v_{x0} v_{x,\psi}+v_{y0} v_{y,\psi}\right)\right. \\
&\quad+\frac{m_s}{v_{s0}} f_{s0}\left(v_{x0} v_{x,\psi}+v_{y0} v_{y,\psi}\right)+\frac{v_{s0} m_s}{h_0} f_{s0} h_\psi+\frac{m_r}{v_{r0}} f_{r0}\left[v_{x0} v_{x,\psi}+\left(v_{y0}-v_t\right) v_{y,\psi}\right] \\
&\quad\left.+\frac{v_{r0} m_r}{h_0} f_{r0} h_\psi\right\}-i \sigma \rho_0 h_0 v_{x,\psi}
\end{aligned}
$$

Simplifying...

$$S_{v_{x,\phi}} = -\frac{\partial p_\psi}{\partial x}h_0 - \rho_0 h_0 v_{x,\psi}\frac{\partial v_{x0}}{\partial x} - \rho_0 h_0 v_{y,\psi}\frac{\partial v_{x0}}{\partial y} - \rho_0 h_\psi v_{x0}\frac{\partial v_{x0}}{\partial x} - \rho_0 h_\psi v_{y0}\frac{\partial v_{x0}}{\partial y} - \frac{\partial p_0}{\partial x}h_\psi$$
$$- 0.5\frac{R}{C}\rho_0 v_{x0}\left\{\frac{(1+m_r)\,f_{r0}}{v_{r0}}\left[v_{x0}v_{x,\psi} + (v_{y0}-v_t)\,v_{y,\psi}\right]\frac{(1+m_s)\,f_{s0}}{v_{s0}}\left(v_{x0}v_{x,\psi} + v_{y0}v_{y,\psi}\right)\right.$$
$$\left. + \frac{v_{s0}m_s}{h_0}f_{s0}h_\psi + \frac{v_{r0}m_r}{h_0}f_{r0}h_\psi\right\} - i\sigma\rho_0 h_0 v_{x,\psi}$$

2.5.3 Circumferential momentum

$$\frac{\partial(\rho_0 h_0 v_{x0}v_{y,\psi})}{\partial x} + \frac{\partial(\rho_0 h_0 v_{y0}v_{y,\psi})}{\partial y} + \left[\frac{\partial(\rho_0 h_0 v_{y0}v_{x,\psi})}{\partial x} + \frac{\partial(\rho_0 h_0 v_{y,\psi}v_{y0})}{\partial y}\right]$$
$$= -\frac{\partial p_\psi}{\partial y}h_0 - 0.5\frac{R}{C}\rho_0\left(f_{r0}v_{r0} + f_{s0}v_{s0}\right)v_{y,\psi} - \left[\frac{\partial(\rho_0 h_\psi v_{x0}v_{y0})}{\partial x} + \frac{\partial\left(\rho_0 h_\psi v_{y0}^2\right)}{\partial y}\right]$$
$$- \frac{\partial p_0}{\partial y}h_\psi - 0.5\frac{R}{C}\rho_0\left(f_{r0}v_{r,\psi}v_{y0} + f_{s0}v_{s,\psi}v_{y0} + f_{s,\psi}v_{s0}v_{y0} + f_{r,\psi}v_{r0}v_{y0}\right)$$
$$+ 0.5\frac{R^2\Omega}{Cu_*}\rho_0\left(f_{r,\psi}v_{r0} + f_{r0}v_{r,\psi}\right) - i\sigma\rho_0 h_0 v_{y,\psi} - \left[i\sigma\rho_0 h_\psi v_{y0}\right]$$

Re-arranging and simplifying the terms in brackets using first-order continuity equation

$$\frac{\partial(\rho_0 h_0 v_{x0}v_{y,\psi})}{\partial x} + \frac{\partial(\rho_0 h_0 v_{y0}v_{y,\psi})}{\partial y} + 0.5\frac{R}{C}\rho_0\left(f_{r0}v_{r0} + f_{s0}v_{s0}\right)v_{y,\psi} = S_{v_{y,\phi}}$$

where

$$S_{v_{y,\phi}} = -\frac{\partial p_\psi}{\partial y}h_0 - \rho_0 h_0 v_{x,\psi}\frac{\partial v_{y0}}{\partial x} - \rho_0 h_0 v_{y,\psi}\frac{\partial v_{y0}}{\partial y} - \rho_0 h_\psi v_{x0}\frac{\partial v_{y0}}{\partial x} - \rho_0 h_\psi v_{y0}\frac{\partial v_{y0}}{\partial y}$$
$$- \frac{\partial p_0}{\partial y}h_\psi - 0.5\frac{R}{C}\rho_0 v_{y0}\left(f_{r0}v_{r,\psi} + f_{s0}v_{s,\psi} + f_{s,\psi}v_{s0} + f_{r,\psi}v_{r0}\right)$$
$$+ 0.5\frac{R^2\Omega}{Cu_*}\rho_0\left(f_{r,\psi}v_{r0} + f_{r0}v_{r,\psi}\right) - i\sigma\rho_0 h_0 v_{y,\psi}$$

Substituting partial derivatives of rotor and stator velocities and assuming Blasius friction factor

$$S_{v_{y,\phi}} = -\frac{\partial p_\psi}{\partial y}h_0 - \rho_0 h_0 v_{x,\psi}\frac{\partial v_{y0}}{\partial x} - \rho_0 h_0 v_{y,\psi}\frac{\partial v_{y0}}{\partial y} - \rho_0 h_\psi v_{x0}\frac{\partial v_{y0}}{\partial x} - \rho_0 h_\psi v_{y0}\frac{\partial v_{y0}}{\partial y}$$
$$- \frac{\partial p_0}{\partial y}h_\psi - 0.5\frac{R}{C}\rho_0 v_{y0}\left\{\frac{f_{r0}}{v_{r0}}\left[v_{x0}v_{x,\psi} + (v_{y0}-v_t)\,v_{y,\psi}\right] + \frac{f_{s0}}{v_{s0}}\left(v_{x0}v_{x,\psi} + v_{y0}v_{y,\psi}\right)\right.$$
$$+ \frac{m_s}{v_{s0}}f_{s0}\left(v_{x0}v_{x,\psi} + v_{y0}v_{y,\psi}\right) + \frac{v_{s0}m_s}{h_0}f_{s0}h_\psi + \frac{m_r}{v_{r0}}f_{r0}\left[v_{x0}v_{x,\psi} + (v_{y0}-v_t)\,v_{y,\psi}\right]$$
$$\left. + \frac{v_{r0}m_r}{h_0}f_{r0}h_\psi\right\} + 0.5\frac{R^2\Omega}{Cu_*}\rho_0\left\{\frac{m_r}{v_{r0}}f_{r0}\left[v_{x0}v_{x,\psi} + (v_{y0}-v_t)\,v_{y,\psi}\right] + \frac{m_r v_{r0}}{h_0}f_{r0}h_\psi\right.$$
$$\left. + \frac{f_{r0}}{v_{r0}}\left[v_{x0}v_{x,\psi} + (v_{y0}-v_t)\,v_{y,\psi}\right]\right\} - i\sigma\rho_0 h_0 v_{y,\psi}$$

Simplifying...

$$
\begin{aligned}
S_{v_{y,\phi}} = & -\frac{\partial p_\psi}{\partial y}h_0 - \rho_0 h_0 v_{x,\psi}\frac{\partial v_{y0}}{\partial x} - \rho_0 h_0 v_{y,\psi}\frac{\partial v_{y0}}{\partial y} - \rho_0 h_\psi v_{x0}\frac{\partial v_{y0}}{\partial x} - \rho_0 h_\psi v_{y0}\frac{\partial v_{y0}}{\partial y} - \frac{\partial p_0}{\partial y}h_\psi \\
& - 0.5\frac{R}{C}\rho_0 v_{y0}\left\{\frac{(1+m_r)\,f_{r0}}{v_{r0}}\left[v_{x0}v_{x,\psi} + (v_{y0}-v_t)\,v_{y,\psi}\right] + \frac{(1+m_s)\,f_{s0}}{v_{s0}}\left(v_{x0}v_{x,\psi} + v_{y0}v_{y,\psi}\right)\right. \\
& \left. + \frac{v_{s0}m_s}{h_0}f_{s0}h_\psi + \frac{v_{r0}m_r}{h_0}f_{r0}h_\psi\right\} \\
& + 0.5\frac{R^2\Omega}{Cu_*}\rho_0\left\{\frac{(1+m_r)}{v_{r0}}f_{r0}\left[v_{x0}v_{x,\psi} + (v_{y0}-v_t)\,v_{y,\psi}\right] + \frac{m_r v_{r0}}{h_0}f_{r0}h_\psi\right\} - i\sigma\rho_0 h_0 v_{y,\psi}
\end{aligned}
$$