1 Bulk-flow energy equation

- x : axial direction
- y : circumferential direction

1.1 Energy equation for general thin film

Need to revisit derivation, i.e. moving from conservation of total energy to conservation of thermal energy starting on pg. 30 of thin film notes.

$$c_p\left(\frac{\partial\left(\rho h T\right)}{\partial t}+\frac{\partial\left(\rho h v_x T\right)}{\partial x}+\frac{\partial\left(\rho h v_y T\right)}{\partial y}\right)=-\left(q_z\right)_{h_1}^{h_2}+\alpha_{\hat{v}}Th\left(\frac{\partial p}{\partial t}+v_x\frac{\partial p}{\partial x}+v_y\frac{\partial p}{\partial y}\right) + R\Omega\,\tau_{zy}\big|^{h_2}-\left(\tau_{zx}v_x+\tau_{zy}v_y\right)_{h_1}^{h_2} \tag{1}$$

where $\alpha_{\hat{v}}$ is a thermal volumetric expansion coefficient.

$$\alpha_{\hat{v}}=\frac{1}{\hat{v}}\left(\frac{\partial\hat{v}}{\partial T}\right)_p \tag{2}$$

$\alpha_{\hat{v}}=0$ for incompressible fluids and $\alpha_{\hat{v}}=1$ for ideal gases.

Note that the volumetric expansion coefficient is sometimes expressed in terms of density...

$$\left(\frac{\partial\log\rho}{\partial\log T}\right)_p=T\underbrace{\left(\frac{1}{\rho}\frac{\partial\rho}{\partial T}\right)_p}_{\beta}=\left(-\rho T\frac{\partial(1/\rho)}{\partial T}\right)_p=\left(-\rho T\frac{\partial v}{\partial T}\right)_p=T\underbrace{\left(-\frac{1}{v}\frac{\partial v}{\partial T}\right)_p}_{\alpha_{\hat{v}}} \tag{3}$$

1.2 Shear stresses

Note that the shear stresses are symmetric, i.e. $\tau_{zx}=\tau_{xz}$ and $\tau_{zy}=\tau_{yz}$. The shear stress differences can be expressed in terms of frictions factors as

$$\tau_{zx}\big|_{h_1}^{h_2}=\tau_{zx}\big|^{h_2}-\tau_{zx}\big|^{h_1}=-0.5\rho\left(f_r v_r+f_s v_s\right)v_x \tag{4}$$

$$\tau_{zy}\big|_{h_1}^{h_2}=\tau_{zy}\big|^{h_2}-\tau_{zy}\big|^{h_1}=-0.5\rho\left(f_r v_r+f_s v_s\right)v_y+0.5\rho f_r v_r R\Omega \tag{5}$$

where

$$v_s=\sqrt{v_x^2+v_y^2} \tag{6}$$

$$v_r=\sqrt{v_x^2+\left(v_y-R\Omega\right)^2} \tag{7}$$

The shear stress differences do not contain pressure terms. However, the shear stress evaluated at a single surface contains both pressure and shear terms. Note that the sign on the pressure term needs to be confirmed as it differs between Hirs and San Andres (notes).

$$\tau_{zy}\big|^{h_2} = \frac{h}{2}\frac{\partial p}{\partial y} + \frac{a\rho}{4}\left(v_y^2 f_s - (v_y - R\Omega)^2 f_r\right) \tag{8}$$

where the weighting factor $a = 1$ for the simplest case.

1.3 Heat transfer

Heat flow to/from the rotor and stator to/from the bulk fluid can be expressed most simply using convective heat transfer coefficients and fixed rotor/stator temperatures (i.e. Newton's law of cooling) as

$$-\,q\big|_{h_1}^{h_2} = -\left(q\big|^{h_2} - q\big|^{h_1}\right) = -\left(q_r - q_s\right) = \left(q_s - q_r\right) \tag{9}$$

$$\left(q_s - q_r\right) = h_s\left(T_s - T\right) - h_r\left(T - T_r\right) \tag{10}$$

$$-\,q\big|_{h_1}^{h_2} = h_s\left(T_s - T\right) + h_r\left(T_r - T\right) \tag{11}$$

Note that the heat flux terms are vector quantities which are assumed to be positive for heat flow in the positive z direction. So q moving from the stator to the fluid is positive while q directed from the rotor into the fluid is negative.

Alternatively, we can simply denote the heat transfer with a generic Q that could be expressed as above

1.4 Energy equation

Inserting the shear stress and heat transfer term defined above into the energy equation results in

$$\begin{aligned} c_p\left(\frac{\partial(\rho h T)}{\partial t} + \frac{\partial(\rho h v_x T)}{\partial x} + \frac{\partial(\rho h v_y T)}{\partial y}\right) = Q &+ \alpha_{\hat{v}} T h\left(\frac{\partial p}{\partial t} + v_x\frac{\partial p}{\partial x} + v_y\frac{\partial p}{\partial y}\right) \\ &+ R\Omega\left[\frac{h}{2}\frac{\partial p}{\partial y} + \frac{a\rho}{4}\left(v_y^2 f_s - (v_y - R\Omega)^2 f_r\right)\right] \\ &+ v_x\left[0.5\rho\left(f_r v_r + f_s v_s\right)v_x\right] \\ &+ v_y\left[0.5\rho\left(f_r v_r + f_s v_s\right)v_y - 0.5\rho f_r v_r R\Omega\right] \end{aligned} \tag{12}$$

Expanding the shear stress terms...

$$\begin{aligned} c_p\left(\frac{\partial(\rho h T)}{\partial t} + \frac{\partial(\rho h v_x T)}{\partial x} + \frac{\partial(\rho h v_y T)}{\partial y}\right) = Q &+ \alpha_{\hat{v}} T h\left(\frac{\partial p}{\partial t} + v_x\frac{\partial p}{\partial x} + v_y\frac{\partial p}{\partial y}\right) + \frac{R\Omega h}{2}\frac{\partial p}{\partial y} \\ &+ \frac{R\Omega a\rho}{4}v_y^2 f_s - \frac{R\Omega a\rho}{4}\left(v_y - R\Omega\right)^2 f_r \\ &+ 0.5\rho v_x^2 f_r v_r + 0.5\rho v_x^2 f_s v_s + 0.5\rho v_y^2 f_r v_r \\ &+ 0.5\rho v_y^2 f_s v_s - 0.5\rho v_y R\Omega f_r v_r \end{aligned} \tag{13}$$

1.5 Non-dimensionalization

Non-dimensional forms of the bulk-flow equations assuming the convective integral approximations provided in the previous section.

$$\begin{aligned}
\tilde{h} &= \frac{h}{C} \\
\tilde{x} &= \frac{x}{R} \\
\tilde{y} &= \frac{y}{R} \\
\tilde{t} &= \omega t \\
\tilde{\rho} &= \frac{\rho}{\rho_*} \\
\tilde{\mu} &= \frac{\mu}{\mu_*} \\
\tilde{v}_x &= \frac{v_x}{u_*} \\
\tilde{v}_y &= \frac{v_y}{u_*} \\
\tilde{p} &= \frac{p}{\rho_* u_*^2} \\
\sigma &= \frac{\omega R}{u_*} \\
\tilde{T} &= \frac{T}{T_*} \\
\tilde{c}_p &= \frac{c_p}{c_p^*}
\end{aligned}$$

Note that $y \in 0, 2\pi R$, so following the non-dimensionalization above, $\tilde{y} = \theta \in 0, 2\pi$

1.6 Dimensionless form of energy equation

Dimensionless energy equation. Tilde $\tilde{}$ dropped from nondimensional variables to simplify notation.

$$\begin{aligned}
&\left[c_p^* \rho_* C T_* \omega\right] c_p \frac{\partial\left(\rho h T\right)}{\partial t} + \left[\frac{c_p^* \rho_* C T_* u_*}{R}\right] c_p \frac{\partial\left(\rho h v_x T\right)}{\partial x} + \left[\frac{c_p^* \rho_* C T_* u_*}{R}\right] c_p \frac{\partial\left(\rho h v_y T\right)}{\partial y} \\
&\quad = Q + \left[\rho_* C u_*^2 \omega\right] \alpha_{\hat{v}} T h \frac{\partial p}{\partial t} + \left[\frac{\rho_* C u_*^3}{R}\right] \alpha_{\hat{v}} T h v_x \frac{\partial p}{\partial x} + \left[\frac{\rho_* C u_*^3}{R}\right] \alpha_{\hat{v}} T h v_y \frac{\partial p}{\partial y} + \left[\frac{C \rho_* u_*^2}{R}\right] \frac{R\Omega h}{2} \frac{\partial p}{\partial y} \\
&\qquad + \left[\rho_* u_*^2\right] \frac{R\Omega a \rho}{4} v_y^2 f_s - \left[\rho_* u_*^2\right] \frac{R\Omega a \rho}{4} \left(v_y - \frac{R\Omega}{u_*}\right)^2 f_r + \left[\rho_* u_*^3\right] 0.5 \rho v_x^2 f_r v_r \\
&\qquad + \left[\rho_* u_*^3\right] 0.5 \rho v_x^2 f_s v_s + \left[\rho_* u_*^3\right] 0.5 \rho v_y^2 f_r v_r + \left[\rho_* u_*^3\right] 0.5 \rho v_y^2 f_s v_s - \left[\rho_* u_*^2\right] 0.5 \rho v_y R\Omega f_r v_r
\end{aligned} \tag{14}$$

Multiplying through by $\dfrac{1}{\rho_* c_p^* T_* u_*}$

$$
\begin{aligned}
&\left[\frac{\omega}{u_*}\right] c_p \frac{\partial\left(\rho h T\right)}{\partial t} + \left[\frac{C}{R}\right] c_p \frac{\partial\left(\rho h v_x T\right)}{\partial x} + \left[\frac{C}{R}\right] c_p \frac{\partial\left(\rho h v_y T\right)}{\partial y} \\
&= \left[\frac{1}{\rho_* c_p^* T_* u_*}\right] Q + \left[\frac{C\omega}{u_*} E_c\right] \alpha_{\hat{v}} T h \frac{\partial p}{\partial t} + \left[\frac{C}{R} E_c\right] \alpha_{\hat{v}} T h v_x \frac{\partial p}{\partial x} + \left[\frac{C}{R} E_c\right] \alpha_{\hat{v}} T h v_y \frac{\partial p}{\partial y} \\
&\quad + \left[\frac{C}{R} E_c\right] \frac{R\Omega}{u_*} \frac{h}{2} \frac{\partial p}{\partial y} + \left[E_c\right] \frac{R\Omega}{u_*} \frac{a\rho}{4} v_y^2 f_s - \left[E_c\right] \frac{R\Omega}{u_*} \frac{a\rho}{4} \left(v_y - \frac{R\Omega}{u_*}\right)^2 f_r + \left[E_c\right] 0.5\rho v_x^2 f_r v_r \\
&\quad + \left[E_c\right] 0.5\rho v_x^2 f_s v_s + \left[E_c\right] 0.5\rho v_y^2 f_r v_r + \left[E_c\right] 0.5\rho v_y^2 f_s v_s - \left[E_c\right] 0.5\rho v_y \frac{R\Omega}{u_*} f_r v_r
\end{aligned}
\tag{15}
$$

where $E_c = \dfrac{u_*^2}{c_p^* T_*}$ is the Eckert number. Expressing the transient terms in terms of the the dimensionless perturbation frequency, σ and multiplying through by $\dfrac{R}{C}$

$$
\begin{aligned}
&\left[\frac{\sigma R^2}{C}\right] c_p \frac{\partial\left(\rho h T\right)}{\partial t} + c_p \frac{\partial\left(\rho h v_x T\right)}{\partial x} + c_p \frac{\partial\left(\rho h v_y T\right)}{\partial y} \\
&= \left[\frac{R}{C \rho_* c_p^* T_* u_*}\right] Q + \left[\sigma E_c\right] \alpha_{\hat{v}} T h \frac{\partial p}{\partial t} + \left[E_c\right] \alpha_{\hat{v}} T h v_x \frac{\partial p}{\partial x} + \left[E_c\right] \alpha_{\hat{v}} T h v_y \frac{\partial p}{\partial y} + \left[E_c\right] \frac{R\Omega}{u_*} \frac{h}{2} \frac{\partial p}{\partial y} \\
&\quad + \left[\frac{R}{C} E_c\right] \frac{R\Omega}{u_*} \frac{a\rho}{4} v_y^2 f_s - \left[\frac{R}{C} E_c\right] \frac{R\Omega}{u_*} \frac{a\rho}{4} \left(v_y - \frac{R\Omega}{u_*}\right)^2 f_r + \left[\frac{R}{C} E_c\right] 0.5\rho v_x^2 f_r v_r \\
&\quad + \left[\frac{R}{C} E_c\right] 0.5\rho v_x^2 f_s v_s + \left[\frac{R}{C} E_c\right] 0.5\rho v_y^2 f_r v_r + \left[\frac{R}{C} E_c\right] 0.5\rho v_y^2 f_s v_s - \left[\frac{R}{C} E_c\right] 0.5\rho v_y \frac{R\Omega}{u_*} f_r v_r
\end{aligned}
\tag{16}
$$

Non-dimensionlize the rotor and stator velocities as well

$$
v_s = \sqrt{v_x^2 + v_y^2} \tag{17}
$$

$$
v_r = \sqrt{v_x^2 + \left(v_y - \frac{R\Omega}{u_*}\right)^2} = \sqrt{v_x^2 + \left(v_y - v_t\right)^2} \tag{18}
$$

where $v_t = R\Omega/u_*$ is the dimensionless rotor surface velocity.

1.7 Simplified form: steady, incompressible

$$
\begin{aligned}
c_p \frac{\partial\left(\rho h v_x T\right)}{\partial x} + c_p \frac{\partial\left(\rho h v_y T\right)}{\partial y} = &\left[\frac{R}{C \rho_* c_p^* T_* u_*}\right] Q + \left[E_c\right] \frac{R\Omega}{u_*} \frac{h}{2} \frac{\partial p}{\partial y} + \left[\frac{R}{C} E_c\right] \frac{R\Omega}{u_*} \frac{a\rho}{4} v_y^2 f_s \\
&- \left[\frac{R}{C} E_c\right] \frac{R\Omega}{u_*} \frac{a\rho}{4} \left(v_y - \frac{R\Omega}{u_*}\right)^2 f_r + \left[\frac{R}{C} E_c\right] 0.5\rho v_x^2 f_r v_r \\
&+ \left[\frac{R}{C} E_c\right] 0.5\rho v_x^2 f_s v_s + \left[\frac{R}{C} E_c\right] 0.5\rho v_y^2 f_r v_r \\
&+ \left[\frac{R}{C} E_c\right] 0.5\rho v_y^2 f_s v_s - \left[\frac{R}{C} E_c\right] 0.5\rho v_y \frac{R\Omega}{u_*} f_r v_r
\end{aligned}
\tag{19}
$$

Re-expressing the heat transfer in terms of convection coefficients

$$c_p \frac{\partial(\rho h v_x T)}{\partial x} + c_p \frac{\partial(\rho h v_y T)}{\partial y} = \frac{R}{C}\left[h_s\left(T_s - T\right) + h_r\left(T_r - T\right)\right] + \left[E_c\right]\frac{R\Omega}{u_*}\frac{h}{2}\frac{\partial p}{\partial y} + \left[\frac{R}{C}E_c\right]\left\{\frac{R\Omega}{u_*}\frac{a\rho}{4}v_y^2 f_s - \frac{R\Omega}{u_*}\frac{a\rho}{4}\left(v_y - \frac{R\Omega}{u_*}\right)^2 f_r + 0.5\rho v_x^2 f_r v_r + 0.5\rho v_x^2 f_s v_s + 0.5\rho v_y^2 f_r v_r + 0.5\rho v_y^2 f_s v_s - 0.5\rho v_y \frac{R\Omega}{u_*} f_r v_r\right\} \tag{20}$$

where the convective heat transfer coefficients have been non-dimensionalized as $\tilde{h} = \dfrac{h}{\rho_* c_p^* T_* u_*}$, but, again, the tilde has been dropped to simplify notation.