Problem

Context: Consider a weather prediction system where the actual weather is hidden, but we can observe certain weather-related phenomena like whether the ground is wet, if there are clouds in the sky, or if people are carrying umbrellas.

Hidden States: The possible weather conditions:

- S_1: Sunny
- S_2: Rainy

Observations: The observable phenomena:

- O_1: Dry ground
- O_2: Wet ground
- O_3: Clouds
- O_4: People carrying umbrellas

Parameters:

1. **Initial State Probabilities** (π):

$$\pi = [\pi_1 \quad \pi_2] = [0.6 \quad 0.4]$$

   This means there is a 60% chance that the day starts sunny and a 40% chance it starts rainy.

2. **Transition Probabilities** (A):

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

   This matrix indicates the probabilities of transitioning from one state to another. For instance, if it's sunny today, there's a 70% chance it will be sunny tomorrow and a 30% chance it will be rainy.

3. **Emission Probabilities** (B):

$$B = \begin{bmatrix} b_1(O_1) & b_1(O_2) & b_1(O_3) & b_1(O_4) \\ b_2(O_1) & b_2(O_2) & b_2(O_3) & b_2(O_4) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.1 & 0.3 & 0.1 \\ 0.1 & 0.5 & 0.3 & 0.1 \end{bmatrix}$$

   This matrix represents the probabilities of observing each phenomenon given the state. For instance, if it’s sunny, there’s a 50% chance the ground will be dry.

Objective: Given the sequence of observations $O = (O_3, O_4, O_2)$ (clouds, people carrying umbrellas, wet ground), determine the most likely sequence of hidden states (weather conditions) that produced these observations.

Solution

1. **Initialization**:

$$\delta_1(i) = \pi_i \cdot b_i(O_1)$$

For $O_1 = O_3$ (clouds),

$$\delta_1(1) = 0.6 \cdot 0.3 = 0.18 \quad \text{(Sunny)}$$

$$\delta_1(2) = 0.4 \cdot 0.3 = 0.12 \quad \text{(Rainy)}$$

2. **Recursion**:

$$\delta_t(j) = \max_i [\delta_{t-1}(i) \cdot a_{ij}] \cdot b_j(O_t)$$

For $O_2 = O_4$ (people carrying umbrellas):

$$\delta_2(1) = \max \begin{cases} \delta_1(1) \cdot a_{11} \cdot b_1(O_4) \\ \delta_1(2) \cdot a_{21} \cdot b_1(O_4) \end{cases}$$

$$= \max \begin{cases} 0.18 \cdot 0.7 \cdot 0.1 \\ 0.12 \cdot 0.4 \cdot 0.1 \end{cases} = \max \begin{cases} 0.0126 \\ 0.0048 \end{cases} = 0.0126 \quad \text{(Sunny)}$$

$$\delta_2(2) = \max \begin{cases} \delta_1(1) \cdot a_{12} \cdot b_2(O_4) \\ \delta_1(2) \cdot a_{22} \cdot b_2(O_4) \end{cases}$$

$$= \max \begin{cases} 0.18 \cdot 0.3 \cdot 0.1 \\ 0.12 \cdot 0.6 \cdot 0.1 \end{cases} = \max \begin{cases} 0.0054 \\ 0.0072 \end{cases} = 0.0072 \quad \text{(Rainy)}$$

For $O_3 = O_2$ (wet ground):

$$\delta_3(1) = \max \begin{cases} \delta_2(1) \cdot a_{11} \cdot b_1(O_2) \\ \delta_2(2) \cdot a_{21} \cdot b_1(O_2) \end{cases}$$

$$= \max \begin{cases} 0.0126 \cdot 0.7 \cdot 0.1 \\ 0.0072 \cdot 0.4 \cdot 0.1 \end{cases} = \max \begin{cases} 0.000882 \\ 0.000288 \end{cases} = 0.000882 \quad \text{(Sunny)}$$

$$\delta_3(2) = \max \begin{cases} \delta_2(1) \cdot a_{12} \cdot b_2(O_2) \\ \delta_2(2) \cdot a_{22} \cdot b_2(O_2) \end{cases}$$

$$= \max \begin{cases} 0.0126 \cdot 0.3 \cdot 0.5 \\ 0.0072 \cdot 0.6 \cdot 0.5 \end{cases} = \max \begin{cases} 0.00189 \\ 0.00216 \end{cases} = 0.00216 \quad \text{(Rainy)}$$

3. **Termination**:
   The highest probability at the end of the sequence gives us the most likely last state.

$$\max(\delta_3(1), \delta_3(2)) = \max(0.000882, 0.00216) = 0.00216 \quad \text{(Rainy)}$$

4. **Path Backtracking**:
   The most likely sequence is obtained by backtracking from the final state.

   - From δ_3, we find the last state is "Rainy".
   - From δ_2, given $\delta_3(2)$ came from $\delta_2(2) \cdot a_{22}$, the second state is "Rainy".
   - From δ_1, given $\delta_2(2)$ came from $\delta_1(2) \cdot a_{21}$, the first state is "Rainy".

   Therefore, the most likely sequence of states is (S_2, S_2, S_2) or "Rainy, Rainy, Rainy".