Relation of GCD and LCM of two numbers

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GCD and LCM Relationship

We want to prove that

$$\gcd(a,b) \times \mathrm{LCM}(a,b) = a \times b.$$

Let the prime factorization of a and b be:

$$a = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}, \quad b = p_1^{f_1} p_2^{f_2} \dots p_k^{f_k}$$

where $p_1, p_2, \dots, p_k$ are primes, and e_i, f_i are non-negative integers. The greatest common divisor and least common multiple of a and b are defined as:

$$\gcd(a,b) = p_1^{\min(e_1,f_1)} p_2^{\min(e_2,f_2)} \dots p_k^{\min(e_k,f_k)}$$

and

$$\mathrm{LCM}(a,b) = p_1^{\max(e_1,f_1)} p_2^{\max(e_2,f_2)} \dots p_k^{\max(e_k,f_k)}.$$

Now, multiplying $\gcd(a,b)$ and $\mathrm{LCM}(a,b)$ gives:

$$\gcd(a,b) \times \mathrm{LCM}(a,b) = \left(p_1^{\min(e_1,f_1)} p_2^{\min(e_2,f_2)} \dots p_k^{\min(e_k,f_k)}\right) \times \left(p_1^{\max(e_1,f_1)} p_2^{\max(e_2,f_2)} \dots p_k^{\max(e_k,f_k)}\right)$$

For each prime p_i, we have:

$$p_i^{\min(e_i,f_i)+\max(e_i,f_i)} = p_i^{e_i+f_i}.$$

Therefore, the product becomes:

$$\gcd(a,b) \times \mathrm{LCM}(a,b) = p_1^{e_1+f_1} p_2^{e_2+f_2} \dots p_k^{e_k+f_k} = a \times b.$$

Thus, we have proven that:

$$\gcd(a,b) \times \mathrm{LCM}(a,b) = a \times b.$$

Another short proof

Let

$$a = dx, \quad b = dy$$

where GCD(x,y) = 1. Then,

$$\mathrm{GCD}(a,b) = d$$

$$\mathrm{LCM}(a,b) = dxy$$

$$\gcd(a,b) \times \mathrm{LCM}(a,b) = d \times dxy = (dx) \times (dy) = a \times b$$