Lucas' Theorem

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Lucas' Theorem

In Lucas' Theorem:
Lucas' Theorem states that:

$$\binom{n}{k} \equiv \prod_{i=0}^{m} \binom{n_i}{k_i} \pmod{p}$$

where n_i and k_i are the digits of n and k in base p, respectively and p is a prime number.

Binomial Coefficient Property:
For any non-negative integers n and k, the binomial coefficient $\binom{n}{k}$ is defined as:

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } n \geq k, \\ 0 & \text{if } n < k. \end{cases}$$

Example

Step-by-step solution for $n = 6$, $k = 3$, and $p = 2$:

1. Convert n and k to base p (base 2 in this case):
We convert $n = 6$ and $k = 3$ to base 2.

$$n = 6 \quad \text{in base 2 is } 110_2 \quad \Rightarrow \quad n_2 = 1, n_1 = 1, n_0 = 0$$

$$k = 3 \quad \text{in base 2 is } 011_2 \quad \Rightarrow \quad k_2 = 0, k_1 = 1, k_0 = 1$$

2. Apply Lucas' Theorem:
Now apply the theorem by calculating the binomial coefficient for each digit:

$$\binom{n_2}{k_2} \times \binom{n_1}{k_1} \times \binom{n_0}{k_0} \pmod{p}$$

$$\binom{n_2}{k_2} = \binom{1}{0} = 1$$

$$\binom{n_1}{k_1} = \binom{1}{1} = 1$$

$$\binom{n_0}{k_0} = \binom{0}{1} = 0$$

3. Multiply the results:
Now, we multiply the results of each binomial coefficient:

$$\binom{n_2}{k_2} \times \binom{n_1}{k_1} \times \binom{n_0}{k_0} = 1 \times 1 \times 0 = 0$$

4. Final result:
Thus, by Lucas' Theorem, the result is:

$$\binom{6}{3} \equiv 0 \pmod{2}$$