Combinatorics: Complete Summary

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Permutation

Permutations count the number of ways to arrange a set of objects.
For a set of n objects, the number of permutations of r objects is given by:

$$P(n, r) = \frac{n!}{(n-r)!}$$

If there are 3 books and you want to arrange 2 of them on a shelf, the number of ways to do this is:

$$P(3, 2) = \frac{3!}{(3-2)!} = \frac{6}{1} = 6$$

The possible combinations (order matters) are: (A, B), (A, C), (B, A), (B, C), (C, A), (C, B)

Combination

Combinations count the number of ways to select r objects from a set of n objects without regard to the order of selection:

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Relation between $P(n, r)$ and $C(n, r)$:

$$C(n, r) = \frac{n!}{r!(n-r)!} = \frac{1}{r!} \times \frac{n!}{(n-r)!} = \frac{1}{r!} \times P(n, r)$$

If you have 5 books and you want to choose 3 to read, the number of ways to choose 3 books is:

$$C(5, 3) = \frac{5!}{3!(5-3)!} = \frac{120}{6 \times 2} = 10$$

The possible combinations (order does not matter) are: (A, B, C), (A, B, D), (A, B, E), (A, C, D), (A, C, E), (A, D, E), (B, C, D), (B, C, E), (B, D, E), (C, D, E)

ABC	ABD	ABE	ACD	ACE	ADE	BCD	BCE	BDE	CDE
ACB	ADB	AEB	ADC	AEC	AED	BDC	BEC	BED	CED
BAC	BAD	BAE	CAD	CAE	DAE	CBD	CBE	DBE	DCE
BCA	BDA	BEA	CDA	CEA	DEA	CDB	CEB	DEB	DEC
CAB	DAB	EAB	DAC	EAC	EAD	DBC	EBC	EBD	ECD
CBA	DBA	EBA	DCA	ECA	EDA	DCB	ECB	EDB	EDC

Theory of n^r

The expression n^r represents the number of ways to choose and arrange r objects from a set of n objects where repetition is allowed. The formula for this is:

$$n^r$$

This is used when we have r positions to fill, and each position can be filled by any of the n objects, allowing repetitions.

Example: If there are 3 objects (A, B, C) and we want to choose 2 objects for a design where repetition is allowed, the number of ways to do this is:

$$3^2 = 9$$

The possible combinations are: (A, A), (A, B), (A, C), (B, A), (B, B), (B, C), (C, A), (C, B), (C, C)
Thus, there are 9 possible combinations when repetition is allowed.

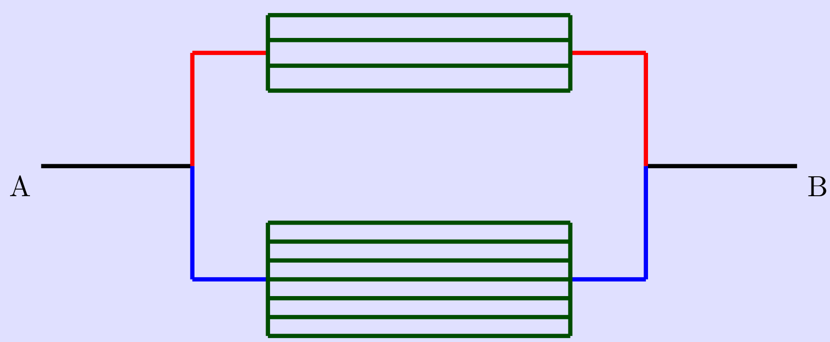
The Addition Principle

If there are two sets of mutually exclusive events (events that cannot happen at the same time), the total number of ways in which one of these events can happen is the sum of the individual possibilities.

$$|A \cup B| = |A| + |B|$$

Example: From A to B, there are 2 routes: Air or Bus. The Air route has 4 ways, and the Bus route has 7 ways. The total number of ways to travel from A to B is:

$$|A \cup B| = 4 + 7 = 11$$



The Multiplication Principle

If there are two events, one of which can happen in m ways and the other in n ways, then the total number of ways both events can happen is the product of m and n.

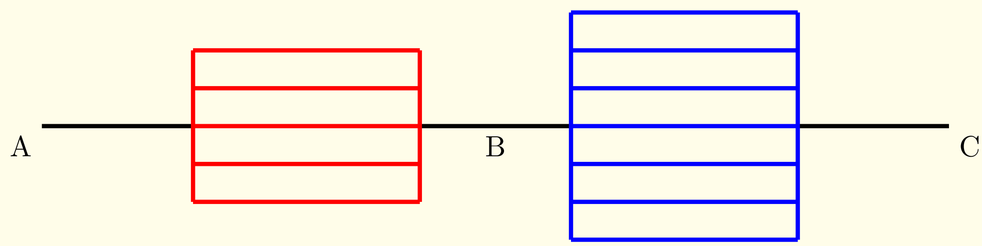
$$|A \times B| = m \times n$$

Example: If there are 3 shirts and 4 pants, the number of ways to choose a shirt and a pair of pants is:

$$|A \times B| = 3 \times 4 = 12$$

Example: There are 5 ways from A to B and there are 7 ways from B to C, so the ways from A to C are:

$$|A \times B| = 5 \times 7 = 35$$



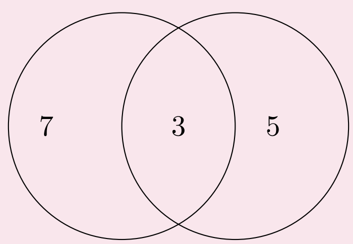
Inclusion and Exclusion Principle

This principle provides a way to count the number of elements in the union of two or more sets, considering their intersections.

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example: If there are 10 students who like basketball, 8 who like football, and 3 who like both, the number of students who like either basketball or football is:

$$|A \cup B| = 10 + 8 - 3 = 15$$

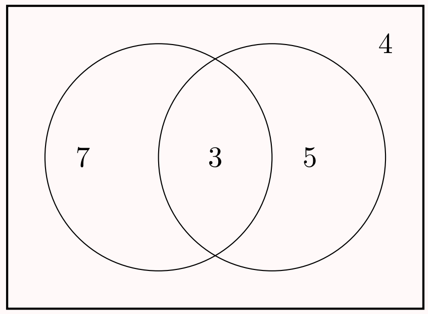


Principle of Complementation

The complement principle states that the number of elements not in a set is the total number of elements minus the number of elements in the set.

Example: If there are 10 students who like basketball, 8 who like football, and 3 who like both, and total student is 19 then, who don't like any games:

$$\text{Don't like any games} = 19 - 15 = 4$$



Over counting

When counting combinations or arrangements, we may accidentally count the same item multiple times. This can occur in problems involving symmetry or repeated objects. We adjust by dividing by the number of indistinguishable objects.

Example: If you have 3 red balls and 2 green balls, and you want to arrange them in a row, the number of ways to do so is:

$$\text{Ways} = \frac{5!}{3!2!} = \frac{120}{6 \times 2} = 10$$

MISSISSIPPI formula

Example: If you want to arrange the letters in the word "MISSISSIPPI", the number of distinct arrangements is calculated as follows:

$$\text{Ways} = \frac{11!}{1!4!4!2!} = \frac{39916800}{1 \times 24 \times 24 \times 2} = 34650$$

Thus, there are 34,650 distinct ways to arrange the letters in "MISSISSIPPI".

Fubini's Principle (Double Counting)

Fubini's principle is used to count in two or more dimensions by summing over smaller cases.

$$\tau(1)+\tau(2)+\cdots+\tau(z)=\left\lfloor\frac{z}{1}\right\rfloor+\left\lfloor\frac{z}{2}\right\rfloor+\cdots+\left\lfloor\frac{z}{z}\right\rfloor$$

$$\tau(1)+\tau(2)+\cdots+\tau(10)=\left\lfloor\frac{10}{1}\right\rfloor+\left\lfloor\frac{10}{2}\right\rfloor+\cdots+\left\lfloor\frac{10}{10}\right\rfloor$$

1										1
1	1									2
1		1								2
1	1		1							3
1				1						2
1	1	1			1					4
1						1				2
1	1		1				1			4
1		1						1		3
1	1			1					1	4
10	5	3	2	2	1	1	1	1	1	27

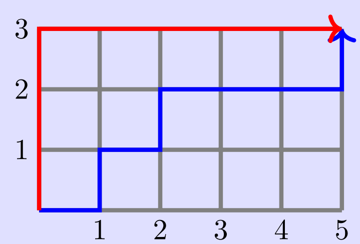
where $\tau(n)$ represents the number of divisors of n.

Grid

In a 5×3 grid, the total number of paths from the bottom-left corner to the top-right corner, using only up and right moves, is:

$$\text{Total paths}=\binom{8}{3}=\binom{8}{5}=\frac{8!}{3!\times5!}=56$$

This comes from choosing 3 "up" moves from 8 total moves (UUURRRRR) (3 "up" and 5 "right").



$$\text{Formula for total paths on a } x\times y \text{ grid}=\binom{x+y}{x}=\binom{x+y}{y}=\frac{(x+y)!}{x!\times y!}$$

Pascal's Triangle

A triangular array that shows the coefficients of the binomial expansions. The n-th row corresponds to the coefficients in the expansion of $(a+b)^n$.

Pascal's Triangle with Real Numbers

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}$$

Pascal's Triangle with Binomial Coefficients

$$\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6}
\end{array}$$

Binomial Expansion

The binomial theorem expresses the expansion of a binomial raised to a power:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

This can also be written as:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n} b^n$$

Where $\binom{n}{k}$ is the binomial coefficient, representing the number of ways to choose k elements from a set of n elements.

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Special Cases:

- If $a = b = 1$, then the expansion becomes:

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

- If $-a = b = -1$, then the expansion becomes:

$$(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{n} (-1)^n = 0$$

Pascal's Triangle Theorems

1. Pascal's Identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2} \quad \Rightarrow \quad 10 = 4 + 6$$

2. The Hockey Stick Theorem (also known as the Christmas Stocking Theorem):

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

$$\binom{3}{3} + \binom{4}{3} + \binom{5}{3} = \binom{6}{2} \quad \Rightarrow \quad 1 + 4 + 10 = 15$$

$$\binom{r}{0} + \binom{r+1}{1} + \binom{r+2}{2} + \cdots + \binom{r+k}{k} = \binom{r+k+1}{k}$$

$$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} = \binom{6}{2} \quad \Rightarrow \quad 1 + 4 + 10 = 15$$

If, $r = k = n$:

$$\sum_{k=0}^{n} \binom{n+k}{k} = \binom{2n+1}{n}$$

$$\sum_{k=0}^{3} \binom{3+k}{k} = \binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} = \binom{7}{3} \quad \Rightarrow \quad 1 + 4 + 10 + 20 = 35$$

3. Subset 2^n:

$$\sum_{k=0}^{n} \binom{n}{k} = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Example: For $n = 3$, we get:

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 1 + 3 + 3 + 1 = 8 = 2^3$$

$$\{\emptyset\}, \{\{A\}, \{B\}, \{C\}\}, \{\{A, B\}, \{B, C\}, \{C, A\}\}, \{\{A, B, C\}\}$$

Pascal's Triangle Theorems (Continued)

4. Chu-Vandermonde Identity:

$$\sum_{k=0}^{r} \binom{n+k}{k}\binom{m+k}{r-k} = \binom{n+m+r}{r}$$

For $n = 3, m = 2,$ and $r = 2,$ $\quad \sum_{k=0}^{2} \binom{3+k}{k}\binom{2+k}{2-k} = \binom{7}{2}$

$$\Rightarrow \binom{3}{0}\binom{2}{2} + \binom{4}{1}\binom{3}{1} + \binom{5}{2}\binom{4}{0} = \binom{7}{2}$$

$$\Rightarrow 1 \times 1 + 4 \times 3 + 10 \times 1 = 21$$

5. Symmetry of Binomial Coefficients:

$$\binom{n}{k} = \binom{n}{n-k}$$

For $n = 5, \binom{5}{2} = \binom{5}{3} = 10$

6. Fibonacci Sequence and Pascal's Triangle:

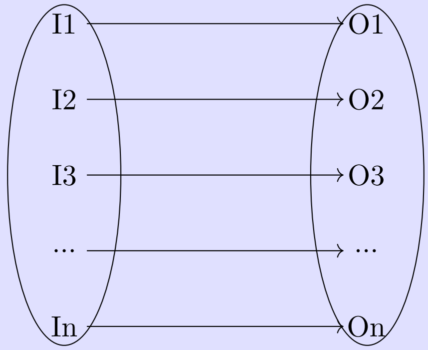
$$F_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \cdots + \binom{n-\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}$$

For $n = 7, F_7 = \sum_{k=0}^{\lfloor \frac{7}{2} \rfloor} \binom{7-k}{k} = \binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3}$

$$\Rightarrow F_7 = 1 + 6 + 10 + 5 = 22$$

The Bijection Principle

Two sets have the same cardinality if and only if there is a one-to-one correspondence (bijection) between them.



Problem

In a 5×3 grid, find out the total number of paths from the bottom-left corner to the top-right corner, using only up and right moves.

Problem

"UUURRRRR" from this word, how many total words can be built? Find out?
Both answer are:

$$\binom{8}{3} = \binom{8}{5} = \frac{8!}{3! \times 5!} = 56$$

Probability Theory

Probability is the measure of the likelihood that an event will occur. The probability of an event E is given by:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

Distribution of Balls into Boxes (Stars and Bars)

Stars and Bars is a method for counting the number of ways to distribute indistinguishable objects (balls) into distinguishable bins (boxes).

Stars and Bars (Non-negative Integer Solutions)

Consider the equation:

$$x_1 + x_2 + \cdots + x_k = n$$

where $x_i \geq 0$ are integers. The number of non-negative integer solutions is given by:

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

For example, if you have the equation $x_1 + x_2 + x_3 = 4$, the number of solutions is:

$$\binom{4+3-1}{3-1} = \binom{6}{2} = \binom{6}{4} = 15$$

Stars and Bars (Positive Integer Solutions)

Consider the equation:

$$x_1 + x_2 + \cdots + x_k = n$$

where $x_i > 0$ are integers.

$$(x_1 - 1) + (x_2 - 1) + \cdots + (x_k - 1) = n - k$$

Consider the equation now as:

$$a_1 + a2 + \cdots + a_k = n - k$$

where $a_i \geq 0$ are integers. The number of non-negative integer solutions is given by:

$$\binom{n-k+k-1}{k-1} = \binom{n-k+k-1}{n-k}$$

$$\binom{n-1}{k-1} = \binom{n-1}{n-k}$$

For example, if you have the equation $x_1 + x_2 + x_3 = 4$, the number of solutions is:

$$\binom{4-1}{3-1} = \binom{4-1}{4-3} = \binom{3}{2} = \binom{3}{1} = 3$$

PHP in details

Pigeonhole Principle

The Pigeonhole Principle is often expressed as:

$$\lceil \frac{n+1}{n} \rceil = 2$$

If n objects are placed in m boxes, and $n > m$, then at least one box must contain more than one object.
If $n+1$ objects are placed in n boxes then at least one box must contain minimum 2 objects.

Extended Pigeonhole Principle

The Extended Pigeonhole Principle states that:

$$\lceil \frac{n}{m} \rceil$$

Where n is the number of objects and m is the number of boxes. It asserts that if n objects are placed into m boxes, at least one box will contain at least $\lceil \frac{n}{m} \rceil$ objects.

Infinite Pigeonhole Principle

When n approaches infinity, we have:

$$\lceil \frac{n}{m} \rceil = \infty$$

This means that if there are infinitely many objects and a finite number of boxes, at least one box will contain infinitely many objects.

Strong Induction

In **Strong Induction**, the process is similar, but instead of assuming the statement holds for just one integer k, we assume it holds for all integers up to k. This is particularly useful when the inductive step requires more than just the previous case. The steps are as follows:

1. **Base Case:** Prove that the statement holds for the initial value (often $k = 1$).
2. **Inductive Hypothesis:** Assume that the statement holds for all integers from the base case up to some integer k.
3. **Inductive Step:** Using the assumption that the statement holds for all values from the base case to k, prove that it holds for $k + 1$.
4. **Conclusion:** By the principle of strong induction, the statement holds for all integers greater than or equal to the base case.

Example of Strong Induction

Example: Prove that every integer greater than or equal to 12 can be written as a sum of 4's and 5's.
Base Case: Prove the statement holds for the first few values, $n = 12$, $n = 13$, and $n = 14$. All of these can be written as sums of 4's and 5's:

$$12 = 4 + 4 + 4 \quad 13 = 5 + 4 + 4 \quad 14 = 5 + 5 + 4$$

Inductive Hypothesis: Assume that the statement holds for all integers from 12 up to k, i.e., $k \geq 12$ can be written as sums of 4's and 5's. **Inductive Step:** Show that $k + 1$ can also be written as a sum of 4's and 5's. Since k can be written as a sum of 4's and 5's, we can add a 4 to this sum to represent $k + 1$.
By strong induction, the statement holds for all $n \geq 12$.

Parity

Parity refers to whether an integer is even or odd. It is often used in combinatorics to solve problems that involve counting elements with certain characteristics.

- **Even + Even = Even**:
- **Even - Even = Even**:
- **Even + Odd = Odd**:
- **Even - Odd = Odd**:
- **Odd + Odd = Even**:
- **Odd - Odd = Even**:
- **Even × Even = Even**:
- **Even × Odd = Even**:
- **Odd × Odd = Odd**:

Invariant

An invariant is a property of a system that remains unchanged under certain operations or transformations. In combinatorics, it is often used to count the number of possible configurations of a problem.

Graph Theory

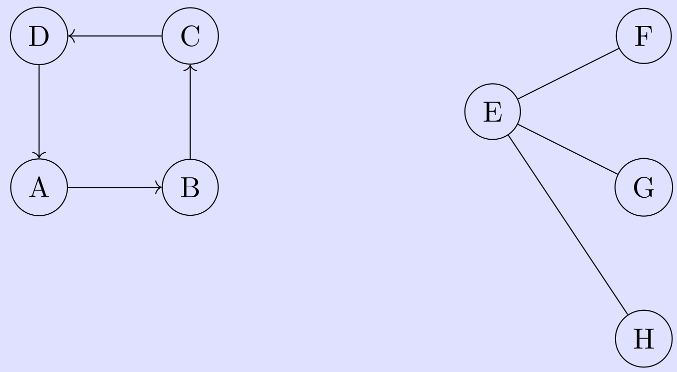


Diagram: Directed Square Graph (Left) and Undirected Graph (Right).

Here inner degree of A,B,C,D are 1 all, outer degree of A,B,C,D are 1 all and total degree of A,B,C,D are 2 all.
And degree of E is 3, and degree of F,G,H are 1 all.

- **Node (Vertex):** A fundamental unit of a graph, represented by a point. Each node can represent an object or an entity.
- **Edge (Arc):** A connection between two nodes. In a directed graph, the edge has a direction, while in an undirected graph, it does not.
- **Inner Degree (In-degree):** The number of edges directed towards a node in a directed graph.
- **Outer Degree (Out-degree):** The number of edges directed away from a node in a directed graph.
- **Total Degree:** The sum of in-degree and out-degree for a node in a directed graph. For an undirected graph, it is the number of edges connected to the node.
- **Directed Graph (Digraph):** A graph in which edges have a direction, represented by arrows.
- **Undirected Graph:** A graph in which edges do not have a direction. They are simply connections between two nodes.
- **Complete Graph:** A graph in which every pair of distinct nodes is connected by a unique edge.
- **Weighted Graph:** A graph in which edges have weights, usually representing cost or distance.
- **Hamiltonian Path:** A path in a graph that visits each vertex exactly once.
- **Euler Path:** A path that uses every edge in a graph exactly once.
- $$\sum \text{degrees} = 2 \times \text{edges}$$