Linear Algebra

1. System of Linear Equation
2. Matrix and Determinant
3. Euclidean Vector
4. Vector
5. Eigen value, vector, Diagonalisation

-Sativa

Linear Equation

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\cdots\cdots \quad \cdots\cdots \quad \cdots \quad \cdots\cdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

If any $b = 0$ like

$$\left[\, a_1x_1 + a_2x_2 + \cdots + a_mx_m = 0 \,\right]$$

then it's called

homogenous linear equation.

which has a trivial solution.

$$x_1 = x_2 = x_3 = \cdots = x_m = 0$$

Linear Combination

$Ax = B$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{bmatrix} = B$$

$$\therefore B = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \cdots \quad \cdots \quad \cdots \quad \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

[Augmented matrix]

$$= x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \cdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \cdots \\ a_{mn} \end{bmatrix}$$

Matrix Multiplication by Row Column Rule

$$AB = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1p} \\ a_{21} & a_{22} & \dots & a_{2p} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ \dots & \dots & \dots & \dots \\ b_{p1} & b_{p2} & \dots & b_{pn} \end{bmatrix}$$

$$(AB)_{23} = a_{21} b_{13} + a_{22} b_{23} + \dots + a_{2p} b_{p3}$$

$$= \sum_{a=1}^{p} a_{2a} b_{a3}$$

$$(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{ip} b_{pj}$$

$$= \sum_{a=1}^{p} a_{ia} b_{aj}$$

Matrix Product as Linear Combination

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7 \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5 \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \cdot \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

RFC

By row & by column:

$$\overset{A}{\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}} \overset{B}{\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}} = \overset{C}{\begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}}$$

RF

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} B$$

$$= \begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 6 & 0 \end{bmatrix} B$$

$$= \begin{bmatrix} 8 & -4 & 26 & 12 \end{bmatrix}$$

FC

$$A \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$A \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$

$$A \begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 30 \\ 26 \end{bmatrix}$$

$$A \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13 \\ 12 \end{bmatrix}$$

Column-Row Expansion

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix}$$

$$c_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad c_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad r_1 = \begin{bmatrix} 2 & 0 & 4 \end{bmatrix}$$

$$r_2 = \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

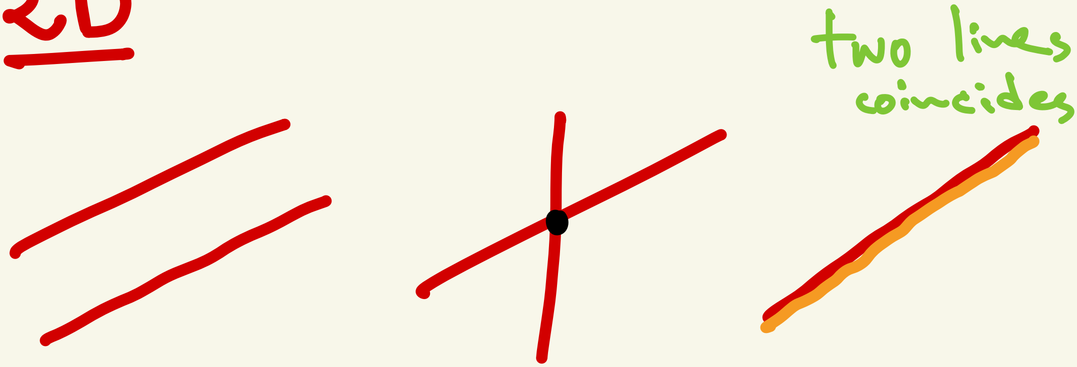
$$AB = c_1 r_1 + c_2 r_2$$

$$= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} \begin{bmatrix} -3 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

2D



no solution — inconsistent

single solution — consistent

infinity solution — consistent

3D

Row-Echelon Form $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$

Pivot

Reduced Row-Echelon / $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Gausian-Jordan Elimination

Here we use Elementary Row Operation

Inverse by Row operations

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

Answer

Rank $[\rho(n)]$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$$

$\det(A) = 0$ $\quad \therefore \rho(A) < 3$

$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \rightarrow 4 - 2 = 2 \neq 0$

$\therefore \rho(A) = 2$

or by row echelon,

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ $\quad \therefore \rho(A) = 2$

rank

Elementry Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

① $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 = R_2 * 7$

det(1) = 7

② $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $R_2 \xleftrightarrow{\text{swap}} R_3$

det(2) = −1

③ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ $R_3 = R_3 + (R_1 * 3)$

det(3) = 1

Row equivalent all are.

Made with Goodnotes

Eauation solving by inverse

$$x_1 + 2x_2 + 3x_3 = 5$$

$$2x_1 + 5x_2 + 3x_3 = 3$$

$$x_1 \qquad + 8x_3 = 17$$

$$\begin{array}{ccc} A & x & = \quad B \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

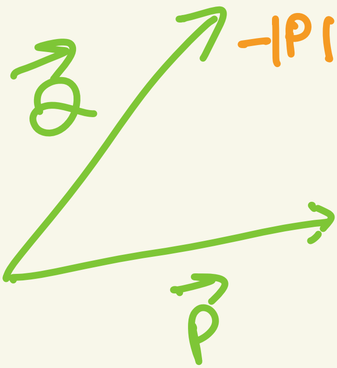
$$\therefore \quad x = A^{-1} B$$

$$= \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 17 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

Vectors

$-1 \leq \cos \theta \leq 1$



$-|P||Q| \leq |P||Q| \cos \theta \leq |P||Q|$

$-|P||Q| \leq P.Q \leq |P||Q|$

$|p_1q_1 + p_2q_2 + p_3q_3 + \dots| \leq$

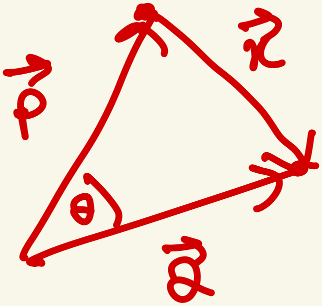
$\sqrt{p_1^2 + p_2^2 + \dots}$

$\cdot \sqrt{q_1^2 + q_2^2 + \dots}$

$\vec{P}.\vec{Q} \leq |\vec{P}|.|\vec{Q}|$

Cauchy-Schwarz Inequality

Dot product



$$\vec{x} = \vec{Q} - \vec{P}$$

$$= (a_1 - p_1)\hat{i} + (a_2 - p_2)\hat{j}$$

$$\| \vec{Q} - \vec{P} \| = \|Q\|^2 + \|P\|^2 - 2\|Q\|\|P\|\cos\theta$$

$$\Rightarrow \|Q\|\|P\|\cos\theta = \frac{\|Q\|^2 + \|P\|^2 - \|\vec{Q} - \vec{P}\|}{2}$$

$$\Rightarrow Q \cdot P = \frac{a_1^2 + a_2^2 + p_1^2 + p_2^2 - (a_1 - p_1)^2 - (a_2 - p_2)^2}{2}$$

$$= \frac{2a_1p_1 + 2a_2p_2}{2} = a_1p_1 + a_2p_2$$

1. $\|u+v\| \leq \|u\| + \|v\|$

$$\|u+v\|^2 = (u+v)\cdot(u+v)$$

$$= u\cdot u + 2u\cdot v + v\cdot v$$

[C.S.I.] $$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$= (\|u\| + \|v\|)^2$$

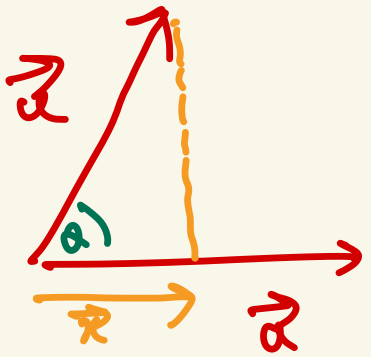
2. $$\|u+v\|^2 + \|u-v\|^2 = (u+v)(u+v) + (u-v)(u-v)$$

$$= 2\,u\cdot u + 2\,v\cdot v$$

$$= 2\|u\|^2 + 2\|v\|^2$$

3. $$\|u+v\|^2 - \|u-v\|^2 = 4(u\cdot v)$$

Orthogonal Projection



$\vec{u} - \vec{x}$

vector component of $\vec{u}$ Along $\vec{a}$

$$\vec{u} \cdot \vec{a} = \|u\| \|a\| \cos\theta$$

$$\|u\| \cos\theta = \frac{u \cdot a}{\|a\|}$$

$$\therefore \vec{x} = \frac{u \cdot a}{\|a\|} * \frac{\vec{a}}{\|a\|}$$

$$\boxed{Au \cdot v = u \cdot A^T v}$$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad u = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad v = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

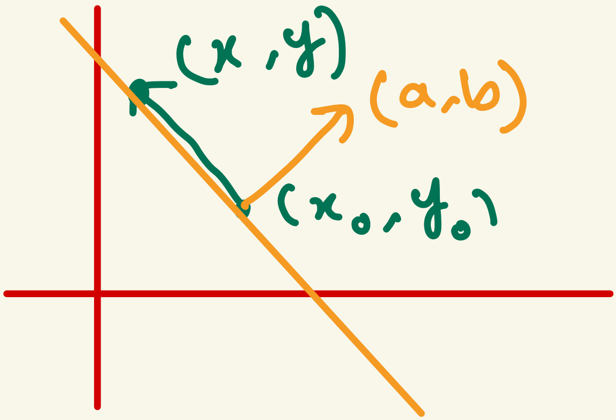
$$Au = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix} \qquad A^T v = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

$$Au \cdot v = -14 + 0 + 25 = 11 \checkmark$$

$$u \cdot A^T v = 7 + 8 - 4 = 11 \checkmark$$

Made with Goodnotes

Point - Normal Equation



$$a(x-x_0)+b(y-y_0)=0$$

for 3D

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$6x+y-11=0$$

$$6(x-a)+(y-b)=0$$

$$-6a-b=-11$$

$$6(x-\overset{3}{3})+(y+\overset{-7}{7})=0$$

Vector Eqn., Parametric Eqn.

$x = tv$, $x = t_1 v_1 + t_2 v_2$

$x = x_0 + tv$

$x = x_0 + t_1 v_1 + t_2 v_2$

origin $\quad v = (-2, 3)$

$(x, y) = t(-2, 3)$

$\therefore x = -2t$, $y = 3t$

$(1, 2, -3) \quad v = (4, -5, 1)$

$(x, y, z) = (1, 2, -3) + t(4, -5, 1)$

$\therefore x = 1 + 4t$, $y = 2 - 5t$, $z = -3 + t$

Distance

$ax+by+c=0$	(x_0, y_0)

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$$

Eigenvalues and Eigenvectors

$$Ax = \lambda x$$

nxn matrix

eigen vector of A

eigen value

Ax is a scaler multiple of x

$$\begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$(\lambda I - A)x = 0$$

$$\det(\lambda I - A) = 0 \leftarrow \text{characteristic eqn.}$$

Date:

M.

$\det(\lambda I - A)$:

$$= \lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3 & 0\\ 8 & -1\end{bmatrix}$$

$$= \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix} - \begin{bmatrix}3 & 0\\ 8 & -1\end{bmatrix}$$

$$= \begin{bmatrix}\lambda-3 & 0\\ -8 & \lambda+1\end{bmatrix}$$

$$\det(M) = \begin{vmatrix}\lambda-3 & 0\\ -8 & \lambda+1\end{vmatrix} = 0$$

$$(\lambda-3)(\lambda+1) = 0$$

$$\therefore \lambda = 3 \, , \quad \lambda = -1$$

Charesteric Polynomial

Date:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

$\det(\lambda I - A)$

$$\Rightarrow \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda - 8 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 8\lambda^2 + 17\lambda + 4 = 0$$

$$\Rightarrow (\lambda - 4)(\lambda^2 - 4\lambda + 1) = 0$$

$$\therefore \lambda = 4,\ 2 \pm \sqrt{3}$$

Eigen space

Date:

$$\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad \begin{vmatrix} \lambda+1 & -3 \\ -2 & \lambda \end{vmatrix} = \lambda(\lambda+1) - 6$$

$$= (\lambda-2)(\lambda+3)$$

$\lambda = 2$ $\qquad\qquad$ $\lambda = -3$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 = x_2 = t$ $\quad \begin{bmatrix} t \\ t \end{bmatrix}$

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -3$

$$\begin{bmatrix} -2 & -3 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix}$$

basis

Basis

Date:

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)^2 = 0$$

$\lambda = 1$ $\lambda = 2$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2s \\ -s \\ -s \end{bmatrix} \qquad \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ -a \end{bmatrix}$$

$$= s \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \qquad \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} = b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

basis

Diagonalization

Date:

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda-2 & -1 \\ -1 & 0 & \lambda-3 \end{vmatrix} = 0$$

$$\Rightarrow (\lambda-2)^2(\lambda-1) = 0$$

$$\therefore \lambda = 2, 1$$

$\lambda = 2$

$$\begin{bmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 1$

$$q \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $P^{-1} \cdot A \cdot P = \begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & \cancel{z} \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$ is not diagonizable!

because $(\lambda-1)(\lambda-2)^2 = 0$

$\lambda = 1$

$P_1 \begin{bmatrix} \frac{1}{8} \\ -\frac{1}{8} \\ 1 \end{bmatrix}$ $P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

নাই, তাই

3x3 matrix, but two basis

so.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}$$

$$\lambda = 4, \quad \lambda = 2 \pm \sqrt{3}$$

$$\therefore P^{-1} A P = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2+\sqrt{3} & 0 \\ 0 & 0 & 2-\sqrt{3} \end{bmatrix}$$

$$D = P^{-1} A P$$

$$\therefore A = P D P^{-1}$$

$$A^k = P D^k D^{-1}$$

Vector Space (Spanning)

$u = (1, 2, -1)$, $v = (6, 4, 2)$

linear combi. $\longrightarrow$ $w_1 = (9, 2, 7)$

not $\longrightarrow$ $w_2 = (4, -1, 8)$

$$\underset{-3}{k_1}(1, 2, -1) + \underset{2}{k_2}(6, 4, 2) = (9, 2, 7)$$

$\therefore$
$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

$k_1 = -3$

$k_2 = 2$

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

inconsistent

* $v_1 = (1,1,2)$, $v_2 = (1,0,1)$, $v_3 = (2,1,3)$

$$b = (b_1, b_2, b_3)$$

$$b = k_1 v_1 + k_2 v_2 + k_3 v_3$$

$$k_1 + k_2 + 2k_3 = b_1$$

$$k_1 + \quad + k_3 = b_2$$

$$2k_1 + k_2 + 3k_3 = b_3$$

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 0$$

$\therefore$ no solu.

$\therefore$ not linear combination

Solution Space

▶ **EXAMPLE 16 Solution Spaces of Homogeneous Systems**

In each part, solve the system by any method and then give a geometric description of the solution set.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Solution

(a) The solutions are

$$x = 2s - 3t, \quad y = s, \quad z = t$$

from which it follows that

$$x = 2y - 3z \quad \text{or} \quad x - 2y + 3z = 0$$

This is the equation of a plane through the origin that has $\mathbf{n} = (1, -2, 3)$ as a normal.

(b) The solutions are

$$x = -5t, \quad y = -t, \quad z = t$$

which are parametric equations for the line through the origin that is parallel to the vector $\mathbf{v} = (-5, -1, 1)$.

(c) The only solution is $x = 0$, $y = 0$, $z = 0$, so the solution space consists of the single point $\{\mathbf{0}\}$.

(d) This linear system is satisfied by all real values of x, y, and z, so the solution space is all of R^3. ◀

Linear Independence

$v_1 = (1,-2,3)$, $v_2 = (5,6,-1)$, $v_3 = (3,2,1)$

$$\boxed{k_1 v_1 + k_2 v_2 + k_3 v_3 = 0}$$

$$k_1 + 5k_2 + 3k_3 = 0$$
$$-2k_1 + 6k_2 + 2k_3 = 0$$
$$3k_1 - k_2 + k_3 = 0$$

$k_1 = -\frac{1}{2}t$, $k_2 = -\frac{1}{2}t$, $k_3 = t$

$\therefore \quad v_1 + v_2 - 2v_3 = 0$

[dependent]

EXAMPLE 3 Linear Independence in R^4

Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), \quad \mathbf{v}_2 = (4, 9, 9, -4), \quad \mathbf{v}_3 = (5, 8, 9, -5)$$

in R^4 are linearly dependent or linearly independent.

Solution The linear independence or linear dependence of these vectors is determined by whether there exist nontrivial solutions of the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{0}$$

or, equivalently, of

$$k_1(1, 2, 2, -1) + k_2(4, 9, 9, -4) + k_3(5, 8, 9, -5) = (0, 0, 0, 0)$$

Equating corresponding components on the two sides yields the homogeneous linear system

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0 \\ 2k_1 + 9k_2 + 8k_3 &= 0 \\ 2k_1 + 9k_2 + 9k_3 &= 0 \\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned}$$

We leave it for you to show that this system has only the trivial solution

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = 0$$

from which you can conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

EXAMPLE 4 An Important Linearly Independent Set in P_n

Show that the polynomials

$$1, \quad x, \quad x^2, \ldots, \quad x^n$$

form a linearly independent set in P_n.

Solution For convenience, let us denote the polynomials as

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \ldots, \quad \mathbf{p}_n = x^n$$

We must show that the only coefficients satisfying the vector equation

$$a_0\mathbf{p}_0 + a_1\mathbf{p}_1 + a_2\mathbf{p}_2 + \cdots + a_n\mathbf{p}_n = \mathbf{0} \tag{5}$$

are

$$a_0 = a_1 = a_2 = \cdots = a_n = 0$$

But (5) is equivalent to the statement that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = 0 \tag{6}$$

for all x in $(-\infty, \infty)$, so we must show that this is true if and only if each coefficient in (6) is zero. To see that this is so, recall from algebra that a nonzero polynomial of degree n has at most n distinct roots. That being the case, each coefficient in (6) must be zero, for otherwise the left side of the equation would be a nonzero polynomial with infinitely many roots. Thus, (5) has only the trivial solution.

EXAMPLE 5 Linear Independence of Polynomials

Determine whether the polynomials

$$\mathbf{p}_1 = 1 - x, \quad \mathbf{p}_2 = 5 + 3x - 2x^2, \quad \mathbf{p}_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in P_2.

Solution The linear independence or dependence of these vectors is determined by whether the vector equation

$$k_1\mathbf{p}_1 + k_2\mathbf{p}_2 + k_3\mathbf{p}_3 = \mathbf{0} \tag{7}$$

can be satisfied with coefficients that are not all zero. To see whether this is so, let us rewrite (7) in its polynomial form

$$k_1(1 - x) + k_2(5 + 3x - 2x^2) + k_3(1 + 3x - x^2) = 0 \tag{8}$$

or, equivalently, as

$$(k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

Since this equation must be satisfied by all x in $(-\infty, \infty)$, each coefficient must be zero (as explained in the previous example). Thus, the linear dependence or independence of the given polynomials hinges on whether the following linear system has a nontrivial solution:

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -\ 2k_2 - k_3 &= 0 \end{aligned} \tag{9}$$

We leave it for you to show that this linear system has nontrivial solutions either by solving it directly or by showing that the coefficient matrix has determinant zero. Thus, the set $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is linearly dependent. ◀

Wronskian

$f_1 = x$, $f_2 = \sin x$

$$w(x) = \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} = x\cos x - \sin x$$

$$w\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \quad \text{linearly ind.}$$

$f_1 = 1$ $\quad f_2 = e^x$ $\quad f_3 = e^{2x}$

$$w(x) = \begin{vmatrix} 1 & e^x & e^{2x} \\ 0 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} = 2e^{3x}$$

the function is non identically zero on $(-\infty, \infty)$ so linearly independent.

Form a basis

[we have to show
* linear independent
* span R^3.]

$v_1 = (1,2,1)$ $v_2 = (2,9,0)$ $v_3 = (3,3,4)$

Let $b = (b_1, b_2, b_3)$

and $b = k_1 v_1 + k_2 v_2 + k_3 v_3$

$(b_1, b_2, b_3) = k_1 (1,2,1) + k_2 (2,9,0) + k_3 (3,3,4)$

$$k_1 + 2k_2 + 3k_3 = b_1$$
$$2k_1 + 9k_2 + 3k_3 = b_2$$
$$k_1 + 0 \cdot k_2 + 4k_3 = b_3$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{vmatrix} = -1 \neq 0$$

$Ax = 0$ has only trivial solution.

$Ax = b$ is consistent.

$\therefore v_1, v_2, v_3$ form a basis for R^3

* $v_1 = (1,2,1)$ $v_2 = (2,9,0)$ $v_3 = (3,3,4)$

find the coordinate vector of $v = (5,-1,9)$ relative to the basis $S = \{v_1, v_2, v_3\}$

$$k_1 + 2k_2 + 3k_3 = 5$$

$$2k_1 + 9k_2 + 3k_3 = 1$$

$$k_1 + 0 . k_2 + 4k_3 = -9$$

$$k_1 = 1, \quad k_2 = -1, \quad k_3 = 2$$

$$\therefore (v)_S = (1,-1,2)$$

If $(v)_S = (-1,3,2)$ $v = ?$

$$v = (-1)v_1 + 3v_2 + 2v_3 = (11,31,7)$$

▶ EXAMPLE 4 The Standard Basis for M_{mn}

Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

Solution We must show that the matrices are linearly independent and span M_{22}. To prove linear independence we must show that the equation

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \mathbf{0} \tag{4}$$

has only the trivial solution, where $\mathbf{0}$ is the 2×2 zero matrix; and to prove that the matrices span M_{22} we must show that every 2×2 matrix

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

can be expressed as

$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = B \tag{5}$$

The matrix forms of Equations (4) and (5) are

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$c_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since the first equation has only the trivial solution

$$c_1 = c_2 = c_3 = c_4 = 0$$

the matrices are linearly independent, and since the second equation has the solution

$$c_1 = a, \quad c_2 = b, \quad c_3 = c, \quad c_4 = d$$

the matrices span M_{22}. This proves that the matrices M_1, M_2, M_3, M_4 form a basis for M_{22}. More generally, the mn different matrices whose entries are zero except for a single entry of 1 form a basis for M_{mn} called the ***standard basis for*** $\boldsymbol{M_{mn}}$. ◀

▶ EXAMPLE 7 Coordinates Relative to the Standard Basis for R^n

In the special case where $V = R^n$ and S is the *standard basis*, the coordinate vector $(\mathbf{v})_S$ and the vector $\mathbf{v}$ are the same; that is,

$$\mathbf{v} = (\mathbf{v})_S$$

For example, in R^3 the representation of a vector $\mathbf{v} = (a, b, c)$ as a linear combination of the vectors in the standard basis $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

so the coordinate vector relative to this basis is $(\mathbf{v})_S = (a, b, c)$, which is the same as the vector $\mathbf{v}$.

▶ EXAMPLE 8 Coordinate Vectors Relative to Standard Bases

(a) Find the coordinate vector for the polynomial

$$\mathbf{p}(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

relative to the standard basis for the vector space P_n.

(b) Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22}.

Solution (a) The given formula for $\mathbf{p}(x)$ expresses this polynomial as a linear combination of the standard basis vectors $S = \{1, x, x^2, \ldots, x^n\}$. Thus, the coordinate vector for $\mathbf{p}$ relative to S is

$$(\mathbf{p})_S = (c_0, c_1, c_2, \ldots, c_n)$$

Solution (b) We showed in Example 4 that the representation of a vector

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

as a linear combination of the standard basis vectors is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so the coordinate vector of B relative to S is

$$(B)_S = (a, b, c, d)$$

▶ EXAMPLE 3 Dimension of a Solution Space

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\quad + 2x_5 \qquad\quad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 + 10x_4 \qquad\quad + 15x_6 &= 0 \\
2x_1 + 6x_2 \qquad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Solution In Example 6 of Section 1.2 we found the solution of this system to be

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

which can be written in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$$

or, alternatively, as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

span the solution space. We leave it for you to check that these vectors are linearly independent by showing that none of them is a linear combination of the other two (but see the remark that follows). Thus, the solution space has dimension 3. ◀

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & 0 \\ 0 & 0 & 5 & 10 & 0 & 15 & 0 \\ 2 & 6 & 0 & 8 & 4 & 18 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 3

dimension = 4

nullity = 1

4

null space $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

column space $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

row space $\begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 & 1 & 2 & 0 & 0 & 0 \end{bmatrix}$
$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$