

Algebraic Geometry

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1 Terminology

The affine space of field k is denoted by \mathbb{A}_k^n which is the Cartesian n-product of k . Let $f \in k[x_1, \dots, x_n]$ be a polynomial. Then, $V(f)$ is the set of zeros of f and is called the hypersurface defined by f . If S is a set of polynomials from $k[x_1, \dots, x_n]$, then $V(S) := \{p \in \mathbb{A}_k^n \mid f(p) = 0, \forall f \in S\}$. One can check that $V(S) = \cap_{f \in S} V(f)$. When $S = \{f_1, \dots, f_r\}$, we write $V(S)$ as $V(f_1, \dots, f_r)$.

A subset $X \subseteq \mathbb{A}_k^n$ is called an affine algebraic set if $X = V(S)$ for some set S of polynomials in $k[x_1, \dots, x_n]$. One can easily show that if I is the ideal in $k[x_1, \dots, x_n]$ generated by polynomials in S , then $V(S) = V(I)$.

For a subset $X \subseteq \mathbb{A}_k^n$, consider the ideal in $k[x_1, \dots, x_n]$ generated by polynomials that vanish on X . This ideal is called the ideal of X , denoted by $I(X)$.

2 Hilbert Basis Theorem

Definition 1. A ring R is called Noetherian if every ideal in R is finitely generated.

Fields and Principal Ideal Domains (PIDs) are Noetherian rings.

Theorem 1. (Hilbert Basis Theorem) If R is a Noetherian ring, then $R[x_1, \dots, x_n]$ is a Noetherian Ring.

Proof. We know $R[x_1, \dots, x_n] \cong R[x_1, \dots, x_{n-1}][x_n]$. So, if we can prove that R Noetherian implies $R[x]$ is Noetherian, by induction we will have proven that $R[x_1, \dots, x_n]$ is also Noetherian.

Suppose R is Noetherian. Let I be an ideal in $R[x]$. Let J denote the set of leading coefficients of polynomials in I . Then, given I is an ideal, J is an ideal in R . Since R is Noetherian, we can write that J is generated by the leading coefficients of $f_1, \dots, f_r \in I$. Suppose $N \in \mathbb{Z}$ such that N is greater than the degrees of all polynomials f_1, \dots, f_r . Then, for any $m \leq N$, we define J_m to be the ideal in R generated by the leading coefficients of all polynomials f in I such that $\deg(f) \leq m$. Once again, since J_m is an ideal in R , we can say that J_m is generated by the finite set of polynomials, $\{f_{mj}\}$, such that each polynomial's degree is less than or equal to m . Finally, define I' be the ideal generated by polynomials $\{f_{jm}\}$ and f_i .

We claim $I' = I$. Suppose not i.e suppose there exists elements in I that are not in I' . Let g be the minimal element such that $g \in I$, $g \notin I'$.

Case 1: $\deg(g) > N$. Then, there exists polynomials Q_i such that $\sum_i Q_i f_i$ has the same leading term as g . Therefore, $\deg(g - \sum_i Q_i f_i) < \deg(g)$. Clearly, $g - \sum_i Q_i f_i$ is in I' . But since g is the minimal element and $\deg(g - \sum_i Q_i f_i) < \deg(g)$, therefore $g - \sum_i Q_i f_i \in I'$, which implies $g \in I'$.

Case 2: $m := \deg(g) \leq N$. Then, there exists polynomials Q_j such that $\sum_j Q_j f_{mj}$ and g have the same leading term. Using a similar argument, we get that $g \in I'$. \square

Theorem 2. *An algebraic set is the intersection of a finite number of hypersurfaces.*

Proof. Let $V(I)$ be an algebraic set. We prove that I is finitely generated since that implies $V(I) = V(f_1, \dots, f_r) = \cap_{i=1}^r V(f_i)$. Given k is a field, k is a Noetherian ring and by the Hilbert Basis Theorem, $k[x]$ is also Noetherian. Therefore, the ideal I in $k[x]$ is finitely generated. \square

Corollary 3. *$k[x_1, \dots, x_n]$ is a Noetherian ring for any field k .*

Proof. Follows from the Hilbert Basis Theorem. \square

3 Modules Revision

Definition 2. *R-Module.*

Let R be a ring. Let M be an abelian group $(M, +)$. Then, an R -module is M with multiplication $R \times M \rightarrow M$ such that for any $a, b \in R$, $m \in M$, $(a + b)m = am + bm$, $a(m + n) = am + an$, $(ab)m = a(bm)$, $1_R m = m$.

Definition 3. *Submodule.*

A submodule N is a subgroup of R -module, M , such that $an \in N$ for any $a \in R, n \in N$.

One can check that $0_R m = 0_M$ by noting that $0_R m = (x - x)m = xm - xm = 0_M$ for any $x \in R, m \in M$. Also, the submodule N of an R -module is an R -module itself.

Definition 4. *Submodule generated by S .*

Let $S := \{s_1, s_2, \dots\}$ be a set of elements of the R -module M . Then the submodule generated by S is $\{\sum_i r_i s_i | r_i \in R, s_i \in S\}$.

When S is finite, we denote the submodule generated by S as $\sum_i R s_i$.

Definition 5. *Finiteness conditions of subrings of a ring.*

Let S be a ring and let R be a subring of S .

(1) S is module-finite over R if S is finitely-generated as an R -module i.e $S = \sum R v_i$ where $v_1, \dots, v_n \in S$.

(2) S is ring-finite over R if $S = R[v_1, \dots, v_n] = \{\sum_i a_i v_1^{i_1} \cdots v_n^{i_n} | a_i \in R\}$ where $v_1, \dots, v_n \in S$.

(3) S is a finitely-generated field extension of R if S and R are fields and $S = R(v_1, \dots, v_n)$ (the quotient field of $R[v_1, \dots, v_n]$) where $v_1, \dots, v_n \in S$.

Properties:

1. If S is module-finite over R , then S is ring-finite over R . (This is straightforwardly seen from the definitions)
2. If $L = K(x)$, then L is a finitely-generated field extension of K but L is not ring-finite over K .

Proof. Using the definition, $K(X)$ is a finitely-generated field extension of K . Now, suppose K is ring-finite over K . Then, $L = K[v_1, \dots, v_n]$. Then, there exists $\frac{s_i}{t_i} \in K(X)$ that generate

L where $i = 1, \dots, n$. Define $p := 1/q$. Then, as $p \in K(X)$, $p = \frac{h}{t_1^{e_1} \dots t_n^{e_n}}$. Now, if we choose q to be an irreducible polynomial that has a higher degree than all t_i 's, we see that p cannot be equal to $\frac{1}{q}$. \square

Definition 6. *Integral elements*

Let R be a subring of the ring S . Then, $v \in S$ is integral over R if there exists a monic polynomial $f = x^n + a_1x^{n-1} + \dots + a_n \in R[x]$ such that $f(v) = 0$ and $a_i \in R$.

When all elements of S is integral over R , we say S is integral over R . When S and R are fields and S is integral over R , we call S an algebraic extension of R .

Theorem 4. Let R be a subring over an integral domain S and let $v \in S$. Then, the following are equivalent:

- (1) v is integral over R .
- (2) $R[v]$ is module-finite over R .
- (3) There exists a subring R' of S such that R' contains $R[v]$ and it is module-finite over R .

Proof. We see (2) implies (3) readily. Now, (1) implies (2): Suppose v is integral over R with the monic polynomial $f(x) = x^n + a_1x^{n-1} + \dots + a_n$. Then, $f(x) = 0 \implies v^n \in \sum_{i=0}^{n-1} Rv^i$. Therefore, for any integer m , $v^m \in \sum_{i=0}^{n-1} Rv^i$. This implies $R[v]$. Lastly, (3) implies (1) as follows: Suppose R' is module-finite over R . Then, $R' = \sum R w_i$, where $w_i \in R'$. Then, $v w_i \in R[v] \subset R'$, so $v w_i = \sum_j a_{ij} w_j$ where $a_{ij} \in R$.

Now, $v w_i - \sum_j a_{ij} w_j = 0$ implies $\sum_{j=1}^n \delta_{ij} v w_j - v w_i = 0$ which then implies $\sum_{j=1}^n (\delta_{ij} v - a_{ij}) w_j = 0$ (here $\delta_{ij} = 1\{i = j\}$). Write this in matrix notation and consider these equations in the quotient field of S and note that (w_1, \dots, w_n) is a non-trivial solution to these equations (as we see, they give 0). Therefore, $\det(\delta_{ij} v - a_{ij}) = 0$ from which we get $v^n + a_1 v^{n-1} + \dots + a_n = 0$. Therefore, v is integral over R . \square

Corollary 5. *The set of elements of S that are integral over R is a subring of R that contains R .*

Proof. Suppose a, b are elements in S that are integral over R . Now, b is integral over R implies b is integral over $R[a]$ as $R \subset R[a]$. Therefore, by the previous theorem, $R[a, b]$ is module-finite over R . Then by the previous theorem $a + b, a - b, ab \in R[a, b]$ and so they are all integral over R . \square

We will require the following results:

Theorem 6. *Suppose an integral domain S is ring-finite over R . Then, S is module-finite over R if and only if S is integral over R .*

Proof. For the forward direction, write $S = \sum Rv_i$. Then consider any $s \in S$. So, $s = \sum Rv_i$. Consider the monic polynomial $f(x) = x - s$. Conversely, suppose S is integral over R . Then consider any $s \in S$ for which we have, using the monic polynomial, $s + a_1s^{n-1} + \dots + a_n = 0$. From this, we write $s = -a_1s^{n-1} - \dots - a_n$. \square

Theorem 7. *Let L be a field and let k be an algebraically closed subfield of L . Then an element of L that is algebraic over k is in k . Furthermore, an algebraically closed field has no module-finite field extension except itself.*

Proof. Proof of the first part - suppose $p \in L$ that is algebraic over k . Therefore, $p^n + a_1p^{n-1} + \dots + a_n = 0$ with $a_i \in k$. This is a polynomial in $k[x]$ with a root p in k , so $p \in k$.

Now, we prove the second part. Suppose L is module-finite over k . Then, by the previous theorem, L is integral over k . Then, by the first part $L = k$. \square

Lastly,

Theorem 8. *Let k be a field. Let $L = k(x)$ be the field of rational functions over k . Then, (a) any element of L that is integral over $k[x]$ is also in $k[x]$. (b) There is no non-zero element $f \in k[x]$ such that $\forall z \in L$, $f^n z$ is integral over $k[x]$ for some $n > 0$.*

Proof. (a) p is integral over $k[x]$ implies there exists the following polynomial $p^n + a_1p^{n-1} + \dots = 0$. Now, since $p \in k(x)$, we may write it as $p = \frac{s}{t}$ where $s, t \in k[x], t \neq 0$. Then, we get $s^n + a_1s^{n-1}t + \dots + a_nt^n = 0$. Rearranging, we get $s^n = -a_1s^{n-1}t - \dots - a_nt^n$. Since t divides the right hand side, t divides s . This means, s/t is a polynomial in $k[x]$. Therefore, $p \in k[x]$.

(b) Suppose, not. Let f be such a function. Let $p(x) \in k[x]$ such that $p(x)$ does not divide f^m for any m . Set $z = \frac{1}{p}$, so $z \in L = k(x)$. Then, $f^n z = \frac{f^n}{p}$ is integral over $k[x]$. This means, there exists $a_i \in k[x]$ such that $(\frac{f^n}{p})^d + \sum_{i=1}^{d-1} a_i(\frac{f^n}{p})^i = 0$. From this, we get $f^{nd} = \sum_{i=1}^{d-1} a_i p^{d-i} f^{in}$. Since p divides the right hand side, we get that p divides f^{nd} which contradicts our definition of p . \square

4 Nullstellensatz Version 1

First, we prove the following:

Theorem 9. (*Zariski*) If a field L is ring-finite over a subfield k , then L is module finite (and, hence, algebraic) over k .

Note that L is module finite over k if and only if L is integral over k which means L is algebraic over k .

Proof. Suppose L is ring-finite over k . Then, $L = k[v_1, \dots, v_n]$ where $v_i \in L$. We proceed by induction.

Suppose $n = 1$. We have that k is a subfield of L and $L = k[v]$. Let $\psi : k[x] \rightarrow L$ be a homomorphism that takes x to v . Now $\ker(\psi) = (f)$ for some f since $k[x]$ is a principal ideal domain. Then, $k[x]/(f) \cong k[v]$ by the first isomorphism theorem. This implies (f) is prime (since $k[v]$ is an integral domain).

Now, if $f = 0$. Then $k[x] \cong k[v]$, so $L \cong k[x]$. However, by the second property following definition 5, this cannot be true. Therefore, $f \neq 0$.

Given $f \neq 0$, we can assume f is monic. Then, (f) prime implies f is irreducible and (f) is a maximal ideal (check Dummit and Foote). This means, $k[v] \cong k[x]/(f)$ is a field (check Dummit and Foote). Therefore, $k[v] = k(v)$. Since $f(v) = 0$, so v is algebraic over k and so, by theorem 4, $L = k[v]$ is module-finite over k . This concludes the proof for $n = 1$.

Now, for the inductive step, assume true for $n - 1$ i.e $k[v_1, \dots, v_{n-1}]$ is module-finite over k . Let $L = k_1[v_2, \dots, v_n]$ where $k_1 = k(v_1)$. Then, by the inductive hypothesis, $k_1[v_2, \dots, v_n]$ is module-finite over k_1 .

We show that v_1 is algebraic over k which would say $k[v_1]$ is module-finite over k concluding the proof. Suppose, v_1 is not algebraic over k . Then, using the inductive hypothesis, for each $i = 2, \dots, n$, we have an equation $v_i^{n_i} + a_{i1}v_i^{n_i-1} + \dots = 0$ where $a_{ij} \in k_1$.

Let $a \in k[v_1]$ such that a is a multiple of all the denominators of $a_{ij} \in k(v_1)$. We get $av_i^{n_i} + aa_{i1}(av_1)^{n_i-1} + \dots = 0$. Then, by corollary 5, for any $z \in L = k[v_1, \dots, v_n]$, there exists N such that $a^N z$ is integral over $k[v_1]$ (since the set of integral elements forms a subring). Since this holds for any $z \in L$, this also holds for any $z \in k(v_1)$. But by theorem 8, this is impossible. This gives us the contradiction. \square

Assume k is algebraically closed.

Theorem 10. (*Nullstellensatz Version I*) If I is a proper ideal in $k[x_1, \dots, x_n]$, then $V(I) \neq \emptyset$.

Proof. For any ideal I , there exists a maximal ideal J containing I (since we are assuming our ring has an identity $1 \neq 0$, see Dummit and Foote). So, for simplicity, we assume I is

the maximal ideal itself since $V(J) \subset V(I)$. Then, $L = k[x_1, \dots, x_n]/I$ is a field (since I is maximal, see Dummit and Foote) and k is an algebraically closed subfield of L . Note that there is a ring-homomorphism from $k[x_1, \dots, x_n]$ onto L , which is the identity. This means, L is ring-finite over k . Then, by theorem 9, L is module-finite over k . Then, by theorem 7, $L = k$ i.e $k = k[x_1, \dots, x_n]/I$.

Now, since $k = L$, in particular this means $k \cong k[x_1, \dots, x_n]/I$. Suppose $x_i \in k[x_1, \dots, x_n]$ is mapped to a_i by the homomorphism ψ whose kernel is I . Then, $x_i - a_i$ is mapped to 0, so $x_i - a_i \in I$. Now, note that $(x_1 - a_1, \dots, x_n - a_n)$ is a maximal ideal as one can easily verify and it contains I , so $I = (x_1 - a_1, \dots, x_n - a_n)$. So, $(a_1, \dots, a_n) \in V(I)$. Therefore, $V(I) \neq \emptyset$. \square