Algebraic Geometry

Jubayer Ibn Hamid

Algebraic geometry is about solutions of polynomial equations and the geometric structures on the space of those solutions. We use the language and techniques from abstract algebra on these geometric objects.

1 Terminology

A field k is algebraically closed if any non-constant polynomial $f \in k[x]$ has a root in k i.e if $f \in k[x]$, then $f(x) = \mu \prod (x - \lambda_i)^{e_i}$ where $\lambda_i \in k$ are the roots. The field $\mathbb{R}$ is not algebraically closed as $f(x)x^+1$ has no root in $\mathbb{R}$, whereas $\mathbb{C}$ is algebraically closed.

The affine space of field k is denoted by $\mathbb{A}_k^n$ which is the Cartesian n-product of k.

Let $f \in k[x_1, ..., x_n]$ be a polynomial. Then, $V(f)$ is the set of zeros of f and is called the hypersurface defined by f. If S is a set of polynomials from $k[x_1, .., x_n]$, then $V(S) := \{p \in \mathbb{A}_k^n | f(p) = 0, \forall f \in S\}$. One can check that $V(S) = \cap_{f \in S} V(f)$. When $S = \{f_1, ..., f_r\}$, we write $V(S)$ as $V(f_1, ..., f_r)$.

A subset $X \subseteq \mathbb{A}_k^n$ is called an affine algebraic set if $X = V(S)$ for some set S of polynomials in $k[x_1, ..., x_n]$. Throughout these notes, we will use the term affine variety to mean the same thing as affine algebraic sets (although some texts refer to only *irreducible* algebraic sets as affine varieties). One can easily show that if I is the ideal in $k[x_1, ..., x_n]$ generated by polynomials in S, then $V(S) = V(I)$. Suppose, $I = (f_1, ..., f_n)$, then, $V(I) = \cap_{i=1}^n V(f_i)$. Some more properties:

(1) If $\{I_\alpha\}$ is a collection of ideals, then $V(\cup_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$. (2) $I \subset J \implies V(J) \subset V(I)$ (3) $V(fg) = V(f) \cup V(g)$ (4) Any finite subset of $\mathbb{A}_k^n$ is an algebraic set (5) $V(A) = V((A))$ where (A) is the ideal generated by A.

The ideal generated by a set of functions $f_1, ..., f_m \in k[x_1, ..., x_n]$ is the set $(f_1, ..., f_m) := \{\sum_{i=1}^m g_i f_i : g_i \in k[x_1, ..., x_n]\}$. For a subset $X \subseteq \mathbb{A}_k^n$, consider the ideal in $k[x_1, ..., x_n]$

generated by polynomials that vanish on X. This ideal is called the vanishing ideal of X, denoted by $I(X)$. So,
$I(X) = \{f \in k[x_1, ..., x_n] : f(a) = 0, \forall a \in X\}$. So, if $f, g \in I$, then $f + g \in I$ and for any $h \in k[x_1, ..., x_n]$, $hf \in I$. Some more properties:

(1) $X \subset Y \implies I(Y) \subset I(X)$ (2) $I(\emptyset) = k[x_1, ..., x_n], I(\mathbb{A}^n) = \emptyset, I(\{a\}) = (x_1 - a_1, ..., x_n - a_n)$.

We say $f_1, ..., f_m$ scheme-theoretically define the affine variety $X \subset \mathbb{A}^n$ if $I(X) = (f_1, ..., f_m)$ i.e the ideal generated by $f_1, ..., f_m$. Furthermore, the ideal I is said to set-theoretically define variety X if $X = V(I)$ if It can be easily shown that $V(I)(X)) = X$. $V(-)$ and $I(-)$ allow us to switch betwen the geometric world and the algebraic world which is a key tool used in algebraic geometry. In particular, later on, we will see that using Hilbert's Nullstellensatz, there is no information lost after we make this switch.

A polynomial mapping $p : V \to W$, where $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ are varieties, is a mapping such that $(x_1, ..., x_n) \to f(x_1, ..., x_n) := (f_1(x_1, ..., x_n), ..., f_m(x_1, ..., x_n))$, $f_i \in k[x_1, ..., x_n]$ and the image of the algebraic set V lies inside the algebraic set W. The mapping set $\text{Map}(V, W)$ is the set of all polynomial maps from V to W and in our case this is the set of all polynomial maps from V to W. We need polynomial mappings in order to investigate the relationships between varieties.

2 Hilbert Basis Theorem

First, we note that for $a := (a_1, ..., a_n) \in \mathbb{A}^n_k$, $I(\{a\}) = (x_1 - a_1, ..., x_n - a_n) \subset k[x_1, ..., x_n]$. To see this, note that $(x_1 - a_1, ..., x_n - a_n) \subset I(\{a\})$ which is straightforward. To see the other direction, suppose $f \in I(\{a\})$. Since $f \in k[x_1, .., x_n]$, we can write it as $f = \sum_{i_1,...i_n \geq 0} a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}$. Since $f(a) = 0$, we can write this as $f(x) = \sum_{i_1,...i_n \geq 0} b_{i_1 \cdots i_n} (x_1 - a_1)^{i_1} \cdots (x_n - a_n)^{i_n}$ and so $f(x) \in (x_1 - a_1, ..., x_n - a_n)$.

Definition 1. *A ring R is called Noetherian if every ideal in R is finitely generated.*

Fields and Principal Ideal Domains (PIDs) are Noetherian rings.

One can easily verify the following:

R is Noetherian if and only if every sequence of ideals $I_1 \subset I_2 \subset \cdots$ stabilizes i.e there exists N such that $I_N \subset I_{N+1} \subset \cdots$.

Proof. Forward direction: If every ideal is finitely generated then the ideal $\cup_i I_i$ is finitely generated and so the generating set of $\cup_i I_i$ must lie in some I_N. Conversely, suppose the

sequence stabilizes but there exists an I that is not finitely generated. Then take a sequence of $f_i \in I$ such that $f_i \notin (f_1, ..., f_{i-1})$ yields an increasing sequence of ideals i.e $(f_1) \subset (f_1, f_2) \subset (f_1, f_2, f_3) \subset \cdots$ that does not stabilize - contradiction. □

Theorem 1. *(Hilbert Basis Theorem) If R is a Noetherian ring, then $R[x_1, ..., x_n]$ is a Noetherian Ring.*

Proof. We know $R[x_1, ..., x_n] \cong R[x_1, ..., x_{n-1}][x_n]$. So, if we can prove that R Noetherian implies $R[x]$ is Noetherian, by induction we will have proven that $R[x_1, ..., x_n]$ is also Noetherian.

Suppose R is Noetherian. Let I be an ideal in $R[x]$. Let J denote the set of leading coefficients of polynomials in I. Then, given I is an ideal, J is an ideal in R. Since R is Noetherian, we can write that J is generated by the leading coefficients of $f_1, ..., f_r \in I$. Suppose $N \in \mathbb{Z}$ such that N is greater than the degrees of all polynomials $f_1, ..., f_r$. Then, for any $m \leq N$, we define J_m to be the ideal in R generated by the leading coefficients of all polynomials f in I such that $deg(f) \leq m$. Once again, since J_m is an ideal in R, we can say that J_m is generated by the finite set of polynomials, $\{f_{mj}\}$, such that each polynomial's degree is less than or equal to m. Finally, define I' be the ideal generated by polynomials $\{f_{jm}\}$ and f_i.

We claim $I' = I$. Suppose not i.e suppose there exists elements in I that are not in I'. Let g be the minimal element such that $g \in I$, $g \notin I'$.

Case 1: $deg(g) > N$. Then, there exists polynomials Q_i such that $\sum_i Q_i f_i$ has the same leading term as g. Therefore, $deg(g - \sum_i Q_i f_i) < deg(g)$. Clearly, $g - \sum_i Q_i f_i$ is in I'. But since g is the minimal element and $deg(g - \sum_i Q_i f_i) < deg(g)$, therefore $g - \sum_i Q_i f_i \in I'$, which implies $g \in I'$.

Case 2: $m := deg(g) \leq N$. Then, there exists polynomials Q_j such that $\sum_j Q_j f_{mj}$ and g have the same leading term. Using a similar argument, we get that $g \in I'$. □

This has the following interesting implication:

Theorem 2. *An algebraic set is the intersection of a finite number of hypersurfaces.*

Proof. Let $V(I)$ be an algebraic set. We prove that I is finitely generated since that implies $V(I) = V(f_1, ..., f_r) = \cap_{i=1}^{r} V(f_i)$. Given k is a field, k is a Noetherian ring and by the Hilbert Basis Theorem, $k[x]$ is also Noetherian. Therefore, the ideal I in $k[x]$ is finitely generated. □

Corollary 3. *$k[x_1, ..., x_n]$ is a Noetherian ring for any field k.*

Proof. Follows from the Hilbert Basis Theorem. □

3 Modules Revision

Definition 2. *R-Module.*

Let R be a ring. Let M be an abelian group $(M,+)$. Then, an R-module is M with multiplication $R \times M \to M$ such that for any $a, b \in R$, $m \in M$, $(a+b)m = am + bm, a(m+n) = am + an, (ab)m = a(bm), 1_R m = m$.

Definition 3. *Submodule.*

A submodule N is a subgroup of R-module, M, such that $an \in N$ for any $a \in R, n \in N$.

One can check that for any $m \in M$, $0_R m = 0_M$ by noting that $0_R m = (x-x)m = xm - xm = 0_M$ for any $x \in R, m \in M$. Also, the submodule N of an R-module is an R-module itself.

Definition 4. *Submodule generated by S.*

Let $S := \{s_1, s_2, ...\}$ be a set of elements of the R-module M. Then the submodule generated by S is $\{\sum_i r_i s_i | r_i \in R, s_i \in S\}$.

When S is finite, we denote the submodule generated by S as $\sum_i Rs_i$.

Definition 5. *Finiteness conditions of subrings of a ring.*

Let S be a ring and let R be a subring of S.

(1) S is <u>module-finite over R</u> if S is finitely-generated as an R-module i.e $S = \sum_{i=1}^n Rv_i$ where $v_1, ..., v_n \in S$. More explicitly, $S = \{\sum_{i=1}^n r_i v_i : r_i \in R\}$, for $v_1, ..., v_n \in S$ fixed.

(2) S is <u>ring-finite over R</u> if $S = R[v_1, ..., v_n] = \{\sum_i a_i v_1^{i_1} \cdots v_n^{i_n} | a_i \in R\}$ where $v_1, ..., v_n \in S$.

(3) S is a <u>finitely-generated field extension of R</u> if S and R are fields and $S = R(v_1, ..., v_n)$ (the quotient field of $R[v_1, ..., v_n]$) where $v_1, ..., v_n \in S$.

Properties:

1. If S is module-finite over R, then S is ring-finite over R. (This is straightforwardly seen from the definitions)

2. If $L = K(x)$, then L is a finitely-generated field extension of K but L is not ring-finite over K.

Proof. Using the definition, L is a finitely generated field extension of K and so $K(X)$ is a finitely-generated field extension of K. Now, suppose L is ring-finite over K. Then, $L = K[v_1, ..., v_n]$ and so $K(x) = K[v_1, ..., v_n]$, where $v_1, ..., v_n \in k(x)$.

Then, there exists $\frac{s_i}{t_i} \in K(X)$ that generate L where $i = 1, ..., n$. Define $p := 1/q$ where q is an irreducible polynomial that has a higher degree than all t_i's. Then, as $p \in K(X)$, $p = \frac{h}{t_1^{e_1} \cdots t_n^{e_n}}$. Since q has a higher degree than all the t_i's, we see that p cannot be equal to $\frac{1}{q}$. □

Definition 6. *Integral elements*

Let R be a subring of the ring S. Then, $v \in S$ is integral over R if there exists a monic polynomial $f = x^n + a_1 x^{n-1} + \cdots + a_n \in R[x]$ such that $f(v) = 0$ and $a_i \in R$. If R and S are fields, we say v is algebraic over R.

When all elements of S is integral over R, we say S is integral over R. When S and R are fields and S is integral over R, we call S an algebraic extension of R.

Theorem 4. *Let R be a subring over an integral domain S and let $v \in S$. Then, the following are equivalent:*

(1) v is integral over R.

(2) $R[v]$ is module-finite over R.

(3) There exists a subring R' of S such that R' contains $R[v]$ and it is module-finite over R.

Proof. We see (2) implies (3) readily. Now, (1) implies (2): Suppose v is integral over R with the monic polynomial $f(x) = x^n + a_1 x^{n-1} + ... + a_n$. Then, $f(x) = 0 \implies v^n \in \sum_{i=0}^{n-1} Rv^i$. Therefore, for any integer m, $v^m \in \sum_{i=0}^{n-1} Rv^i$. This implies $R[v]$. Lastly, (3) implies (1) as follows: Suppose R' is module-finite over R. Then, $R' = \sum Rw_i$, where $w_i \in R'$. Then, $vw_i \in R[v] \subset R'$, so $vw_i \sum_j a_{ij} w_j$ where $a_{ij} \in R$.

Now, $vw_i - vw_i = 0$ implies $\sum_{j=1}^{n} \delta_i jvw_j - vw_i = 0$ which then implies $\sum_{j=1}^{n} (\delta_{ij} v - a_{ij}) w_j = 0$ (here $\delta_{ij} = 1\{i = j\}$. Write this in matrix notation and consider these equations in the quotient field of S and note than $(w_1, ..., w_n)$ is a non-trivial solution to these equations (as we see, they give 0). Therefore, $det(\delta_{ij} v - a_{ij}) = 0$ from which we get $v^n + a_1 v^{n-1} + ... + a_n = 0$. Therefore, v is integral over R. □

Corollary 5. *The set of elements of S that are integral over R is a subring of R that contains R.*

Proof. Suppose a, b are elements in S that are integral over R. Now. b is integral over R implies b is integral over $R[a]$ as $R \subset R[a]$. Therefore, by the previous theorem, $R[a, b]$ is module-finite over R. Then by the previous theorem $a + b, a - b, ab \in R[a, b]$ and so they are all integral over R. □

We will require the following results:

Theorem 6. *Suppose an integral domain S is ring-finite over R. Then, S is module-finite over R if and only if S is integral over R.*

Proof. For the forward direction, write $S = \sum Rv_i$. Then consider any $s \in S$. So, $s = \sum Rv_i$. Consider the monic polynomial $f(x) = x - s$. Conversely, suppose S is integral over R. Then consider any $s \in S$ for which we have, using the monic polynomial, $s + a_1 s^{n-1} + \cdots + a_n = 0$. From this, we write $s = -a_1 s^{n-1} + ... - a_n$. □

Theorem 7. *Let L be a field and let k be an algebraically closed subfield of L. Then an element of L that is algebraic over k is in k. Furthermore, an algebraically closed field has no module-finite field extension except itself.*

Proof. Proof of the first part - suppose $p \in L$ that is algebraic over k. Therefore, $p^n + a_1 p^{n-1} + \cdots + a_n = 0$ with $a_i \in k$. This is a polynomial in $k[x]$ with a root p in k, so $p \in k$.

Now, we prove the second part. Suppose L is module-finite over k. Then, by the previous theorem, L is integral over k. Then, by the first part $L = k$. □

Lastly,

Theorem 8. *Let k be a field. Let $L = k(x)$ be the field of rational functions over k. Then, (a) any element of L that is integral over $k[x]$ is also in $k[x]$. (b) There is no non-zero element $f \in k[x]$ such that $\forall z \in L$, $f^n z$ is integral over $k[x]$ for some $n > 0$.*

Proof. (a) p is integral over $k[x]$ implies there exists the following polynomial $p^n + a_1 p^{n-1} + ... = 0$. Now, since $p \in k(x)$, we may write it as $p = \frac{s}{t}$ where $s, t \in k[x], t \neq 0$. Then, we get $s^n + a_1 s^{n-1} t + \cdots + a_n t^n = 0$. Rearranging, we get $s^n = -a_1 s^{n-1} t - \cdots - a_n t^n$. Since t divides the right hand side, t divides s. This means, s/t is a polynomial in $k[x]$. Therefore, $p \in k[x]$.

(b) Suppose, not. Let f be such a function. Let $p(x) \in k[x]$ such that $p(x)$ does not divide f^m for any m. Set $z = \frac{1}{p}$, so $z \in L = k(x)$. Then, $f^n z = \frac{f^n}{p}$ is integral over $k[x]$. This means, there exists $a_i \in k[x]$ such that $(\frac{f^n}{p})^d + \sum_{i=1}^{d-1} a_i (\frac{f^n}{p})^i = 0$. From this, we get $f^{nd} = \sum_{i=1}^{d-1} a_i p^{d-i} f^{in}$. Since p divides the right hand side, we get that p divides f^{nd} which contradicts our definition of p. □

4 Nullstellensatz Version 1

First, we prove the following:

Theorem 9. *(Zariski) If a field L is ring-finite over a subfield k, then L is module finite (and, hence, algebraic) over k.*

Note that L is module finite over k if and only if L is integral over k which means L is algebraic over k.

Proof. Suppose L is ring-finite over k. Then, $L = k[v_1, ..., v_n]$ where $v_i \in L$. We proceed by induction.

Suppose $n = 1$. We have that k is a subfield of L and $L = k[v]$. Let $\psi : k[x] \to L$ be a homomorphism that takes x to v. Now $ker(\psi) = (f)$ for some f since $k[x]$ is a principal ideal domain. Then, $k[x]/(f) \cong k[v]$ by the first isomorphism theorem. This implies (f) is prime (since $k[v]$ is an integral domain).

Now, if $f = 0$. Then $k[x] \cong k[v]$, so $L \cong k[x]$. However, by the second property following definition 5, this cannot be true. Therefore, $f \neq 0$.

Given $f \neq 0$, we can assume f is monic. Then, (f) prime implies f is irreducible and (f) is a maximal ideal (check Dummit and Foote). This means, $k[v] \cong k[x]/(f)$ is a field (check Dummit and Foote). Therefore, $k[v] = k(v)$. Since $f(v) = 0$, so v is algebraic over k and so, by theorem 4, $L = k[v]$ is module-finite over k. This concludes the proof for $n = 1$.

Now, for the inductive step, assume true for $n - 1$ i.e $k[v_1, ..., v_{n-1}]$ is module-finite over k. Let $L = k_1[v_2, ..., v_n]$ where $k_1 = k(v_1)$. Then, by the inductive hypothesis, $k_1[v_2, \cdots, v_n]$ is module-finite over k_1.

We show that v_1 is algebraic over k which would say $k[v_1]$ is module-finite over k concluding the proof. Suppose, v_1 is not algebraic over k. Then, using the inductive hypothesis, for each $i = 2, ..., n$, we have an equation $v_i^{n_i} + a_{i1}v_i^{n_i-1} + \cdots = 0$ where $a_{ij} \in k_1$.

Let $a \in k[v_1]$ such that a is a multiple of all the denominators of $a_{ij} \in k(v_1)$. We get $av_i^{n_i} + aa_{i1}(av_1)^{n_i-1} + \cdots = 0$. Then, by corollary 5, for any $z \in L = k[v_1, \cdots, v_n]$, there exists N such that $a^N z$ is integral over $k[v_1]$ (since the set of integral elements forms a subring). Since this holds for any $z \in L$, this also holds for any $z \in k(v_1)$. But by theorem 8, this is impossible. This gives us the contradiction. □

Assume k is algebraically closed.

Theorem 10. *(Nullstellensatz Version I) If I is a proper ideal in $k[x_1, ..., x_n]$, then $V(I) \neq \emptyset$.*

Proof. For any ideal I, there exists a maximal ideal J containing I (since we are assuming our ring has an identity $1 \neq 0$, see Dummit and Foote). So, for simplicity, we assume I is the maximal ideal itself since $V(J) \subset V(I)$. Then, $L = k[x_1, \cdots, x_n]/I$ is a field (since I is maximal, see Dummit and Foote) and k is an algebraically closed subfield of L. Note that there is a ring-homomorphism from $k[x_1, ..., x_n]$ onto L, which is the identity. This means, L is ring-finite over k. Then, by theorem 9, L is module-finite over k. Then, by theorem 7, $L = k$ i.e $k = k[x_1, ..., x_n]/I$.

Now, since $k = L$, in particular this means $k \cong k[x_1, ..., x_n]/I$. Suppose $x_i \in k[x_1, ..., x_n]$ is mapped to a_i by the homormorphism ψ whose kernel is I. Then, $x_i - a_i$ is mapped to 0, so $x_i - a_i \in I$. Now, note that $(x_1 - a_1, ..., x_n - a_n)$ is a maximal ideal as one can easily verify and it contains I, so $I = (x_1 - a_1, ..., x_n - a_n)$. So, $(a_1, ..., a_n) \in V(I)$. Therefore, $V(I) \neq \emptyset$. □

The fact that every maximal ideal in the polynomial ring over n variables is of the form $(x_1 - a_1, ..., x_n - a_n)$ is an important takeaway.

We recall some definitions before moving to Hilbert's Nullstellensatz. The radical of an ideal I in R is $\sqrt{I} := \{a \in R : a^n \in I, \text{for some} n \in \mathbb{Z}, n > 0\}$. It can be easily shown that $\sqrt{I}$ is an ideal itself and $I \subseteq \sqrt{I}$. I is called a radical ideal if $I = \sqrt{I}$.

For any ideal I in $k[x_1...x_n]$, $V(I) = V(\sqrt{I})$. To see this, note that $I \subseteq \sqrt{I}$ implies $V(\sqrt{I}) \subseteq V(I)$. Conversely, let $v \in V(I)$ and let $f \in \sqrt{I}$. Then, $f^n \in I$ for some $n > 0$. This implies $f^n(v) = 0$ which implies $f(v) = 0$ as k has no zero divisor. Therefore, $v \in V(\sqrt{I})$.

Lastly, $\sqrt{I} \subset I(V(I))$. To see this, suppose $s \in \sqrt{I}$. Then, $s^n \in I$ for some n. Now, let $v \in V(I)$. Then, $s^n(v) = 0$ implies $s(v) = 0$, so $s \in I(V(I))$.

Now, we prove Hilbert's Nullstellensatz:

Theorem 11. *(Hilbert's Nullstellensatz) Let I be an ideal in $k[x_1, ..., x_n]$ where k is algebraically closed. Then, $I(V(I)) = \sqrt{I}$.*

Proof. We already know $\sqrt{I} \subset I(V(I))$. So, we only need to prove the other direction. Let $I = (f_1, ..., f_r)$ where $f_i \in k[x_1, ..., x_n]$. Suppose, $G \in I(V(f_1, ..., f_r))$. Define $J := (f_1, ..., f_r, x_{n+1}G - 1) \subset k[x_1, ..., x_n, x_{n+1}]$. Then, $V(J) \subset \mathbb{A}^n_k$ is $\emptyset$ since G is 0 whenever all f_i are 0 and therefore, $x_{n+1}G - 1 \neq 0$ at those points.

Since $V(J) = \emptyset$, J is not a proper ideal by the previous theorem. Therefore, $J = k[x_1, \cdots, x_{n+1}]$. So, $1 \in J$ (check Dummit and Foote; an ideal in R is all of R iff it contains a unit). So $1 = \sum_i a_i(x_1, ..., x_{n+1})f_i + b(x_1, .., x_{n+1})(x_{n+1}G - 1)$.

In particular, if $x_{n+1} = \frac{1}{G}$, then, $1 = \sum_i a_i f_i + b(1-1) = \sum_i a_i f_i$. Therefore, $G^N = G^N \sum_i a_i f_i$, so $G^N \in (I)$. Therefore, $G \in \sqrt{I}$. Therefore, $I(V(I)) \subseteq \sqrt{I}$. □

This has a series of interesting applications.

Corollary 12. *If I is a radical ideal in $k[x_1, ..., x_n]$, then $I(V(I)) = I$. Therefore, there is a one-to-one correspondence between radical ideals and algebraic sets.*

Corollary 13. *If I is a prime ideal, then $V(I)$ is irreducible. There is a one-to-one correspondence between prime ideals and irreducible algebraic sets. The maximal ideals correspond to points.*

Corollary 14. *Let F be a non-constant polynomial in $k[x_1, \cdots, x_n]$ with the irreducible decomposition of F being $F = F_1^{n_1} F_2^{n_2} \cdots F_r^{n_r}$. Then, $V(F) = V(F_1) \cup \cdots \cup V(F_r)$ is the decomposition of $V(F)$ into irreducible components and $I(V(F)) = (F_1 \cdots F_r)$. Therefore, there is a one-to-one correspondence between irreducible polynomials $F \in k[x_1, \cdots, x_n]$ (up to multiplication by a non-zero element of k) and irreducible hypersurfaces in $\mathbb{A}_k^n$.*

Corollary 15. *Let I be an ideal in $k[x_1, \cdots, x_n]$. Then, $V(I)$ is a finite set if and only if $k[x_1, \cdots, x_n]/I$ is a finite dimensional vector space over k. If this occurs, then, the number of points in $V(I)$ is at most $dim_k(k[x_1, \cdots, x_n]/I)$.*

Proof. Let $p_1, \cdots, p_r \in V(I)$. Choose $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$ such that $f_i(p_j) = 0$ if $i \neq j$ and $f_i(p_i) = 1$ and let $\bar{f}_i$ be the residue class of f_i. Now, if $\sum_i \lambda_i \bar{f}_i = 0$ with $\lambda_i \in k$, then, $\sum_i \lambda_i f_i \in I$. Therefore, $\lambda_j = (\sum_i \lambda_i f_i)(p_j) = 0$. Therefore, $\bar{f}_i$ are linearly independent over k. So $r \leq dim_k(k[x_1, \cdots, x_n]/I)$.

Conversely, suppose $V(I) = (p_1, \cdots, p_r)$ and so is finite. Let $p_i = (a_{1i}, \cdots, a_{1n})$ and define $f_j := \prod_{i=1}^r (x_j - a_{ij}), j = 1, \cdots, n$. Then, $f_j \in I(V(I))$, so for all j, $f_j^N \in I$ for some large enough $N > 0$. Now, taking I-residues, $\bar{f_j}^N = 0$. By expanding f_j^N, we get that $\bar{x_j}^{rN}$ is a k-linear combination of $\bar{1}, \bar{x_j}, \cdots, \bar{x_j}^{rN-1}$. So, for all s, $\bar{x_j}^s$ is a k-linear combination of $\bar{1}, \bar{x_j}, \cdots, \bar{x_j}^{rN-1}$. Therefore, the set $\{\bar{x_1}^{m_1}, \cdots, \bar{x_n}^{m_n} : m_i < rN\}$ generates $k[x_1, \cdots, x_n]/I$ as a vector space over k. □

Next, we find irreducible decompositions of algebraic sets of an affine space.

5 Irreducible Components of Algebraic Sets

So far, we have seen polynomials and the varieties defined over them. Now, we bring in topological invariants.

Definition 7. *Irreducible decomposition of a set. Let $V \in \mathbb{A}_k^n$ be an algebraic set. Then, V is reducible if $V = V_1 \cup V_2$ where V_1, V_2 are non-empty, algebraic sets in $\mathbb{A}_k^n$ i.e $V_i \neq V$ for $i = 1, 2$. If V is not irreducible, we call it reducible.*

Theorem 16. *The algebraic set V is irreducible if and only if $I(V)$ is prime.*

Proof. Suppose, V is irreducible. Now, suppose for contradiction, $I(V)$ is not prime. Therefore, by definition of prime, there exists $f_1 f_2 \in I(V)$ such that $f_1 \notin I(V)$ and $f_2 \notin I(V)$. Now, $V = (V \cap V(f_1)) \cup (V \cap V(f_2))$ and $V \cap V(f_i) \subset V, V \cap V(f_i) \neq V$- to see this, note that for any $p \in V$ such that p is a zero of $f_1 f_2$, p has to be a root of either f_1 or f_2 since f_i belong to an integral domain, therefore, $p \in (V \cap V(f_1)) \cup (V \cap V(f_2))$ (the other direction is obvious). Then, $V = (V \cap V(f_1)) \cup (V \cap V(f_2))$ is decomposition of V which means V is not irreducible - contradiction.

Conversely, suppose $I(V)$ is prime. For contradiction, suppose V is reducible with $V = V_1 \cup V_2$, V_i non-empty. Then, consider $f_i \in I(V_i)$ such that $f_i \notin V$. Clearly, $f_1 f_2 \in I(V)$, so $I(V)$ is not prime - contradiction. □

Theorem 17. *Let A be a non-empty collection of ideals in a Noetherian ring R. Then, A has a maximal ideal i.e an ideal I such that $I \in A$ and no other ideal in A contains I.*

Proof. Given our collection of ideals, A, choose an ideal $I_0 \in A$. Then, define $A_1 = \{I \in A : I_0 \subsetneq I\}$ and $I_1 \in A_1$, $A_2 = \{I \in A : I_1 \subsetneq I\}$ and $I_2 \in A_2$ and so on. Then, the statement in the theorem is equivalent to saying that there exists positive integer n such that A_n is empty since that would mean there exists no ideal containing I_{n-1}. Suppose this is not true. Then, with $I := \cup_{n=0}^{\infty} I_n$, since R is Noetherian, therefore there exists $f_1, ..., f_m$ that generates the ideal I where each $f_i \in I_n$ for n sufficiently large. But since the generates are all in I_n, $I = I_n$ and so $I_{n'} = I_n$ for any $n' > n$ (since $I = \cup_{n=0}^{\infty} I_n$ by definition) - contradiction. □

We finally prove the main result. Note that this is pretty closely tied to the Hilbert Basis Theorem which says that every algebraic set is the intersection of a finite number of algebraic sets/hypersurfaces:

Theorem 18. *Let V be an algebraic set in $\mathbb{A}_k^n$. Then, there exists unique, irreducible algebraic sets $V_1, ..., V_r$ such that $V = V_1 \cup V_2 \cdots \cup V_r$ and $V_i \subsetneq V_i$ for any $i \neq j$.*

Proof. Proving this statement is equivalent to disproving that $\mathcal{F}$ is non-empty where $\mathcal{F} := \{\text{algebraic set} V \in \mathbb{A}_k^n : V \text{is not the union of finitely many irreducible algebraic sets}\}$.

Suppose, $\mathcal{F}$ is not empty. Let $V \in \mathcal{F}$ such that V is the minimal member of $\mathcal{F}$ i.e V cannot be written as the union of sets in $\mathcal{F}$.

Now, since $V \in \mathcal{F}$, V is reducible (if V is irreducible, then it is trivially the union of 1 irreducible subsets). Since V is reducible, $V = V_1 \cup V_2$ where $V_i \neq \emptyset$. Since V is the minimal member of $\mathcal{F}$, $V_i \notin \mathcal{F}$. Since $V_i \notin \mathcal{F}$, it is the union of finitely many irreducible algebraic sets, so let $V_i = V_{i1} \cup V_{i2} \cdots \cup V_{im_i}$. Then, $V = \cup_{i,j} V_{ij}$, so $V \notin \mathcal{F}$. So, we have shown that V can be written as $V = V_1 \cup \cdots \cup V_m$ where each V_i is irreducible. First, remove any V_i such that $V_i \subset V_j$. Now we prove uniqueness. Suppose $V = W_1 \cup \cdots W_m$ be another such decomposition. Then, $V_i = \cup_j (W_j \cap V_i)$. Now, $W_j \cap V_i = V_i$ since otherwise we will have found a decomposition of the irreducible set V_i. Therefore, $V_i \subset W_{j(i)}$ for some $j(i)$. Similarly, by symmetry, $W_{j(i)} \subset V_k$ for some k. But then, $V_i \subset V_k$ implies $i = k$ and so $V_i = W_{j(i)}$. Continuing this for each $i \in \{1, ..., m\}$, we get that the two decompositions are equal. □

Furthermore, we use the following terms:

Definition 8. *An ideal $I \subset k[x_1, ..., x_n]$ set-theoretically defines a variety V if $V = V(I)$. An ideal $J \subset \mathbb{A}^n$ scheme-theoretically defines a variety V if $J = I(V)$.*

Here's a pretty straightforward result:

Theorem 19. *For an affine variety X, if $f_1, ..., f_m$ scheme-theoretically define X, then $V(I(X)) = X$*

Two affine-varities can be isomorphic in the usual sense using the language of polynomial maps:

Definition 9. *Isomorphic affine varieties. Two affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ are isomorphic if there exists polynomial maps $f : V \to W$ and $g : W \to V$ such that $f \circ g = g \circ f = i_d$.*

Lastly, we will require the following useful result for the section on topology.

Theorem 20. *Let $Z \subset \mathbb{A}^n$ be an affine variety and let $x \in \mathbb{A}^n - Z$. Then, there exists $f \in k[x_1, ..., x_n]$ such that $f(Z) = 0$ and $f(x) \neq 0$.*

Proof. Suppose this is not true. Then, $f \in I(Z) \implies f \in I(Z \cup \{x\})$. Then, $I(Z) = I(Z \cup \{x\})$. Therefore, $Z = Z \cup \{x\}$ since $V(I(X)) = X$. This is contradiction since this implies $x \in Z$. □

6 Zariski Topology

Definition 10. *Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then, $Z \subseteq X$ is closed if $Z \subseteq X \subseteq \mathbb{A}^n$ is an affine variety i.e there exists $f_1, ..., f_m \in k[x_1, ..., x_n]$ such that $Z = V(f_1, ..., f_m) \subset X$.*

This forms a topology. $\emptyset$ is closed as $\emptyset = V(1)$. X itself is closed since $X = V(g_1, ..., g_m)$ by definition (since it's an affine variety). Now, suppose $\{Z_i\}_{i \in A}$ are affine varieties. Then, $\cap_{i \in A} Z_i = V(\sum_i I(Z_i))$. Lastly, $V(f_1, ..., f_m) \cup V(h_1, ..., h_r) = V(\sum_{i,j} f_i h_j)$

Theorem 21. *The pre-image of an affine variety under a polynomial map $p : V \to W$ is a variety.*

Proof. Let $V \subseteq \mathbb{A}^n_k$, $W \subseteq \mathbb{A}^m_k$ be affine varieties. Write p as $p = (p_1, ..., p_m)$ where the image of each p_i is in k. Now, suppose $Z := V(g_1, ..., g_m) \subseteq W$ is closed. We show $f^{-1}(Z)$ is closed. $f^{-1}(Z) = \{x = (x_1, ..., x_n) \in V : (p_1(x), ..., p_m(x) \in Z\} = \{x \in V : g_j(f(x)) = 0, \forall j\} \implies f^{-1}(Z)$ is closed. □

For an example, consider the Zariski topology on $\mathbb{A}^1_k$ and let $V(f_1, ..., f_m) \subset \mathbb{A}^1_k$. Now, given K is a field, $k[x]$ is a principal ideal domain so $(f_1, ..., f_m) = (g)$ for some $g \in k[x]$. Then, the closed subset i.e variety of $\mathbb{A}^1_k$ is of the form $V(g) = \{x \in k : g(x) = 0\}$ which is finite since g is a polynomial of some degree. This means that the closed subsets of $\mathbb{A}^1_k$ are of the form $\emptyset, \mathbb{A}^1_k$ and finite subsets of $\mathbb{A}^1_k$.

Definition 11. *Coordinate Ring. Let $X \subset \mathbb{A}^n$ be an affine variety. The coordinate ring of functions on V is*

$$O(X) := k[x_1, .., x_n]/I(X)$$

is the quotient ring of polynomials in n-variables.

Note that, for a point $a = (a_1, ..., a_n) \in X$ and $f \in O(X)$, the value of $f(a) \in k$ is well-defined. This is because for any $f' \in I(X)$, $f'(a) = 0$, so the value $f(a)$ is independent of our choice of function from $I(V)$.

The coordinate ring $O(X)$ can be thought of as a ring of polynomials such that we only care about their values on X since we identify two polynomials that are equal on X to be the same.

Definition 12. *First, we define $V(f)_X := V(f) \cap X$ where $X \subset \mathbb{A}^n$ is an affine variety. Now, we define basic closed sets of X be sets of the form $V(f)_X$. Note that $V(\{f_i\}_{i \in I}) = \cap_i V(f_i)$. Any closed set in the Zariski topology is a union of basic closed sets for some set of functions. On the other hand, the basic open sets of X are of the form $D(f)_X := \{x \in X :$*

$f(x) \neq 0\}$ *i.e* $D(f)_X = X - V(f)$. *Any open set in Zariski topology is the union of some basic open sets.*

Note that, by Hilbert Basis Theorem, every closed subset of X *is a finite intersection of basic closed sets. Similarly, every open set is a finite union of basic open sets.*

There is a particularly local nature of algebraic geometry as evident by the following:

Corollary 22. *Let* $U \subseteq X$ *be a basic open subset of an affine variety* X*. Then, for any* $x \in U$*, there exists a basic open subset* $D(f) \subset X$ *and* $f \in k[x_1, ..., x_n]$ *such that* $x \in D(f) \subseteq U$*.*

Proof. Let $Z = X - U$ be the closed subset of X i.e an affine variety. Then, Theorem 15 allows us to conclude the statement. □

Definition 13. *For a subset* X *of* V*, the Zariski closure of* X *in* V *is the minimal closed subset of* V *that contains* X *which we denote by* $\bar{X} \subseteq V$*.*

Note: S is irreducible if and only if $\bar{S} \subseteq V$ is irreducible.

7 Coordinate Rings

First, we recall that given $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ are varieties, $f : V \to W$ is a polynomial map if $f(x_1, ..., x_n) = (f_1(x_1, ..., x_n), \cdots, f_m(x_1, ..., x_n))$, $f_i \in k[x_1, ..., x_n]$ and $f(V) \subset W$.

Furthermore, given the definition of coordinate ring and $I(\mathbb{A}^n_k) = 0$, $O(\mathbb{A}^n_k = k[x_1, \cdots, x_n]$ is the true coordinate ring of $\mathbb{A}^n_k$.

Definition 14. *k-algebra. Let k be a field (i.e a commutative division ring). A ring R is a k-algebra if $k \subseteq Z(R) := \{x \in R : xy = yx, \forall y \in R\}$ and the identity of k is the same as the identity of R.*

Note, $Z(R)$ is the center of the ring R.

Definition 15. *Finitely generated k-algebra. A finitely generated k-algebra is a ring that is isomorphic to a quotient of a polynomial ring $k[x_1, ..., x_n]/I$.*

Equivalently, a ring R is a finitely-generated k-algebra if R is generated as a ring by k with some finite set $r_1, ..., r_n$ of elements of R i.e $k[r_1, ..., r_n]$.

These definitions are equivalent. Suppose, R is a finitely generated k-algebra i.e $R = k[r_1, ..., r_n]$. Then, by the first isomorphism theorem, $R \cong k[r_1, ..., r_n]/I$. Conversely, suppose $R \cong k[r_1, .., r_n]/I$ i.e $\varphi : k[r_1, .., r_n]/I \to R$. Then, with $\pi : k[r_1, ..., r_n] \to k[r_1, ..., r_n]/I$, $f := \varphi \circ \pi : k[x_1, ..., x_n] \to R$ is a surjective homomorphism. Since f is a homomorphism, $f(p(x_1, ..., x_n)) = p(f(x_1), f(x_2), ..., f(x_n))$ and so all elements of R is a polynomial in $f(x_1), ..., f(x_n)$ with coefficients in R so they are generated by these n elements as a k-algebra.

Definition 16. *$Mor_k(R, S)$. Let R and S be k-algebras. Then, $\psi : R \to S$ is a k-algebra homomorphism, $\psi \in Mor_k(R, S)$ if ψ is a ring homomorphism that is identity on k.*

Note that if $\phi : R \to k$ is a k-algebra homomorphism, then ϕ is surjective.

Theorem 23. *$O(X) \cong Map(X, \mathbb{A}^1)$. Here, $Map(X, \mathbb{A}^1)$ is a commutative k-algebra under addition and multiplication on $\mathbb{A}^1$. Furthermore, $O(X)^m \cong Map(X, \mathbb{A}^m)$*

Proof. Let $\varphi : O(X) \to \text{Map}(X, \mathbb{A}^1)$. Then, define $\varphi(f)(a) = f(a)$ for any $a \in X$. This is a homomorphism by design. To show surjectivity, by definition of $\text{Map}(X, \mathbb{A}^1)$, $f \in \text{Map}(X, \mathbb{A}^1)$ implies $f(x) \in k[x_1, ..., x_n]$ so $\bar{f} \in O(X)$ is mapped to f. To show injectivity, suppose $f \in O(X)$ is mapped to 0. Then, $f(x) = 0$ for all $x \in X$. This means, $f \in I(X)$ implying $f = 0$ in $O(X)$. □

Corollary 24. *Given X and Y are affine varieties, $X \cong Y$ implies $O(X) \cong O(Y)$.*

With these results in mind, we note that a key idea in algebraic geometry is to characterize an affine variety X by the ring of functions $O(X) \cong \text{Map}(X, \mathbb{A}^1)$.

Let $\text{Mor}_k(R_1, R_2)$ be th set of morphisms between 2 k-algebras R_1 and R_2. More strictly:

With this, we can define the pullback function:

Definition 17. *Given $X \in \mathbb{A}^n$, $Y \in \mathbb{A}^m$ are affine varieties, $p \in Map(X, Y)$, define p^* to be the map $p^* : Mor_k(O(Y), O(X))$, $p^*(f) = f \circ p$.*

Note that p is a map from X to Y whereas p^* is a morphism from $O(Y)$ to $O(X)$. In light of the previous theorem, we can also say $p^* : \text{Map}(Y, \mathbb{A}^1 \to \text{Map}(X, \mathbb{A}^1)$.

Next, we prove that there is a one-to-one correspondence between p and p^*:

Theorem 25. *Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be affine varieties. There exists a natural 1-1 correspondence between $Map(V, W)$ and $Mor_k(O(W), O(V))$.*

Proof. Define p and p^* as in the definition of pullbacks. We claim that the map $p \to p^*$ is injective.

Let $s, s' \in \text{Map}(V, W)$ with $s = (f_1, ..., f_m)$ and $s' = (f'_1, ..., f'_m)$. We want to show that if $s^* = s'^*$ i.e $s^*(f) = s'^*(f)$ for all $f \in O(W)$, then $s = s'$. To see this, note that $f_i = x_i \circ s = s^*(x_1) = s'^*(x_i) = x_i \circ s' = f'_i$. Given $f_i = f'_i$ for all $i = 1, ..., m$, therefore $s = s'$.

Now we claim that the map $p \to p^*$ is surjective. Let $\lambda \in \text{Mor}_k(O(W), O(V))$. We construct a map $s \in \text{Map}(V, W)$ such that $\lambda = s^*$.

Let $f_i \in k[x_1, ..., x_n]$ such that $\lambda(y_i) = f_i$ for $i = 1, ..., m$. Deine $s : \mathbb{A}^n \to \mathbb{A}^m$ such that $s(a_1, ..., a_n) = (f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n))$. Now, if $g \in I(W)$, then $g(f_1, ..., f_m) = g(\lambda(y_1), ..., \lambda(y_m)) = \lambda g(y_1, ..., y_m) = 0$, where we got the last inequality by noting that $g \in I(W)$ so it is 0 in $O(W)$ and λ is a homomorphism so it must send 0s to 0s. Note that for any $g \in k[y_1, ..., y_m]$, $\lambda(g) = g(f_1, ..., f_m)$; to see this, write $g(y_1, ..., y_m) = \sum_i c_i y_1^{i_1} \cdots y_m^{i_m}$, so $\lambda(g(y_1, ..., y_m)) = \lambda(\sum_i c_i y_1^{i_1} \cdots y_m^{i_m}) = \sum_i \lambda(c_i y_1^{i_1} \cdots y_m^{i_m}) = \lambda(c_i)\lambda(y_1^{i_1} \cdots y_m^{i_m}) =$
$\sum_i c_i \lambda(y_1^{i_1} \cdots y_m^{i_m}) = g(f_1, ..., f_m)$

This means, for any $a = (a_1, ..., a_n) \in V$, $g(s(a)) = g(f_1(a), ..., f_m(a)) = 0$. Therefore, all $g \in I(W)$ vanish on $s(a), a \in V$. So, $s(a) \in W, \forall a \in V$. This means s restricted to V is a polynomial map i.e $s|_V \in \text{Map}(V, W)$.

Note that $\lambda = s^*$ on $y_1, ..., y_m$ because if $s = (f_1, ..., f_m)$, then $s^*(y_i) = y_i \circ s = y_i \circ (f_1, ..., f_m) = y_i \circ (\lambda(y_1), ..., \lambda(y_m)) = \lambda(y_i)$. Since they agree on $y_1, ..., y_m$, they agree on all of $O(W)$. □

Definition 18. *Kernel ideal. For a k-algebra homomorphism $\phi : R \to S$, the ideal $ker(\phi) = \{x \in R : phi(x) = 0\}$ is called the kernel ideal.*

Definition 19. *m_a. Given $a = (a_1, ..., a_n)$,*

$$m_a = (x_1 - a_1, ..., x_n - a_n)$$

.

One can check that m_a is the kernel of the k-algebra homomorphism $\psi : k[x_1, ..., x_n] \to k$ such that $\psi(x_i) = a_i$.

Therefore, $m_a = I(\{a_1, ..., a_n\})$.

Theorem 26. *For $I \subset k[x_1, \cdots, x_n]$, the vanishing ideal $I(V(I))$ is given by the intersection of all kernels of k-algebra homorphisms $k[x_1, \cdots, x_n] \to k$ such that I is mapped to 0 i.e*

$$I(V(I)) = \cap_{a \in V(I)} m_a$$

Proof. This is pretty straightforward. Suppose $f \in I(V(I))$. Then, $f(a) = 0, \forall a \in V(I)$. Therefore, $f \in m_a, \forall a \in V(I)$. Therefore, $f \in \cap_{a \in V(I)} m_a$. On the other hand, suppose $f \in \cap_{a \in V(I)} m_a$. Then, $f(a) = 0$ for all $a \in V(I)$. So, $f \in I(V(I))$. □

Corollary 27. *A finitely generated k-algebra R is the coordinate ring of an affine variety if and only if for all $f \neq 0$, $f \in R$, there exists a k-algebra homomorphism $\phi : R \to k$ such that $\phi(f) \neq 0$.*

Proof. Suppose R is the coordinate ring of an affine variety, i.e $R = O(X)$. Let $X := V(I)$. Then, $I(X) = I(V(I)) = \cap_{a \in V(I)} m_a$. Now, if $f \neq 0$ in $R = O(X)$, then $f \notin I(V(I))$ implying $f \notin m_a$ for some $a \in V(I)$. Therefore, there exists some k-algebra homomorphism $\psi : k[x_1, ..., x_n] \to k$ such that $\psi(x_i) = a_i$ and $\psi(f) \neq 0$.

Conversely, suppose for any $f \neq 0$, $f \in R$, there exists a k-algebra homomorphism $\psi : R \to k$ such that $\psi(f) \neq 0$.

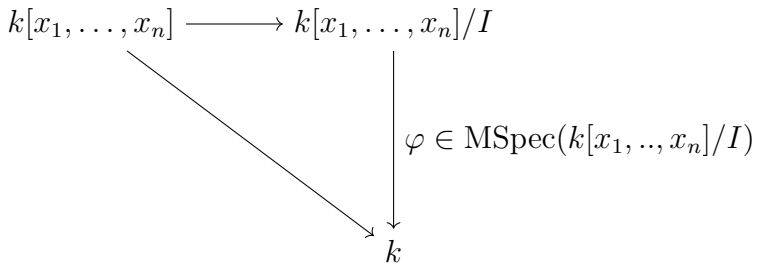
Now, suppose $f \neq 0$. Then, $f \notin m_a$ for some $a \in V := V(I)$. So, $f \in m_a$ for all $a \in V = V(I)$ implies $f = 0$. So, $f \in I(V(I))$ implies $f = 0$. Therefore, $I(V(I)) = \{0\}$ and we can write $R = O(V)$ i.e a coordinate ring of the affine variety V. □

With these results, we can redefine coordinate ring: a finitely generated k-algebra R is a coordinate ring if for any $f \neq 0$, $f \in R$, there exists a k-algebra homomorphism $\psi : R \to k$ such that $\psi(f) \neq 0$.

We can now define three abstract structures that are ubiquitous in algebraic geometry:

Definition 20. *Maximal space MSpec(R). For R a coordinate ring, the maximal space MSpec(R) of R is the set of k-algebra homomorphisms $p : R \to k$ which we call points. For $f \in R$, we say f vanishes at point p if $p(f) = 0$.*

Let's make this more clear. Let $R = k[x_1, ..., x_n]/I(X)$. Let $\varphi : R \to k$ be a k-algebra homomorphism where I is the kernel. Then, we can visualize φ as:



With this in mind, we note that we can associate an element of $\text{MSpec}(k[x_1, ..., x_n]/I)$ with a point in $(a_1, \cdots, a_n) \in X$ in k. This is because $\varphi(f_i) = 0, \forall f_i \in I$, so $x_i \to a_i \in k$ and since x_i generate all of the quotient ring, therefore, we can send all polynomials to k.

Therefore, we will often write:

$$MSpec(k[x_1, ..., x_n]/(f_1, .., f_m)) = \{(a_1, ..., a_n) \in \mathbb{A}^n : f_i(a_1, .., a_n) = 0, \forall i = 1, .., r\}$$

We can turn $\text{MSpec}(R)$ into a topological space by letting the basic closed sets be $V_{\text{MSpec}(R)}(f) = \{p \in \text{Mspec}(R) : p(f) = 0\}$.

Definition 21. *Abstract affine variety and ring of functions. A pair $(V, R$ is an abstract affine variety if R is a coordinate ring and V is identified with the topological space $MSpec(R)$.*

We often just write V instead of (V, R). We call R the ring of functions on V.

Here's an intuition for why abstract affine varieties are required. We want to study polynomials in $R = O(X) = k[x_1, \cdots, x_n]$. We can move to the geometric world by studying the affine variety of a polynomial in R. Given a polynomial, we can fully determine the set of zeros and therefore attain the variety. But given a just variety in the geometric world which is just a set of elements in $\mathbb{A}^n$, we cannot hope to recover $R = O(X)$. Therefore, in an almost silly manner, we remember the data by just letting the abstract affine variety be (V, R).

Definition 22. *Morphism of abstract varieties. For (V, R) and (W, S) abstract varieties, the morphism of (V, R) and (W, S) is a k-algebra homomorphism $\psi^* : W \to V$ with the induced map $\psi : V = MSpec(R) \to V = MSpec(S) = W$.*

Theorem 28. *(Rabinowitch Trick) The solutions $(a_1, ..., a_n)$ to $f_1 = f_2 = \cdots = f_n = 0$ and $f \neq 0$ are in bijection with the solutions $(a_1, \cdots, a_n, a_{n+1})$ to $f_1 = f_2 = \cdots = f_n = 0$ and $x_{n+1} f(x_1, \cdots, x_n) - 1 = 0$.*

Proof. The bijections take $a_1, \cdots, a_n$ to $a_1, ...a_n, \frac{1}{f(a_1,...,a_n)}$ and (reverse) $a_1, ...a_n, a_{n+1}$ to $a_1, ...a_n$. □

Now, we define something central to a lot of the techniques.

Definition 23. *Localization of ring R at element $g \in R$. We define*

$$R_g := R[x]/(xg - 1).$$

Because $xg - 1 = 0$ in R_g, we refer to x as g^{-1} (here g is a unit).

Since $x = g^{-1}$ in R_g, therefore, we will often write R_g as $R[g^{-1}]$. Localization is very useful; for example, one can notice that we had used localization in the proof of Hilbert's Nullstellensatz (although we used it as an arbitrary "trick" without really looking deep into the construction).

Here are some immediate properties of the localization of a ring at an element.

Lemma 29. *For any ring R, we have:*
(a) Every element of R_g can be written as rg^{-i} for some $r \in R$ and $i \geq 0$.
(b) $rg^{-i} = sg^{-i} \in R_g$ if and only if $g^N(r - s) = 0 \in R_g$ for some N.
(c) A ring map $R_g \to S$ is the same as a ring map $R \to S$ such that $Im(g) \in S^\times$ i.e g is mapped to a unit in S. In other words,

$$Mor(R_g, S) = \{\varphi \in Mor(R, S) : \varphi(g) \in S^\times\}$$

(d) The map $R \to R_g$ is an isomorphism exactly when $g \in R^\times$.

Proof. (a) Suppose $s \in R_g$. Then, $s = f(x)$. Given f is a polynomial, for i large enough, sg^i not have any x (since $x = g^{-1}$), so $sg^i = r \in R$. This implies the result.
(b) Suppose $g^N(r - s) = 0$. Then, since g is a unit, this implies $r - s = 0$, so $r = s$ and $rg^{-i} = sg^{-i}$. Conversely, suppose $rg^{-i} = sg^{-i}$. Then, $(r - s)g^{-i} = 0$. Since we are operating in R_g, this implies $(r - s)g^{-i} = (1 - xg)(a_0 + a_1x + \cdots + a_nx^n)$. Expanding the right hand side and comparing the coefficients of $x^0, x^1, \cdots, x^n$, we see that $a_0 = r - s$, $a_1 = (r - s)g$ and $a_n = (r - s)g^n$. Now, comparing the coefficient of x^{n+1}, we get that $a_ng = 0 \implies (r-s)g^{n+1} = 0$. // (c) A ring map $\psi : R_g \to S$ is the same as a map $\phi : R \to S$ such that ϕ sends $(xg - 1)$ to 0. But then, $\phi(x)\phi(g) - \phi(1) = 0$, so $\phi(x) = \phi(g)^{-1} = \phi(g^{-1})$. Therefore, g has to be a unit. If g is not a unit, these maps cannot be equivalent.
(d) If g is a unit, then $R[x]/(xg - 1) = R[x](x - g^{-1}) = R$. Conversely, if g is not a unit, then R is not isomorphic to R_g. □

Now, we get the following result:

Theorem 30. *For R a coordinate ring i.e $R = O(V)$, R_f is also a coordinate ring with $MSpec(R_f) = D(f)$ as topological spaces. If $V \subset \mathbb{A}^n$ is a variety with cooridnate ring R, then, $\{(v, f(v)^{-1} : v \in V, f(v) \neq 0\} \subset \mathbb{A}_k^{n+1}$ is a variety with coordiante ring R_f.*

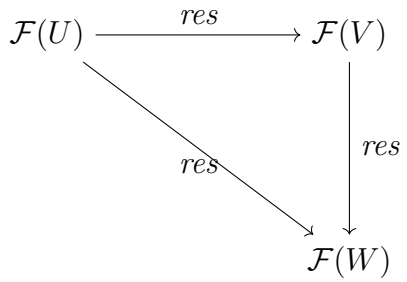
Proof. First, we prove that for $R = O(V)$, R_f is a coordinate ring. Given R is the quotient of a polynomial ring, it is a finitely generated algebra. Now, using corollary 22, R_f is a coordinate ring if and only if for all $g \in R_f$, $g \neq 0$, there exists a k-algebra homomorphism $q : R_f \to k$ such that $q(g) \neq 0$. Let $g = hf^{-i}$ for $i \geq 0$, $h \in R$, $g \neq 0$ in R_f. Then, $hf \neq 0$, since otherwise $g = 0$. Now, we know there exists a k-algebra homomorphism $p : R := O(v) \to k$ such that $p(hf) = p(h)p(f) \neq 0$. As $p(f) \neq 0$, therefore, we get a map from $R_f \to k$ such that $p(g) \neq 0$. Therefore, R_f is a coordinate ring.

Furthermore, $MSpec(R_f) = D(f)$. To see this, suppose $V = (f_1, ..., f_m)$, so
$R_f = (k[x_1, ..., x_n]/(f_1, ..., f_m))_f = (k[x_1, ..., x_n]/(f_1, ..., f_m))[f^{-1}]$. Therefore, $MSpec(R_f)$ is associated with points such that $(f_1, ..., f_m)$ are 0 but $1/f(x_1, ..., x_n)$ is not 0. This is the same as $D(f)_V$. Furthermore, $D(f)$ can be considered an affine variety too by considering it as $V(f_1, ..., f_m, xf - 1)$. □

8 Sheaves

Although sheaves can defined using any category, we will define it using rings.

Definition 24. *Sheaf. Let X be a topological space. A sheaf of rings $\mathcal{F}$ satisfies the following:*
*(1) For every open $U \subset X$, $\mathcal{F}(U)$ is a ring whose elements are called **sections** or **functions** over U. $\mathcal{F}(X)$ is called the ring of global sections or functions.*
(2) Restriction. If $V \subset U$, we can restrict $\mathcal{F}(U) \xrightarrow{res} \mathcal{F}(V)$ i.e if $f \in \mathcal{F}(U)$, then $res(f) = f|_V$. Restrictions on sections commute:



(3) Identity. Given a collection of open sets $\{U_i\}_i$ covering U and $f, g \in \mathcal{F}(U)$,

$$f|_{U_i} = g|_{U_i}, \forall i \implies f = g$$

(4) Gluability. Given $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}, \forall i, j$, there exists a unique $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$

9 References

1. William Fulton. *Algebraic Curves. An Introduction to Algebraic Geometry.* 2008.

2. Dummit and Foote. *Abstract Algebra, 3rd Ed.*

3. Ravi Vakil. *Math 216 Lectures at Stanford University.*

4. Zhiyu Zhang. *Math 145 Lectures at Stanford University.*

5. Ravi Vakil. *The Rising Sea: Foundations of Algebraic Geometry*

A Category Theory

These notes are directly taken from (5).

Definition: Category. A **category** consists of a collection of **objects** and for each pair of objects, a set of **morphisms** or **arrows** between them (which are often called **maps**). The collection of objects of a category, C, is denoted as $\text{obj}(C)$ but we will often denote this also by C. If $A, B \in C$, the set of morphisms from A to B are denoted by $\mathbf{Mor}(A, B)$ where a morphism is often written as $f : A \to B$. A is the **source** of f whereas B is the **target** of f.

Morphisms compose as expected; $\text{Mor}(B, C) \times \text{Mor}(A, B) \to \text{Mor}(A, C)$. Composition is associative, i.e $(f \circ g) \circ h = f \circ (g \circ h)$.

For each object $A \in C$, there exists an **identity morphism** $\text{id}_A : A \to A$ such that $f \circ \text{id}_A = f$ and $\text{id}_A \circ f = f$. Identity morphism is unique.

Definition: Isomorphism. An isomorphism between two objects is a morphism $f : A \to B$ such that there exists a unique morphism $g : B \to A$ such that $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$

Definition: Automorphism. The set of invertible elements of $\text{Mor}(A, A)$ forms a group called the **automorphism groupp of** A.

Examples:
(1) Category of sets. The objects are sets and the morphisms are maps of sets.

(2) Another good example is the category Vec_k of vector spaces over a given field k. The objects are k-vector spaces, and the morphisms are linear transformations.

(3) Category of Abelian groups. The objects are Abelian groups and the morphisms are the

group homomorphisms. This category is denoted as Ab.

(4) Category of modules over a ring. If A is a ring, then the A-modules form a category $Mod(A)$

(5) Category of rings. Objects are rings and morphisms are ring homomorphisms.

Defintion: Subcategory. A subcategory A of a category C includes some of the objects and morphisms of C such that the objects of A include the sources and targets of morphisms of A and the morphisms of A include the identity morphisms of the objects in A and are preserved by composition.

Defintion: Covariant function from category $\mathcal{A}$ to category $\mathcal{B}$, denoted by $\mathrm{F} : \mathcal{A} \to \mathcal{B}$. This is a map of objects $\mathrm{F} : \mathrm{obj}(\mathcal{A}) \to \mathrm{obj}(\mathcal{B})$ and for each $A_1, A_2 \in \mathcal{A}$ and morphism $m : A_1 \to A_2$, a morphism $\mathrm{F}(m) : \mathrm{F}(A_1) \to \mathrm{F}(A_2)$. We require F preserves identity morphisms i.e $\mathrm{F}(\mathrm{id})_A = \mathrm{id}_{\mathrm{F}(A)}, \forall A \in \mathcal{A}$.