

Introduction to Real and Complex Projective Spaces

Jubayer Ibn Hamid

1 Real Projective Space

We define the real projective space as follows: for $n \geq 1$, define $\mathbb{RP}^n = S^n / \sim$ with the equivalence relation $x \sim y$ if and only if $x = y$ or $x = -y$. It can also be seen as the space attained by quotienting $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.

Therefore, \mathbb{RP}^n identifies each direction through the origin of the n -dimensional sphere as unique points.

An interesting observation is the appearance of the mobius band inside \mathbb{RP}^2 (see figure 1). To see this, let D be a closed disk of radius 1 in \mathbb{R}^2 i.e $D := \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Then, it is clear that $D \setminus \sim$ is homeomorphic to \mathbb{RP}^2 (via projection). Now, let $r \in (0, 1)$. We first cut out the disk D_r of radius r from inside D to get an annulus A . Now, $A \setminus \sim$ is homeomorphic to the mobius band [1].

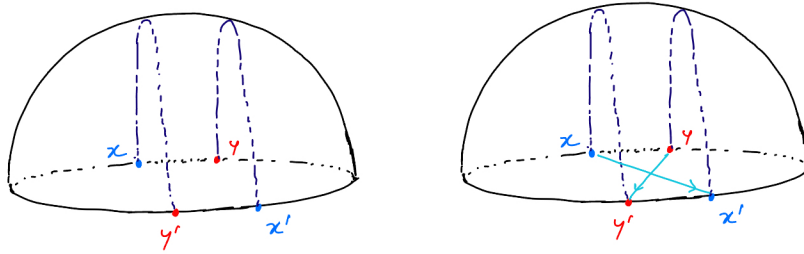


Figure 1: Möbius strip inside \mathbb{RP}^2

Theorem 1. *The real projective space, \mathbb{RP}^n , is a compact, n -dimensional manifold.*

Proof. First, we show that \mathbb{RP}^n is compact. Note that S^n is compact. Consider the quotient map $p : S^n \rightarrow S^n / \sim$. Note that this mapping is continuous. To see this, let I be the identity function on S^n / \sim . Then, $(I \circ p)(x) = p(x)$. Now, given I is continuous, then $I \circ p$ is also continuous $\implies p$ is continuous. Since $p : S^n \rightarrow S^n / \sim$ is continuous and S^n is compact, therefore, \mathbb{RP}^n is compact.

Next, we show \mathbb{RP}^n is Hausdorff. Consider any $[x], [y]$ in \mathbb{RP}^n such that $[x] \neq [y]$. This means $x \neq y, x \neq -y$ in S^n . Now, in S^n , consider the following open sets - U_x which contains x , U_{-x} which contains $-x$, U_y which contains y and U_{-y} which contains $-y$. Given S^n is Hausdorff, we can let these sets be pairwise disjoint. Furthermore, $p(U_x), p(U_{-x}), p(U_y), p(U_{-y})$ are all open since. Furthermore, $p(U_x) \cup p(U_{-x})$ contains x and is open. We claim $(p(U_x) \cup p(U_{-x})) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$. This is because $p(U_x) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$ and $p(U_{-x}) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$. For the first part, suppose $[z] \in p(U_x) \cap (p(U_y) \cup p(U_{-y})) \implies \exists a \in U_x$ such that $p(a) = z$ and $\exists b \in U_y$ such that $p(b) = z$ and $\exists c \in U_{-y}$ such that $p(c) = z$. Now $p(a) = p(b)$ implies $a = b$ or $a = -b$. If $a = b$, then $U_x \cap U_y \neq \emptyset$. So $a = -b$. By similar logic $a = -b' \implies -b = -b' \implies U_y \cap U_{-y} \neq \emptyset$ which is also a contradiction.

Next, we know \mathbb{RP}^n is second countable since S^n is second countable.

Now, we show \mathbb{RP}^n is locally Euclidean and has dimension n . Let $[x] \in \mathbb{RP}^n$. Without loss of generality, suppose $x_0 \neq 0$ (if it is, then we can always rotate the space to ensure it is not 0). Then consider the following function $\pi([(x_0, x_1, \dots, x_n)]) = \left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)$. This function is bijective since, for any $y \in \mathbb{R}^n$ with $y := (y_1, \dots, y_n)$, we have $\pi^{-1}((y_1, \dots, y_n)) = \left(\frac{1}{\sqrt{1+y_1^2+\dots+y_n^2}}, \frac{y_1}{\sqrt{1+y_1^2+\dots+y_n^2}}, \dots, \frac{y_n}{\sqrt{1+y_1^2+\dots+y_n^2}}\right)$. Therefore π maps \mathbb{RP}^n to all of \mathbb{R}^n . Given π is continuous, \mathbb{RP}^n is locally Euclidean with dimension n . \square

2 Complex Projective Space

Let $X = \mathbb{C}^{n+1} \setminus \{0\}$. Now, define the following equivalence class on X : $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then, the complex projective space is defined as $\mathbb{CP}^n = X / \sim$.

Note that \mathbb{C}^{n+1} is isomorphic to \mathbb{R}^{2n+2} . Therefore, if $p \in \mathbb{C}^{n+1}$ with $(p_1 + ip_2, p_3 + ip_4, \dots, p_{2n+1} + ip_{2n+2})$, then we can write p in \mathbb{R}^{2n+2} as $p = (p_1, p_2, \dots, p_{2n+2})$. Now, suppose $p \sim p'$ with $p = \lambda p'$ where $\lambda = \lambda_1 + i\lambda_2$. Then, in \mathbb{R}^{2n+2} , after expanding and simplifying, we see that $(p_1\lambda_1 - p_2\lambda_2, p_2\lambda_1 + p_1\lambda_2, \dots) = (p'_1, p'_2, \dots)$. This tells us that, for the first two coordinates, we have the following relation:

$$\begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix}. \quad (1)$$

Similarly, for the third and fourth coordinates, we also have the similar relation. Therefore, the equivalence class of p in \mathbb{R}^{2n+2} can be written as the set consisting of

$$\left[\begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{2n+1} \\ p_{2n+2} \end{bmatrix} \right] \quad (2)$$

for any $\lambda_1, \lambda_2 \in \mathbb{R}$.

The complex projective space is the space of all complex lines through the origin or the set of all one-dimensional subspace. Furthermore, if we consider λ to have unit length, we can also write \mathbb{CP}^n as $S^{2n+1}/U(1)$ where $U(1)$ is the unitary group.

Theorem 2. *The complex projective space, \mathbb{CP}^n , is a compact $2n$ -dimensional manifold.*

Proof. Let $\pi : X \rightarrow X/\sim$.

First, we show that this space is compact. Consider the sphere S^{2n+1} . This sphere is compact. The function that maps points on the sphere to X/\sim is continuous (by the same reasoning as provided in the previous proof). Furthermore, the function is surjective, since for any equivalence class in X/\sim , there exists a point that is on the surface of the sphere. Therefore, the image space of this function, X/\sim is compact too.

Next, we show that this space is Hausdorff. Consider any $x \in S^{2n+1}$. Then, its equivalence class is $[x] = \{y \in X | y = \lambda x, \forall \lambda \in \mathbb{C} \setminus \{0\}\} =: O_x$. Given the function $(\lambda, x) \rightarrow \lambda x$ is continuous and since S^{2n+1} is compact, therefore, $[x]$ is compact. Given $[x] \neq [y]$, this means $O_x \cap O_y = \emptyset$ with both O_x, O_y being compact. On the other hand, since S^{2n+1} is Hausdorff, there exists open sets $O_x \subset U_x$ and $O_y \subset U_y$ with $y \in U_y, x \in U_x$ and $U_x \cap U_y = \emptyset$.

Now, \bar{U}_y is closed (given this is the closure of U_y) $\implies \pi(\bar{U}_y)$ is closed. On the other hand, define $U'_x := \mathbb{CP}^n \setminus \pi(\bar{U}_y)$, which must be open in \mathbb{CP}^n . Furthermore, define $U'_y := \pi(U_y)$, which must be open. Clearly, $U'_x \cap U'_y = \emptyset$. All that's left to show is $[x] \in U'_x$ and $[y] \in U'_y$. Let us show the first one. S^{2n+1} is compact and Hausdorff, while \bar{U}_x is closed, so \bar{U}_x is

compact. On the other hand, O_x is compact. Given both O_x and \bar{U}_y are compact, we find two disjoint open sets U and W such that $O_x \subset U$ and $\bar{U}_y \subset W$ and $U \cap W = \emptyset$. Therefore, $O_x \cap W = \emptyset$. Now, $[x] \in O_x$ implies $[x] \notin \pi(W)$. Therefore, $[x] \notin \pi(\bar{U}_y)$. This means, $[x] \in \mathbb{CP}^n \setminus \pi(\bar{U}_y)$. Therefore, $[x] \in U'_x$. Similarly, $[y] \in U'_y$.

Now, we show that this space is locally Euclidean. Consider the following function:

$$[x_1, \dots, x_{n+1}] \rightarrow \frac{1}{x_k} [x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n].$$

Then, this is a function from the set $A_k := \{[x] \in \mathbb{CP}^n | x_k \neq 0\}$ to \mathbb{C}^n . The inverse of this function is:

$$(x_1, \dots, x_n) \rightarrow [x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n].$$

Therefore, this is a homeomorphism. Now, $\mathbb{CP}^n = \cup_{k=1, \dots, n+1} A_k$ is open and so our functions maps all of the space too all of \mathbb{C}^n which is isomorphic to \mathbb{R}^{2n} . Therefore, \mathbb{CP}^n is locally Euclidean with dimension n .

Lastly, X/\sim is second countable too since \mathbb{R}^{2n+2} is second countable.

□

3 References

[1] <https://math.stackexchange.com/questions/2963241>