Introduction to Real and Complex Projective Spaces

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1 Real Projective Space

We define the real projective space as follows: for $n \geq 1$, define $\mathbb{RP}^n = S^n/\sim$ with the equivalence relation $x \sim y$ if and only if $x = y$ or $x = -y$. It can also be seen as the space attained by quotienting $\mathbb{R}^{n+1}\backslash\{0\}$ under the equivalence relation $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.

Therefore, $\mathbb{RP}^n$ identifies each direction through the origin of the n-dimensional sphere as unique points.

An interesting observation is the appearance of the mobius band inside $\mathbb{RP}^2$ (see figure 1). To see this, let D be a closed disk of radius 1 in $\mathbb{R}^2$ i.e $D := \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Then, it is clear that $D\backslash\sim$ is homeomorphic to $\mathbb{RP}^2$ (via projection). Now, let $r \in (0, 1)$. We first cut out the disk D_r of radius r from inside D to get an annulus A. Now, $A\backslash\sim$ is homeomorphic to the mobius band [1].

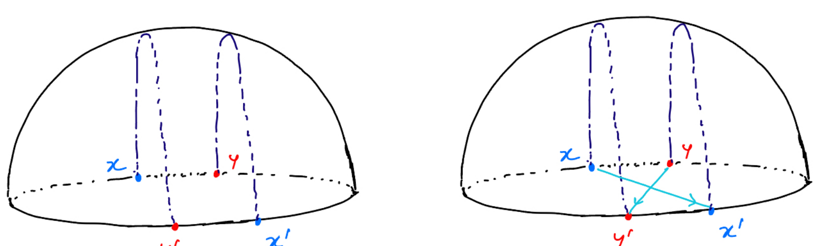

Figure 1: Mobius strip inside $\mathbb{RP}^2$

Theorem 1. *The real projective space, $\mathbb{RP}^n$, is a compact, n-dimensional manifold.*

Proof. First, we show that $\mathbb{RP}^n$ is compact. Note that S^n is compact. Consider the quotient map $p : S^n \to S^n \backslash \sim$. Note that this mapping is continuous. To see this, let I be the identity function on $S^n \backslash \sim$. Then, $(I \circ p)(x) = p(x)$. Now, given I is continuous, then $I \circ p$ is also continuous $\implies$ p is continuous. Since $p : S^n \to S^n \backslash \sim$ is continuous and S^n is compact, therefore, $\mathbb{RP}^n$ is compact.

Next, we show $\mathbb{RP}^n$ is Hausdorff. Consider any $[x], [y]$ in $\mathbb{RP}^n$ such that $[x] \neq [y]$. This means $x \neq y$, $x \neq -y$ in S^n. Now, in S^n, consider the following open sets - U_x which contains x, U_{-x} which contains $-x$, U_y which contains y and U_{-y} which contains $-y$. Given S^n is Hausdorff, we can let these sets be pairwise disjoint. Furthermore, $p(U_x), p(U_{-x}), p(U_y), p(U_{-y})$ are all open since. Furthermore, $p(U_x) \cup p(U_{-x})$ contains x and is open. We claim $(p(U_x) \cup p(U_{-x})) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$. This is because $p(U_x) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$ and $p(U_{-x}) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$. For the first part, suppose $[z] \in p(U_x) \cap (p(U_y) \cup p(U_{-y})) \implies \exists a \in U_x$ such that $p(a) = z$ and $\exists b \in U_y$ such that $p(b) = z$ and $\exists c \in U_{-y}$ such that $p(c) = z$. Now $p(a) = p(b)$ implies $a = b$ or $a = -b$. If $a = b$, then $U_x \cap U_y \neq \emptyset$. So $a = -b$. By similar logic $a = -b' \implies -b = -b' \implies U_y \cap U_{-y} \neq \emptyset$ which is also a contradiction.

Next, we know $\mathbb{RP}^n$ is second countable since S^n is second countable.

Now, we show $\mathbb{RP}^n$ is locally Euclidean and has dimension n. Let $[x] \in \mathbb{RP}^n$. Without loss of generality, suppose $x_k \neq 0$ (if it is, then we can always rotate the space to ensure it is not 0). Then consider the following function $\pi([(x_0, x_1, ..., x_n]) = \left(\frac{x_1 x_k}{|x_k|}, ..., \frac{x_n x_k}{|x_k|}\right)$. This function is bijective from the set $A_k := \{[x] \in \mathbb{RP}^n | x_k \neq 0\}$ to $\mathbb{D}^n := \{x \in \mathbb{R}^n | |x| < 1\}$. Its inverse is given by $\pi^{-1}((x_1, ..., x_n)) = (x_1, ..., x_{k-1}, \sqrt{1 - |x|^2}, x_k, ..., x_n)$. Note that $\mathbb{RP}^n = \cup_{k=1,..,n+1} A_k$. Therefore π maps $\mathbb{RP}^n$ to all of R^n. Given π is continuous, $\mathbb{RP}^n$ is locally Euclidean with dimension n. ∎

Theorem 2. *The real projective space, $\mathbb{RP}^n$, is a smooth manifold.*

Proof. We denote points in the real projective space as $[x_0 : x_1 : \cdots : x_n]$ which represents the equivalence class of $(x_0, x_1, ..., x_n) \in \mathbb{R}^{n+1} \backslash \{0\}$. We will construct an atlast on $\mathbb{RP}^n$ with $n+1$ charts (U_i, ϕ_i) for $i = 0, .., n$. Define $U_i = \{[x_0 : x_1 : \cdots : x_n] | x_i \neq 0\}$. Then, as before, $\mathbb{RP}^n = \cup_i U_i$. Now, we define the homeomorphism:

$$\phi_i : U_i \to \mathbb{R}^n$$

such that $\phi_i([x_0 : .... : x_n]) = (\frac{x_0}{x_i}, \frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i})$. We prove, now, that the transition functions, i.e $\phi_j \circ \phi_i^{-1}$, are smooth. Let $\phi_i([x_0 : .... : x_n]) = (\frac{x_0}{x_i}, \frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_n}{x_i}) =: (y_0, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)$ and let

$\phi_j([x_0 : \cdots : x_n]) = (\frac{x_0}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j}) = (\frac{y_0}{y_j}, \cdots, \frac{1}{y_j}, \cdots, \frac{y_n}{y_j})$. Therefore, our transition function becomes:

$$(\phi_j \circ \phi_i^{-1})(y_0, \cdots, y_n) = \left(\frac{y_0}{y_j}, \cdots, \frac{1}{y_j}, \cdots, \frac{y_n}{y_j}\right)$$

which is smooth. This gives us an atlas and therefore, implicitly defines a smooth structure on $\mathbb{RP}^n$. $\square$

2 Complex Projective Space

Let $X = \mathbb{C}^{n+1} \backslash 0$. Now, define the following equivalence class on X: $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C} \backslash \{0\}$. Then, the complex projective space is defined as $\mathbb{CP}^n = X/\sim$.

Note that $\mathbb{C}^{n+1}$ is isomorphic to $\mathbb{R}^{2n+2}$. Therefore, if $p \in \mathbb{C}^{n+1}$ with $(p_1+ip_2, p_3+ip_4, ..., p_{2n+1}+ip_{2n+2})$, then we can write p in $\mathbb{R}^{2n+2}$ as $p = (p_1, p_2, ..., p_{2n_2})$. Now, suppose $p \sim p'$ with $p = \lambda p'$ where $\lambda = \lambda_1 + i\lambda_2$. Then, in $\mathbb{R}^{2n+2}$, after expanding and simplifying, we see that $(p_1\lambda_1 - p_2\lambda_2, p_2\lambda_1 + p_1\lambda_2, ...) = (p_1', p_2', ...)$. This tells us that, for the first two coordinates, we have the following relation:

$$\begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p_1' \\ p_2' \end{bmatrix}. \tag{1}$$

Similarly, for the third and fourth coordinates, we also have the similar relation. Therefore, the equivalence class of p in $\mathbb{R}^{2n+2}$ can be written as the set consisting of

$$\begin{bmatrix} \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} & & & \\ & \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \\ & & & & \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ p_{2n+1} \\ p_{2n+2} \end{bmatrix} \tag{2}$$

for any $\lambda_1, \lambda_2 \in \mathbb{R}$.

The complex projective space is the space of all complex lines through the origin or the set of all one-dimensional subspace. Furthermore, if we consider λ to have unit length, we can also write $\mathbb{CP}^n$ as $S^{2n+1}/U(1)$ where $U(1)$ is the unitary group.

Theorem 3. *The complex projective space, $\mathbb{CP}^n$, is a compact $2n$-dimensional manifold.*

Proof. Let $\pi : X \to X/\sim$.

First, we show that this space is compact. Consider the sphere S^{2n+1}. This sphere is compact. The function that maps points on the sphere to $X/\sim$ is continuous (by the same reasoning as provided in the previous proof). Furthermore, the function is surjective, since for any equivalence class in $X/\sim$, there exists a point that is on the surface of the sphere. Therefore, the image space of this function, $X/\sim$ is compact too.

Next, we show that this space is Hausdorff. Consider any $x \in S^{2n+1}$. Then, its equivalence class is $[x] = \{y \in X | y = \lambda x, \forall \lambda \in \mathbb{C}\backslash\{0\}\} =: O_x$. Given the function $(\lambda, x) \to \lambda x$ is continuous and since S^{2n+1} is compact, therefore, $[x]$ is compact. Given $[x] \neq [y]$, this means $O_x \cap O_y = \emptyset$ with both O_x, O_y being compact. On the other hand, since S^{2n+1} is Hausdorff, there exists open sets $O_x \subset U_x$ and $O_y \subset U_y$ with $y \in U_y, x \in U_x$ and $U_x \cap U_y = \emptyset$.

Now, $\bar{U_y}$ is closed (given this is the closure of U_y) $\implies \pi(\bar{U_y})$ is closed. On the other hand, define $U'_x := \mathbb{CP}^n \backslash \pi(\bar{U_y})$, which must be open in $\mathbb{CP}^n$. Furthermore, define $U'_y := \pi(U_y)$, which must be open. Clearly, $U'_x \cap U'_y = \emptyset$. All that's left to show is $[x] \in U'_x$ and $[y] \in U'_y$. Let us show the first one. S^{2n+1} is compact and Hausdorff, while $\bar{U_x}$ is closed, so $\bar{U_x}$ is compact. On the other hand, O_x is compact. Given both O_x and $\bar{U_y}$ are compact, we find two disjoint open sets U and W such that $O_x \subset U$ and $\bar{U_y} \subset W$ and $U \cap W = \emptyset$. Therefore, $O_x \cap W = \emptyset$. Now, $[x] \in O_x$ implies $[x] \notin \pi(W)$. Therefore, $[x] \notin \pi(\bar{U_y})$. This means, $[x] \in \mathbb{CP}^n \backslash \pi(\bar{U_y})$. Therefore, $[x] \in U'_x$. Similarly, $[y] \in U'_y$.

Now, we show that this space is locally Euclidean. Consider the following function:

$$[x_1, \cdots, x_{n+1}] \to \frac{1}{x_k}[x_1, ..., x_{k-1}, x_{k+1}, .., x_n].$$

Then, this is a function from the set $A_k := \{[x] \in \mathbb{CP}^n | x_k \neq 0\}$ to $\mathbb{C}^n$. The inverse of this function is:

$$(x_1, ..., x_n) \to [x_1, ..., x_{k-1}, 1, x_{k+1}, ..., x_n].$$

Therefore, this is a homeomorphism. Now, $\mathbb{CP}^n = \cup_{k=1,...,n+1} A_k$ is open and so our functions maps all of the space too all of C^n which is isomorphic to $\mathbb{R}^{2n}$. Therefore, $\mathbb{CP}^n$ is locally Euclidean with dimension n.

Lastly, $X/\sim$ is second countable too since $\mathbb{R}^{2n+2}$ is second countable. □

Theorem 4. *The complex projective space, $\mathbb{CP}^n$, is a smooth manifold.*

Proof. This can be proven with the same atlas as we defined for the real projective space. □

3 References

[1] https://math.stackexchange.com/questions/2963241