Introduction to Whitney's Theorems for Embeddings

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The Smooth Partition of Unity

Let X be a smooth manifold with open cover $\{U_\alpha\}_{\alpha\in I}$. A smooth partition of unity subordinate to this open cover is a sequence of smooth functions $\{\theta_i : X \to \mathbb{R}\}_{i=1,2,...}$ such that:

(a) $0 \leq \theta_i(x) \leq 1$ for any $x \in X$.

(b) For any $x \in X$, there exists a neighbourhood V_x such that $\theta_i(y) = 0$ for any $y \in V_x$ holds for at most finitely many i.

(c) For any i, $supp(\theta_i) := \overline{\theta_i^{-1}(\mathbb{R}\backslash\{0\})} \subset U_\alpha$ for some $\alpha \in I$.

(d) For any $x \in X$, $\sum_{i=1}^{\infty} \theta_i(x) = 1$.

It can be proven that every open cover of a smooth manifold admits a smooth partition of unity subordinate to that cover. Additionally, if $\{U_i\}_{i=1,..,N}$ is a finite open cover, we can take $\{\theta_i\}_{i=1,..,n}$ such that $supp(\theta_i) \subset U_i$ for each i and $\theta_i = 0$ for $i > n$ in our original infinite set of smooth functions.

The Bump Function

We want to show the following: given X is a smooth manifold with (U, ϕ) smooth chart and $p \in U$, then there exists a smooth bump function $\beta : X \to \mathbb{R}$ and open neighbourhoods $p \in W \subseteq V \subseteq U$ and $\overline{V} \subseteq U$ such that $\beta(x) = 1$ for $x \in W$, $\beta(x) = 0$ for $x \notin V$ and $\beta(x) \in [0, 1]$ for $x \in X$.

We first construct $f_1(x)$ to be

$$f_1(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{1}$$

Now, let

$$f_2(x) = \frac{f_1(2-x)}{f_1(2-x) - f_1(x-1)}.$$

Note that $f_2(x) = 0$ for any $x \geq 2$, $f_2(x) = 1$ for any $x \leq 1$ and $f_2(x) \in [0,1]$ for any $x \in [1,2]$.

Now, suppose X is a smooth manifold. Then, for any $p \in X$, there exists a chart (U, ϕ). WLOG, suppose $\phi(p) = 0$. Also suppose we have open neighbourhoods such that $W \subseteq V \subseteq U$ with $x \in W$ and $\overline{V} \subseteq U$. Now, select $\epsilon > 0$ such that $B_{3\epsilon}(0)$ is inside $\tilde{U}$ (which is the image of U under ϕ). Then, if $W = \phi^{-1}(B_\epsilon(0))$ and $V = \phi^{-1}(B_{2\epsilon}(0))$, our bump function is defined to be

$$\beta(x) = \begin{cases} h(\frac{||\phi(x)||}{\epsilon}) & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

With this function, $\overline{B_{2\epsilon}(0)} \subset B_{3\epsilon}(0) \subseteq \tilde{U}$ which implies $\tilde{V} \subseteq U$.

Note that $W \subseteq V \subseteq U$ with $x \in W$ and $\overline{V} \subseteq U$. We can easily check that $\beta(x) = 1$ for $x \in W$, $\beta(x) = 0$ for $x \notin V$ and $\beta(x) \in [0,1]$ for $x \in X$.

Now, we move on to the first important result.

Theorem 1. *Let X be a compact, smooth manifold of dimension m. Then, there exists $N \geq m$ and a smooth embedding $f : X \to \mathbb{R}^N$.*

Proof. Pick any $x \in X$ with the smooth chart (U_x, g_x) near it. Then, there exists $\epsilon_x > 0$ such that $B_{\epsilon_x}(g_x(x)) \subset \tilde{U_x}$. Define $W_x := g_x^{-1}(B_{\epsilon_x/2}(g_x(x)))$ and $V_x := g_x^{-1}(B_{\epsilon_x}(g_x(x)))$ - both of these are subsets of $\tilde{U_x}$. Now, $\{W_x\}_{x \in X}$ is a covering of X and since X is compact, there is a finite subcover given by $W_1, ..., W_n$ where $W_i = W_{x_i}$. For each W_i, let V_i and g_i be the corresponding V_{x_i} and g_{x_i}.

Now, we use the following bump function:

$$\phi : X \to [0,1] \text{ such that } \phi_i(x) = \begin{cases} 1, & \text{if } x \in W_i \\ 0, & \text{if } x \in X - V_i \\ 0 \leq \phi_i(x) \leq 1 & \text{otherwise} \end{cases}.$$

$$\text{Then, we define } h_i(x) = \begin{cases} \phi_i(x) g_i(x), & \text{on } V_i \\ 0 & \text{outside } V_i \end{cases}.$$

Using these two functions, we define $f : X \to \mathbb{R}^N$ where $N = n(1+m)$ to be $f(x) = (\phi_1(x), ..., \phi_n(x), h_1(x), ..., h_n(x))$. This map is smooth. Furthermore, we claim that f is injective. This is because, if $f(x) = f(y)$, then $\phi_i(x) = \phi_i(y)$ and $h_i(x) = h_i(y)$ for each i. Given $x \in W_j$ for some j, $\phi_j(x) = 1$ and so $\phi_j(y) = 1$ implying $y \in W_j$. Therefore, $g_j(x) = \phi_j(x)g_j(x) = h_j(x) = h_j(y) = \phi_j(y)g_j(y) = g_j(y)$. Given g is a homeomorphism, $x = y$, showing that f is injective.

Given X is compact and f is injective and continuous, therefore f is a topological embedding too. All that is left is to show that f is an immersion.

Given $x \in X$, $x \in W_i$ for some i. Now, for any $y \in W_i$, given $\phi_i(y) = 1$ and $h_i(y) = g_i(y)$, therefore, $f(y) = (1, ..., 1, g_1(y), ..., g_n(y))$. Now consider the chart (W_i, g_i) where g_i is restricted to W_i. In this chart, g_i looks like the identity, so its derivative also looks like the identity which implies that Df_y has a non-zero $m \times m$ minor. Therefore, Df_x is injective, implying f is a smoother immersion which tells us that f is a smooth embedding. □

Lemma 2. *Let X be a smooth manifold of dimension n. Then, there exists a smooth, proper function from X to $\mathbb{R}$.*

Proof. For any open set of X, we can get a compact closure by mapping the open set to the euclidean space using the chart function, then taking the closed ball around it and then mapping it back to X. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X made up of subsets of X with compact closure i.e $\overline{U_\alpha}$ is compact for each α.

Let $\{\theta_i\}$ be a subordinate partition of unity s.t $supp(\theta_i) \subset U_{\alpha_i}$ for $i = 1, 2, ...$ Now we define the following smooth function: $\rho : X \to \mathbb{R}$ to be $\rho = \sum_{i=1}^{\infty} i\theta_i$. Given (b) in our definition of partition of unity, $\rho(x)$ is finite.

We claim ρ is a proper map. Suppose $K \subseteq \mathbb{R}$ is compact. We want to show that $\rho^{-1}(K)$ is compact.

Since K is compact, it is closed and bounded, meaning there exists some $j > 0$ such that $K \subset [-j, j]$. Then, $\rho^{-1}(K)$ is also closed (since ρ is continuous) and is contained in the set $\{x \in X | \rho(x) \leq j\}$. We claim that if $\rho(x) \leq j$, then at least one of the function $\theta_1, ..., \theta_j$ must take x to a non-zero value. If not, then:

$$\begin{aligned}
\rho(x) &= \sum_{i=1}^{\infty} i\theta_i(x) \\
&= \sum_{i=j+1}^{\infty} i\theta_i(x) \\
&\geq \sum_{i=j+1}^{\infty} (j+1)\theta_i(x) \\
&= (j+1)\sum_{i=1}^{\infty} \theta_i(x) \\
&= (j+1)
\end{aligned}$$

This means, $\rho(x) \geq j+1$ which is a contradiction.

With this, we can now write $\rho^{-1}(K) \subseteq \{x \in X | \rho(x) \leq j\} \subseteq \cup_{i=1}^{j}\{x \in X | \theta_i(x) \neq 0\} \subseteq \cup_{i=1}^{j} U_{\alpha_i} \subseteq \cup_{i=1}^{j} \overline{U_{\alpha_i}}$. Since $\cup_{i=1}^{j} \overline{U_{\alpha_i}}$ is compact, we see that $\rho^{-1}(K)$ is a closed subset of a compact set, so it is compact. □

Theorem 3. *Let X be a smooth manifold of dimension n. Then, there exists $N \geq m$ and a proper, smooth embedding $f : X \to \mathbb{R}^n$.*

Proof. By Theorem 1, we have a smooth embedding $g : X \to \mathbb{R}^p$ and by Lemma 2, we have a proper, smooth function $\rho : X \to \mathbb{R}$. Now, with $N := p+1$, define $f : X \to \mathbb{R}^N$ such that $f(x) = (g(x), \rho(x))$. This is a smooth embedding - f is clearly smooth and since g is a smooth embedding, therefore, the derivative of f at any x is injective and f is a topological embedding.

We now claim f is proper. Suppose $K \subset \mathbb{R}^{p+1}$ is compact, which implies it is closed and bounded - therefore, $K \subset \mathbb{R}^p \times [-j, j]$ for some $j > 0$. Then, $f^{-1}(K) \subseteq \rho^{-1}([-j, j])$. Note that since ρ is compact, $\rho^{-1}([-j, j])$ is compact, so $f^{-1}(K)$ is a closed subset of a compact set which means it is compact. □

Whitney's Theorem While Whitney proved the following theorem for to embed X in $\mathbb{R}^{2n}$, we will prove it for $2n+1$ instead because it is significantly simpler.

Theorem 4. *Let X be a smooth, n-dimensional manifold. Then, X admits a proper, smooth embedding into $\mathbb{R}^{2n+1}$.*

We will prove this by coming up with a proper, smooth immersion $f : X \to \mathbb{R}^{2n+1}$ which automatically allows us to deduce that f is a smooth embedding and $f(X)$ is therefore a smooth submanifold.

Proof. First, we construct $f : X \to \mathbb{R}^{2n+1}$ to be an injective immersion:

By Theorem 1, we can find an injective immersion $f : X \to \mathbb{R}^N$. Now, consider any $a \in \mathbb{R}^{2N}$. Let H_a be the hyperplane that is orthogonal to a and let $\pi_a : \mathbb{R}^N \to H_a$ be the orthogonal projection i.e $\pi_a(x) = x - (x \cdot a)a$. Note that $\frac{\partial \pi_a(x)_i}{\partial x_j} = \delta_{ij} - (a_i a_j)$, which means $D(\pi_a)v = \pi_a$. We claim that $\pi_a \circ f : X \to H_a \cong \mathbb{R}^{N-1}$ is our injective immersion for almost all a in $\mathbb{R}^N$.

To prove this, construct $h : X \times X \times \mathbb{R} \to \mathbb{R}^N$ s.t. $h(x, y, t) = t(f(x) - f(y))$ and $g : TX \to \mathbb{R}^N$ s.t. $g(x, v) = Df_x(v) =: D_v f_x$ with $x \in X$, $v \in T_x X$. Note that g is a function from a $2n$ dimensional space to N and h is a function from $2n + 1$ dimensional space to N.

Now, by Sard's theorem, the set of critical values of g and h have measure zero and therefore their union is also measure zero. Therefore, we can select an arbitrary $a \in \mathbb{R}^N$ such that a is a regular value for both h and g. By the definition of regular values, $Dg_(x', v')$ and $Dh_{x',y',t'}$ are both surjective where the derivatives are taken at $g^{-1}(a)$ and $h^{-1}(a)$. However, since domain of g and h are of dimensions $2n + 1 < N$ and $2n < N$ respectively, therefore, the derivatives cannot be surjective. This means, $a \notin Im(g)$ and $a \notin Im(h)$.

Now, we show that $\pi_a \circ f$ is injective. Suppose $(\pi_a \circ f)(x) = (\pi_a \circ f)(y)$. Then, $(\pi_a)(f(x) - f(y)) = 0$. Given π_a is the projection map, this means, $f(x) - f(y) = ta$ for some t. Furthermore, $t = 0$ because if $t \neq 0$, then $h(x, y, \frac{1}{t}) = \frac{1}{t}(f(x) - f(y)) = \frac{1}{t}ta = a \in Im(h)$. Given $t = 0$, therefore $f(x) = f(y)$ and since f is injective, therefore, $x = y$.

Next, we show $\pi_a \circ f$ is an immersion i.e we show that $D(\pi_a \circ f)$ is injective. Suppose not. Then, there exists $v \neq 0$ such that $D(\pi_a \circ f)_x(v) = 0$. Then,

$$\begin{aligned}
D(\pi_a \circ f)_x(v) &= 0 \\
D(\pi_a)_{f(x)}(Df_x(v)) &= 0 \\
\pi_a \circ Df_x(v) &= 0 \\
Df_x(v) &= ta
\end{aligned}$$

for some t. Given f is an immersion, its derivative is injective and so $t \neq 0$. This means $Df_x(\frac{v}{t}) = \frac{1}{t}(ta) = a \in Im(g)$ which is a contraction, so $\pi_a \circ f$ is an immersion.

So far, we have shown that $\pi_a \circ f$ is an injective immersion from X to $\mathbb{R}^{N-1}$ for $N > 2n+1$. Continuing this way and composing our immersions, we will get an immersion from X to R^{2n+1}.

Next, we will make f a proper map.

Note that $\mathbb{R}^{2n+1} \cong B^{2n+1} = B_1(0)$ by some diffeomorphism s. consider $s \circ f : X \to B_1(0)$. For simplicity in our notation, we will refer to $s \circ f$ as just f. Since the image of f is in $B_1(0)$, therefore, $||f(x)|| < 1$ for any $x \in X$. Furthermore, by Lemma 2, there exists $\rho : X \to \mathbb{R}$ that is smooth and proper.

Define $F : X \to \mathbb{R}^{2n+2}$ s.t. $F(x) = (f(x), \rho(x))$. Then, consider the map $\pi_a \circ F : X \to H_a \cong \mathbb{R}^{2n+2}$ for some a such that the map is an injective immersion as we showed before and $||a|| = 1$. Then, $a \in S^{2n+1}$. Furthermore, suppose $a \neq (0, ..., 0, \pm 1)$ which we can assume given Sard's Theorem tells us almost all points are regular.

We claim $\pi_a \circ F$ is a proper map.

$(\pi_a \circ F)(x) = \pi_a(f(x), \rho(x)) = F(x) - (F(x) \cdot a)a$. Write a as $a = (v, \alpha)$ where $\alpha \in \mathbb{R}$. Then, $F(x) \cdot a = f(x) \cdot v + \rho(x) \cdot \alpha$ and therefore, the last coordinate of $(\pi_a \circ F)(x)$ is $\rho(x) - (f(x) \cdot v + \rho(x) \cdot \alpha)\alpha) = \rho(x)(1 - \alpha^2) - \alpha f(x) \cdot v$.

Now, suppose $K \subset \mathbb{R}^{2n+1}$ is compact. We claim $C := (\pi_a \circ f)^{-1}(K)$ is also compact. We know that K compact means K is closed and bounded. Since our function is smooth, C is also closed.

For any $x \in C$ s.t. $(\pi_a \circ F)(x) \in K$, the last coordinate is $\rho(x)(1 - \alpha^2) - \alpha f(x) \cdot v$. Since K is bounded, this coordinate is also bounded. note that since $|f(x)| < 1$ and α, v are constants, $-\alpha f(x) \cdot v$ is bounded. Therefore, $\rho(x)(1 - \alpha^2)$ is bounded. Furthermore, since $\alpha^2 \neq 1$ (given the last coordinate of a is neither +1 nor -1), so $\rho(x)$ is bounded.

This means, $\rho(C)$ is bounded. Then, $\overline{\rho(C)}$ is closed and bounded and therefore, compact. Given ρ is proper, $\rho^{-1}(\overline{\rho(C)})$ is compact. Now, $C \subseteq \rho^{-1}(\overline{\rho(C)})$ is a closed subset, so C is compact. Therefore, $\pi_a \circ F$ is a proper, injective immersion which implies $\pi_a \circ F$ is a smooth, proper embedding. $\square$

Whitney Immersion Theorem

Theorem 5. *Every n-dimensional, smooth manifold can be immersed in $\mathbb{R}^{2n}$.*

Proof. Suppose, X is a smooth manifold of dimension n. By Whitney's Theorem, we can immerse this into R^{2n+1}. Suppose the immersion is f and suppose it takes X to $M \subset \mathbb{R}^{2n+1}$. Now, we define $g : TX \to \mathbb{R}^{2n+1}$ by $g(x, v) = D_v f_x$. Given f is an immersion, it is smooth and therefore, by Sard's Theorem, we know that almost all values of g are regular. Therefore, we can choose $a \in \mathbb{R}^{2n+1}$ such that a is a regular value. However, note that g's domain is TX is $2n$ dimensional which is less than $2n + 1$. This means, $Dg_{(x', v')}$ (where (x', v') is in the preimage of a under g) cannot be surjective. Therefore, a is not in the image of g i.e $a \notin Im(g)$.

Now, with a as a regular value of g, we claim $\pi_a \circ f$ is a smooth immersion from X to $\mathbb{R}^{2n}$. To show this, we will show that $D(\pi_a \circ f)_x$ is injective.

Suppose, there existed a non-zero $v \in \mathbb{R}^n$ such that $D(\pi_a \circ f)_x(v) = 0$. Now, $D(\pi_a)_{f(x)}(Df_x(v)) = \pi_a \circ Df_x(v)$. Given this is equal to 0, therefore, $Df_x(v) = ta$ for some t. Now, given f is an immersion, $t \neq 0$. But then, $Df_x(\frac{v}{t}) = \frac{1}{t}(ta) = a \in Im(g)$. This is a contradiction. Therefore, $D(\pi_a \circ f)_x$ is injective. Furthermore, $\pi_a \circ f$ is smooth. This gives us our immersion. □

Exhaustion Function on a topological space M

Let M be a topological space. An exhaustion function $f : M \to \mathbb{R}$ is a continuous function such that $f^{-1}((-\infty, c])$ is compact in M for each $c \in \mathbb{R}$.

Turns out we can construct such a function for any smooth manifold M as shown below:

Lemma 6. *Every smooth manifold admits a smooth, positive exhaustion function.*

Proof. Given M is a smooth manifold, we can build a countable open cover of M. Let that be $\{V_j\}_{j=1}^{\infty}$. Furthermore, let $\{\psi_j\}$ be the smooth partitition of unity subordinate to this over cover. Now, we construct the following function:

$$f(p) = \sum_{j=1}^{\infty} j\psi_j(p).$$

Note that this is well-defined and smooth since for any neighbourhood that p is in, there exists only finitely many ψ_j that give non-zero terms. Furthermore, f is positive since $f(p) = \sum_j j\psi_j(p) \geq \sum_j \psi_j(p) = 1$.

Now we claim f is an exhaustion function. To show this, we will prove that for any $c \in \mathbb{R}$, $f^{-1}((-\infty, c])$ is compact.

Choose any arbitrary $c \in \mathbb{R}$. Let $N > c$ be a positive integer.

Now, suppose $p \notin \cup_{j=1}^{N}\overline{V_j}$. Then, $\psi_j(p) = 0$ using the definition of partition of unity for any $j \in [1, N]$. This means, $f(p) = \sum_{j=N+1}^{\infty} j\psi_j(p) \geq \sum_{j=N+1}^{\infty} N\psi_j(p) = N\sum_{j=1}^{\infty} \psi_j(p) = N > c$.

Therefore, if $p \notin \cup_{j=1}^{N}\overline{V_j}$, then $f(p) > c$. So, if $f(p) \leq c$, then $p \in \cup_{j=1}^{N}\overline{V_j}$. Therefore, $f^{-1}((-\infty, c])$ is a closed subset of a compact set $\cup_{j=1}^{N}\overline{V_j}$, which means it is compact. □

Now, we can prove Whitney's Embedding Theorem for the non-compact manifolds.

Theorem 7. *Every non-compact smooth manifold X of dimension n admits a proper smooth embedding into $\mathbb{R}^{2n+1}$.*

Proof. Let $f : X \to \mathbb{R}$ be a smooth exhaustion function on X. Then, by Sard's Theorem, for each non-negative integer i, there exists regular values a_i, b_i of f such that $i < a_i < b_i < i+1$.

Now, we define the following sets: $D_i, E_i \subset X$ such that $D_0 = f^{-1}((-\infty, 1])$, $E_0 = f^{-1}((-\infty, a_1])$, $D_i = f^{-1}([i, i+1])$ and $E_i = f^{-1}([b_{i-1}, a_{i+1}])$ for $i \geq 1$.

Now, given f is a smooth exhaustion, each E_i is compact. Furthermore, one can show that each E_i is a submanifold with a boundary. Therefore, we can embed it into $\mathbb{R}^{2n+1}$ by Theorem 4. Let $\psi_i : E_i \to \mathbb{R}^{2n+1}$

Now, $D_i \subset \operatorname{Int}(E_i)$. Then, $X = \cup_i D_i$ with $E_i \cap E_j = \emptyset$ for any $j = i_1, i$ or $i+1$.

For each i, let $\rho_i : X \to \mathbb{R}$ be a smooth bump function such that $\rho_i = 1$ on an open neighbourhood of D_i and $supp(\rho_i) \subset \operatorname{Int}(E_i)$.

Now, we define
$F : X \to \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R}$ by $F(p) = (\sum_{i \text{ even}} \rho_i(p)\psi_i(p), \sum_{i \text{ odd}} \rho_i(p)\psi_i(p), f(p))$.

(a) F is smooth and well-defined since for each p, there is only one term in each summation that is non-zero.

(b) F is proper as f is.

Furthermore F is injective since $F(x) = F(y)$ implies $f(x) = f(y)$ and using a similar argument as in theorem 4, we can show that $x = y$. F is an immersion too. Let $x \in X$ and let j such that $p \in D_j$. Then, $\rho_j = 1$ on a neighbourhood of p since $p \in D_j$ and D_j has an open neighbourhood on which ρ_j is 1. Suppose j is odd. Then, for any q in this neighbourhood, $F(q) = (\psi_j(q), ...)$. Then, dF_q is injective since ψ_j is an immersion.

Having found an immersion into Euclidean space, we can then use projection like we did with the compact case (using π_a) to find an immersion into $\mathbb{R}^{2n+1}$ $\square$