Algebraic Topology
Notes

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These notes are a combination of notes from Allen Hatcher's book "Algebraic Topology" and Prof. Ciprian Manolescu's lectures from MATH 215a at Stanford University.

Basic Constructions

Def : Homeomorphism

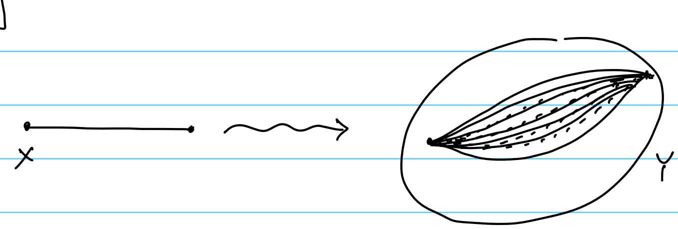
Let X and Y be topological spaces.
$f : X \to Y$ is a homeomorphism if f is bijective and both f and f^{-1} are continuous.
We say $\boxed{X \cong Y}$

Def : Homotopy

A family of maps, $f_t : X \to Y$ where $t \in I = [0,1]$ s.t the associated map $F : X \times [0,1] \to Y$ given by $F(x,t) = f_t(x)$ is continuous.
Two maps $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy $F : X \times [0,1] \to Y$ s.t

$$F(x,0) = f_0(x) \quad \forall x \in X$$
$$F(x,1) = f_1(x)$$

We say $\boxed{f_0 \simeq f_1}$



Def: Homotopy Equivalence

A map $f : X \to Y$ is a homotopy equivalence if $\exists\, g : Y \to X$ s.t $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$
We say the spaces X and Y are homotopy equivalent and $\boxed{X \simeq Y}$

→ can prove easily that this is an equivalence relation.

Examples of homotopy equivalence

(1) $\mathbb{R}^n \simeq$ a point (even though $\underset{\text{infinite}}{\mathbb{R}^n} \not\cong \underset{\text{finite}}{\text{a point}}$)

Why?

$f : \mathbb{R}^n \longrightarrow \{0\}$

and take $g : \{0\} \longrightarrow \mathbb{R}^n$ by $g(0) = 0$

Then $f \circ g = id_{\{0\}}$ and $(g \circ f)(x) = 0$ $\forall x \in \mathbb{R}^n$

Now $g \circ f \sim id_{\mathbb{R}^n}$ by $f_t(x) = tx$ where $f_0 = 0$ and $f_1 = id_{\mathbb{R}^n}$

(2) $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \simeq$ a point

$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \simeq$ a point

Def : Contractive

We say the space X is contractible if $X \simeq$ point.

Def : Retractions

Let X be a space and let $A \subset X$.
Then, a retraction is a map $r : X \longrightarrow X$ s.t
$r(X) = A$ and $r|_A = id_A$.

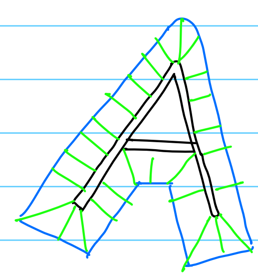
Def : Deformation Retraction

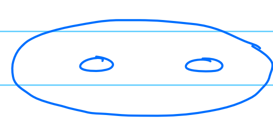
A deformation retraction of X onto a subspace A is
a family of maps $f_t : X \to X$, with $t \in I$ s.t
$f_0 = id_X$ and $f_1(X) = A$ and $f_t|_A = id_A$ for $\forall t \in I$.
The family f_t must also be continuous
→ an example of a homotop from id_X to a retraction of X onto $A \subset X$.

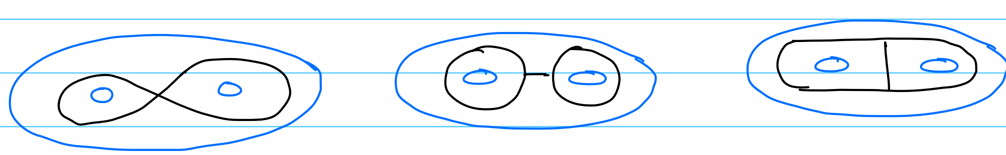
→ in this case, $A \simeq X$ as $f_0 : A \to X$ by id_X
$f_1 : X \longrightarrow A$ as above
then $f_0 \circ f_1 = id_X$
and $f_1 \circ f_0 = id_A$

Examples of deformation retraction:

(1)



(2) Look at deformations of 



(3) $X = \mathbb{R}^2 - \{0\}$, $A = S^1$
Use $F(x,t) = (1-t)x + t\frac{x}{\|x\|}$.

Def: Mapping Cylinder ("the structure through which the def. retraction occurs")

for a map $f: X \to Y$, the mapping cylinder M_f is the quotient space of the disjoint union $(X \times I) \amalg Y$ obtained by the equivalence $(x, 1) \in X \times I \sim f(x) \in Y$

Makes the endpoint of the deformation equivalent to the image of the map.

Mapping cylinders are continuous.

Def: Homotopy relative to A (homotopy rel. A)

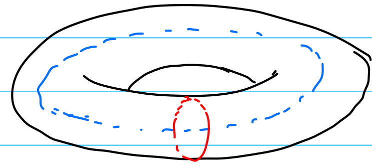
A homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t.

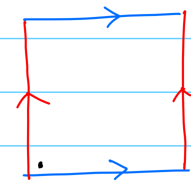
In other words, f_t is a homotopy and $f_t|_A$ is independent of t.

$\to$ def. retraction of X onto A is a homotopy rel. A from id_X to a retraction of X onto $A \subset X$.

Cell Complexes

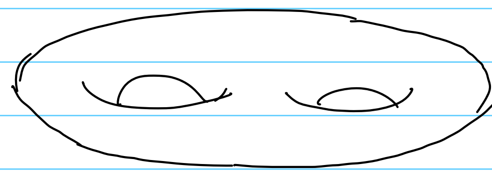
Examples :

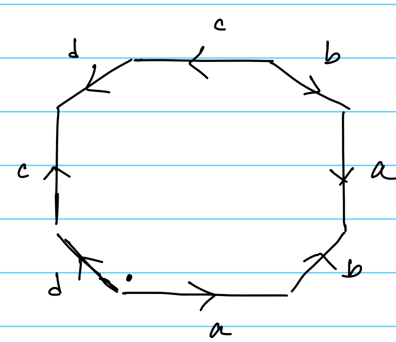




The torus $S^1 \times S^1$ can be constructed from the square

Generally, an orientable surface M_g of genus g can be constructed from a polygon of $4g$ sides, by identifying pairs of edges.





2 cell : interior of a polygon which is an open disk

1 cell : an open interval like $(0,1)$

3 cell : an open ball.

n - cell : open - disk

$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$

Def: Cell Complex (or CW complex)

A space constructed as follows:

(1) Start with discrete set X^0 → the points are 0-cells

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\varphi_\alpha : S^{n-1} \to X^{n-1}.$$

lives in $\mathbb{R}^n$ & is the boundary of D^n

→ So, X^n is the quotient space of $X^{n-1} \sqcup_\alpha D^n_\alpha$ under the equivalence $x \sim \varphi_\alpha(x)$ $\forall x \in \partial D^n_\alpha$

X^{n-1}: (n-1)-skeleton; D^n_α: n-disks

i.e. attach boundaries of the n-disk to the (n-1)-skeleton

$\therefore$ $X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$ where e^n_α is an open n-disk

(3) Either stop this induction at a finite stage and set $X = X^n$ for $n < \infty$

or continue indefinitely, setting

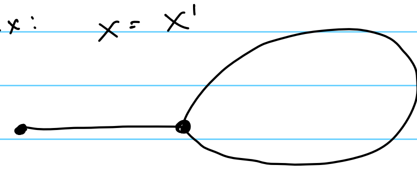
$$X = \bigcup_n X^n$$

↳ in this case, X has the weak topology:

$A \subset X$ is open iff $A \cap X^n$ is open in X^n for each n

Examples of Cell Complexes :

(1) 1-dimensional cell complex : $X = X^1$
graph



(2) The sphere S^n has a cell complex with two cells, e^0 and e^n. where e^n is attached by $\varphi : S^{n-1} \longrightarrow e^0$

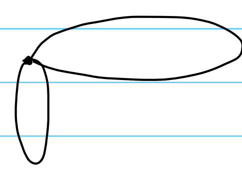
$\therefore$ S^n is being regarded as the quotient space $D^n / \partial D^n$

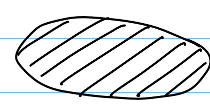
(3) Cell Complex of a torus :

Step 1 : X^0 is just 1 point $\longrightarrow \bullet$

Step 2 : Attach two 1-cells to this point

$X^1 =$



Step 3 : Attach a disk  to X^1 by attaching its boundary to X^1.

Def: Characteristic Map

Each cell e^n_α in a cell complex X has a characteristic map

$$\Phi_\alpha : D^n_\alpha \longmapsto X$$

which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α

$\rightarrow$ Φ_α is the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \overset{q}{\longmapsto} X^n \hookrightarrow X$$

q the quotient map that defines X^n

Example of characteristic map:

(1) Recall: S^n can be constructed by two cells: e^0 and e^n just one point

where e^n is attached to e^0 by

$$\varphi_\alpha : S^{n-1} \longrightarrow e^0$$

Then, the characteristic map of e^n is

$$\Phi_\alpha : D^n_\alpha \longrightarrow S^n \quad \text{which collapses } \partial D^n_\alpha \text{ to } e^0$$

Def: Subcomplex

A subcomplex of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X.

→ As A is closed, for each cell in A,

the image of its characteristic map
the image of its attaching map } contained in A

$\therefore$ A is a cell complex as well

Def: CW pair

A cell complex X and a subcomplex A forms a pair (X, A)

Example of subcomplex

→ Each skeleton, X^n, is a subcomplex.

→ in $\mathbb{R}P^n$ and $\mathbb{C}P^n$, the only subcomplexes are $\mathbb{R}P^k$ and $\mathbb{C}P^k$, $\forall k \leq n$

Properties of subcomplexes

(i) Closure of a collection of cells is a subcomplex.

Operations on Spaces

Products

$X, Y \longrightarrow$ cell complexes

$X \times Y \longrightarrow$ cell complex with the cells $\underbrace{e^m_\alpha}_{\text{cells of } X} \times \underbrace{e^n_\beta}_{\text{cells of } Y}$

Quotients

cell complex

cell subcomplex

Given (X, A) a CW pair,

the quotient space X/A also has a cell complex structure:

$\rightarrow$ the cells of X/A are the cells of $X - A$ and a new 0-cell which is the image of A in X/A.

$\rightarrow$ for a cell e^n_α of $X - A$ attached by $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$, the attaching map for the corresponding cell in X/A is the composition

$$S^{n-1} \longrightarrow X^{n-1} \longrightarrow X^{n-1}/A^{n-1}$$

Wedge Sum

Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, the wedge sum $X \vee Y$ is the quotient of $X \amalg Y$ by identifying x_0 and y_0 to a single point

$\longrightarrow$ Examples: $S^1 \vee S^1 = \infty$

$\longrightarrow$ $\bigvee_\alpha X_\alpha$ for an arbitrary collection of spaces X_α: start with $\amalg_\alpha X_\alpha$ and then identify $x_\alpha \in X_\alpha$ to one point.

$\longrightarrow$ If X_α are cell complexes and the points x_α are 0-cells, then $\bigvee_\alpha X_\alpha$ is a cell complex because we obtain it from the cell complex $\amalg_\alpha X_\alpha$ and attach by collapsing a subcomplex to a point.

→ For a cell complex X, the quotient X^n / X^{n-1} is a wedge sum of n-spheres $\bigvee_\alpha S^n_\alpha$ with one sphere for each n-cell of X

Smash Product

Inside the product space $X \times Y$, there are copies of X and Y : $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $y_0 \in Y$ and $x_0 \in X$.

These copies of X and Y intersect only at (x_0, y_0) so their union can be identifies with the wedge sum $X \vee Y$

ie $\left(X \times \{y_0\}\right) \cup \left(\{x_0\} \times Y\right) = X \vee Y$
$= \left(X \amalg Y\right) / (x_0 \sim y_0)$

The smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$

↳ i.e we are collapsing away the separate factors X and Y.

Suspension

For a space X, the suspension SX is the quotient of $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another.

Example

(i) $X = S^n$

$SX = S^{n+1}$ with the two suspension points at North and South of S^{n+1}