Algebraic Topology
Notes
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These notes are a combination of notes from Allen Hatcher's book "Algebraic Topology" and Prof. Ciprian Manolescu's lectures from MATH 215a at Stanford University. and my own.

Basic Constructions

Def : Homeomorphism

Let X and Y be topological spaces.

$f : X \to Y$ is a homeomorphism if f is bijective and both f and f^{-1} are continuous.

We say $\boxed{X \cong Y}$

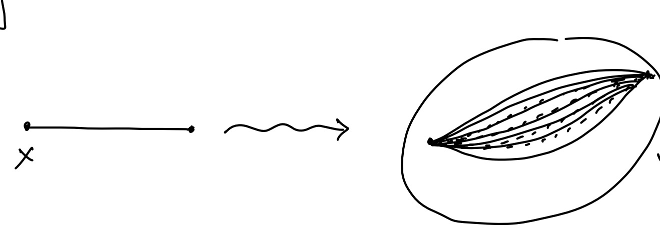
Def : Homotopy

A family of maps, $f_t : X \to Y$ where $t \in I = [0,1]$ s.t the associated map $F : X \times [0,1] \longrightarrow Y$ given by $F(x,t) = f_t(x)$ is continuous.

Two maps $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy $F : X \times [0,1] \to Y$ s.t

$$F(x,0) = f_0(x) \quad \forall x \in X$$
$$F(x,1) = f_1(x)$$

we say $\boxed{f_0 \simeq f_1}$



Def : Homotopy Equivalence

A map $f : X \to Y$ is a homotopy equivalence if $\exists\, g : Y \to X$ s.t $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

We say the spaces X and Y are homotopy equivalent and $\boxed{X \simeq Y}$

$\to$ can prove easily that this is an equivalence relation.

Examples of homotopy equivalence

(1) $\mathbb{R}^n \simeq$ a point (even though $\mathbb{R}^n \not\cong$ a point)

infinite finite

Why?

$f : \mathbb{R}^n \longrightarrow \{0\}$

and take $g : \{0\} \longrightarrow \mathbb{R}^n$ by $g(0) = 0$

Then $f \circ g = id_{\{0\}}$ and $(g \circ f)(x) = 0 \quad \forall x \in \mathbb{R}^n$

Now $g \circ f \sim id_{\mathbb{R}^n}$ by $f_t(x) = tx$ where $f_0 = 0$ and $f_1 = id_{\mathbb{R}^n}$

(2) $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \simeq$ a point

$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \simeq$ a point

Def : Contractible

We say the space X is contractible if $X \simeq$ point.

$\Updownarrow$

Equivalent definition : the identity map of X is null homotopic

homotopic to a constant map.

Def : Retractions

Let X be a space and let $A \subset X$.
Then, a retraction is a map $r : X \to X$ s.t
$r(X) = A$ and $r|_A = id_A$.

Def : Deformation Retraction

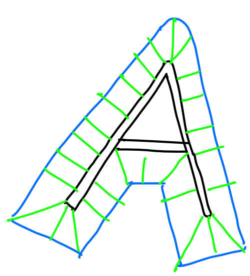
A deformation retraction of X onto a subspace A is
a family of maps $f_t : X \to X$, with $t \in I$ s.t
$f_0 = id_X$ and $f_1(X) = A$ and $f_t|_A = id_A$ for $\forall t \in I$.
The family f_t must also be continuous
→ an example of a homotopy from id_X to a retraction of X onto $A \subset X$.

→ in this case, $A \simeq X$ as $f_0 : A \to X$ by id_X
$f_1 : X \to A$ as above

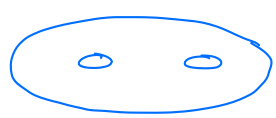
then $f_0 \circ f_1 = id_X$
and $f_1 \circ f_0 = id_A$

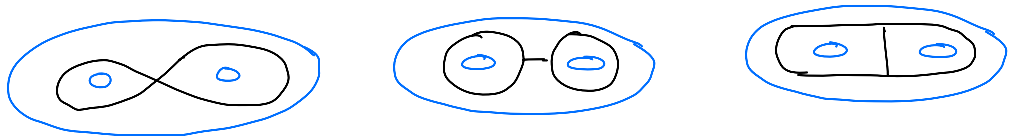
Examples of deformation retraction:

(1)



(2) Look at deformations of





(3) $X = \mathbb{R}^2 - \{0\}$. $A = S^1$
Use $F(x,t) = (1-t)x + t\dfrac{x}{\|x\|}$.

Proposition :

If X def. retracts to a point $x \in X$, then for any $U \subset X$, $x \in U$.
$\exists\, V \subset U$ with $x \in V$ s.t the inclusion map $V \hookrightarrow U$ is nullhomotopic.

$\longrightarrow$ homotopic to constant map

Def : Deformation Retraction in the weak sense :

Let $A \subset X$.
Then, this is the homotopy $f_t : X \mapsto X$ s.t $f_0 = id_X$ and $f_1(X) \subset A$ with $f_t(A) \subset A$, $\forall t \in I$.

Lemma :

If X deformation retracts to A in the weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Proof :

Let the weak def. ret be f_t.

Let $\iota : A \mapsto X$ by inclusion i.e $\iota(a) = a$, $\forall a \in A$.

Then, $(\iota \circ f_1)(x) = \iota(f_1(x)) = f_1(x)$. $\forall x \in X$

But $f_1 \simeq f_0 = id_X$

So, $i \circ f \simeq id_X$

Also, $(f_1 \circ \iota)(a) = f_1(a)$ ~~and~~ $\forall a \in A$

But $f_1|_A \simeq f_0|_A = id_X|_A = id_A$

$\Rightarrow f_1 \circ \iota \simeq id_A$

Def: Mapping Cylinder ("the structure through which the def. retraction occurs")

for a map $f : X \to Y$, the mapping cylinder M_f is the quotient space of the disjoint union $(X \times I) \amalg Y$ obtained by the equivalence $(x, 1) \in X \times I \sim f(x) \in Y$

Makes the endpoint of the deformation equivalent to the image of the map.

Mapping cylinders are continuous.

Def: Homotopy relative to A (homotopy rel. A)

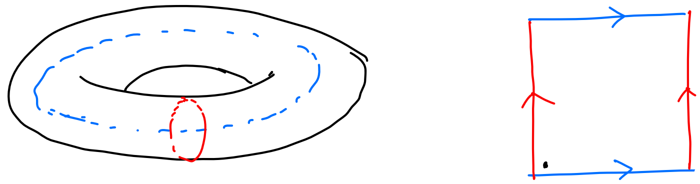
A homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t.

In other words, f_t is a homotopy and $f_t|_A$ is independent of t.

$\rightarrow$ def. retraction of X onto A is a homotopy rel. A from id_X to a retraction of X onto $A \subset X$.

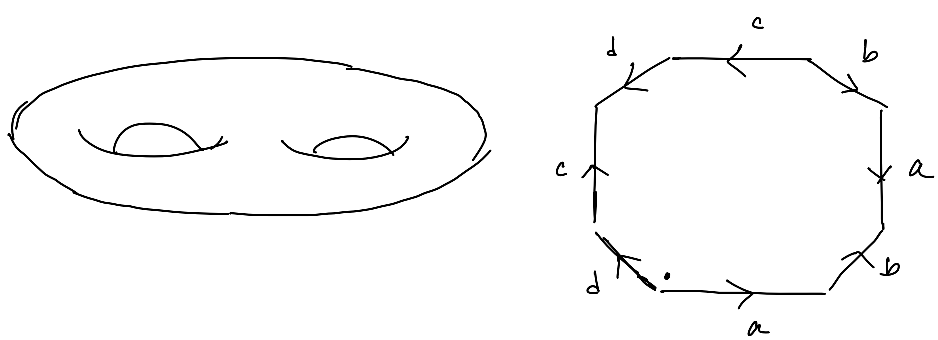
Cell Complexes

Examples :



The torus $S^1 \times S^1$ can be constructed from the square

Generally, an orientable surface M_g of genus g can be constructed from a polygon of $4g$ sides, by identifying pairs of edges.



2 cell : interior of a polygon which is an open disk
1 cell : an open interval like $(0,1)$
3 cell : an open ball.
n-cell : open-disk

$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$

Def : Cell Complex (or CW complex)

A space constructed as follows:

(1) Start with discrete set X^0 → the points are 0-cells

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\varphi_\alpha : S^{n-1} \to X^{n-1}.$$

lives in $\mathbb{R}^n$ & is the boundary of D^n!

→ So, X^n is the quotient space of $X^{n-1} \sqcup_\alpha D^n_\alpha$ under the equivalence $x \sim \varphi_\alpha(x)$ $\forall x \in \partial D^n_\alpha$

(n-1)-skeleton n-disks

i.e. attach boundaries of the n-disk to the (n-1)-skeleton

$\therefore X^n = X^{n-1} \bigsqcup_\alpha e^n_\alpha$ where e^n_α is an open n-disk

(3) Either stop this induction at a finite stage and set $X = X^n$ for $n < \infty$

or continue indefinitely, setting

$$X = \bigcup_n X^n$$

↳ in this case, X has the weak topology :

$A \subset X$ is open iff $A \cap X^n$ is open in X^n for each n

Vocabulary :

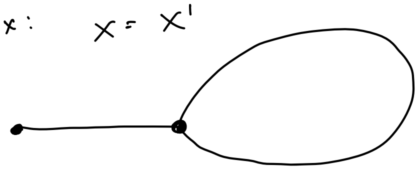
① X^n → n-skeleton

② Dimension of X → largest n s.t an n-cell exists

Examples of Cell Complexes :

(1) 1-dimensional cell complex : $X = X^1$

multigraphs

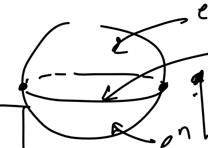


(2) The sphere S^n has a cell complex with two cells, e^0 and e^n. where e^n is attached by $\varphi : S^{n-1} \to e^0$

$\therefore$ S^n is being regarded as the quotient space $D^n / \partial D^n$

$S^n = e^0 \cup e^n$.

Alternatively,



$S^n = S^{n-1} \cup e^n_+ \cup e^n_-$
$= e^0_+ \cup e^0_- \cup$
$\vdots$
$\cup e^n_+ \cup e^n_-$

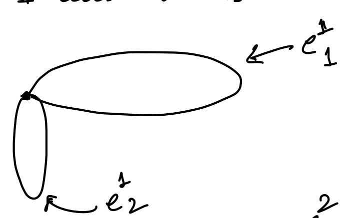
$\therefore S^\infty = \bigcup_n S^n$

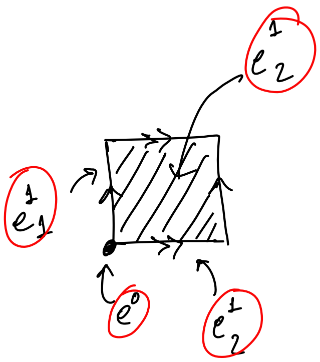
(3) Cell Complex of a torus :

Step 1 : X^0 is just 1 point $\longrightarrow \bullet \leftarrow e^0$

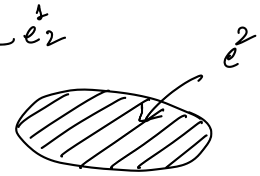
Step 2 : Attach two 1-cells to this point

$X^1 =$





Step 3 : Attach a disk



to X^1 by attaching its boundary to X^1.

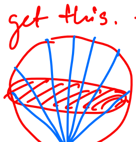
(4) Real Projective Space, $\mathbb{RP}^n$

$(\mathbb{R}^{n+1} - \{0\}) / (v \sim \lambda v,\ \forall v \in \mathbb{R}^{n+1}.\ \lambda \neq 0)$

→ Restricting to vectors of length 1, $S^n / (v \sim -v)$

$\Rightarrow$ D^n with antipodal points of ∂D^n identified

To get this, think of



i.e. for the upper hemisphere's points, find where the line to south pole intersects with D^n

∂D^n with antipodal points equivalent is $\mathbb{RP}^{n-1}$

$\therefore$ $\mathbb{RP}^n$ can be formed from $\mathbb{RP}^{n-1}$ by attaching an n-cell. and the attaching map $\varphi : S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$

$\therefore$ $\mathbb{RP}^n$ has the cell complex structure $e^0 \cup e^1 \cup \dots \cup e^n$ with one cell e^i in each dimension $i \leq n$.

(5) <u>Complex Projective Space. $\mathbb{CP}^n$</u>

Space of all complex lines through the origin in $\mathbb{C}^{n+1}$

ie $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\}) / (v \sim \lambda v, \ \forall v \in \mathbb{C}^{n+1}, \lambda \neq 0)$

Equivalent to $S^{2n+1} / (v \sim \lambda v, \ |\lambda| = 1)$ $\quad (S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1})$

Equivalent to $D^{2n} / (v \sim \lambda v, \ v \in \partial D^{2n})$

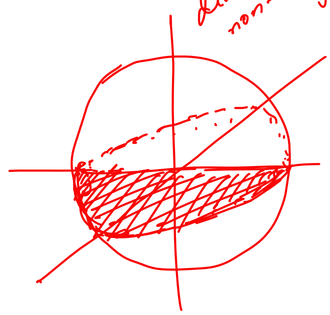
↳ <u>Why?</u>

$S^{2n+1} \subset \mathbb{C}^{n+1}$ → consider vectors in $\mathbb{C}^{n+1}$ whose last coordinate is ~~are~~ real and non-negative

These vectors are of the form $(w, \sqrt{1-w^2}) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$

They form the graph of the function $w \longmapsto \sqrt{1-w^2}$ with $|w| \leq 1$, $w \in \mathbb{C}^n$

in $\mathbb{R}^{2n+2}$, this kills one dimension & makes another dimension non-negative



Note : $w \in \mathbb{C}^n$ and $|w| \leq 1 \Rightarrow w \in D^{2n}$

This is a disk D^{2n}_+ bounded by the spheres S^{2n-1}.

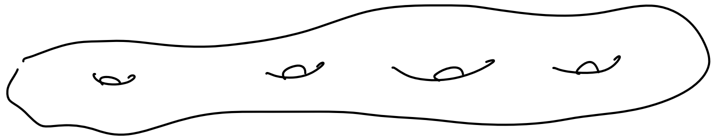
By adding another dimension and viewing them as $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$, we ~~view th~~ view them as vectors in $(D^{2n}_+, 0)$ bounded by $S^{2n-1} \subset S^{2n+1}$

Now each vector in S^{2n+1} is equivalent to a vector in D^{2n}_+ by identifying $v \sim \lambda v$. In particular, if the last coordinate is zero, we have $v \sim \lambda v$, $\forall v \in S^{2n-1}$.

$\therefore$ $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a cell e^{2n} using the attaching map $\varphi : S^{2n-1} \longmapsto \mathbb{CP}^{n-1}$

$\therefore$ $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions

(6)



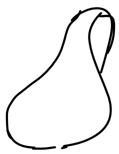
Orientable surface $\longrightarrow$ Σ_g $\longrightarrow$ g holes

g ← genus

Can be constructed from a $4g$ polygon

↳ Start with one e^0

(7)

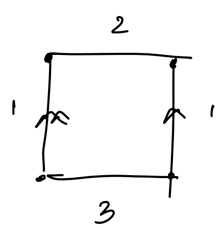


Non-orientable surface $\longrightarrow$ N_g

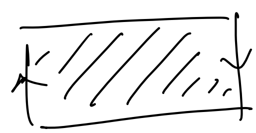
Eg: $N_2 \longrightarrow$ Klein bottle

$N_1 \longrightarrow \mathbb{RP}^2$

(8) Annulus :

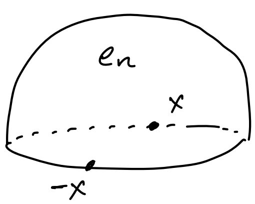


(9) Mobius band



(9) $\mathbb{RP}^n$ revisited

$$\mathbb{RP}^n = S^n / (x \sim -x, \forall x)$$



$$\Rightarrow \mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n$$

$$\therefore \mathbb{RP}^n = e^0 \cup e^1 \cup \cdots\cdots \cup e^n$$

Then, $\mathbb{RP}^\infty = e^0 \cup e^1 \cup e^2 \cup \cdots\cdots = \bigcup_n \mathbb{RP}^n$

(10) <u>$\mathbb{CP}^n$ revisited</u>

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - \{0\}\right) \Big/ \left(x \sim \lambda x,\ \lambda \in \mathbb{C}^*\right)$$

norm 1

$$\therefore\ z \sim \frac{z}{\|z\|} \Rightarrow \mathbb{CP}^n \cong S^{2n+1} \big/ \left(z \sim \lambda z,\ \lambda \in S^1\right)$$

Divide everything by x_1 i.e last coordinate in $\mathbb{R}_{\geq 0}$

$$z = (\underbrace{z_0, \ldots\ldots, z_n}_{w}, \underbrace{z_{n+1}}_{\sqrt{1-\|w\|^2}})$$

with $\|w\| \leq 1$

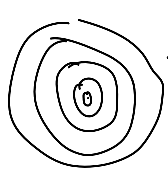
$$D^{2n}_+ = \text{graph}\left(w \longmapsto \sqrt{1-\|w\|^2}\right)$$

$$\therefore\ \mathbb{CP}^n = D^{2n}_+ \big/ \left(w \sim \lambda w \quad \text{if } w \in S^{2n-1}\right)$$

$$= e^{2n} \cup \left(S^{2n-1} \big/ (w \sim \lambda w)\right)$$

$$= \mathbb{CP}^{n-1} \cup e^{2n}$$

$$= e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

(11)  $\leftarrow$ Circles of Radii $\frac{1}{n}$ $\longrightarrow$ <u>NOT a cell complex</u>

Properties of CW Complexes

(1) They are normal ($\therefore$ also Hausdorff)

(2) Any finite cell complex is compact

(3) A compact subspace of a cell cx is contained in a finite subcomplex

(4) Closure finiteness $\longrightarrow$ The closure of each cell E meets only finitely many cells.

(5) Locally contractible:

$$\forall x \in X,\ U \ni x \text{ open},\ \exists\ V \subset U \text{ with } x \in V \text{ s.t. } V \text{ is contractible}$$

(6)

Recall:
Top manifolds $\longrightarrow$ 2^{ND} Countable, Hausdorff, locally Euclidean
Smooth manifolds $\longrightarrow$

Theorem: Every smooth manifold is homeomorphic to a cell complex.

Theorem: Every topological manifold is homotopy equivalent to a cell complex.

Theorem: Every top. manifold of dimension $\neq 4$ is homeomorphic to a cell complex (unknown in dim 4)

Def: Characteristic Map

Each cell e^n_α in a cell complex X has a characteristic map

$$\Phi_\alpha : D^n_\alpha \longmapsto X$$

which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α

→ Φ_α is the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \xmapsto{q} X^n \hookrightarrow X$$

q the quotient map that defines X^n

Example of characteristic map:

(1) Recall: S^n can be constructed by two cells: e^0 and e^n ← just one point

where e^n is attached to e^0 by

$$\varphi_\alpha : S^{n-1} \longrightarrow e^0$$

Then, the characteristic map of e^n is

$$\Phi_\alpha : D^n_\alpha \longrightarrow S^n \quad \text{which collapses } \partial D^n_\alpha \text{ to } e^0$$

Def : Subcomplex

A subcomplex of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X.

→ As A is closed, for each cell in A,

the image of its characteristic map
the image of its attaching map
} contained in A

$\therefore$ A is a cell complex as well

Def : CW pair

A cell complex X and a subcomplex A forms a pair (X, A)

Example of subcomplex

→ Each skeleton, X^n, is a subcomplex.

→ in $\mathbb{RP}^n$ and $\mathbb{CP}^n$, the only subcomplexes are $\mathbb{RP}^k$ and $\mathbb{CP}^k$, $\forall k \leq n$

Properties of subcomplexes

(i) Closure of a collection of cells is a subcomplex.

Operations on Spaces

↳ Most of them preserves cell complexes i.e. if you start with CW complex, you get a CW complex.

Products

$X, Y \longrightarrow$ cell complexes

$X \times Y \longrightarrow$ cell complex with the cells $\underbrace{e^m_\alpha}_{\text{cells of } X} \times \underbrace{e^n_\beta}_{\text{cells of } Y}$

Quotients

$\longrightarrow A \subset X \rightsquigarrow X/A$

Given (X, A) a CW pair, (X: cell complex, A: cell subcomplex)

the quotient space X/A also has a cell complex structure:

→ the cells of X/A are the cells of $X-A$ and a new 0-cell which is the image of A in X/A.

→ for a cell e^n_α of $X-A$ attached by $\varphi_\alpha : S^{n-1} \to X^{n-1}$, the attaching map for the corresponding cell in X/A is the composition

$$S^{n-1} \longrightarrow X^{n-1} \longrightarrow X^{n-1}/A^{n-1}$$

Eg: (1) $D^n/S^{n-1} = S^n$

Wedge Sum (for based spaces)

Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, the wedge sum $X \vee Y$ is the quotient of $X \amalg Y$ by identifying x_0 and y_0 to a single point

$$X \vee Y = X \amalg Y / (x_0 \sim y_0)$$

$\longrightarrow$ Examples: $S^1 \vee S^1 = \infty$

$\longrightarrow$ $\bigvee_\alpha X_\alpha$ for an arbitrary collection of spaces X_α: start with $\amalg_\alpha X_\alpha$ and then identify $x_\alpha \in X_\alpha$ to one point.

$\longrightarrow$ If X_α are cell complexes and the points x_α are 0-cells, then $\bigvee_\alpha X_\alpha$ is a cell complex because we obtain it from the cell complex $\amalg_\alpha X_\alpha$ and attach by collapsing a subcomplex to a point.

→ For a cell complex X, the quotient X^n / X^{n-1} is a wedge sum of n-spheres $\bigvee_\alpha S^n_\alpha$ with one sphere for each n-cell of X

Smash Product

$$X \wedge Y = (X \times Y) / ((x_0 \times Y) \cup (X \times y_0))$$

Inside the product space $X \times Y$, there are copies of X and Y : $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $y_0 \in Y$ and $x_0 \in X$.

These copies of X and Y intersect only at (x_0, y_0) so their union can be identifies with the wedge sum $X \vee Y$

ie $(X \times \{y_0\}) \cup (\{x_0\} \times Y) = X \vee Y$
$= (X \amalg Y) / (x_0 \sim y_0)$

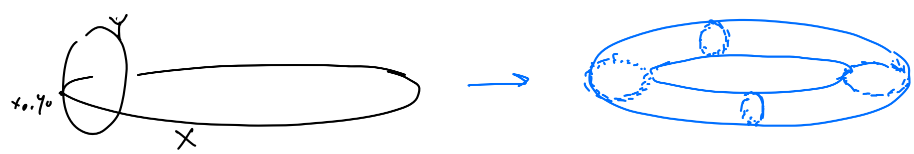
The smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$

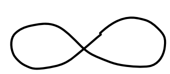
↳ i.e we are collapsing away the separate factors X and Y.

Eg : $S^1 \wedge S^1 = S^2$ ⟶ $S^1 = I / (0 \sim 1)$
$S^m \wedge S^n = S^{m+n}$

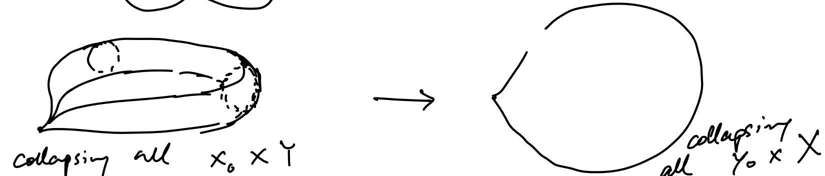
Why ?

Firstly, $S^1 \times S^1$ results in a torus T^2



Secondly, $S^1 \vee S^1 =$ 

Then, quotienting :



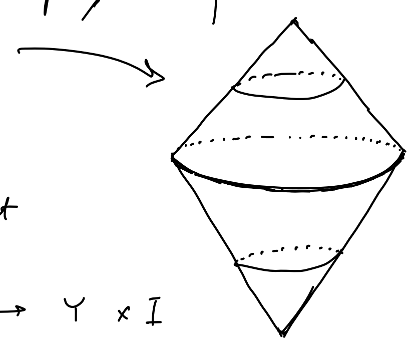
Suspension

For a space X, the suspension SX is the quotient of $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another.

Example

(i) $X = S^n$

$SX = S^{n+1}$ with the two suspension points at North and South of S^{n+1}



→ We can suspend maps too

$f : X \to Y \rightsquigarrow Sf : SX \to SY$

which is the quotient map of

$f \times \mathbb{1} : X \times I \to Y \times I$

Cone :

$$CX = (X \times I) / (X \times \{0\})$$

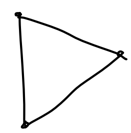
→ If X is a CW complex, then so are SX and CX as quotients of $X \times I$ with its product cell structure with I given the standard cell structure of ~~to~~ two 0-cells joined by one 1-cell.

Join

Given X and Y, we can define the space of all line segments joining points in X to points in Y.

$$X * Y = (X \times Y \times I) \Big/ \left(\begin{array}{ll} (x,y_1,0) \sim (x,y_2,0) & \forall\, x, x_1, x_2 \in X \\ (x_1,y,1) \sim (x_2,y,1) & \forall\, y, y_1, y_2 \in Y \end{array} \right)$$

$\longrightarrow$ $pt * pt \longrightarrow$

$pt * pt * pt \longrightarrow$ 

$$\underbrace{pt * pt * \cdots * pt}_{n+1 \text{ points}} = \Delta^n \longrightarrow \text{n-simplex}$$

Df Reduced Suspension:

$X \to$ CW complex
$\{x_0\} \to$ base point

$$SX = (X \times I) / (X \times \{0\}) \cup (X \times \{1\})$$

$$\Sigma X = SX / (\{x_0\} \times I)$$

Thm

Criteria for Homotopy Equivalence

Recall:

Def: Homotopy Equivalence

A map $f: X \to Y$ is a homotopy equivalence if $\exists\, g: Y \to X$ s.t $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

We say the spaces X and Y are homotopy equivalent and $\boxed{X \simeq Y}$

→ can prove easily that this is an equivalence relation.

Collapsing Subspaces

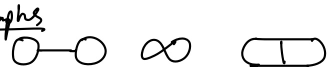
Theorem:

If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \mapsto X/A$ is a homotopy equivalence.

$A \simeq$ point

Example

(i) Graphs

(a) 

A B C

→ they are homotopy equivalent

→ collapsing the middle edge of A and C produces B

(b) Let X be a graph with finitely many vertices and edges.

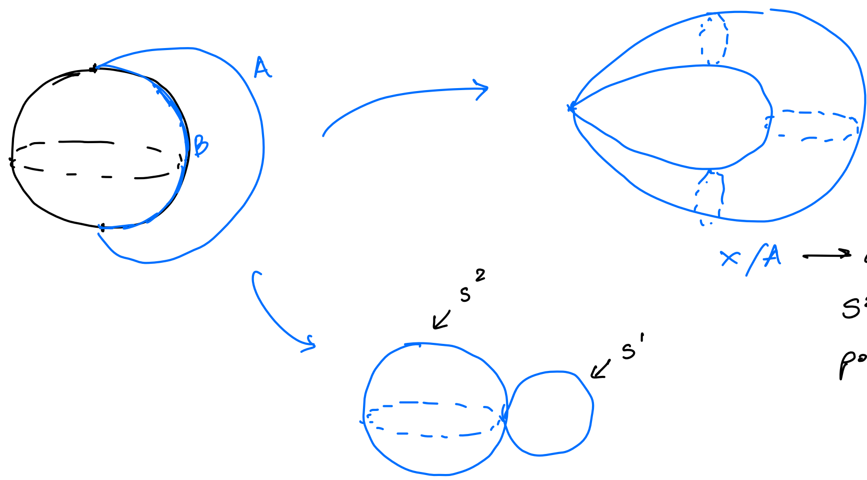
→ if the two endpoints of any edge are distinct, we can collapse it to a pt.

↓

Leads to a homotopy equivalent graph with one less edge.

Can Repeat until all edges are loops.

(2) $X \longrightarrow S^2$ but attach 2 ends of an arc A to N and S pole



$X/A \longrightarrow$ quotient of S^2/S^0 with 2 points identified

$X/B \longrightarrow S^1 \vee S^2$

Reduced Suspension

$$\Sigma X \simeq SX$$

Attaching Spaces

Start with space X_0 and another space X_1 which we will attach to X_0 by identifying points in a subspace $A \subset X_1$ with points of X_0.

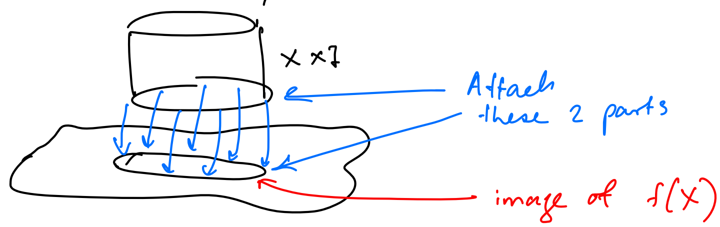
We do this using a map $f : A \longmapsto X_0$ and then forming a quotient space of $X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A)$

We denote

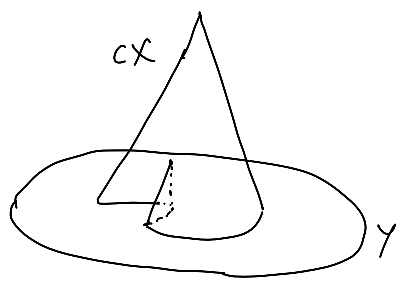
$$X_0 \sqcup_f X_1 = X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A) \quad \text{where } f : A \longmapsto X_0, \; A \subset X_1$$

Example :

(1) Mapping cylinder of a map $f : X \longmapsto Y$ is $M_f \longrightarrow$ the space obtained from Y by attaching $X \times I$ along $X \times \{1\}$ via f.



(2) $\boxed{\text{Mapping Cone}} \longrightarrow$ $C_f = Y \sqcup_f CX$ where ~~C_f~~
CX is the cone $(X \times I)/(X \times \{0\})$
and we attach this to Y along $X \times \{1\}$
via $(x, 1) \sim f(x)$



Example : $X = S^{n-1}$

$C_f \longrightarrow$ attaching to Y the n-cell via $f : S^{n-1} \longmapsto Y$

Proposition

If (X_1, A) is a CW pair and the two attaching maps $f, g : A \longmapsto X_0$ are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$$

Homotopy Extension Property

Intuition:

Consider the map $f_0 : X \mapsto Y$. Let $A \subset X$ and consider the homotopy on A $f_t : A \mapsto Y$ with $f_0 = f|_A$. We would like to extend this to a homotopy on X as a whole with f_0.

Def: Homotopy Extension

$A \subset X$

(X, A) has the homotopy extension property (h.e.p)

if $\forall Y$, $\forall f_0 : X \longmapsto Y$, $\forall$ homotopy $G : A \times I \longmapsto Y$,

$$G(a,0) = f_0(a)$$

we can extend G to a homotopy $F : X \times I \longmapsto Y$

s.t $f(x,0) = f_0(x)$

$$\Updownarrow$$

(X, A) has the h.e.p if every pair of maps $\overbrace{X \times \{0\} \longmapsto Y}^{f_0(x) = y}$ and $\underbrace{A \times I \longmapsto Y}_{f_t(a)}$ that agree on $\underbrace{A \times \{0\}}_{f_0 = f_0|_A}$ can be extended to a map $\underbrace{X \times I \longmapsto Y}_{f_t : X \longmapsto Y}$.

Lemma:

$$\begin{tikzcd} & X \arrow[dr, dotted] & \\ A \arrow[ur] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

$A \subset X$ top space.

$\forall Y$, any map $f : A \longmapsto Y$ extends to $X \longmapsto Y$ if and only if A is a retract of X

Proof:

$\Leftarrow$ Suppose A is a retract of X via $r : X \longmapsto A$ s.t $r|_A = id_A$

Then $(f \circ r) : X \longmapsto Y$ is our extension

$\Rightarrow$ Suppose, $\forall Y$ and any map $f : A \longmapsto Y$ extends to $X \longmapsto Y$.

i.e $f_t : X \longmapsto Y$ s.t $f_0|_A = f$

Then. let $Y = A$ and $f = id_A$ i.e $id_A : A \longmapsto A$ extends to $f_t : X \longmapsto A$ s.t $f_0|_A = id_A \Rightarrow A$ is a retract of X

Lemma :

A pair (X, A) has the h.e.p if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof : By hep, the identity map

$\Rightarrow$: $X \times \{0\} \cup A \times I \longmapsto X \times \{0\} \cup A \cup I$ extends to a map

$X \times I \longmapsto X \times \{0\} \cup A \times I$

$\therefore$ $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

$\Leftarrow$ if A is closed : consider any two maps $X \times \{0\} \longmapsto Y$ and $A \times I \longmapsto Y$ that agree on $A \times \{0\}$. They combine to give a map $X \times \{0\} \cup A \times I \longmapsto Y$ which is continuous by continuity on the closed sets $X \times \{0\}$ and $A \times I$.

Compose this map $X \times \{0\} \cup A \times I \longmapsto Y$ with a retraction $X \times I \longmapsto X \times \{0\} \cup A \times I$ (we have this via hypothesis)

We get an extension $X \times I \longmapsto Y$

$\therefore$ (X, A) has the h.e.p.

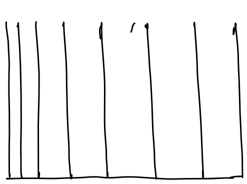
Properties

(i) $\left.\begin{array}{l} \text{H.e.p} \\ X \text{ - hausdorff} \end{array}\right\} \Rightarrow A$ is closed in X

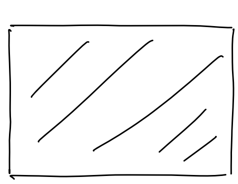
Non-example : (X, A) does not have h.e.p

(1) (I, A) where $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

There is no continuous retraction $I \times I \longmapsto I \times \{0\} \cup A \times I$ because of the structure of (I, A) near 0.



$I \times \{0\} \cup A \times I$



$I \times I$

Consider the ball ~~x_0~~ $B = B(x_0, \frac{1}{2})$

Then $\exists \delta > 0$ s.t $r(B(x_0, \delta)) \subset B$

$\gamma \longrightarrow$ path in B from x_0 to x_1

but x_0 and x_1 are in diff components of $C \cap B$

path at $t=1$

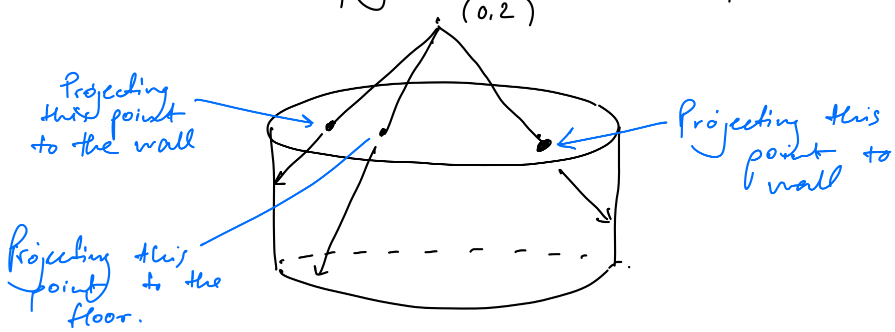
$B(x_0, \delta)$

Proposition

If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the h.e.p.

Proof:

First, note that $\exists$ a retraction $r: D^n \times I \longrightarrow D^n \times \{0\} \cup \partial D^n \times I$ for ex- radial projection from the point $(0,2) \in D^n \times \mathbb{R}$



Now, set $r_t = tr + (1-t)\mathbb{1}$ is a deformation retraction of $D^n \times I$ onto $D^n \times \{0\} \cup \partial D^n \times I$.

Now, with this, we have a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ since $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$.

If we perform the def. ret. of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \times A^n) \times I$ during the t-interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, this infinite concatenation of homotopies is a def. ret. of $X \times I$ onto $X \times \{0\} \cup A \times I$.

Proposition

If the pair (X, A) satisfies h.e.p and A is contractible, then the quotient map $q : X \longrightarrow X/A$ is a homotopy equivalence.

Proof :

Let $f_t : X \to X$ be the homotopy extending a contraction of A with $f_0 = \mathbb{1}$.

Now, $f_t(A) \subset A \quad \forall t$, so the composition

$$q \circ f_t : X \longrightarrow X/A$$

sends A to a point and so factors as a composition

$$X \xrightarrow{q} \underbrace{X/A \longrightarrow X/A}$$

(Denote this by $\overline{f_t} : X/A \longrightarrow X/A$)

So, $q f_t = \overline{f_t}\, q$

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_t}"'] & X/A \end{tikzcd}$$

When $t=1$, $f_1(A)$ equals to a point (since f_t is homotopy extension of the contraction of A), so f_1 induces a map

$$g : X/A) \longrightarrow X \quad \text{with} \quad gq = f_1$$

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\overline{f_1}"'] \arrow[ur, "g"] & X/A \end{tikzcd}$$

So, $qg = \overline{f_1}$ since

$$\begin{aligned} qg(\bar{x}) &= qgq(x) \\ &= qf_1(x) \\ &= \overline{f_1}\, q(x) \\ &= \overline{f_1}(\bar{x}) \end{aligned}$$

The maps g and q are inverse homotopy equivalences as

$gq = f_1 \simeq f_0 = \mathbb{1}$ via f_t and
$qg = \bar{f}_1 \simeq \bar{f}_0 = \mathbb{1}$ via $\bar{f}_t$.

Def: $W \simeq Z$ rel Y

for (W, Y) and (Z, Y), there are maps $\varphi : W \mapsto Z$ and $\psi : Z \mapsto W$ restricting to identity on Y s.t $\psi\varphi \simeq \mathbb{1}_W$ and $\varphi\psi \simeq \mathbb{1}_Z$ via homotopies that restrict to the identity on Y at all times.

Proposition

If (X_1, A) is a CW pair and we have attaching maps $f, g : A \mapsto X_0$ that are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

Proof:

Let $F : A \times I \mapsto X_0$ is a homotopy from f to g, consider the space $X_0 \sqcup_F (X_1 \times I)$. which has both $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ as subspaces

We can deformation retract $X_1 \times I$ onto $X_1 \times \{0\} \cup A \times I$ which induces a def retraction of

$$X_0 \sqcup_F (X_1 \times I) \text{ onto } X_0 \sqcup_f X_1$$

Similarly, $X_0 \sqcup_F (X_1 \times I)$ def retracts onto $X_0 \sqcup_g X_1$

Both of them are identity on X_0 so we get the homotopy equivalence

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$