

Algebraic Topology

Notes

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These notes are a combination of notes from Allen Hatcher's book "Algebraic Topology" and Prof. Ciprian Manolescu's lectures from MATH 215a at Stanford University, and my own.

Introduction

Our goal is to develop algebraic invariants associated with topological spaces.

We will look at

(1) Fundamental Group:

$$\pi_1(X) = \{\text{loops in } X\} / \text{homotopy}$$

(2) Homology Group:

$$H_n(X), n \in \mathbb{N} \text{ and abelian}$$

Intuitively, they count "holes" in X

(3) Cohomology Group:

$$H^n(X) = \text{Dual to } H_n(X)$$

$\oplus H^n(X)$ is a ring!

Basic Constructions

Def : Homeomorphism

Let X and Y be topological spaces.

$f: X \rightarrow Y$ is a homeomorphism if f is bijective and both f and f^{-1} are continuous.

We say $\boxed{X \cong Y}$

Def : Homotopy

A family of maps, $f_t: X \rightarrow Y$ where $t \in I = [0, 1]$ s.t

the associated map $F: X \times [0, 1] \rightarrow Y$ given by $F(x, t) = f_t(x)$ is continuous.

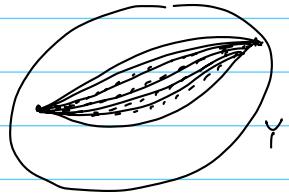
Two maps $f_0, f_1: X \rightarrow Y$ are homotopic if there exists a homotopy $F: X \times [0, 1] \rightarrow Y$ s.t

$$F(x, 0) = f_0(x) \quad \forall x \in X$$

$$F(x, 1) = f_1(x)$$

We say

$$\boxed{f_0 \simeq f_1}$$



Def : Homotopy Equivalence

A map $f: X \rightarrow Y$ is a homotopy equivalence if $\exists g: Y \rightarrow X$ s.t $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$

We say the spaces X and Y are homotopy equivalent and

$$\boxed{X \simeq Y}$$

→ can prove easily that this is an equivalence relation.

Examples of homotopy equivalence

(1) $\mathbb{R}^n \simeq$ a point (even though $\mathbb{R}^n \not\simeq$ a point)
infinite finite

Why?

$$f: \mathbb{R}^n \rightarrow \{0\}$$

and take $g: \{0\} \rightarrow \mathbb{R}^n$ by $g(0) = 0$

Then $f \circ g = \text{id}_{\{0\}}$ and $(g \circ f)(x) = 0 \quad \forall x \in \mathbb{R}^n$

Now $g \circ f \sim \text{id}_{\mathbb{R}^n}$ by $f_t(x) = tx$ where $f_0 = 0$ and $f_1 = \text{id}_{\mathbb{R}^n}$

(2) $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \simeq$ a point

$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \simeq$ a point

Def : Contractible

We say the space X is contractible if $X \simeq$ point.



Equivalent definition : the identity map of X is nullhomotopic

homotopic to a
constant map.

Def : Retractions

Let X be a space and let $A \subset X$.

then, a retraction is a map $r: X \rightarrow X$ s.t
 $r(X) = A$ and $r|_A = id_A$.

Def : Deformation Retraction

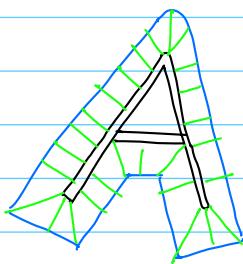
A deformation retraction of X onto a subspace A is
a family of maps $f_t: X \rightarrow X$, with $t \in I$ s.t
 $f_0 = id_X$ and $f_1(X) = A$ and $f_t|_A = id_A$ for $\forall t \in I$.

The family f_t must also be continuous
 \rightarrow an example of a homotopy from id_X to a retraction of X onto $A \subset X$.

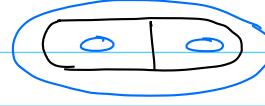
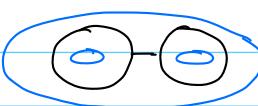
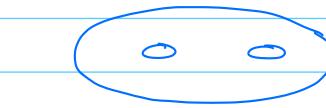
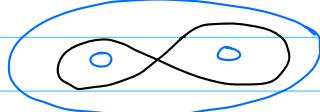
\rightarrow in this case, $A \cong X$ as $f_0: A \rightarrow X$ by id_X
 $f_1: X \rightarrow A$ as above
then $f_0 \circ f_1 = id_X$
and $f_1 \circ f_0 = id_A$

Examples of deformation retraction:

(1)



(2) Look at deformations of



(3) $X = \mathbb{R}^2 - \{0\}$. $A = S^1$

$$(4) f(x, t) = (1-t)x + t \frac{x}{\|x\|}.$$

Proposition:

If X def. retracts to a point $x \in X$, then for any $U \subset X$, $x \in U$.
 $\exists V \subset U$ with $x \in V$ s.t. the inclusion map $V \hookrightarrow U$ is
nullhomotopic.

homotopic to constant map

'Def: Deformation Retraction in the weak sense:

Let $A \subset X$.

Then, this is the homotopy $f_t : X \rightarrow X$ s.t $f_0 = \text{id}_X$
and $f_t(X) \subset A$ with $f_t(A) \subset A$, $\forall t \in I$.

Lemma:

If X deformation retracts to A in the weak sense, then
the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Proof:

Let the weak def. ret be f_t .

Let $i : A \hookrightarrow X$ by inclusion i.e. $i(a) = a$, $\forall a \in A$.

Then, $(i \circ f_t)(x) = i(f_t(x)) = f_t(x)$, $\forall x \in X$

But $f_t \simeq f_0 = \text{id}_X$

so, $i \circ f_t \simeq \text{id}_X$

Also, $(f_t \circ i)(a) = f_t(i(a)) = f_t(a)$ ~~and~~ $\forall a \in A$

But $f_t|_A \simeq f_0|_A = \text{id}_X|_A = \text{id}_A$

$\Rightarrow f_t \circ i \simeq \text{id}_A$

Def: Mapping Cylinder ("the structure through which the def. retraction occurs")

for a map $f: X \rightarrow Y$, the mapping cylinder M_f is the quotient space of the disjoint union $(X \times I) \sqcup Y$ obtained by the equivalence $(x, 1) \in X \times I \sim f(x) \in Y$

$\nwarrow \nearrow$
Make the endpoint of the deformation equivalent to the image of the map.

Mapping cylinders are continuous.

Def: Homotopy relative to A (homotopy rel. A)

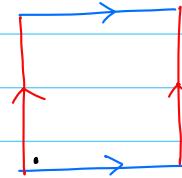
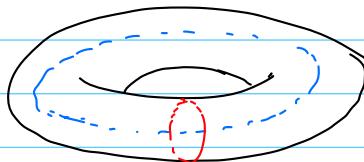
A homotopy $f_t: X \rightarrow Y$ whose restriction to a subspace $A \subset X$ is independent of t .

In other words, f_t is a homotopy and $f_t|_A$ is independent of t .

\rightarrow def. retraction of X onto A is a homotopy rel. A from id_X to a retraction of X onto $A \subset X$.

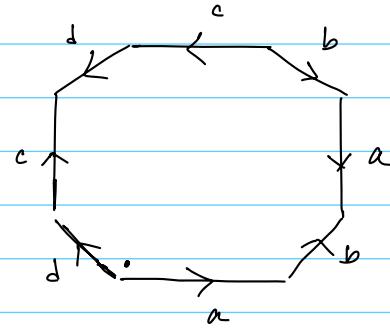
Cell Complexes

Examples :



The torus $S^1 \times S^1$ can be constructed from the square

Generally, an orientable surface M_g of genus g can be constructed from a polygon of $4g$ sides by identifying pairs of edges.



2 cell: interior of a polygon which is an open disk

1 cell: an open interval like $(0, 1)$

3 cell: an open ball.

n -cell: open-disk

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

Def : Cell Complex (or CW complex)

A space constructed as follows:

- (1) Start with discrete set $X^0 \rightarrow$ the points are 0-cells
- (2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e_α^n via maps

$\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$.

So, X^n is the quotient space of $X^{n-1} \coprod_\alpha D_\alpha^n$ under the equivalence $x \sim \varphi_\alpha(x) \forall x \in \partial D_\alpha^n$

\nwarrow (n-1)-skeleton \nearrow n-disks

: i.e attach boundaries of the n-disk to the (n-1)-skeleton

$$\therefore X^n = X^{n-1} \coprod_\alpha e_\alpha^n \text{ where } e_\alpha^n \text{ is an open n-disk}$$

(3) Either stop this induction at a finite stage and set $X = X^n$ for $n < \infty$

or continue indefinitely, setting

$$X = \bigcup_n X^n$$

↳ in this case, X has the weak topology:

$A \subset X$ is open iff $A \cap X^n$ is open in X^n for each n

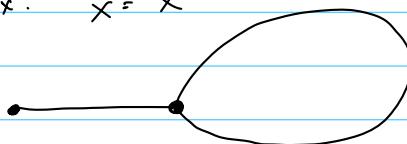
Vocabulary :

① $X^n \rightarrow$ n-skeleton

② Dimension of $X \rightarrow$ largest n s.t. an n-cell exists

Examples of Cell Complexes:

(1) 1-dimensional cell complex: $X = X^1$
 (multigraphs)



(2) The sphere S^n has a cell complex with two cells, e^0 and e^n , where e^n is attached by $\varphi: S^{n-1} \rightarrow e^0$.

$\therefore S^n$ is being regarded as the quotient space

$$D^n / \partial D^n$$

$$S^n = e^0 \cup e^n.$$

Alternatively,

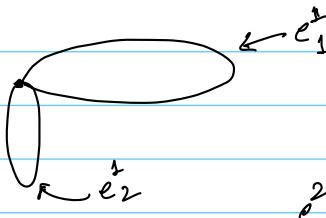
$$\begin{aligned} S^n &= S^{n-1} \cup e_+^n \cup e_-^n \\ &= e_+^0 \cup e_-^0 \cup \dots \cup e_+^n \cup e_-^n \end{aligned}$$

(3) Cell Complex of a torus:

Step 1: X^0 is just a point $\rightarrow \bullet \leftarrow e^0$

Step 2: Attach two 1-cells to this point

$$X^1 =$$



$$\therefore S^\infty = \bigcup_n S^n$$

Step 3: Attach a disk to X^1 by attaching its boundary to X^1 .

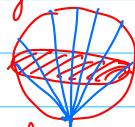
(4) Real Projective Space, $\mathbb{R}P^n$

$$(\mathbb{R}^{n+1} - \{0\}) / (\nu \sim \lambda \nu, \forall \nu \in \mathbb{R}^{n+1}, \lambda \neq 0)$$

\rightarrow Restricting to vectors of length 1, $S^n / (\nu \sim -\nu)$

$\Rightarrow D^n$ with antipodal points of ∂D^n identified

To get this, think of



∂D^n with antipodal points equivalent is $\mathbb{R}P^{n-1}$

$\therefore \mathbb{R}P^n$ can be formed from $\mathbb{R}P^{n-1}$ by attaching an n -cell. and the attaching map $\varphi: S^{n-1} \rightarrow \mathbb{R}P^{n-1}$

$\therefore \mathbb{R}P^n$ has the cell complex structure $e^0 \cup e^1 \cup \dots \cup e^n$ with one cell e^i in each dimension $i \leq n$.

i.e. for the upper hemisphere's points, find where the line to south pole intersects with D^n

(5) Complex Projective Space. $\mathbb{C}P^n$

Space of all complex lines through the origin in \mathbb{C}^{n+1}

$$\text{i.e. } \mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\}) / (v \sim \lambda v, \forall v \in \mathbb{C}^{n+1}, \lambda \neq 0)$$

Equivalent to $S^{2n+1} / (v \sim \lambda v, |\lambda|=1)$ ($S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$)

Equivalent to $D^{2n} / (v \sim \lambda v, v \in \partial D^{2n})$

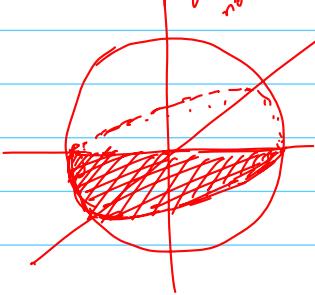
↳ Why?

$S^{2n+1} \subset \mathbb{C}^{n+1}$ → consider vectors in \mathbb{C}^{n+1} whose last coordinate is ~~one~~ real.

and non-negative

These vectors are of the form $(w, \sqrt{1-w^2}) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$

They form the graph of the function $w \mapsto \sqrt{1-w^2}$
with $|w| \leq 1, w \in \mathbb{C}^n$



Note: $w \in \mathbb{C}^n$ and $|w| \leq 1 \Rightarrow w \in D^{2n}$

This is a disk D^{2n}_+ bounded by the spheres S^{2n-1} .

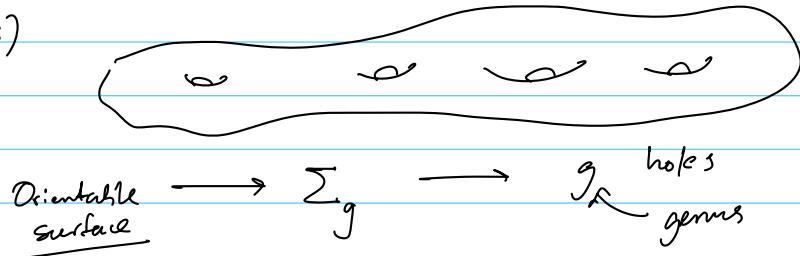
By adding another dimension and viewing them as $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$, we ~~view~~ them as vectors in $(D^{2n}_+, 0)$ bounded by $S^{2n-1} \subset S^{2n+1}$

Now, each vector in S^{2n+1} is equivalent to a vector in D^{2n}_+ by identifying $v \sim \lambda v$. In particular, if the last coordinate is zero, we have $v \sim \lambda v, \forall v \in S^{2n-1}$.

$\therefore \mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell e^{2n} using the attaching map $\varphi : S^{2n-1} \rightarrow \mathbb{C}P^{n-1}$

$\therefore \mathbb{C}P^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$ with cells only in even dimensions

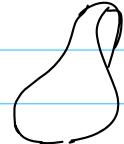
(6)



Can be constructed from a $4g$ polygon

↳ Start with one e^0

(7)

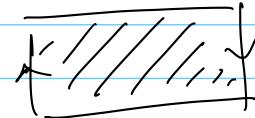
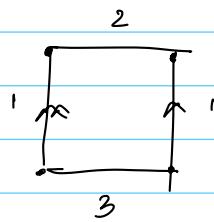


Non-orientable
surface

$$\begin{aligned} \text{e.g.: } N_2 &\longrightarrow \text{Klein bottle} \\ N_1 &\longrightarrow \mathbb{RP}^2 \end{aligned}$$

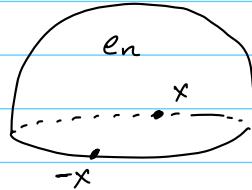
(8) Annulus :

(9) Möbius band



(a) \mathbb{RP}^n revisited

$$\mathbb{RP}^n = S^n / (x \sim -x, \forall x)$$



$$\Rightarrow \mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n$$

$$\therefore \mathbb{RP}^n = e^0 \cup e^1 \cup \dots \cup e^n$$

$$\text{Then, } \mathbb{RP}^\infty = e^0 \cup e^1 \cup e^2 \cup \dots = \bigcup_n \mathbb{RP}^n$$

(i) $\mathbb{C}\mathbb{P}^n$ revisited

$$\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / (x \sim \lambda x, \lambda \in \mathbb{C}^*)$$

$$\therefore z \sim \frac{z}{\|z\|} \Rightarrow \mathbb{C}\mathbb{P}^n \cong S^{2n+1} / (z \sim \lambda z, \lambda \in S^1)$$

Divide everything by x_1 , i.e. last coordinate in $\mathbb{R}_{\geq 0}$

$$z = \underbrace{(z_0, \dots, z_n)}_{w} \underbrace{z_{n+1}}_{\sqrt{1-\|w\|^2}}$$

with $\|w\| \leq 1$

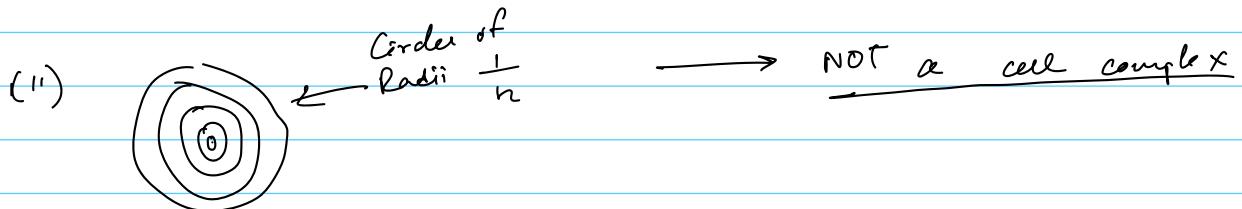
$$D_+^{2n} = \text{graph } (w \mapsto \sqrt{1-\|w\|^2})$$

$$\therefore \mathbb{C}\mathbb{P}^n = D_+^{2n} / (w \sim \lambda w \text{ if } w \in S^{2n-1})$$

$$= \mathbb{C}^{2n} \cup \left(S^{2n-1} / (w \sim \lambda w) \right)$$

$$= \mathbb{C}\mathbb{P}^{n-1} \cup \mathbb{C}^{2n}$$

$$= e^0 \cup e^2 \cup \dots \cup e^{2n}$$



Properties of CW Complexes

- (1) They are normal (\therefore also Hausdorff)
- (2) Any finite cell complex is compact
- (3) A compact subspace of a cell cx is contained in a finite subcomplex
- (4) Closure finiteness \rightarrow The closure of each cell ℓ meets only finitely many cells.
- (5) Locally contractible:
 $\forall x \in X, \exists x \text{ open}, \exists V \subset U \text{ with } x \in V$
s.t. V is contractible

(6)

Recall:

Top manifolds \rightarrow 2nd Countable, Hausdorff, locally Euclidean
Smooth manifolds \rightarrow

Theorem: Every smooth manifold is homeomorphic to a cell complex.

Theorem: Every topological manifold is homotopy equivalent to a cell complex.

Theorem: Every top manifold of dimension $\neq 4$ is homeomorphic to a cell complex
(unknown in dim 4)

Def: Characteristic Map

Each cell e_α^n in a cell complex X has a characteristic map

$$\varphi_\alpha : D_\alpha^n \xrightarrow{\sim} X$$

which extends the attaching map φ_α and is a homeomorphism from the interior of D_α^n onto e_α^n

→ φ_α is the composition

$$D_\alpha^n \hookrightarrow X^{n-1} \coprod_\alpha D_\alpha^n \xrightarrow{\qquad\qquad\qquad} X^n \hookrightarrow X$$

↑
the quotient
map that
defines X^n

Example of characteristic map:

(i) Recall: S^n can be constructed by two cells: e^0 and e^n ← just one point

where e^n is attached to e^0 by

$$\varphi_\alpha : S^{n-1} \rightarrow e^0$$

Then, the characteristic map of e^n is

$$\varphi_\alpha : D_\alpha^n \rightarrow S^n \text{ which collapses } \partial D_\alpha^n \text{ to } e^0$$

Def: Subcomplex

A subcomplex of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X .

\rightarrow As A is closed, for each cell in A ,

the image of its characteristic map } contained in A
the image of its attaching map }

$\therefore A$ is a cell complex as well

Def : CW pair

A cell complex X and a subcomplex A forms a pair (X, A)

Example of subcomplex

\rightarrow Each skeleton, X^n , is a subcomplex.

\rightarrow in $\mathbb{R}\mathbb{P}^n$ and $\mathbb{C}\mathbb{P}^n$, the only subcomplexes are $\mathbb{R}\mathbb{P}^k$ and $\mathbb{C}\mathbb{P}^k$, $\forall k \leq n$

Properties of subcomplexes

(1) Closure of a collection of cells is a subcomplex.

(2) Any union and intersection of subcomplexes is a subcomplex.

Operations on Spaces

III Products

$X, Y \rightarrow \text{cell complexes}$

$X \times Y \rightarrow \text{cell complex with the cells } e_\alpha^m \times e_\beta^n$

IV Quotients

$\longrightarrow A \subset X \rightsquigarrow X/A$

Given (X, A) a CW pair,
the quotient space X/A also has a cell complex structure:

→ the cells of X/A are the cells of $X-A$ and
a new 0-cell which is the image of
 A in X/A .

→ for a cell e_α^n of $X-A$ attached by
 $\varphi_\alpha: S^{n-1} \rightarrow X^{n-1}$, the attaching map for the
corresponding cell in X/A is the composition

$$S^{n-1} \rightarrow X^{n-1} \rightarrow X^{n-1}/A^{n-1}$$

Eg: ① $D^n/S^{n-1} = S^n$

V Wedge Sum (for based spaces)

Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$,
the wedge sum $X \vee Y$ is the quotient of $X \coprod Y$ by
identifying x_0 and y_0 to a single point

→ Example: $S^1 \vee S^1 = \infty$

$$X \vee Y = X \coprod Y / (x_0 \sim y_0)$$

→ $\bigvee_\alpha X_\alpha$ for an arbitrary collection of spaces X_α :
start with $\coprod_\alpha X_\alpha$ and then identify $x_\alpha \in X_\alpha$
to one point.

→ If X_α are cell complexes and the points x_α
are 0-cells, then $\bigvee_\alpha X_\alpha$ is a cell complex
because we obtain it from the cell complex $\coprod_\alpha X_\alpha$ and attach by

collapsing a subcomplex to a point.

→ For a cell complex X , the quotient X^n/X^{n-1} is a wedge sum of n -spheres $\bigvee_{\alpha} S_{\alpha}^n$ with one sphere for each n -cell of X

7) Smash Product $X \wedge Y = (X \times Y) / ((x_0 \times Y) \cup (X \times y_0))$

Inside the product space $X \times Y$, there are copies of X and Y : $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $y_0 \in Y$ and $x_0 \in X$.

These copies of X and Y intersect only at (x_0, y_0) so their union can be identified with the wedge sum $X \vee Y$

$$\begin{aligned} \text{i.e. } (X \times \{y_0\}) \vee (\{x_0\} \times Y) &= X \vee Y \\ &= (X \amalg Y) / (x_0 \sim y_0) \end{aligned}$$

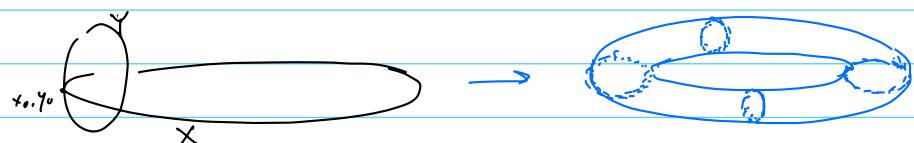
The smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$

↳ i.e. we are collapsing away the separate factors X and Y .

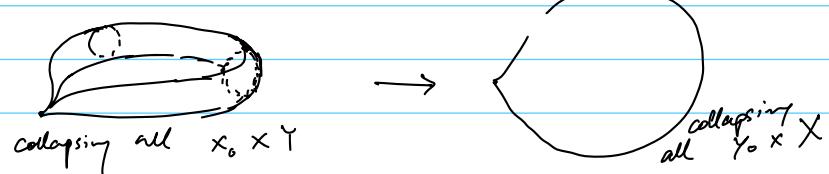
Eg: $S^1 \wedge S^1 = S^2$ \longrightarrow $S^1 = I / (0 \sim 1)$
 $S^m \wedge S^n = S^{m+n}$

Why?

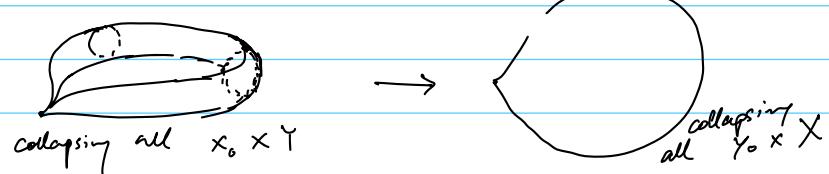
Firstly, $S^1 \times S^1$ results in a torus T^2



Secondly, $S^1 \vee S^1 = \infty$



Then, quotienting:



II

Suspension

for a space X , the suspension SX is the quotient of $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another.

Example

(i) $X = S^n$

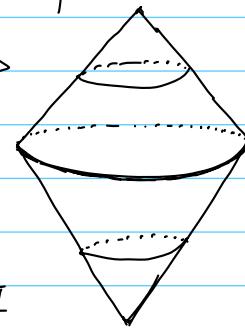
$SX = S^{n+1}$ with the two suspension points at North and South of S^{n+1}

→ We can suspend maps too

$$f: X \rightarrow Y \rightsquigarrow Sf: SX \rightarrow SY$$

which is the quotient map of

$$f \times 1 : X \times I \rightarrow Y \times I$$



III

Cone

$$CX = (X \times I) / (X \times \{0\})$$



→ If X is a CW complex, then so are SX and CX as quotients of $X \times I$ with its product cell structure with I given the standard cell structure of ~~two~~ two 0-cells joined by one 1-cell.

7

Join

Given X and Y , we can define the space of all line segments joining points in X to points in Y .

$$X * Y = (X \times Y \times I) / \left(\begin{array}{l} (x, y_1, 0) \sim (x, y_2, 0) \quad \forall x, x_1, x_2 \in X \\ (x_1, y_1, 1) \sim (x_2, y_1, 1) \quad \forall y, y_1, y_2 \in Y \end{array} \right)$$

$$\rightarrow pt * pt \longrightarrow \bullet \longrightarrow$$

$$pt * pt * pt \longrightarrow \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$pt * pt * \dots * pt = \Delta^n \rightarrow n\text{-simplex}$$

$\underbrace{\hspace{10em}}$
 $n+1$ points

④

Reduced Suspension:

$X \rightarrow \text{CW complex}$
 $\{x_0\} \rightarrow \text{base point}$

$$SX = (X \times I) / (X \times \{0\}) \cup (X \times \{1\})$$

$$\Sigma X = SX / (\{x_0\} \times I)$$

⑤

Criterion for Homotopy Equivalence

Recall:

Def: Homotopy Equivalence

A map $f: X \rightarrow Y$ is a homotopy equivalence if $\exists g: Y \rightarrow X$
s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

We say the spaces X and Y are homotopy equivalent
and

$$X \simeq Y$$

→ can prove easily that this is an equivalence relation.



Collapsing Subspaces

Theorem:

If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A , then the quotient map $X \rightarrow X/A$ is a homotopy equivalence.

Example

(1) Graphs



→ they are homotopy equivalent

→ collapsing the middle edge of A and C produces B

(b) Let X be a graph with finitely many vertices and edges.

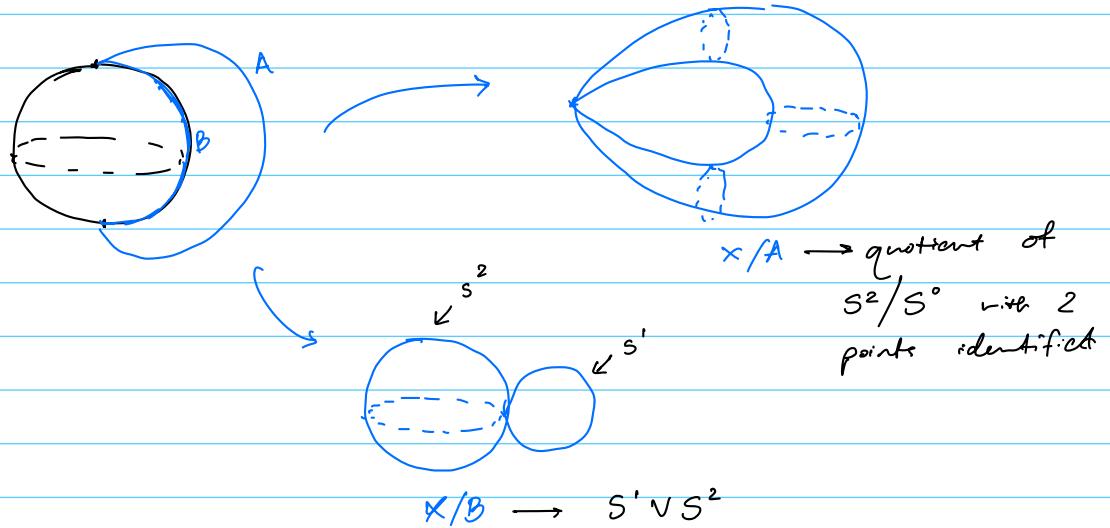
→ if the two endpoints of any edge are distinct, we can collapse it to a pt.



Leads to a homotopy equivalent graph with one less edge.

Can repeat until all edges are loops.

(2) $X \rightarrow S^2$ but attach 2 ends of an arc A to N and S pole



7.1 Reduced Suspension

$$\Sigma X \cong SX$$



Attaching spaces

Start with space X_0 and another space X_1 , which we will attach to X_0 by identifying points in a subspace $A \subset X_1$, with points of X_0 .

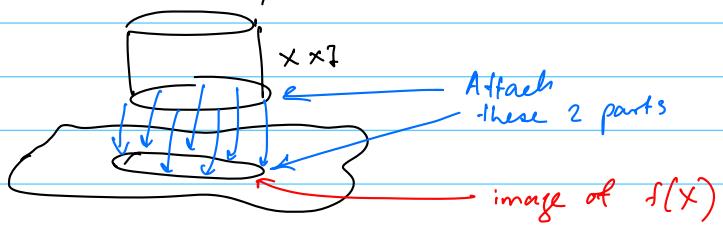
We do this using a map $f: A \hookrightarrow X_0$ and then forming a quotient space of $X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A)$

We denote

$$X_0 \sqcup_f X_1 = X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A) \text{ where } f: A \hookrightarrow X_0, A \subset X_1$$

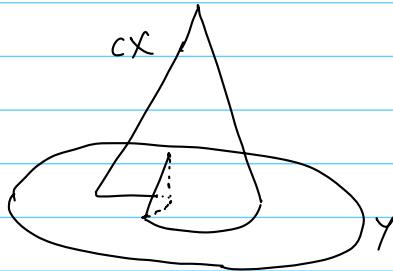
Example :

- (1) Mapping cylinder of a map $f: X \rightarrow Y$ is $M_f \rightarrow$ the space obtained from Y by attaching $X \times I$ along $X \times \{1\}$ via f .



- (2) Mapping Cone $\rightarrow C_f = Y \sqcup_f CX$ where CX is the cone $(X \times I) / (X \times \{0\})$

and we attach this to Y along $X \times \{1\}$
via $(x, 1) \sim f(x)$



Example : $X = S^{n-1}$

$C_f \rightarrow$ attaching to Y the n -cell
via $f: S^{n-1} \rightarrow Y$

Proposition

If (X_1, A) is a CW pair and the two attaching maps
 $f, g: A \rightarrow X_0$ are homotopic, then
 $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$

Homotopy Extension Property

Intuition:

Consider the map $f_0 : X \rightarrow Y$. Let $A \subset X$ and consider the homotopy on A $f_t : A \rightarrow Y$ with $f_0 = f|_A$. We would like to extend this to a homotopy on X as a whole with f_t .

Def: Homotopy Extension

$A \subset X$

(X, A) has the homotopy extension property (h.e.p)

if $\forall Y, \forall f_0 : X \rightarrow Y, \forall$ homotopy $g : A \times I \rightarrow Y,$
 $g(a, 0) = f_0(a)$

we can extend g to a homotopy $F : X \times I \rightarrow Y$

$$\text{i.e. } F(x, 0) = f_0(x)$$



$$f_0(x) = y$$

(X, A) has the h.e.p if every pair of maps $X \times \{0\} \rightarrow Y$ and

$A \times I \rightarrow Y$ that agree on $A \times \{0\}$ can be extended to

$$f_t(a)$$

a map $X \times I \rightarrow Y$

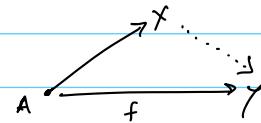
$$\hookrightarrow f_t : X \rightarrow Y$$

$$f_0 = f|_A$$

Lemma:

$A \subset X$ top space.

$\forall Y, \text{ any map } f : A \rightarrow Y \text{ extends to } X \rightarrow Y \text{ if and only if } A$
 $\text{is a retract of } X$



Proof:

\Leftarrow Suppose A is a retract of X via $r : X \rightarrow A$ s.t. $r|_A = \text{id}_A$
 Then $(f \circ r) : X \rightarrow Y$ is our extension

\Rightarrow Suppose, $\forall Y$ and any map $f : A \rightarrow Y$ extends to $X \rightarrow Y$.
 i.e. $f_t : X \rightarrow Y$ s.t. $f|_A = f$

Then, let $Y = A$ and $f = \text{id}_A$ i.e. $\text{id}_A : A \rightarrow A$ extends
 to $f_t : X \rightarrow A$ s.t. $f|_A = \text{id}_A \Rightarrow A$ is a retract of X

Lemma :

A pair (X, A) has the h.e.p if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof : By hypothesis the identity map

$$\Rightarrow : X \times \{0\} \cup A \times I \hookrightarrow X \times \{0\} \cup A \times I \text{ extends to a map}$$

$$X \times I \hookrightarrow X \times \{0\} \cup A \times I$$

$\therefore X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

\Leftarrow if A is closed: consider any two maps $X \times \{0\} \hookrightarrow Y$ and $A \times I \hookrightarrow Y$ that agree on $A \times \{0\}$. They combine to give a map $X \times \{0\} \cup A \times I \hookrightarrow Y$ which is continuous by continuity on the closed sets $X \times \{0\}$ and $A \times I$.

Compose this map $X \times \{0\} \cup A \times I \hookrightarrow Y$ with a retraction $X \times I \hookrightarrow X \times \{0\} \cup A \times I$ (we have this via hypothesis)

We get an extension $X \times I \hookrightarrow Y$

$\therefore (X, A)$ has the h.e.p.

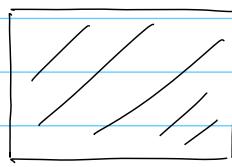
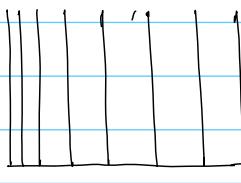
Properties

$$(1) \begin{array}{l} \text{H.e.p} \\ X - \text{normal iff} \end{array} \} \Rightarrow A \text{ is closed in } X$$

Non-example: (X, A) does not have h.e.p

(1) (I, A) where $A = \{0, 1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\}$

There is no continuous retraction $I \times I \hookrightarrow I \times \{0\} \cup A \times I$ because of the structure of (I, A) near 0.



Consider the ball $B = B(x_0, \epsilon)$
Then $\exists \delta > 0$ s.t. $B(x_0, \delta) \subset B$

γ — path in B from x_0 to x_1

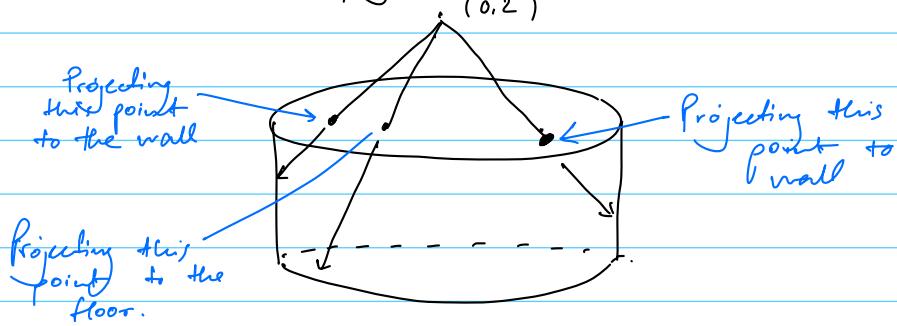
but x_0 and x_1 are in diff components
 \downarrow path at $t=1$ $B(x_0, \delta)$ of $C \cap B$

Proposition

If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the h.e.p.

Proof :

First, note that \exists a retraction $r: D^n \times I \rightarrow D^n \times \{0\} \cup \partial D^n \times I$ for ex - radial projection from the point $(0, 2) \in D^n \times \mathbb{R}$



Now, set $r_t = tr + (1-t)\mathbb{1}$ is a deformation retraction of $D^n \times I$ onto $D^n \times \{0\} \cup \partial D^n \times I$.

Now, with this, we have a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ since $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$.

If we perform the def. ret. of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ during the t -interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, this infinite concatenation of homotopies is a def. ret. of $X \times I$ onto $X \times \{0\} \cup A \times I$.

Proposition

If the pair (X, A) satisfies h.e.p and A is contractible, then the quotient map $q: X \rightarrow X/A$ is a homotopy equivalence.

Proof :

Let $f_t: X \rightarrow X$ be the homotopy extending a contraction of A with $f_0 = \text{id}$.

Now, $f_t(A) \subset A \quad \forall t$, so the composition

$$q \circ f_t: X \rightarrow X/A$$

sends A to a point and so factors as a composition

$$X \xrightarrow{q} X/A \longrightarrow X/A$$



Denote this by $\bar{f}_t: X/A \rightarrow X/A$)

$$\text{So, } q \bar{f}_t = \bar{f}_t q$$

$$X \xrightarrow{\bar{f}_t} X$$

$$\begin{array}{ccc} & & \\ q & \downarrow & \downarrow q \\ X/A & \xrightarrow{\bar{f}_t} & X/A \end{array}$$

When $t=1$, $f_1(A)$ equals to a point (since f_t is homotopy extension of the contraction of A), so f_1 induces a map $g: X/A \rightarrow X$ with $gq = f_1$

$$\begin{array}{ccc} & f_1 & \\ X & \xrightarrow{\quad} & X \\ q \downarrow & \nearrow g & \downarrow q \\ X/A & \xrightarrow{\bar{f}_1} & X/A \end{array}$$

$$\begin{aligned} \text{So, } qg &= \bar{f}_1 & \text{since } qg(\bar{x}) &= qg(x) \\ &&&= qf_1(x) \\ &&&= f_1(q(x)) \\ &&&= f_1(\bar{x}) \end{aligned}$$

The maps g and q are inverse homotopy equivalences as

$$gq = f_1 \simeq f_0 = 1 \text{ via } f_t \text{ and}$$

$$qg = f_1 \simeq \overline{f_0} = 1 \text{ via } \overline{f_t}.$$

Def: $W \simeq Z \text{ rel } Y$

for (W, Y) and (Z, Y) , there are maps $\varphi: W \rightarrow Z$ and $\psi: Z \rightarrow W$ restricting to identity on Y s.t. $\psi\varphi \simeq 1_W$ and $\varphi\psi \simeq 1_Z$ via homotopies that restrict to the identity on Y at all times.

Proposition

If (X_1, A) is a CW pair and we have attaching maps $f, g: A \hookrightarrow X_0$ that are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

Proof:

Let $F: A \times I \rightarrow X_0$ is a homotopy from f to g , consider the space $X_0 \sqcup_F (X_1 \times I)$, which has both $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ as subspaces

We can deformation retract $X_1 \times I$ onto $X_1 \times \{0\} \cup A \times I$ which induces a def retraction of $X_0 \sqcup_F (X_1 \times I)$ onto $X_0 \sqcup_f X_1$

Similarly, $X_0 \sqcup_f (X_1 \times I)$ def retracts onto $X_0 \sqcup_g X_1$

Both of them are identity on X_0 so we get the homotopy equivalence

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

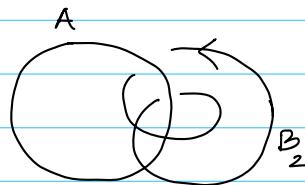
Fundamental Group

Intuition

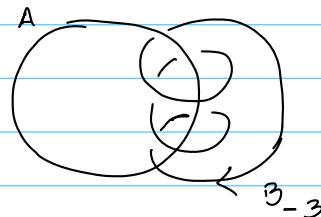
Two linked circles in \mathbb{R}^3 :



Link B with A two times
in the forward direction :



Link B with A three times
in the backward direction :

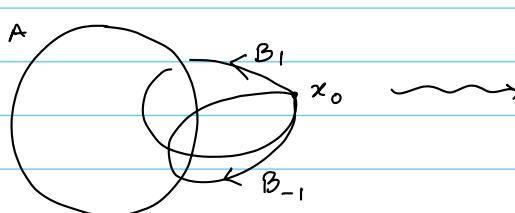
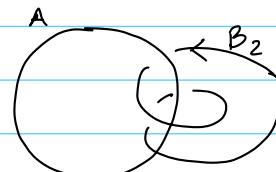
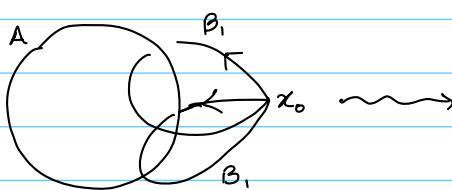


B_2 and B_{-3} are oriented circles/loops.

Two loops, B and B' , starting and ending at the same point x_0 can be added to form a new loop that travels around B and B' .

$$\text{So, } B_1 + B_1 = B_2$$

$$B_1 + B_{-1} = B_0 \leftarrow \text{unlinked from A}$$



$$\text{More generally, } B_m + B_n = B_{m+n}$$

Paths and Homotopy of paths

Def: Path in X

A continuous map $f: I \rightarrow X$ where $I = [0, 1]$

→ Notice that any path is nullhomotopic.

Def: Homotopy

A family $f_t: I \rightarrow X$ where $t \in I$ s.t

(1) $f_t(0) = x_0$ and $f_t(1) = x_1 \forall t$

(2) The associated map $F: I \times I \rightarrow X$
is continuous

We say $f_b \simeq f_1$.

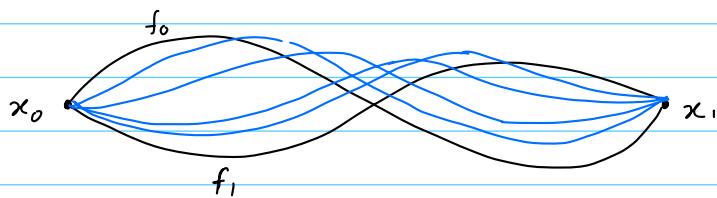
→ $f_0 \simeq f_1$ means homotopic rel. $\partial I = \{0, 1\}$ as the endpoints are fixed.

Examples

(1) Linear homotopies:

Any 2 paths f_0 and f_1 in \mathbb{R}^n with endpoints x_0 and x_1 ,
are homotopic by $f_t(x) = (1-t)f_0(x) + tf_1(x)$

Here, $F(x, t) = f_t(x) = (1-t)f_0(x) + tf_1(x)$ is continuous
since f_0 and f_1 are continuous, and sum & scalar
multiplication preserve continuity.



Non-example

$$f_0, f_1 : I \rightarrow S^1$$

$$\left. \begin{array}{l} f_0(t) = 1 \\ f_1(t) = e^{2\pi i t} \end{array} \right\} \text{They are not path homotopic}$$

Proposition

The relation of homotopy on paths with fixed endpoints

in any space is an equivalence relation.

We denote the equivalence class of f by $[f]$ and is called the homotopy class of f .

Proof:

Reflexivity: $f \simeq f$ by homotopy $f_t = f$

Symmetry: If $f_0 \simeq f_1$ via f_t , then $f_1 \simeq f_0$ via f_{1-t} .

Transitivity: Suppose $f_0 \simeq f_1$ via f_t . and if $f_1 = g_0$ with $g_0 \simeq g_1$ via g_t , then the homotopy

$$h_t = \begin{cases} f_{2t}, & t \in [0, \frac{1}{2}] \\ g_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}$$

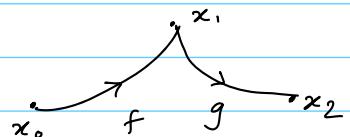
The associated function $H(s, t) = h_t(s)$ is continuous.

A function on the union of 2 closed sets is continuous if it is continuous restricted to each of the 2 sets separately.

Def: Product path

Given two paths $f, g: I \rightarrow X$ s.t $f(1) = g(0)$, the product path $f \cdot g$ first traverses f and then g :

$$f \cdot g(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$



This product path preserves homotopy classes:

if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via f_t and g_t homotopies respectively

and if $f_0(1) = g_0(0)$ so that $f \cdot g_0$ is well-defined

then $f_t \cdot g_t$ provides the homotopy

$$f \cdot g_0 \simeq f \cdot g_1$$

Def : Loop

Paths $f: I \rightarrow X$ s.t $f(0) = f(1) = x_0 \in X$

$x_0 \rightarrow$ basepoint

→ The set of all homotopy classes $[f]$ of loops $f: I \rightarrow X$ at the basepoint $x_0 \in X$ is denoted $\pi_1(X, x_0)$

Proposition :

$\pi_1(X, x_0)$ is a group w.r.t the product
 $[f][g] = [f \cdot g]$

This group is called the fundamental group of X at basepoint x_0 .

Proof :

Since the basepoint $x_0 \in X$ is fixed, the product of any two paths, f and g in $\pi_1(X, x_0)$ is defined.

Firstly, define reparametrisation of a path f to be a composition $f\varphi$ where $\varphi: I \rightarrow X$ is a continuous map r.t $\varphi(0) = 0$ and $\varphi(1) = 1$.

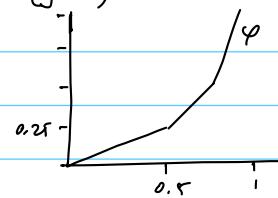
Reparametrisation preserves homotopy class of f since $f\varphi \simeq f$ via homotopy $f\varphi_t$ where $\varphi_t(x) = (1-t)\varphi(x) + tx$ so $\varphi_0(x) = \varphi(x)$ and $\varphi_1(x) = x$

We often show that f is a reparametrisation of g to prove $f \simeq g$.

Given the paths f, g and h with $f(1) = g(0)$ and $g(1) = h(0)$, then both $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are defined.

Note $(f \cdot g) \cdot h$ is a reparametrisation of $f \cdot (g \cdot h)$ via $f \cdot (g \cdot h) = (f \cdot g) \cdot h \varphi$ where φ is a continuous map s.t $\varphi: [0, \frac{1}{2}] \rightarrow [0, \frac{1}{4}]$ $\varphi: [\frac{1}{2}, 1] \rightarrow [\frac{1}{4}, 1]$

$$\text{So, } (f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$$



Given a path $f: I \hookrightarrow X$, let c be the constant path at $f(1)$ defined by $c(s) = f(1)$, $\forall s \in I$. Then, $f \cdot c$ is a reparametrisation of f :

$$f \cdot c(x) = \begin{cases} f(2x) & \text{for } x \in [0, \frac{1}{2}] \\ c(2x-1) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$\text{So, } f \cdot c = f \varphi \text{ where } \varphi: [0, \frac{1}{2}] \longrightarrow [0, 1]$$

$$\varphi: [\frac{1}{2}, 1] \longrightarrow \text{ }$$

$$\therefore f \cdot c \simeq f$$

Similarly $c \cdot f \simeq f$ where c is constant path at $f(0)$.

Taking f to be a loop, the homotopy class of the constant path is a two-sided identity.

Now, let f be a path from x_0 to x_1 . Its inverse path is \bar{f} from x_1 to x_0 defined by $\bar{f}(s) = f(1-s)$

Then, $f \cdot \bar{f}$ is homotopic to a constant path via homotopy

$$h_t = f_t \cdot g_t$$

where $f_t = f$ on $[0, 1-t]$ and $f_t = f(1-t)$ on $[1-t, 1]$

$$\text{and } g_t = \bar{f}_t$$

Then, $f_0 = f$ and $f_1 = \text{constant path } c \text{ at } x_0$

So, h_t is a homotopy from $f \cdot \bar{f}$ to $c \cdot \bar{c}$

$$\text{as } h_0 = f_0 \cdot g_0 = \begin{cases} f & \text{for } x \in [0, \frac{1}{2}] \\ \bar{f} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$h_1 = f_1 \cdot g_1 = \begin{cases} c, & x \in [0, \frac{1}{2}] \\ \bar{c}, & x \in [\frac{1}{2}, 1] \end{cases}$$

$$\therefore f \cdot \bar{f} \simeq c \quad (\text{defining } c \cdot \bar{c} = c) \text{ where } c = x_0$$

Replacing f by \bar{f} gives $\bar{f} \cdot f = c$

Take f to be the loop at x_0 , then $[\bar{f}]$ is a 2-sided inverse for $[f]$ in $\pi_1(X, x_0)$.

Fundamental Group of X at x_0 : $\pi_1(X, x_0)$

$\pi_1(X, x_0) = \{ \text{loops home } x_0 \text{ to itself in } X \} / (\text{path homotopy})$

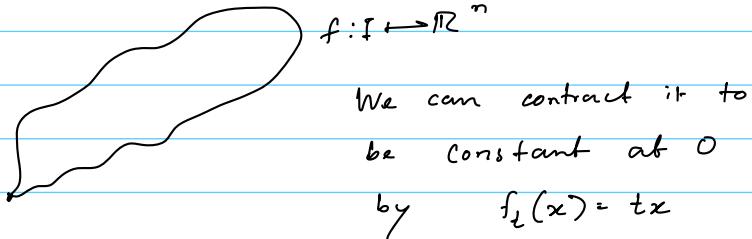
$$\rightarrow [f] \cdot [g] = [fg]$$

$$\rightarrow [f]^{-1} = [\bar{f}] \quad \text{where} \quad \bar{f}(t) = f(1-t)$$

$$\rightarrow [\text{constant}_{x_0}] = 1$$

Examples

(i) $\pi_1(\mathbb{R}^n, 0) = 1$



We say $\pi_1(X) = 1$ if X is contractible

$$f_t(x) = r_t \circ f$$

homotopy from id_X to constant map

Properties

(1)

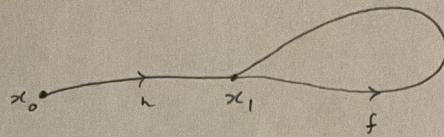
Gta

Change of basepoint

Let x_0 and x_1 lie in the same path-component of X .

Let $h: I \rightarrow X$ be a path from x_0 to x_1 , with the inverse path $\bar{h}(s) = h(1-s)$ from x_1 to x_0 .

Then, for each loop f based at x_1 , define the loop $h \cdot f \cdot \bar{h}$ based at x_0 .



Alternatively, we can define a general n -fold product f_1, \dots, f_n in which the path f_i is traversed in $[\frac{i-1}{n}, \frac{i}{n}]$.

Then, define the change of basepoint map $\beta_h: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ by $\underline{\beta_h[f] = [h \cdot f \cdot \bar{h}]}$

Proposition: The map $\beta_h: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$ is an isomorphism.
So, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Proof: Homomorphism as

$$\begin{aligned}\beta_h[f \cdot g] &= [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}] \\ &= \beta_h[f] \beta_h[g]\end{aligned}$$

This has the inverse $\beta_{\bar{h}}$ as

$$\begin{aligned}\beta_h \beta_{\bar{h}}[f] &= \beta_h[\bar{h} \cdot f \cdot h] = [h \cdot \bar{h} \cdot f \cdot h \cdot \bar{h}] \\ &= [f]\end{aligned}$$

$$\text{Similarly, } \beta_{\bar{h}} \beta_h[f] = [f]$$

Def: Simply connected

A space is simply connected if it is path connected

and has trivial fundamental group.

i.e. the constant path

Proposition

A space X is simply connected iff there is a unique homotopy class of paths connecting any two points in X .

Proof :

\Rightarrow : Need to show uniqueness.

Suppose

let f and g be 2 paths from x_0 to x_1 .

Then $f \simeq f \cdot \bar{g} \cdot g \simeq g$ since the loops $\bar{g} \cdot g$ and $f \cdot \bar{g}$ are each homotopic to constant loops, given $\pi_1(X) = 0$

\Leftarrow : If there is only one homotopy class of paths loops at x_0 , then all loops at x_0 are homotopic to the constant loop
 $\therefore \pi_1(X, x_0) = \pi_1(X) = 0$

If X is path connected, then $\pi_1(X, x_0)$ is independent of x_0 . We write it as $\pi_1(X)$.

Def: Induced Homomorphism

Suppose, $\varphi: X \rightarrow Y$ is a map taking basepoint $x_0 \in X$ to the basepoint $y_0 \in Y$

We say $\varphi: (X, x_0) \mapsto (Y, y_0)$

Then, φ induces a homomorphism

$$\varphi_*: \pi_1(X, x_0) \mapsto \pi_1(Y, y_0)$$

defined by composing the loops $f: I \rightarrow X$ based at x_0 with φ :

$$\varphi_*[f] = [\varphi f]$$

→ Well-defined:

Homotopy f_t of loops at x_0 yields a homotopy φf_t of loops based at y_0 .

$$\therefore \varphi_*[f_0] = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$$

→ φ_* is a homomorphism:

$$\begin{aligned} \varphi(f \cdot g) &= \varphi f \cdot \varphi g && \text{both functions have values} \\ &&& \varphi f(2s), \quad 0 \leq s \leq \frac{1}{2} \\ &&& \varphi g(2s-1), \quad \frac{1}{2} \leq s \leq 1 \end{aligned}$$

Properties of induced homomorphisms

$$(1) \quad (X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$$

$$(\varphi\varphi)_* = \varphi_*\varphi_*$$

Proof:

$$(\varphi\varphi)_f = \varphi(\varphi_f)$$

$$(2) \quad \mathbb{1}_* = \mathbb{1} \quad \text{which is saying } \mathbb{1}: X \rightarrow X \text{ induces } \mathbb{1}: \pi_1(X, x_0) \mapsto \pi_1(X, x_0)$$

(3) If φ is a homeomorphism with inverse ψ

then φ_* is an isomorphism with inverse ψ_* since

$$\varphi_*\psi_* = (\varphi\psi)_* = \mathbb{1}_* = \mathbb{1} \quad \text{and similarly } \psi_*\varphi_* = \mathbb{1}$$

(4) Let $\varphi, \psi: X \rightarrow Y$.

If φ and ψ are homotopic, then $\varphi_* = \psi_*$

Proof:

$$\varphi_*[f] = [\varphi f]$$

= $[\psi f]$ (via homotopy of φ and ψ)

$$= \psi_*[f]$$

(5) Proposition:

If a space X retracts onto a subspace A , then the induced homomorphism $i_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$ induced by the inclusion $i: A \hookrightarrow X$ is injective. If A is a deformation retract of X , then i_* is an isomorphism.

Proof:

Suppose, X retracts onto $A \subset X$ via $r: X \rightarrow A$

Then $r_i = \text{id}_A$

$$\text{So, } (ri)_* = r_* i_* = \text{id}$$

Suppose $i_*(f) = \text{id}$ for some $f \in \pi_1(A, x_0)$
Therefore i_* is injective

$$\text{Then, } (r_* i_*)(f) = r_*(\text{id}) = \text{id}. \text{ But } r_* i_* = \text{id}$$

$$\text{so } f = \text{id}.$$

Now, suppose X def. retracts onto A via $r_t: X \rightarrow X$

$$\text{so, } r_0 = \text{id}_X, r_t|_A = \text{id}_A \text{ and } r_t(X) \subset A$$

then, for any loop $f: I \rightarrow X$ based at $x_0 \in A$,

the composition $r_t f$ gives a homotopy of f to a loop in A , so i_* is also surjective.

$$\hookrightarrow \text{as } r_t(X) \subseteq A$$

i.e. for any $f: I \rightarrow X$,
first def retract to $f': I \rightarrow A$
where $f' = r_t f$. Then $i_*(f') = f' \in \pi_1(X, x_0)$
and $[f'] = [f]$ by
the homotopy r_t .

Def : Basepoint Preserving Homotopy

Consider a homotopy φ_t taking $A \subset X$ to a subspace $B \subset Y$ for all t , then we speak of maps of pairs

$$\varphi_t : (X, A) \rightarrow (Y, B)$$

A basepoint-preserving homotopy $\varphi_t : (X, x_0) \rightarrow (Y, y_0)$ is the case where $\varphi_t(x_0) = y_0 \quad \forall t$.

(6) If $\varphi_t : (X, x_0) \rightarrow (Y, y_0)$ is a basepoint preserving homotopy, then $\varphi_{0*} = \varphi_{1*}$

$$\begin{aligned} \text{Proof : } \varphi_{0*}[f] &= [\varphi_0 f] \\ &= [\varphi, f] \quad (\text{via homotopy } \varphi_t f) \\ &= \varphi_{1*}[f] \end{aligned}$$

Def : Homotopy Equivalence for spaces with basepoints

We say $(X, x_0) \simeq (Y, y_0)$ if there are maps $\varphi : (X, x_0) \rightarrow (Y, y_0)$ and $\psi : (Y, y_0) \rightarrow (X, x_0)$ with homotopies $\varphi \psi \simeq \text{id}_{(Y, y_0)}$ and $\psi \varphi \simeq \text{id}_{(X, x_0)}$ through maps that fix the basepoint.

In this case, the induced maps on π_1 satisfy

$$\varphi_* \psi_* = (\varphi \psi)_* = \text{id}_* = \text{id}$$

$$\psi_* \varphi_* = (\psi \varphi)_* = \text{id}_* = \text{id}$$

$\therefore \varphi_*$ and ψ_* are inverse isomorphisms

$$\therefore \pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

What if φ_t does not send x_0 to a fixed $y_0 \in Y$ for all t ? This means the basepoint in X is not always mapped to the same point by a homotopy.

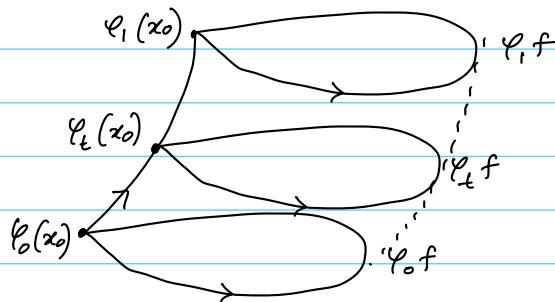
Lemma:

If $\varphi_t : X \rightarrow Y$ is a homotopy and h is the path $\varphi_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the three maps in the diagram satisfy $\varphi_{0*} = \beta_h \varphi_{1*}$

$$\begin{array}{ccc} & \varphi_{1*} & \rightarrow \pi_1(Y, \varphi_1(x_0)) \\ \pi_1(X, x_0) & \swarrow & \downarrow \beta_h \\ & \varphi_{0*} & \rightarrow \pi_1(Y, \varphi_0(x_0)) \end{array}$$

Proof:

Let h_t be the restriction of h to the interval $[0, t]$ (with a reparametrization so that domain of h_t is $[0, 1]$):
So. $h_t(s) = h(ts)$ where $h : I \rightarrow Y$ with $h(\tilde{t}) = \varphi_{\tilde{t}}(x_0)$



Then, if f is a loop in X at basepoint x_0 , then the product $h_t \cdot (\varphi_t f) \cdot \bar{h}$ gives a homotopy of loops at $\varphi_0(x_0)$.

Restricting this to $t = 0$ and $t = 1$,

$$\text{we see } \varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f]))$$

Theorem :

If $\varphi: X \rightarrow Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, \varphi(x_0))$ is an isomorphism for $\forall x_0 \in X$.

Proof :

Let $\varphi: X \rightarrow Y$ be a homotopy equivalence \Rightarrow
So, Let $\psi: Y \rightarrow X$ be the homotopy inverse

$$\begin{aligned} \text{So, } \varphi \psi &\simeq \text{Id} \\ \psi \varphi &\simeq \text{Id} \end{aligned}$$

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi \varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi \psi \varphi(x_0))$$

Given $\psi \varphi \simeq \text{Id}$, then $\psi_* \varphi_* = \text{Id}$ for some h by the previous lemma. $\Rightarrow \psi_* \varphi_*$ is an isomorphism

Since $\psi_* \varphi_*$ is an isomorphism

φ_* is injective.

Similarly, with $\psi_* \varphi_*$, we conclude ψ_* is injective.

$\therefore \varphi_*, \psi_*$ are injections and $\psi_* \varphi_*$ is an isomorphism. so φ_* is a surjection too.

$$(4) \quad \pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Proof :

A path $I \rightarrow X \times Y$
is a pair of paths $(f: I \rightarrow X, g: I \rightarrow Y)$

Fundamental Group of the Circle

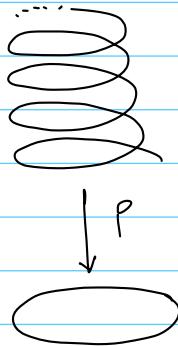
Some preliminary tools:

- (1) Let $w(s) = (\cos 2\pi s, \sin 2\pi s)$ for $s \in I$ be a loop based at $(1,0)$.
 Then, $[w]^n = [w_n]$ where $w_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$ for $n \in \mathbb{Z}$.
 by the definition of product path and the fact that product preserves homotopy.
- (2) Compare paths in S^1 with paths in \mathbb{R} :
 → Let $p: \mathbb{R} \rightarrow S^1$ via $p(s) = (\cos 2\pi s, \sin 2\pi s)$

Visualization: first, consider the helix $s \mapsto (\cos 2\pi s, \sin 2\pi s, s)$

Then, project \mathbb{R}^3 onto \mathbb{R}^2 by $(x,y,z) \mapsto (x,y)$

So, projecting the helix onto \mathbb{R}^2 gives p



$$\rightarrow w_n(s) = p \tilde{w}_n(s) \quad \text{where } \underbrace{\tilde{w}_n: I \rightarrow \mathbb{R}}_{\text{is the path } \tilde{w}_n(s) = ns} \text{ starts at } 0 \text{ and ends at } n$$

\tilde{w}_n is called the lift of w_n .

$\tilde{w}_n(s)$ winds around the helix $|n|$ times \rightarrow upwards if $n > 0$ and downwards if $n < 0$.

- (3) Def: Covering Space

Given a space X , a covering space of X consists of a

space \tilde{X} and a map $p: \tilde{X} \rightarrow X$ satisfying:

- (a) for each $x \in X$, \exists open neighbourhood $U \ni x$ in X st

$p^{-1}(U) = \coprod_{x \in U} V_x$ where each V_x is open and each

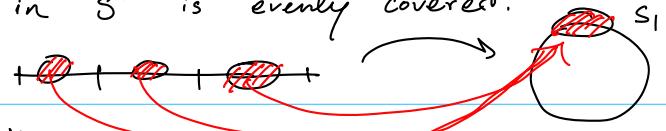
V_x is mapped homeomorphically onto U by p .
is a union of disjoint open sets (each of

We say U is evenly covered.

$p|_{V_x}: V_x \rightarrow U$
is a homeomorphism

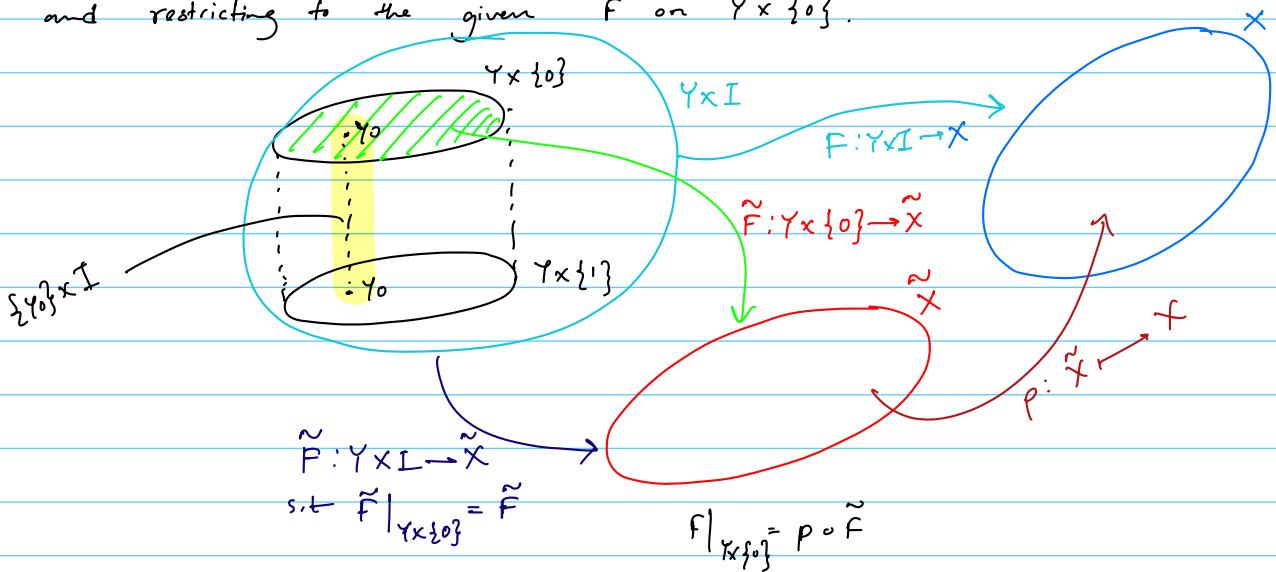
Example:

- (1) $p: \mathbb{R} \rightarrow S^1$, an open arc in S^1 is evenly covered.
Define it by $p(\theta) = e^{2\pi i \theta}$.



Lemma: Consider covering spaces $p: \tilde{X} \rightarrow X$.

Given a map $f: Y \times I \rightarrow X$ and a map $\tilde{F}: Y \times \{0\} \rightarrow \tilde{X}$ lifting $F|_{Y \times \{0\}}$, then there is a unique map $\tilde{f}: Y \times I \rightarrow \tilde{X}$ lifting f and restricting to the given \tilde{F} on $Y \times \{0\}$.



Proof:

First, construct a lift $\tilde{F}: N \times I \rightarrow \tilde{X}$ for some neighbourhood N of $y_0 \in Y$.

Given f is continuous, $\forall (y_0, t) \in Y \times I$ has a product

neighbourhood $N_t \times (a_t, b_t)$ s.t. $f(N_t \times (a_t, b_t))$ is contained in an evenly covered neighbourhood of $f(y_0, t)$.

around $f(y_0, t)$, \exists an evenly covered neighbourhood, since $p: \tilde{X} \rightarrow X$ is a covering space.

By continuity, we can always shrink $N_t \times (a_t, b_t)$ so that $f(N_t \times (a_t, b_t))$ is inside this evenly covered nbd.

By compactness of $\{y_0\} \times I$, finitely many such $N_t \times (a_t, b_t)$ products cover $\{y_0\} \times I$. Thus, we can choose one neighbourhood N of $\{y_0\}$ and a partition $0 = t_0 < t_1 < \dots < t_m = 1$ of I s.t. for each i , $f(N \times [t_i, t_{i+1}])$ is contained in an evenly covered neighbourhood U_i .

Assume, inductively, \tilde{F} has been constructed on $N \times [0, t_i]$ starting with our given \tilde{F} on $N \times \{0\}$. Thus, $f(N \times [t_i, t_{i+1}]) \subset U_i$, so since U_i is evenly covered, \exists open set $\tilde{U}_i \subset \tilde{X}$ projecting homeomorphically onto U_i by p and containing $\tilde{F}(y_0, t_i)$ because

$\tilde{f}|_{N \times [0, t_i]}$ is a lift of $f|_{N \times [0, t_i]}$
so $p(\tilde{f}(y_0, t_i)) = f(y_0, t_i)$
we know it is a lift
or we have already
constructed the lift
on $N \times [0, t_i]$.

We can extend \tilde{f} on
 $N \times [t_i, t_{i+1}]$ by
composing $p^{-1}: U_i \rightarrow \tilde{U}_i$
(since p is a homeomorphism)
with f . For this to
be continuous, \tilde{f} must
agree with \tilde{f} on
 $N \times [0, t_i]$, in particular at
 (y_0, t_i) .

Replace N by a small enough nbd of y_0 , we can get that
 $\tilde{f}(N \times \{t_i\})$ is contained in \tilde{U}_i by replacing $N \times \{t_i\}$ by
its intersection with $(\tilde{f}|_{N \times \{t_i\}})^{-1}(\tilde{U}_i)$. Then define \tilde{f} on
 $N \times [t_i, t_{i+1}]$ to be the composition of f
with $p^{-1}: U_i \rightarrow \tilde{U}_i$.

Continuing, we get $\tilde{f}: N \times I \rightarrow \tilde{X}$, a lift, for
some neighbourhood N of y_0 .

Next we show uniqueness of this lift. We prove for when
 Y is a point. Since Y is a point, we suppress
it from our notation.

Let \tilde{f} and \tilde{f}' be 2 lifts of $f: I \rightarrow X$
s.t. $\tilde{f}(0) = \tilde{f}'(0)$.

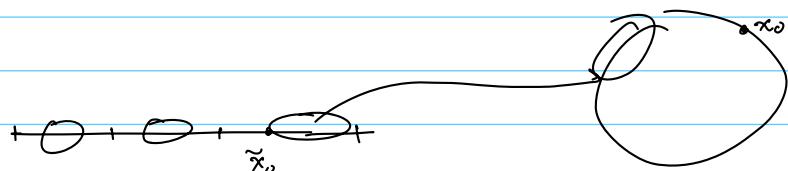
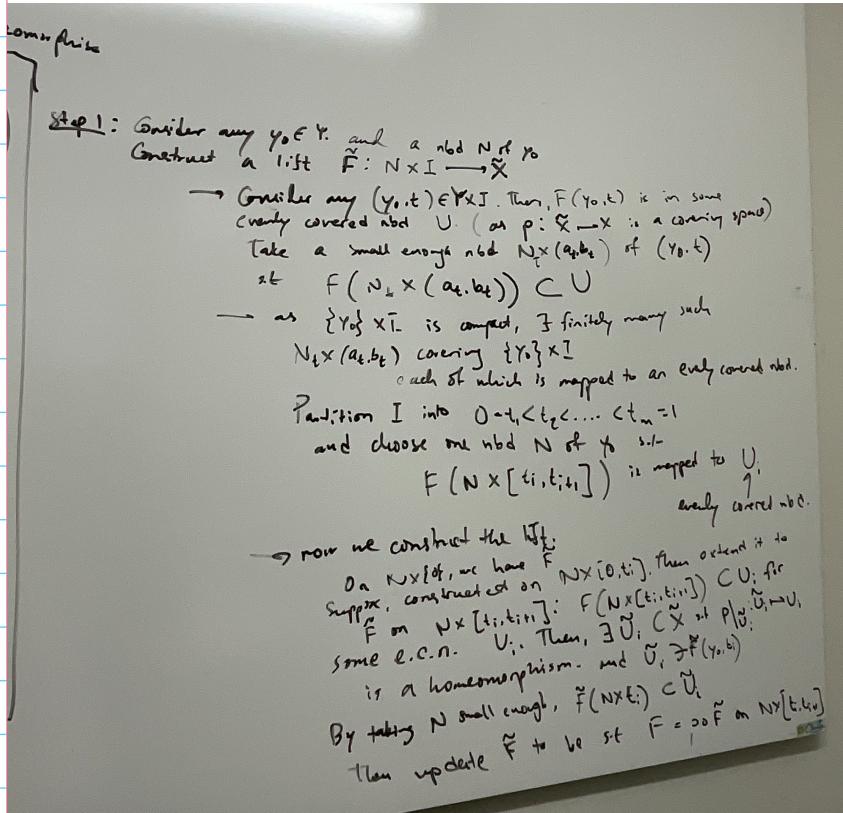
Again, we choose a partition $\mathcal{D} = t_1 < t_2 < \dots < t_m = 1$
of I s.t. for each i , $f([t_i, t_{i+1}])$ is contained in
evenly covered nbd U_i .

Assume inductively that $\tilde{f} = \tilde{f}'$ on $[0, t_i]$. As $[t_i, t_{i+1}]$
is connected, so is $\tilde{f}([t_i, t_{i+1}]) \Rightarrow$ it must lie in single
one of the disjoint open sets \tilde{U}_i projecting homeomorphically
to U_i . By same logic, $\tilde{f}'([t_i, t_{i+1}])$ lies in a
single \tilde{U}_i and it must be the same one as
 $\tilde{f}'(t_i) = \tilde{f}(t_i)$. As p is injective on \tilde{U}_i and
 $p \tilde{f} = p \tilde{f}'$, we get $\tilde{f} = \tilde{f}'$ on $[t_i, t_{i+1}]$. Continuing this way,
 $\tilde{f} = \tilde{f}'$.

Lastly, observe that since the lift \tilde{f} constructed on sets of the form
 $N \times I$ is unique when restricted to each segment $\{y\} \times I$,
they must agree when two such sets $N \times I$ overlap.

∴ we have a well-defined lift \tilde{F} on all of $Y \times I$.

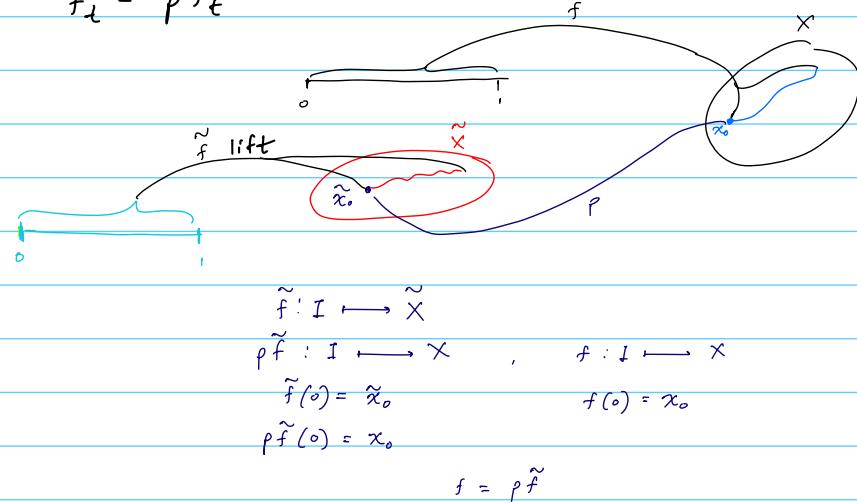
\tilde{F} is continuous as it is continuous on each $N \times I$, and unique as it is unique on each segment $\{y\} \times I$.



Path Lifting Property

!! Lemma : Consider covering spaces $p: \tilde{X} \rightarrow X$.

- (1) For each path $f: I \rightarrow X$ s.t $f(0) = x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, \exists a unique lift $\tilde{f}: I \rightarrow \tilde{X}$ starting at \tilde{x}_0 .
Hence, $f = p\tilde{f}$.
- (2) For each homotopy $f_t: I \rightarrow X$ of paths s.t $f_t(0) = x_0$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, \exists a unique lifted homotopy $\tilde{f}_t: I \rightarrow \tilde{X}$ of paths starting at \tilde{x}_0 .
Hence, $f_t = p\tilde{f}_t$



Proof :

(1) follows from prev. lemma when Y is a point

(2) Let $Y = I$.

Then for the homotopy $f_t: I \rightarrow X$, we have a map $F: I \times I \rightarrow X$ with $F(s, t) = f_t(s)$.

We get a unique lift $\tilde{F}: I \times \{0\} \rightarrow \tilde{X}$ using part (1).

Then, by prev. lemma, we get a unique lift

$$\tilde{F}: I \times I \rightarrow \tilde{X}$$

The restrictions $\tilde{F}|_{\{0\} \times I}$ and $\tilde{F}|_{\{1\} \times I}$ are paths

lifting constant paths, so they must also be constant by uniqueness of part (1).

So, $\tilde{f}_t(s) = \tilde{F}(s, t)$ is a homotopy of paths

and \tilde{f}_t lifts f_t as $F = p\tilde{F}$

We set \tilde{X} to be \mathbb{R}
 here or
 $p: \mathbb{R} \rightarrow S'$ is a
 covering space.

Theorem: $\pi_1(S')$ is an infinite cyclic group generated by the homotopy class of the loop $w(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1, 0)$. So, $\pi_1(S') \cong \mathbb{Z}$ as a group.

Proof: Let $f: I \rightarrow S'$ be a loop at the basepoint $x_0 = (1, 0)$ which is one element of the group $\pi_1(S', x_0)$.

Then, \tilde{f} starting at 0 and must end at some integer n since $p\tilde{f}(1) = f(1) = x_0$ and $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$
 recall $p(s) = e^{2\pi i s}$.

Another path in \mathbb{R} from 0 to n is \tilde{w}_n and $\tilde{f} \simeq \tilde{w}_n$ via the

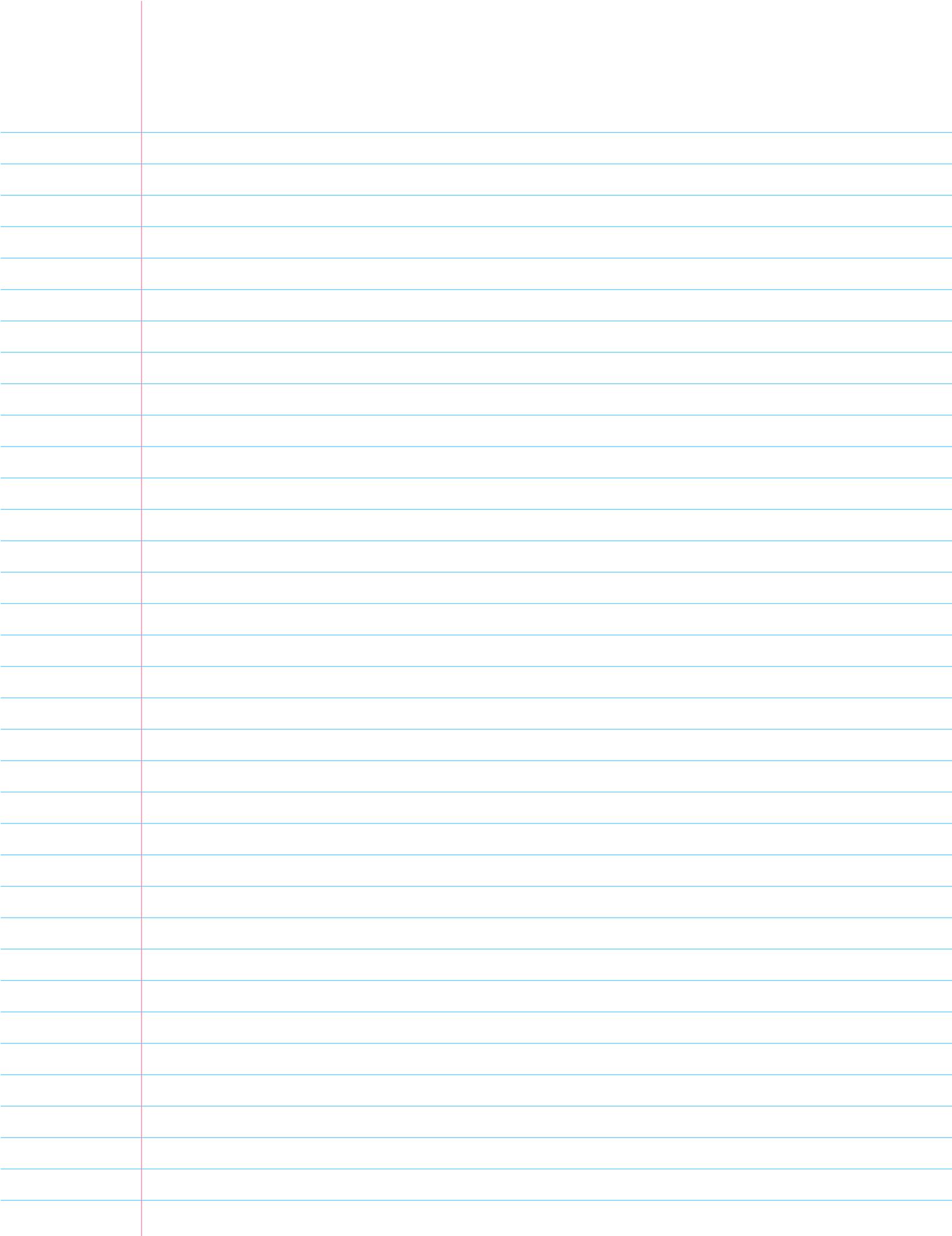
linear homotopy $(1-t)\tilde{f} + t\tilde{w}_n$. Compose the homotopy with p gives the homotopy $\tilde{f} \simeq \tilde{w}_n$ so $[f] = [w_n]$.
 ∴ for any loop f , $f = [w_n]$. Is n fixed here? Yes.

Next, we show that n is uniquely determined by $[f]$:
 Suppose $f \simeq w_m$ and $f \simeq w_n$. Let f_t be a homotopy from $w_m = f_0$ to $w_n = f_1$.

Then, f_t lifts to a homotopy \tilde{f}_t of paths starting at 0
 by previous lemma (2)

The uniqueness of \tilde{f} (by prev. lemma) implies that $\tilde{f}_0 = \tilde{w}_m$ and $\tilde{f}_1 = \tilde{w}_n$. Since \tilde{f}_t is a homotopy of paths, the endpoint $\tilde{f}_t(1)$ is independent of t . for $t=0$, the endpoint is m and for $t=1$, it is n so $m=n$.

The fact that this group is generated by $w(s)$ is obvious from noting that $[w]^n = [w_n]$



Theorem: Fundamental Theorem of Algebra

Every non-constant polynomial with coefficients in \mathbb{C} has a root in \mathbb{C} .

Proof:

Consider an arbitrary polynomial $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n$. Suppose, $p(z)$ has no roots in \mathbb{C} (for contradiction)

Since $p(z)$ has no roots in \mathbb{C} , then $\forall r \in \mathbb{R}$,

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|} \quad \text{is a loop in } S^1 \subset \mathbb{C} \text{ based at 1.}$$

$$\hookrightarrow f_r(0) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1$$

$$f_r(1) = \frac{p(r \cos(2\pi) + ri \sin(2\pi))/p(r)}{(\dots)} = 1$$

Then, as r varies, f_r is a homotopy of loops with basepoint 1.

for $r=0$, f_0 is the trivial loop constant at 1.

$$\therefore [f_r] = 0 \quad \forall r \text{ in } \pi_1(S^1)$$

$$\therefore p(z) \xleftarrow{\partial \in \pi_1(S^1)} \longrightarrow \textcircled{1}$$

Now, consider a large r s.t. $r > |a_1| + \dots + |a_n|$ and $r > 1$

Then, for $|z|=r$, $p(z)$ has no solution in $|z|=r$:

$$|z^n| > (|a_1| + \dots + |a_n|) |z^{n-1}| > |a_1 z^{n-1}| + \dots + |a_n| \geq |a_1 z^{n-1} + \dots + a_n|$$

$$\Rightarrow |z|^n > |a_1 z^{n-1} + \dots + a_n|$$

$$\Rightarrow p_t(z) := z^n + (a_1 z^{n-1} + \dots + a_n) \cdot \forall t \in I \text{ has no}$$

root on the circle $|z|=r \longrightarrow$ this is a deformation of our polynomial to z^n

Then, redefine $f_r(s) := \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|}$

Let t go from 1 to 0, we find a homotopy from the loop f_r to $w_n(s) = e^{2\pi i n s}$.

$$\text{But } [\omega_n] = [\omega]^n \therefore p(z) \longleftrightarrow n \in \pi_1(S') \longrightarrow (2)$$

$$\Rightarrow [\omega_n] = [f_r] = 0 \quad \text{using (1) and (2)}$$

$\therefore n=0$. \rightarrow contradiction as we assumed the degree was n .

Theorem: Brower Fixed Point Theorem in 2 dimensions

Every continuous map $h: D^2 \rightarrow D^2$ has a fixed point i.e
 $x \in D^2$ s.t $h(x) = x$.

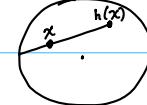
Proof:

Suppose, $\forall x \in D^2, h(x) \neq x$.

Define $r: D^2 \rightarrow S^1$ (where $\partial D^2 = S^1$)

to be the point where the line from $h(x)$ through x meets S^1 :

$$r(x) = \frac{x - h(x)}{\|x - h(x)\|}$$



Clearly, r is continuous. Also, $r(x) = x \quad \forall x \in S^1$.

Thus, r is a retraction of D^2 onto S^1 .

However, no such retraction exists.

Let $f_0 \in \pi_1(S')$

In D^2 , $f_0 \cong$ constant loop by linear homotopy

$$f_t(x) = (1-t)f_0(x) + tx_0 \quad \nwarrow \text{basepoint of } f_0$$

Since $r = \text{id}$ on S^1 , $r \circ f_t$ is a homotopy in S^1 from $r \circ f_0 = f_0$ to the constant loop at x_0 , since r is a retraction of D^2 onto S^1 .

But this contradicts the fact that $\pi_1(S')$ is non-zero.

Theorem: Borsuk-Ulam Theorem in 2 dimensions

for every continuous map $f: S^2 \mapsto \mathbb{R}^2$, \exists a pair of antipodal points x and $-x$ in S^2 s.t. $f(x) = f(-x)$.

OR

Weather Theorem

At any moment, there exists a pair of antipodal points on Earth s.t. they have

Proof :

Suppose not for $f: S^2 \mapsto \mathbb{R}^2$

Define $g: S^2 \mapsto S^1$ by $g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$. Notice: $-g(x) = g(-x)$.

Let the loop η in $S^2 \subseteq \mathbb{R}^3$ be $\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)$

and let $h: I \mapsto S^1$ be the composed loop $h = g \circ \eta$

Now, $g(-x) = -g(x) \Rightarrow h(s + \frac{1}{2}) = -h(s) \quad \forall s \in [0, \frac{1}{2}]$.

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Now, the loop h can be lifted to $\tilde{h}: I \mapsto \mathbb{R}$.

Since $h(s + \frac{1}{2}) = -h(s)$, $\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$ for some odd integer.

Now, q is independent of s : q depends on $s \in [0, \frac{1}{2}]$ continuously but can take on odd integer values \Rightarrow it must be constant.

$$\text{Also, } \tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$$

$\therefore h$ represents q times the generator of $\pi_1(S^1)$

Since q is odd, h is not nullhomotopic

But $h \circ g \circ \eta: I \mapsto S^2 \mapsto S^1$ and η is nullhomotopic in S^2



Shrink it along the surface.

$\therefore g \circ \eta$ is nullhomotopic

$\therefore h$ is nullhomotopic \rightarrow contradiction.