

Algebraic Topology

Notes

Jabayer Ibn Hamid

These notes are a combination of notes from Allen Hatcher's book "Algebraic Topology" and Prof. Ciprian Manolescu's lectures from MATH 215a at Stanford University.

Basic Constructions

Def : Homeomorphism

Let X and Y be topological spaces.

$f: X \rightarrow Y$ is a homeomorphism if f is bijective and both f and f^{-1} are continuous.

We say $\boxed{X \cong Y}$

Def : Homotopy

A family of maps, $f_t: X \rightarrow Y$ where $t \in I = [0, 1]$ s.t

the associated map $F: X \times [0, 1] \rightarrow Y$ given by $F(x, t) = f_t(x)$ is continuous.

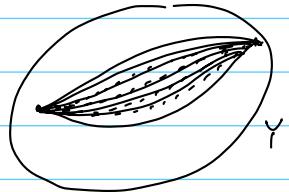
Two maps $f_0, f_1: X \rightarrow Y$ are homotopic if there exists a homotopy $F: X \times [0, 1] \rightarrow Y$ s.t

$$F(x, 0) = f_0(x) \quad \forall x \in X$$

$$F(x, 1) = f_1(x)$$

We say

$$\boxed{f_0 \simeq f_1}$$



Def : Homotopy Equivalence

A map $f: X \rightarrow Y$ is a homotopy equivalence if $\exists g: Y \rightarrow X$ s.t $f \circ g \simeq \text{id}_Y$ and $g \circ f \simeq \text{id}_X$

We say the spaces X and Y are homotopy equivalent and

$$\boxed{X \simeq Y}$$

→ can prove easily that this is an equivalence relation.

Examples of homotopy equivalence

(1) $\mathbb{R}^n \simeq$ a point (even though $\mathbb{R}^n \not\simeq$ a point)
↑ infinite ↑ finite

Why?

$$f: \mathbb{R}^n \rightarrow \{0\}$$

and take $g: \{0\} \rightarrow \mathbb{R}^n$ by $g(0) = 0$

Then $f \circ g = \text{id}_{\{0\}}$ and $(g \circ f)(x) = 0 \quad \forall x \in \mathbb{R}^n$

Now $g \circ f \sim \text{id}_{\mathbb{R}^n}$ by $f_t(x) = tx$ where $f_0 = 0$ and $f_1 = \text{id}_{\mathbb{R}^n}$

(2) $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \simeq$ a point

$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \simeq$ a point

Def : Contractible

We say the space X is contractible if $X \simeq$ point.

Def : Retractions

Let X be a space and let $A \subset X$.

then, a retraction is a map $r: X \rightarrow X$ s.t
 $r(X) = A$ and $r|_A = id_A$.

Def : Deformation Retraction

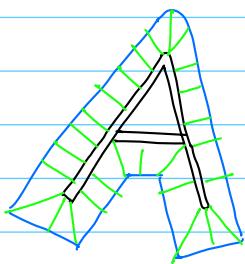
A deformation retraction of X onto a subspace A is
a family of maps $f_t: X \rightarrow X$, with $t \in I$ s.t
 $f_0 = id_X$ and $f_1(X) = A$ and $f_t|_A = id_A$ for $\forall t \in I$.

The family f_t must also be continuous
→ an example of a homotopy from id_X to a retraction of X onto $A \subset X$.

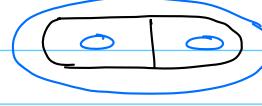
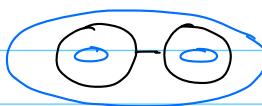
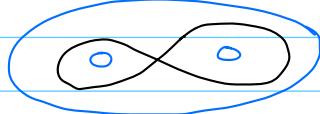
→ in this case, $A \cong X$ as $f_0: A \rightarrow X$ by id_X
 $f_1: X \rightarrow A$ as above
then $f_0 \circ f_1 = id_X$
and $f_1 \circ f_0 = id_A$

Examples of deformation retraction:

(1)



(2) Look at deformations of



(3) $X = \mathbb{R}^2 - \{0\}$. $A = S^1$

$$(4) f(x, t) = (1-t)x + t \frac{x}{\|x\|}.$$

Def: Mapping Cylinder ("the structure through which the def. retraction occurs")

for a map $f: X \rightarrow Y$, the mapping cylinder M_f is the quotient space of the disjoint union $(X \times I) \sqcup Y$ obtained by the equivalence $(x, 1) \in X \times I \sim f(x) \in Y$

$\nwarrow \nearrow$
Make the endpoint of the deformation equivalent to the image of the map.

Mapping cylinders are continuous.

Def: Homotopy relative to A (homotopy rel. A)

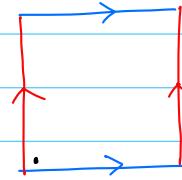
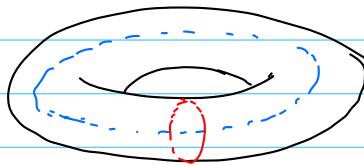
A homotopy $f_t: X \rightarrow Y$ whose restriction to a subspace $A \subset X$ is independent of t .

In other words, f_t is a homotopy and $f_t|_A$ is independent of t .

\rightarrow def. retraction of X onto A is a homotopy rel. A from id_X to a retraction of X onto $A \subset X$.

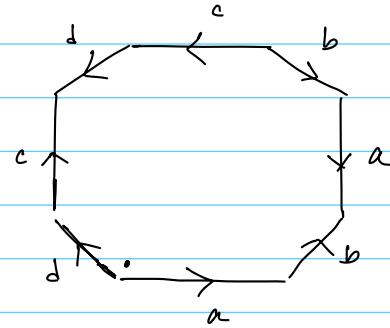
Cell Complexes

Examples :



The torus $S^1 \times S^1$ can be constructed from the square

Generally, an orientable surface M_g of genus g can be constructed from a polygon of $4g$ sides by identifying pairs of edges.



2 cell: interior of a polygon which is an open disk

1 cell: an open interval like $(0, 1)$

3 cell: an open ball.

n -cell: open-disk

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

Def : Cell Complex (or CW complex)

A space constructed as follows:

- (1) Start with discrete set $X^0 \rightarrow$ the points are 0-cells
- (2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e_α^n via maps

$\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$.

So, X^n is the quotient space of $X^{n-1} \coprod_\alpha D_\alpha^n$ under the equivalence $x \sim \varphi_\alpha(x) \forall x \in \partial D_\alpha^n$

\nwarrow (n-1)-skeleton \nearrow n-disks

: i.e attach boundaries of the n-disk to the (n-1)-skeleton

$$\therefore X^n = X^{n-1} \coprod_\alpha e_\alpha^n \text{ where } e_\alpha^n \text{ is an open n-disk}$$

(3) Either stop this induction at a finite stage and set $X = X^n$ for $n < \infty$

or continue indefinitely, setting

$$X = \bigcup_n X^n$$

↳ in this case, X has the weak topology:

$A \subset X$ is open iff $A \cap X^n$ is open in X^n for each n

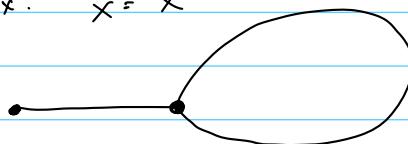
Vocabulary :

① $X^n \rightarrow$ n-skeleton

② Dimension of $X \rightarrow$ largest n s.t. an n-cell exists

Examples of Cell Complexes:

(1) 1-dimensional cell complex: $X = X^1$
 (multigraphs)



(2) The sphere S^n has a cell complex with two cells, e^0 and e^n , where e^n is attached by $\varphi: S^{n-1} \rightarrow e^0$.

$\therefore S^n$ is being regarded as the quotient space

$$D^n / \partial D^n$$

$$S^n = e^0 \cup e^n.$$

Alternatively,

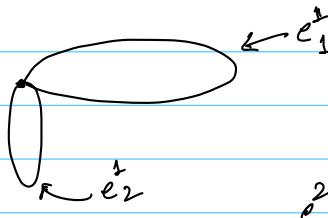
$$\begin{aligned} S^n &= S^{n-1} \cup e_+^n \cup e_-^n \\ &= e_+^0 \cup e_-^0 \cup \dots \cup e_+^n \cup e_-^n \end{aligned}$$

(3) Cell Complex of a torus:

Step 1: X^0 is just a point $\rightarrow \bullet \leftarrow e^0$

Step 2: Attach two 1-cells to this point

$$X^1 =$$



$$\therefore S^\infty = \bigcup_n S^n$$

Step 3: Attach a disk to X^1 by attaching its boundary to X^1 .

(4) Real Projective Space, $\mathbb{R}P^n$

$$(\mathbb{R}^{n+1} - \{0\}) / (\nu \sim \lambda \nu, \forall \nu \in \mathbb{R}^{n+1}, \lambda \neq 0)$$

\rightarrow Restricting to vectors of length 1, $S^n / (\nu \sim -\nu)$

$\Rightarrow D^n$ with antipodal points of ∂D^n identified

To get this, think of



∂D^n with antipodal points equivalent is $\mathbb{R}P^{n-1}$

$\therefore \mathbb{R}P^n$ can be formed from $\mathbb{R}P^{n-1}$ by attaching an n -cell. and the attaching map $\varphi: S^{n-1} \rightarrow \mathbb{R}P^{n-1}$

$\therefore \mathbb{R}P^n$ has the cell complex structure $e^0 \cup e^1 \cup \dots \cup e^n$ with one cell e^i in each dimension $i \leq n$.

i.e. for the upper hemisphere's points, find where the line to south pole intersects with D^n

(5) Complex Projective Space. $\mathbb{C}P^n$

Space of all complex lines through the origin in \mathbb{C}^{n+1}

$$\text{i.e. } \mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\}) / (v \sim \lambda v, \forall v \in \mathbb{C}^{n+1}, \lambda \neq 0)$$

Equivalent to $S^{2n+1} / (v \sim \lambda v, |\lambda|=1)$ ($S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$)

Equivalent to $D^{2n} / (v \sim \lambda v, v \in \partial D^{2n})$

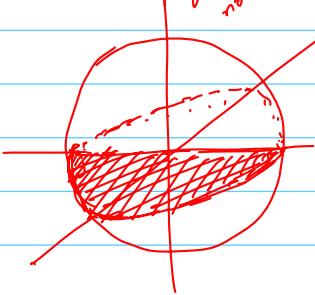
↳ Why?

$S^{2n+1} \subset \mathbb{C}^{n+1}$ → consider vectors in \mathbb{C}^{n+1} whose last coordinate is ~~one~~ real.

and non-negative

These vectors are of the form $(w, \sqrt{1-w^2}) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$

They form the graph of the function $w \mapsto \sqrt{1-w^2}$
with $|w| \leq 1, w \in \mathbb{C}^n$



Note: $w \in \mathbb{C}^n$ and $|w| \leq 1 \Rightarrow w \in D^{2n}$

This is a disk D^{2n}_+ bounded by the spheres S^{2n-1} .

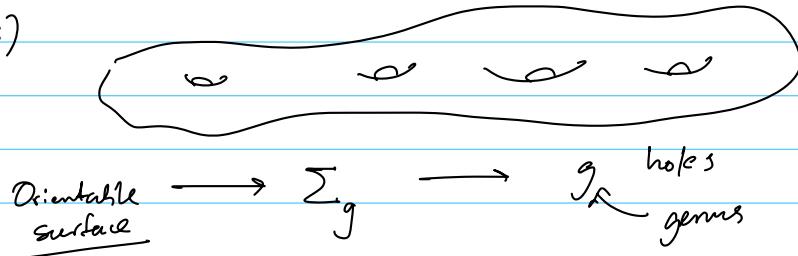
By adding another dimension and viewing them as $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$, we ~~view~~ them as vectors in $(D^{2n}_+, 0)$ bounded by $S^{2n-1} \subset S^{2n+1}$

Now, each vector in S^{2n+1} is equivalent to a vector in D^{2n}_+ by identifying $v \sim \lambda v$. In particular, if the last coordinate is zero, we have $v \sim \lambda v, \forall v \in S^{2n-1}$.

$\therefore \mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a cell e^{2n} using the attaching map $\varphi : S^{2n-1} \rightarrow \mathbb{C}P^{n-1}$

$\therefore \mathbb{C}P^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$ with cells only in even dimensions

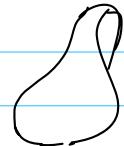
(6)



Can be constructed from a $4g$ polygon

↳ Start with one e^0

(7)

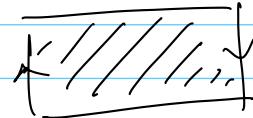
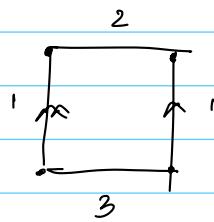


Non-orientable
surface

E.g.: $N_2 \rightarrow$ Klein bottle
 $N_1 \rightarrow RP^2$

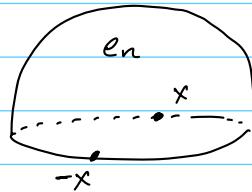
(8) Annulus:

(9) Möbius band



(a) RP^n revisited

$$RP^n = S^n / (x \sim -x, \forall x)$$



$$\Rightarrow RP^n = RP^{n-1} \cup e^n$$

$$\therefore RP^n = e^0 \cup e^1 \cup e^2 \cup \dots \cup e^n$$

$$\text{Then, } RP^\infty = e^0 \cup e^1 \cup e^2 \cup \dots = \bigcup_n RP^n$$

(i) $\mathbb{C}\mathbb{P}^n$ revisited

$$\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / (x \sim \lambda x, \lambda \in \mathbb{C}^*)$$

$$\therefore z \sim \frac{z}{\|z\|} \Rightarrow \mathbb{C}\mathbb{P}^n \cong S^{2n+1} / (z \sim \lambda z, \lambda \in S^1)$$

Divide everything by x_1 , i.e. last coordinate in $\mathbb{R}_{\geq 0}$

$$z = \underbrace{(z_0, \dots, z_n)}_{w} \underbrace{z_{n+1}}_{\sqrt{1-\|w\|^2}}$$

with $\|w\| \leq 1$

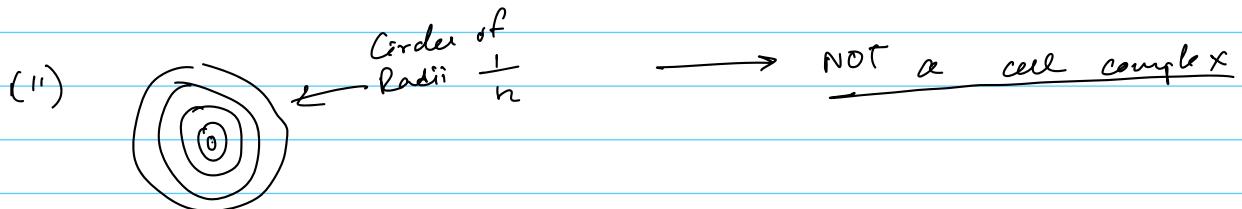
$$D_+^{2n} = \text{graph } (w \mapsto \sqrt{1-\|w\|^2})$$

$$\therefore \mathbb{C}\mathbb{P}^n = D_+^{2n} / (w \sim \lambda w \text{ if } w \in S^{2n-1})$$

$$= \mathbb{C}^{2n} \cup \left(S^{2n-1} / (w \sim \lambda w) \right)$$

$$= \mathbb{C}\mathbb{P}^{n-1} \cup \mathbb{C}^{2n}$$

$$= e^0 \cup e^2 \cup \dots \cup e^{2n}$$



Properties of CW Complexes

- (1) They are normal (\therefore also Hausdorff)
- (2) Any finite cell complex is compact
- (3) A compact subspace of a cell cx is contained in a finite subcomplex
- (4) Closure finiteness \rightarrow The closure of each cell ℓ meets only finitely many cells.
- (5) Locally contractible:
 $\forall x \in X, \exists x \text{ open}, \exists V \subset U \text{ with } x \in V$
s.t. V is contractible

(6)

Recall:

Top manifolds \rightarrow 2nd Countable, Hausdorff, locally Euclidean
Smooth manifolds \rightarrow

Theorem: Every smooth manifold is homeomorphic to a cell complex.

Theorem: Every topological manifold is homotopy equivalent to a cell complex.

Theorem: Every top manifold of dimension $\neq 4$ is homeomorphic to a cell complex
(unknown in dim 4)

Def: Characteristic Map

Each cell e_α^n in a cell complex X has a characteristic map

$$\varphi_\alpha : D_\alpha^n \xrightarrow{\sim} X$$

which extends the attaching map φ_α and is a homeomorphism from the interior of D_α^n onto e_α^n

→ φ_α is the composition

$$D_\alpha^n \hookrightarrow X^{n-1} \coprod_\alpha D_\alpha^n \xrightarrow{\qquad\qquad\qquad} X^n \hookrightarrow X$$

↑
the quotient
map that
defines X^n

Example of characteristic map:

(i) Recall: S^n can be constructed by two cells: e^0 and e^n ← just one point

where e^n is attached to e^0 by

$$\varphi_\alpha : S^{n-1} \rightarrow e^0$$

Then, the characteristic map of e^n is

$$\varphi_\alpha : D_\alpha^n \rightarrow S^n \text{ which collapses } \partial D_\alpha^n \text{ to } e^0$$

Def: Subcomplex

A subcomplex of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X .

\rightarrow As A is closed, for each cell in A ,

the image of its characteristic map } contained in A
the image of its attaching map }

$\therefore A$ is a cell complex as well

Def : CW pair

A cell complex X and a subcomplex A forms a pair (X, A)

Example of subcomplex

\rightarrow Each skeleton, X^n , is a subcomplex.

\rightarrow in $\mathbb{R}\mathbb{P}^n$ and $\mathbb{C}\mathbb{P}^n$, the only subcomplexes are $\mathbb{R}\mathbb{P}^k$ and $\mathbb{C}\mathbb{P}^k$, $\forall k \leq n$

Properties of subcomplexes

(i) Closure of a collection of cells is a subcomplex.

Operations on Spaces

III Products

$X, Y \rightarrow \text{cell complexes}$

$X \times Y \rightarrow \text{cell complex with the cells } e_\alpha^m \times e_\beta^n$

IV Quotients

$\longrightarrow A \subset X \rightsquigarrow X/A$

Given (X, A) a CW pair,
the quotient space X/A also has a cell complex structure:

→ the cells of X/A are the cells of $X-A$ and
a new 0-cell which is the image of
 A in X/A .

→ for a cell e_α^n of $X-A$ attached by
 $\varphi_\alpha: S^{n-1} \rightarrow X^{n-1}$, the attaching map for the
corresponding cell in X/A is the composition

$$S^{n-1} \rightarrow X^{n-1} \rightarrow X^{n-1}/A^{n-1}$$

Eg: ① $D^n/S^{n-1} = S^n$

V Wedge Sum (for based spaces)

Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$,
the wedge sum $X \vee Y$ is the quotient of $X \coprod Y$ by
identifying x_0 and y_0 to a single point

→ Example: $S^1 \vee S^1 = \infty$

$$X \vee Y = X \coprod Y / (x_0 \sim y_0)$$

→ $\bigvee_\alpha X_\alpha$ for an arbitrary collection of spaces X_α :
start with $\coprod_\alpha X_\alpha$ and then identify $x_\alpha \in X_\alpha$
to one point.

→ If X_α are cell complexes and the points x_α
are 0-cells, then $\bigvee_\alpha X_\alpha$ is a cell complex
because we obtain it from the cell complex $\coprod_\alpha X_\alpha$ and attach by

collapsing a subcomplex to a point.

→ For a cell complex X , the quotient X^n/X^{n-1} is a wedge sum of n -spheres $\bigvee_{\alpha} S_{\alpha}^n$ with one sphere for each n -cell of X

7) Smash Product $X \wedge Y = (X \times Y) / ((x_0 \times Y) \cup (X \times y_0))$

Inside the product space $X \times Y$, there are copies of X and Y : $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $y_0 \in Y$ and $x_0 \in X$.

These copies of X and Y intersect only at (x_0, y_0) so their union can be identified with the wedge sum $X \vee Y$

$$\begin{aligned} \text{i.e. } (X \times \{y_0\}) \vee (\{x_0\} \times Y) &= X \vee Y \\ &= (X \amalg Y) / (x_0 \sim y_0) \end{aligned}$$

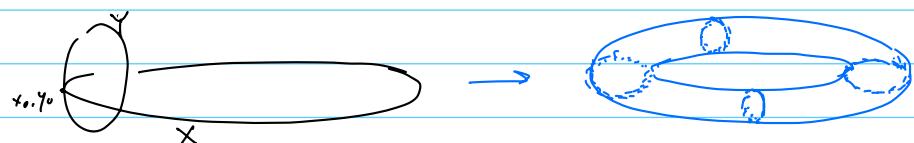
The smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$

↳ i.e. we are collapsing away the separate factors X and Y .

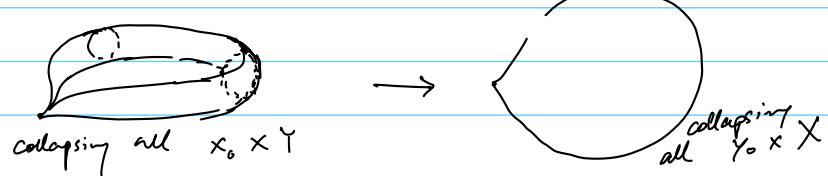
Eg: $S^1 \wedge S^1 = S^2$ \longrightarrow $S^1 = I / (0 \sim 1)$
 $S^m \wedge S^n = S^{m+n}$

Why?

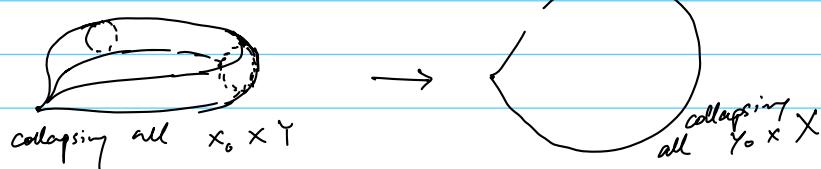
Firstly, $S^1 \times S^1$ results in a torus T^2



Secondly, $S^1 \wedge S^1 = \infty$



Then, quotienting:



II

Suspension

for a space X , the suspension SX is the quotient of $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another.

Example

(i) $X = S^n$

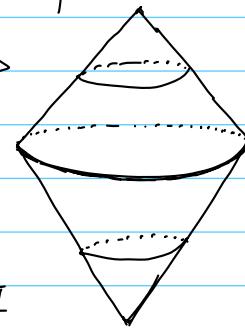
$SX = S^{n+1}$ with the two suspension points at North and South of S^{n+1}

→ We can suspend maps too

$$f: X \rightarrow Y \rightsquigarrow Sf: SX \rightarrow SY$$

which is the quotient map of

$$f \times 1 : X \times I \rightarrow Y \times I$$



III

Cone

$$CX = (X \times I) / (X \times \{0\})$$



→ If X is a CW complex, then so are SX and CX as quotients of $X \times I$ with its product cell structure with I given the standard cell structure of ~~two~~ two 0-cells joined by one 1-cell.

7

Join

Given X and Y , we can define the space of all line segments joining points in X to points in Y .

$$X * Y = (X \times Y \times I) / \left(\begin{array}{l} (x, y_1, 0) \sim (x, y_2, 0) \quad \forall x, x_1, x_2 \in X \\ (x_1, y_1, 1) \sim (x_2, y_1, 1) \quad \forall y, y_1, y_2 \in Y \end{array} \right)$$

$$\rightarrow pt * pt \longrightarrow \bullet \longrightarrow$$

$$pt * pt * pt \longrightarrow \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$pt * pt * \dots * pt = \Delta^n \rightarrow n\text{-simplex}$$

$\underbrace{\hspace{10em}}$
 $n+1$ points

④

Reduced Suspension:

$X \rightarrow \text{CW complex}$
 $\{x_0\} \rightarrow \text{base point}$

$$SX = (X \times I) / (X \times \{0\}) \cup (X \times \{1\})$$

$$\Sigma X = SX / (\{x_0\} \times I)$$

⑤

Criterion for Homotopy Equivalence

Recall:

Def: Homotopy Equivalence

A map $f: X \rightarrow Y$ is a homotopy equivalence if $\exists g: Y \rightarrow X$
s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

We say the spaces X and Y are homotopy equivalent
and

$$\boxed{X \simeq Y}$$

→ can prove easily that this is an equivalence relation.



Collapsing Subspaces

Theorem:

If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A , then the quotient map $X \rightarrow X/A$ is a homotopy equivalence.

Example

(1) Graphs



→ they are homotopy equivalent

→ collapsing the middle edge of A and C produces B

(b) Let X be a graph with finitely many vertices and edges.

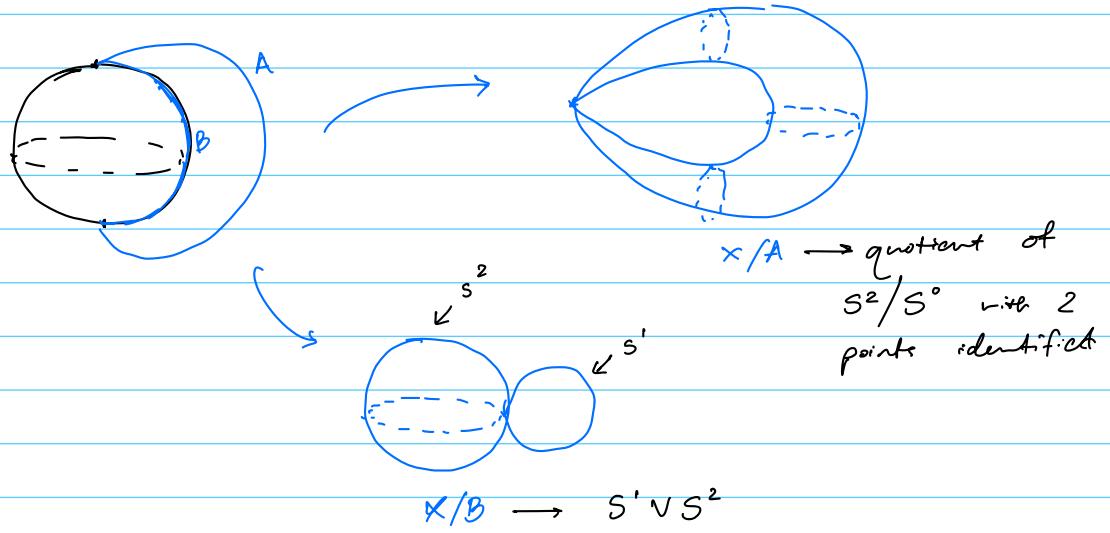
→ if the two endpoints of any edge are distinct, we can collapse it to a pt.



Leads to a homotopy equivalent graph with one less edge.

Can repeat until all edges are loops.

(2) $X \rightarrow S^2$ but attach 2 ends of an arc A to N and S pole



Reduced Suspension

$$\Sigma X \simeq SX$$