

Algebraic Topology

Notes

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These notes are a combination of notes from Allen Hatcher's book "Algebraic Topology" and Prof. Ciprian Manolescu's lectures from MATH 215a at Stanford University, and my own.

## Introduction

Our goal is to develop algebraic invariants associated with topological spaces.

We will look at

(1) Fundamental Group:

$$\pi_1(X) = \{\text{loops in } X\} / \text{homotopy}$$

(2) Homology Group:

$$H_n(X), n \in \mathbb{N} \text{ and abelian}$$

Intuitively, they count "holes" in  $X$

(3) Cohomology Group:

$$H^n(X) = \text{Dual to } H_n(X)$$

$\oplus H^n(X)$  is a ring!

## Basic Constructions

### Def : Homeomorphism

Let  $X$  and  $Y$  be topological spaces.

$f: X \rightarrow Y$  is a homeomorphism if  $f$  is bijective and both  $f$  and  $f^{-1}$  are continuous.

We say  $\boxed{X \cong Y}$

### Def : Homotopy

A family of maps,  $f_t: X \rightarrow Y$  where  $t \in I = [0, 1]$  s.t

the associated map  $F: X \times [0, 1] \rightarrow Y$  given by  $F(x, t) = f_t(x)$  is continuous.

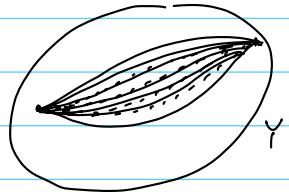
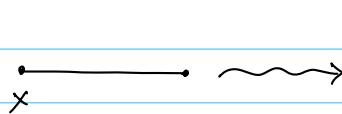
Two maps  $f_0, f_1: X \rightarrow Y$  are homotopic if there exists a homotopy  $F: X \times [0, 1] \rightarrow Y$  s.t

$$F(x, 0) = f_0(x) \quad \forall x \in X$$

$$F(x, 1) = f_1(x)$$

We say

$$\boxed{f_0 \simeq f_1}$$



### Def : Homotopy Equivalence

A map  $f: X \rightarrow Y$  is a homotopy equivalence if  $\exists g: Y \rightarrow X$  s.t  $f \circ g \simeq \text{id}_Y$  and  $g \circ f \simeq \text{id}_X$

We say the spaces  $X$  and  $Y$  are homotopy equivalent and

$$\boxed{X \simeq Y}$$

→ can prove easily that this is an equivalence relation.

### Examples of homotopy equivalence

(1)  $\mathbb{R}^n \simeq$  a point (even though  $\mathbb{R}^n \not\simeq$  a point)  
↑ infinite      ↑ finite

Why?

$$f: \mathbb{R}^n \rightarrow \{0\}$$

and take  $g: \{0\} \rightarrow \mathbb{R}^n$  by  $g(0) = 0$

Then  $f \circ g = \text{id}_{\{0\}}$  and  $(g \circ f)(x) = 0 \quad \forall x \in \mathbb{R}^n$

Now  $g \circ f \sim \text{id}_{\mathbb{R}^n}$  by  $f_t(x) = tx$  where  $f_0 = 0$  and  $f_1 = \text{id}_{\mathbb{R}^n}$

(2)  $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \simeq$  a point

$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \simeq$  a point

Def : Contractible

We say the space  $X$  is contractible if  $X \simeq$  point.



Equivalent definition : the identity map of  $X$  is nullhomotopic

homotopic to a constant map.

### Def : Retractions

Let  $X$  be a space and let  $A \subset X$ .

then, a retraction is a map  $r: X \rightarrow X$  s.t  
 $r(X) = A$  and  $r|_A = id_A$ .

### Def : Deformation Retraction

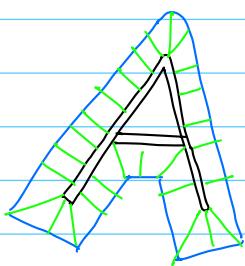
A deformation retraction of  $X$  onto a subspace  $A$  is  
a family of maps  $f_t: X \rightarrow X$ , with  $t \in I$  s.t  
 $f_0 = id_X$  and  $f_1(X) = A$  and  $f_t|_A = id_A$  for  $\forall t \in I$ .

The family  $f_t$  must also be continuous  
 $\rightarrow$  an example of a homotopy from  $id_X$  to a retraction of  $X$  onto  $A \subset X$ .

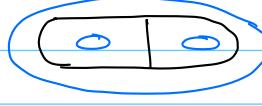
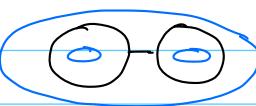
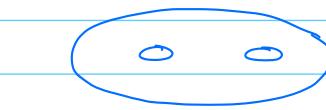
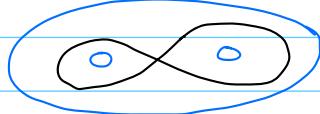
$\rightarrow$  in this case,  $A \cong X$  as  $f_0: A \rightarrow X$  by  $id_X$   
 $f_1: X \rightarrow A$  as above  
then  $f_0 \circ f_1 = id_X$   
and  $f_1 \circ f_0 = id_A$

### Examples of deformation retraction:

(1)



(2) Look at deformations of



(3)  $X = \mathbb{R}^2 - \{0\}$ .  $A = S^1$

$$(4) f(x, t) = (1-t)x + t \frac{x}{\|x\|}.$$

Proposition:

If  $X$  def. retracts to a point  $x \in X$ , then for any  $U \subset X$ ,  $x \in U$ .  
 $\exists V \subset U$  with  $x \in V$  s.t. the inclusion map  $V \hookrightarrow U$  is  
nullhomotopic.

homotopic to constant map

'Def: Deformation Retraction in the weak sense:

Let  $A \subset X$ .

Then, this is the homotopy  $f_t : X \rightarrow X$  s.t  $f_0 = \text{id}_X$   
and  $f_t(X) \subset A$  with  $f_t(A) \subset A$ ,  $\forall t \in I$ .

Lemma:

If  $X$  deformation retracts to  $A$  in the weak sense, then  
the inclusion  $A \hookrightarrow X$  is a homotopy equivalence.

Proof:

Let the weak def. ret be  $f_t$ .

Let  $i : A \hookrightarrow X$  by inclusion i.e.  $i(a) = a$ ,  $\forall a \in A$ .

Then,  $(i \circ f_t)(x) = i(f_t(x)) = f_t(x)$ ,  $\forall x \in X$

But  $f_t \simeq f_0 = \text{id}_X$

so,  $i \circ f \simeq \text{id}_X$

Also,  $(f_1 \circ i)(a) = f_1(i(a)) = f_1(a)$  ~~and~~  $\forall a \in A$

But  $f_1|_A \simeq f_0|_A = \text{id}_X|_A = \text{id}_A$

$\Rightarrow f_1 \circ i \simeq \text{id}_A$

Def: Mapping Cylinder ("the structure through which the def. retraction occurs")

for a map  $f: X \rightarrow Y$ , the mapping cylinder  $M_f$  is the quotient space of the disjoint union  $(X \times I) \sqcup Y$  obtained by the equivalence  $(x, 1) \in X \times I \sim f(x) \in Y$

$\nwarrow \nearrow$   
Make the endpoint of the deformation equivalent to the image of the map.

Mapping cylinders are continuous.

Def: Homotopy relative to A (homotopy rel. A)

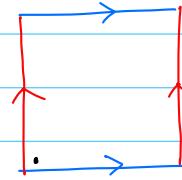
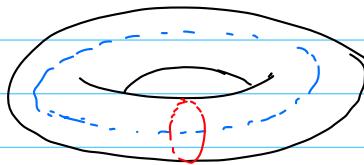
A homotopy  $f_t: X \rightarrow Y$  whose restriction to a subspace  $A \subset X$  is independent of  $t$ .

In other words,  $f_t$  is a homotopy and  $f_t|_A$  is independent of  $t$ .

$\rightarrow$  def. retraction of  $X$  onto  $A$  is a homotopy rel.  $A$  from  $\text{id}_X$  to a retraction of  $X$  onto  $A \subset X$ .

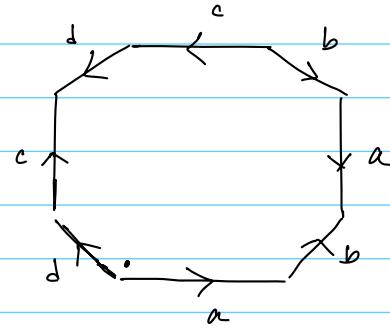
## Cell Complexes

Examples :



The torus  $S^1 \times S^1$  can be constructed from the square

Generally, an orientable surface  $M_g$  of genus  $g$  can be constructed from a polygon of  $4g$  sides by identifying pairs of edges.



2 cell: interior of a polygon which is an open disk

1 cell: an open interval like  $(0, 1)$

3 cell: an open ball.

$n$ -cell: open-disk

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

Def : Cell Complex (or CW complex)

A space constructed as follows:

- (1) Start with discrete set  $X^0 \rightarrow$  the points are 0-cells
- (2) Inductively, form the n-skeleton  $X^n$  from  $X^{n-1}$  by attaching n-cells  $e_\alpha^n$  via maps

$\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$ .

So,  $X^n$  is the quotient space of  $X^{n-1} \coprod_\alpha D_\alpha^n$  under the equivalence  $x \sim \varphi_\alpha(x) \forall x \in \partial D_\alpha^n$

$\nwarrow$  (n-1)-skeleton  $\nearrow$  n-disks

: i.e attach boundaries of the n-disk to the (n-1)-skeleton

$$\therefore X^n = X^{n-1} \coprod_\alpha e_\alpha^n \text{ where } e_\alpha^n \text{ is an open n-disk}$$

(3) Either stop this induction at a finite stage and set  $X = X^n$  for  $n < \infty$

or continue indefinitely, setting

$$X = \bigcup_n X^n$$

↳ in this case,  $X$  has the weak topology:

$A \subset X$  is open iff  $A \cap X^n$  is open in  $X^n$  for each  $n$

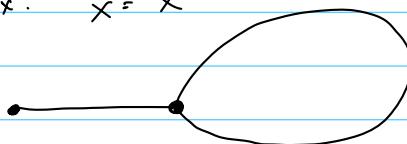
Vocabulary :

①  $X^n \rightarrow$  n-skeleton

② Dimension of  $X \rightarrow$  largest  $n$  s.t. an n-cell exists

### Examples of Cell Complexes:

(1) 1-dimensional cell complex:  $X = X^1$   
 (multigraphs)



(2) The sphere  $S^n$  has a cell complex with two cells,  $e^0$  and  $e^n$ , where  $e^n$  is attached by  $\varphi: S^{n-1} \rightarrow e^0$ .

$\therefore S^n$  is being regarded as the quotient space

$$D^n / \partial D^n$$

$$S^n = e^0 \cup e^n.$$

Alternatively,

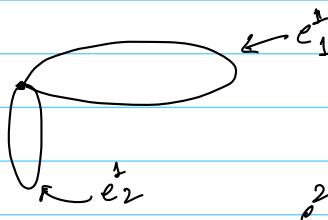
$$\begin{aligned} S^n &= S^{n-1} \cup e_+^n \cup e_-^n \\ &= e_+^0 \cup e_-^0 \cup \dots \cup e_+^n \cup e_-^n \end{aligned}$$

(3) Cell Complex of a torus:

Step 1:  $X^0$  is just a point  $\rightarrow \bullet \leftarrow e^0$

Step 2: Attach two 1-cells to this point

$$X^1 =$$



$$\therefore S^\infty = \bigcup_n S^n$$

Step 3: Attach a disk to  $X^1$  by attaching its boundary to  $X^1$ .

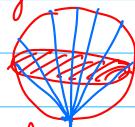
(4) Real Projective Space,  $\mathbb{R}P^n$

$$(\mathbb{R}^{n+1} - \{0\}) / (\nu \sim \lambda \nu, \forall \nu \in \mathbb{R}^{n+1}, \lambda \neq 0)$$

$\rightarrow$  Restricting to vectors of length 1,  $S^n / (\nu \sim -\nu)$

$\Rightarrow D^n$  with antipodal points of  $\partial D^n$  identified

To get this, think of



$\partial D^n$  with antipodal points equivalent is  $\mathbb{R}P^{n-1}$

$\therefore \mathbb{R}P^n$  can be formed from  $\mathbb{R}P^{n-1}$  by attaching an  $n$ -cell. and the attaching map  $\varphi: S^{n-1} \rightarrow \mathbb{R}P^{n-1}$

$\therefore \mathbb{R}P^n$  has the cell complex structure  $e^0 \cup e^1 \cup \dots \cup e^n$  with one cell  $e^i$  in each dimension  $i \leq n$ .

i.e. for the upper hemisphere's points, find where the line to south pole intersects with  $D^n$

## (5) Complex Projective Space. $\mathbb{C}P^n$

Space of all complex lines through the origin in  $\mathbb{C}^{n+1}$

$$\text{i.e. } \mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\}) / (v \sim \lambda v, \forall v \in \mathbb{C}^{n+1}, \lambda \neq 0)$$

Equivalent to  $S^{2n+1} / (v \sim \lambda v, |\lambda|=1)$  ( $S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ )

Equivalent to  $D^{2n} / (v \sim \lambda v, v \in \partial D^{2n})$

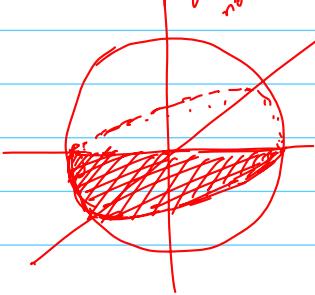
↳ Why?

$S^{2n+1} \subset \mathbb{C}^{n+1}$  → consider vectors in  $\mathbb{C}^{n+1}$  whose last coordinate is ~~one~~ real.

and non-negative

These vectors are of the form  $(w, \sqrt{1-w^2}) \in \mathbb{C}^n \times \mathbb{C}$  with  $|w| \leq 1$

They form the graph of the function  $w \mapsto \sqrt{1-w^2}$   
with  $|w| \leq 1, w \in \mathbb{C}^n$



Note:  $w \in \mathbb{C}^n$  and  $|w| \leq 1 \Rightarrow w \in D^{2n}$

This is a disk  $D^{2n}_+$  bounded by the spheres  $S^{2n-1}$ .

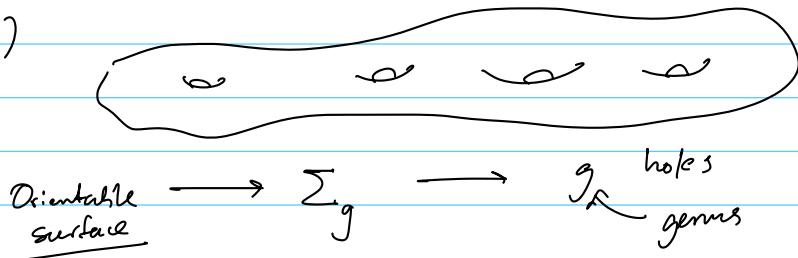
By adding another dimension and viewing them as  $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ , we ~~view~~ them as vectors in  $(D^{2n}_+, 0)$  bounded by  $S^{2n-1} \subset S^{2n+1}$

Now, each vector in  $S^{2n+1}$  is equivalent to a vector in  $D^{2n}_+$  by identifying  $v \sim \lambda v$ . In particular, if the last coordinate is zero, we have  $v \sim \lambda v, v \in S^{2n-1}$ .

$\therefore \mathbb{C}P^n$  is obtained from  $\mathbb{C}P^{n-1}$  by attaching a cell  $e^{2n}$  using the attaching map  $\varphi : S^{2n-1} \rightarrow \mathbb{C}P^{n-1}$

$\therefore \mathbb{C}P^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$  with cells only in even dimensions

(6)

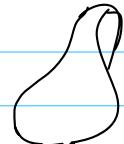


$\Sigma_g$   $\xrightarrow{\text{holes}}$   
g genus

Can be constructed from a  $4g$  polygon

↳ Start with one  $e^o$

(7)

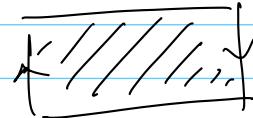
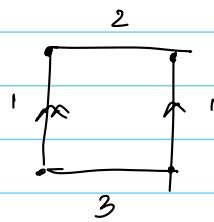


Non-orientable  
surface  $\longrightarrow N_g$

Eg:  $N_2 \longrightarrow$  Klein bottle  
 $N_1 \longrightarrow RP^2$

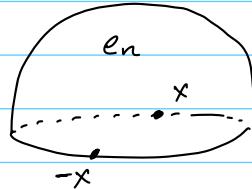
(8) Annulus:

(9) Möbius band



(a)  $RP^n$  revisited

$$RP^n = S^n / (x \sim -x, \forall x)$$



$$\Rightarrow RP^n = RP^{n-1} \cup e^n$$

$$\therefore RP^n = e^0 \cup e^1 \cup \dots \cup e^n$$

$$\text{Then, } RP^\infty = e^0 \cup e^1 \cup e^2 \cup \dots = \bigcup_n RP^n$$

(i)  $\mathbb{C}\mathbb{P}^n$  revisited

$$\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / (x \sim \lambda x, \lambda \in \mathbb{C}^*)$$

$$\therefore z \sim \frac{z}{\|z\|} \Rightarrow \mathbb{C}\mathbb{P}^n \cong S^{2n+1} / (z \sim \lambda z, \lambda \in S^1)$$

Divide everything by  $x_1$ , i.e. last coordinate in  $\mathbb{R}_{\geq 0}$

$$z = \underbrace{(z_0, \dots, z_n)}_{w} \underbrace{z_{n+1}}_{\sqrt{1-\|w\|^2}}$$

with  $\|w\| \leq 1$

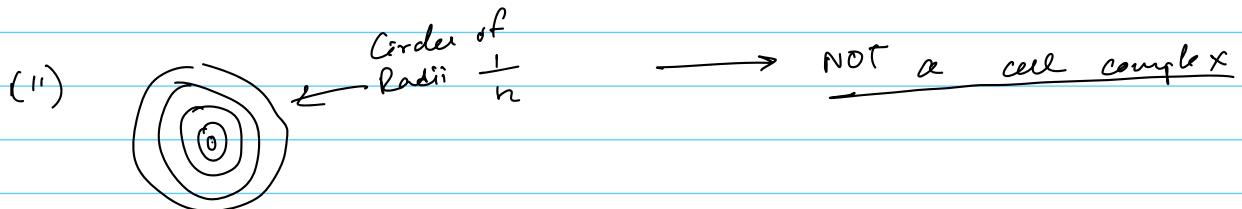
$$D_+^{2n} = \text{graph } (w \mapsto \sqrt{1-\|w\|^2})$$

$$\therefore \mathbb{C}\mathbb{P}^n = D_+^{2n} / (w \sim \lambda w \text{ if } w \in S^{2n-1})$$

$$= \mathbb{C}^{2n} \cup \left( S^{2n-1} / (w \sim \lambda w) \right)$$

$$= \mathbb{C}\mathbb{P}^{n-1} \cup \mathbb{C}^{2n}$$

$$= e^0 \cup e^2 \cup \dots \cup e^{2n}$$



## Properties of CW Complexes

- (1) They are normal ( $\therefore$  also Hausdorff)
- (2) Any finite cell complex is compact
- (3) A compact subspace of a cell cx is contained in a finite subcomplex
- (4) Closure finiteness  $\rightarrow$  The closure of each cell  $\ell$  meets only finitely many cells.
- (5) Locally contractible:  
 $\forall x \in X, \exists x \text{ open}, \exists V \subset U \text{ with } x \in V$   
s.t.  $V$  is contractible

(6)

Recall:

Top manifolds  $\rightarrow$  2<sup>nd</sup> Countable, Hausdorff, locally Euclidean  
Smooth manifolds  $\rightarrow$

Theorem: Every smooth manifold is homeomorphic to a cell complex.

Theorem: Every topological manifold is homotopy equivalent to a cell complex.

Theorem: Every top manifold of dimension  $\neq 4$  is homeomorphic to a cell complex  
(unknown in dim 4)

Def: Characteristic Map

Each cell  $e_\alpha^n$  in a cell complex  $X$  has a characteristic map

$$\varphi_\alpha : D_\alpha^n \xrightarrow{\sim} X$$

which extends the attaching map  $\varphi_\alpha$  and is a homeomorphism from the interior of  $D_\alpha^n$  onto  $e_\alpha^n$

→  $\varphi_\alpha$  is the composition

$$D_\alpha^n \hookrightarrow X^{n-1} \coprod_\alpha D_\alpha^n \xrightarrow{\qquad\qquad\qquad} X^n \hookrightarrow X$$

↑  
the quotient  
map that  
defines  $X^n$

Example of characteristic map:

(i) Recall:  $S^n$  can be constructed by two cells:  $e^0$  and  $e^n$  ← just one point

where  $e^n$  is attached to  $e^0$  by

$$\varphi_\alpha : S^{n-1} \rightarrow e^0$$

Then, the characteristic map of  $e^n$  is

$$\varphi_\alpha : D_\alpha^n \rightarrow S^n \text{ which collapses } \partial D_\alpha^n \text{ to } e^0$$

Def: Subcomplex

A subcomplex of a cell complex  $X$  is a closed subspace  $A \subset X$  that is a union of cells of  $X$ .

$\rightarrow$  As  $A$  is closed, for each cell in  $A$ ,

the image of its characteristic map } contained in  $A$   
the image of its attaching map }

$\therefore A$  is a cell complex as well

Def : CW pair

A cell complex  $X$  and a subcomplex  $A$  forms a pair  $(X, A)$

Example of subcomplex

$\rightarrow$  Each skeleton,  $X^n$ , is a subcomplex.

$\rightarrow$  in  $\mathbb{R}\mathbb{P}^n$  and  $\mathbb{C}\mathbb{P}^n$ , the only subcomplexes are  $\mathbb{R}\mathbb{P}^k$  and  $\mathbb{C}\mathbb{P}^k$ ,  $\forall k \leq n$

Properties of subcomplexes

(i) Closure of a collection of cells is a subcomplex.

## Operations on Spaces

### Products

$X, Y \rightarrow \text{cell complexes}$

$X \times Y \rightarrow \text{cell complex with the cells } e_\alpha^m \times e_\beta^n$

cells of X      cells of Y

### Quotients

Given  $(X, A)$  a CW pair,  
 ↗ cell complex  
 ↗ cell subcomplex

$\longrightarrow A \subset X \rightsquigarrow X/A$

the quotient space  $X/A$  also has a cell complex structure:  
 → the cells of  $X/A$  are the cells of  $X-A$  and  
 a new 0-cell which is the image of  
 $A$  in  $X/A$ .

→ for a cell  $e_\alpha^n$  of  $X-A$  attached by  
 $\varphi_\alpha: S^{n-1} \rightarrow X^{n-1}$ , the attaching map for the  
 corresponding cell in  $X/A$  is the composition

$$S^{n-1} \rightarrow X^{n-1} \rightarrow X^{n-1}/A^{n-1}$$

Eg: ①  $D^n/S^{n-1} = S^n$

### Wedge Sum (for based spaces)

Given spaces  $X$  and  $Y$  with chosen points  $x_0 \in X$  and  $y_0 \in Y$ ,  
 the wedge sum  $X \vee Y$  is the quotient of  $X \coprod Y$  by  
 identifying  $x_0$  and  $y_0$  to a single point

→ Example:  $S^1 \vee S^1 = \infty$

$X \vee Y = X \coprod Y / (x_0 \sim y_0)$

→  $\bigvee_\alpha X_\alpha$  for an arbitrary collection of spaces  $X_\alpha$ :  
 start with  $\coprod_\alpha X_\alpha$  and then identify  $x_\alpha \in X_\alpha$   
 to one point.

→ If  $X_\alpha$  are cell complexes and the points  $x_\alpha$   
 are 0-cells, then  $\bigvee_\alpha X_\alpha$  is a cell complex  
 because we obtain it from the cell complex  $\coprod_\alpha X_\alpha$  and attach by

collapsing a subcomplex to a point.

→ For a cell complex  $X$ , the quotient  $X^n/X^{n-1}$  is a wedge sum of  $n$ -spheres  $\bigvee_{\alpha} S_{\alpha}^n$  with one sphere for each  $n$ -cell of  $X$

7) Smash Product  $X \wedge Y = (X \times Y) / ((x_0 \times Y) \cup (X \times y_0))$

Inside the product space  $X \times Y$ , there are copies of  $X$  and  $Y$ :  $X \times \{y_0\}$  and  $\{x_0\} \times Y$  for points  $y_0 \in Y$  and  $x_0 \in X$ .

These copies of  $X$  and  $Y$  intersect only at  $(x_0, y_0)$  so their union can be identified with the wedge sum  $X \vee Y$

$$\begin{aligned} \text{i.e. } (X \times \{y_0\}) \vee (\{x_0\} \times Y) &= X \vee Y \\ &= (X \amalg Y) / (x_0 \sim y_0) \end{aligned}$$

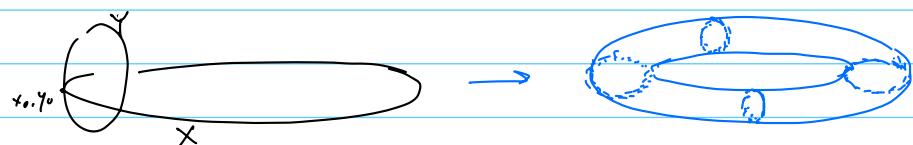
The smash product  $X \wedge Y$  is the quotient  $X \times Y / X \vee Y$

↳ i.e. we are collapsing away the separate factors  $X$  and  $Y$ .

Eg:  $S^1 \wedge S^1 = S^2$   $\longrightarrow$   $S^1 = I / (0 \sim 1)$   
 $S^m \wedge S^n = S^{m+n}$

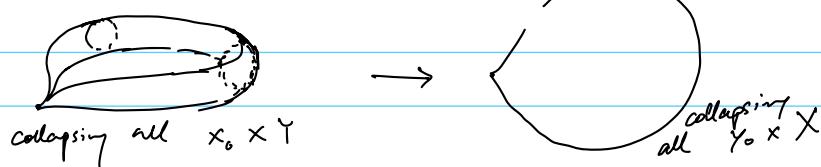
Why?

Firstly,  $S^1 \times S^1$  results in a torus  $T^2$



Secondly,  $S^1 \wedge S^1 = \infty$

Then, quotienting:



II

### Suspension

for a space  $X$ , the suspension  $SX$  is the quotient of  $X \times I$  by collapsing  $X \times \{0\}$  to a point and  $X \times \{1\}$  to another.

#### Example

(i)  $X = S^n$

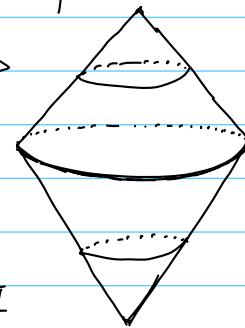
$SX = S^{n+1}$  with the two suspension points at North and South of  $S^{n+1}$

→ We can suspend maps too

$$f: X \rightarrow Y \rightsquigarrow Sf: SX \rightarrow SY$$

which is the quotient map of

$$f \times 1 : X \times I \rightarrow Y \times I$$



III

### Cone

$$CX = (X \times I) / (X \times \{0\})$$



→ If  $X$  is a CW complex, then so are  $SX$  and  $CX$  as quotients of  $X \times I$  with its product cell structure with  $I$  given the standard cell structure of ~~two~~ two 0-cells joined by one 1-cell.

7

### Join

Given  $X$  and  $Y$ , we can define the space of all line segments joining points in  $X$  to points in  $Y$ .

$$X * Y = (X \times Y \times I) / \left( \begin{array}{l} (x, y_1, 0) \sim (x, y_2, 0) \quad \forall x, x_1, x_2 \in X \\ (x_1, y_1, 1) \sim (x_2, y_1, 1) \quad \forall y, y_1, y_2 \in Y \end{array} \right)$$

$$\rightarrow pt * pt \longrightarrow \bullet \longrightarrow$$

$$pt * pt * pt \longrightarrow \begin{array}{c} \diagup \\ \diagdown \end{array}$$

$$pt * pt * \dots * pt = \Delta^n \rightarrow n\text{-simplex}$$

$\underbrace{\hspace{10em}}$   
 $n+1$  points

④

Reduced Suspension:

$X \rightarrow \text{CW complex}$   
 $\{x_0\} \rightarrow \text{base point}$

$$SX = (X \times I) / (X \times \{0\}) \cup (X \times \{1\})$$

$$\Sigma X = SX / (\{x_0\} \times I)$$

⑤

## Criterion for Homotopy Equivalence

Recall:

Def: Homotopy Equivalence

A map  $f: X \rightarrow Y$  is a homotopy equivalence if  $\exists g: Y \rightarrow X$   
s.t.  $f \circ g \simeq id_Y$  and  $g \circ f \simeq id_X$

We say the spaces  $X$  and  $Y$  are homotopy equivalent  
and

$$X \simeq Y$$

→ can prove easily that this is an equivalence relation.



## Collapsing Subspaces

Theorem:

If  $(X, A)$  is a CW pair consisting of a CW complex  $X$  and a contractible subcomplex  $A$ , then the quotient map  $X \rightarrow X/A$  is a homotopy equivalence.

Example

(1) Graphs



→ they are homotopy equivalent

→ collapsing the middle edge of A and C produces B

(b) Let  $X$  be a graph with finitely many vertices and edges.

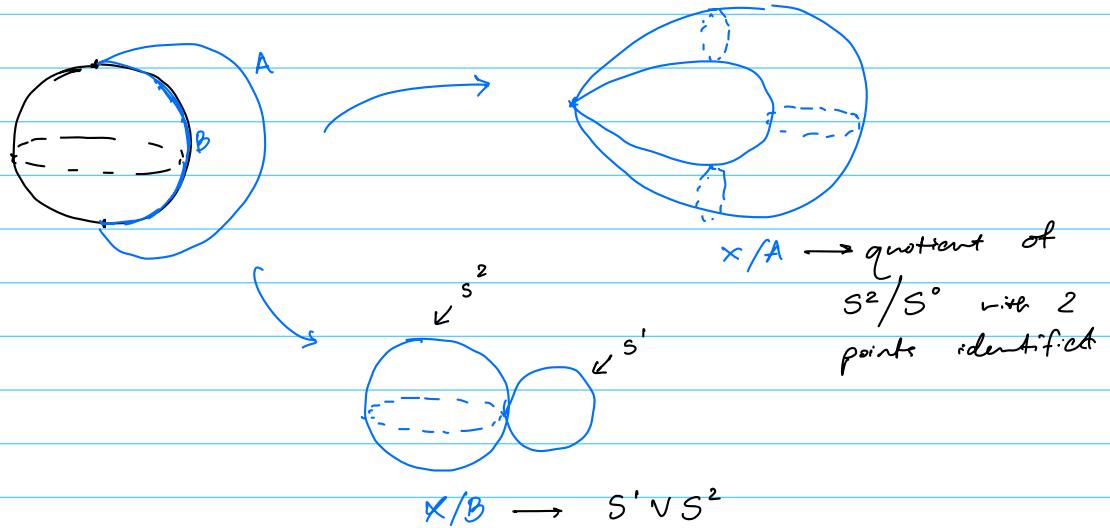
→ if the two endpoints of any edge are distinct, we can collapse it to a pt.



Leads to a homotopy equivalent graph with one less edge.

Can repeat until all edges are loops.

(2)  $X \rightarrow S^2$  but attach 2 ends of an arc  $A$  to  $N$  and  $S$  pole



### 7.1 Reduced Suspension

$$\Sigma X \cong SX$$



#### Attaching spaces

Start with space  $X_0$  and another space  $X_1$ , which we will attach to  $X_0$  by identifying points in a subspace  $A \subset X_1$ , with points of  $X_0$ .

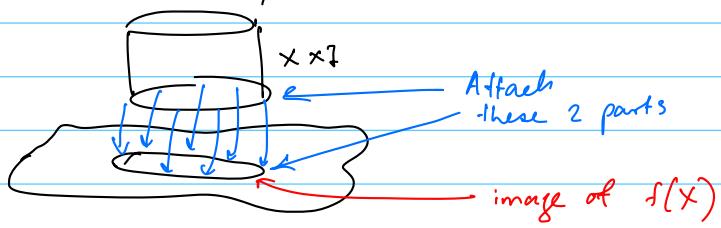
We do this using a map  $f: A \hookrightarrow X_0$  and then forming a quotient space of  $X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A)$

We denote

$$X_0 \sqcup_f X_1 = X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A) \text{ where } f: A \hookrightarrow X_0, A \subset X_1$$

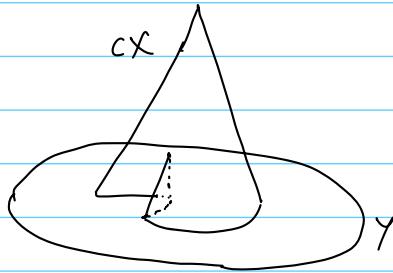
Example :

- (1) Mapping cylinder of a map  $f: X \rightarrow Y$  is  $M_f \rightarrow$  the space obtained from  $Y$  by attaching  $X \times I$  along  $X \times \{1\}$  via  $f$ .



- (2) Mapping Cone  $\rightarrow C_f = Y \sqcup_f CX$  where  $CX$  is the cone  $(X \times I) / (X \times \{0\})$

and we attach this to  $Y$  along  $X \times \{1\}$   
via  $(x, 1) \sim f(x)$



Example :  $X = S^{n-1}$

$C_f \rightarrow$  attaching to  $Y$  the  $n$ -cell  
via  $f: S^{n-1} \rightarrow Y$

Proposition

If  $(X_1, A)$  is a CW pair and the two attaching maps  
 $f, g: A \rightarrow X_0$  are homotopic, then  
 $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$

## Homotopy Extension Property

Intuition:

Consider the map  $f_0 : X \rightarrow Y$ . Let  $A \subset X$  and consider the homotopy on  $A$   $f_t : A \rightarrow Y$  with  $f_0 = f|_A$ . We would like to extend this to a homotopy on  $X$  as a whole with  $f_t$ .

Def: Homotopy Extension

$A \subset X$

$(X, A)$  has the homotopy extension property (h.e.p)

if  $\forall Y, \forall f_0 : X \rightarrow Y, \forall$  homotopy  $g : A \times I \rightarrow Y,$   
 $g(a, 0) = f_0(a)$

we can extend  $g$  to a homotopy  $F : X \times I \rightarrow Y$

$$\text{i.e. } F(x, 0) = f_0(x)$$



$$f_0(x) = y$$

$(X, A)$  has the h.e.p if every pair of maps  $X \times \{0\} \rightarrow Y$  and

$A \times I \rightarrow Y$  that agree on  $A \times \{0\}$  can be extended to

$$f_t(a)$$

a map  $X \times I \rightarrow Y$

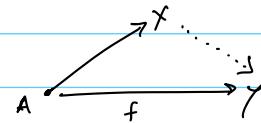
$$\hookrightarrow f_t : X \rightarrow Y$$

$$f_0 = f|_A$$

Lemma:

$A \subset X$  top space.

$\forall Y, \text{ any map } f : A \rightarrow Y \text{ extends to } X \rightarrow Y \text{ if and only if } A$   
 $\text{is a retract of } X$



Proof:

$\Leftarrow$  Suppose  $A$  is a retract of  $X$  via  $r : X \rightarrow A$  s.t.  $r|_A = \text{id}_A$   
 Then  $(f \circ r) : X \rightarrow Y$  is our extension

$\Rightarrow$  Suppose,  $\forall Y$  and any map  $f : A \rightarrow Y$  extends to  $X \rightarrow Y$ .  
 i.e.  $f_t : X \rightarrow Y$  s.t.  $f|_A = f$

Then, let  $Y = A$  and  $f = \text{id}_A$  i.e.  $\text{id}_A : A \rightarrow A$  extends  
 to  $f_t : X \rightarrow A$  s.t.  $f|_A = \text{id}_A \Rightarrow A$  is a retract of  $X$

Lemma :

A pair  $(X, A)$  has the h.e.p if and only if  $X \times \{0\} \cup A \times I$  is a retract of  $X \times I$ .

Proof : By hypothesis the identity map

$$\Rightarrow : X \times \{0\} \cup A \times I \hookrightarrow X \times \{0\} \cup A \times I \text{ extends to a map}$$

$$X \times I \hookrightarrow X \times \{0\} \cup A \times I$$

$\therefore X \times \{0\} \cup A \times I$  is a retract of  $X \times I$ .

$\Leftarrow$  if  $A$  is closed: consider any two maps  $X \times \{0\} \hookrightarrow Y$  and  $A \times I \hookrightarrow Y$  that agree on  $A \times \{0\}$ . They combine to give a map  $X \times \{0\} \cup A \times I \hookrightarrow Y$  which is continuous by continuity on the closed sets  $X \times \{0\}$  and  $A \times I$ .

Compose this map  $X \times \{0\} \cup A \times I \hookrightarrow Y$  with a retraction  $X \times I \hookrightarrow X \times \{0\} \cup A \times I$  (we have this via hypothesis)

We get an extension  $X \times I \hookrightarrow Y$

$\therefore (X, A)$  has the h.e.p.

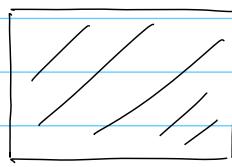
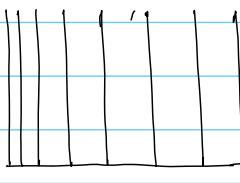
Properties

$$(1) \begin{array}{l} \text{H.e.p} \\ X - \text{normal iff} \end{array} \} \Rightarrow A \text{ is closed in } X$$

Non-example:  $(X, A)$  does not have h.e.p

(1)  $(I, A)$  where  $A = \{0, 1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\}$

There is no continuous retraction  $I \times I \hookrightarrow I \times \{0\} \cup A \times I$  because of the structure of  $(I, A)$  near 0.



Consider the ball  $B = B(x_0, \epsilon)$   
Then  $\exists \delta > 0$  s.t.  $B(x_0, \delta) \subset B$

$\gamma$  — path in  $B$  from  $x_0$  to  $x_1$

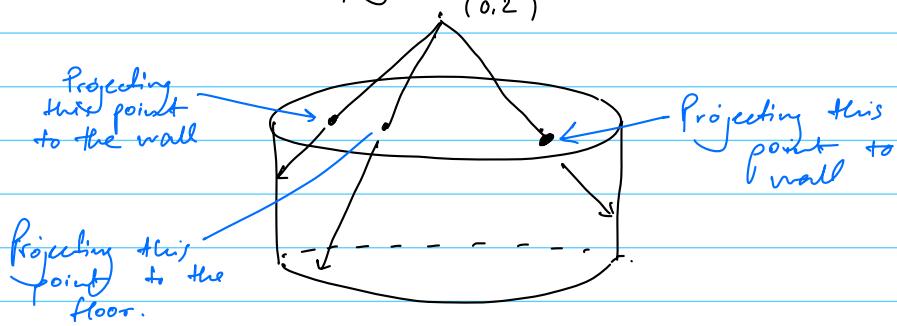
but  $x_0$  and  $x_1$  are in diff components  
 $\downarrow$  path at  $t=1$   $B(x_0, \delta)$  of  $C \cap B$

Proposition

If  $(X, A)$  is a CW pair, then  $X \times \{0\} \cup A \times I$  is a deformation retract of  $X \times I$ , hence  $(X, A)$  has the h.e.p.

Proof :

First, note that  $\exists$  a retraction  $r: D^n \times I \rightarrow D^n \times \{0\} \cup \partial D^n \times I$  for ex - radial projection from the point  $(0, 2) \in D^n \times \mathbb{R}$



Now, set  $r_t = tr + (1-t)\mathbb{1}$  is a deformation retraction of  $D^n \times I$  onto  $D^n \times \{0\} \cup \partial D^n \times I$ .

Now, with this, we have a deformation retraction of  $X^n \times I$  onto  $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$  since  $X^n \times I$  is obtained from  $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$  by attaching copies of  $D^n \times I$  along  $D^n \times \{0\} \cup \partial D^n \times I$ .

If we perform the def. ret. of  $X^n \times I$  onto  $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$  during the  $t$ -interval  $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$ , this infinite concatenation of homotopies is a def. ret. of  $X \times I$  onto  $X \times \{0\} \cup A \times I$ .

### Proposition

If the pair  $(X, A)$  satisfies h.e.p and  $A$  is contractible, then the quotient map  $q: X \rightarrow X/A$  is a homotopy equivalence.

### Proof :

Let  $f_t: X \rightarrow X$  be the homotopy extending a contraction of  $A$  with  $f_0 = \text{id}$ .

Now,  $f_t(A) \subset A \quad \forall t$ , so the composition

$$q \circ f_t: X \rightarrow X/A$$

sends  $A$  to a point and so factors as a composition

$$X \xrightarrow{q} X/A \longrightarrow X/A$$



Denote this by  $\bar{f}_t: X/A \rightarrow X/A$ )

$$\text{So, } q \bar{f}_t = \bar{f}_t q$$

$$X \xrightarrow{\bar{f}_t} X$$

$$\begin{array}{ccc} & & \\ q & \downarrow & \downarrow q \\ X/A & \xrightarrow{\bar{f}_t} & X/A \end{array}$$

When  $t=1$ ,  $f_1(A)$  equals to a point (since  $f_t$  is homotopy extension of the contraction of  $A$ ), so  $f_1$  induces a map  $g: X/A \rightarrow X$  with  $gq = f_1$

$$\begin{array}{ccc} & f_1 & \\ X & \xrightarrow{\quad} & X \\ q \downarrow & \nearrow g & \downarrow q \\ X/A & \xrightarrow{\bar{f}_1} & X/A \end{array}$$

$$\begin{aligned} \text{So, } qg &= \bar{f}_1 & \text{since } qg(\bar{x}) &= qg(x) \\ & & &= qf_1(x) \\ & & &= f_1(q(x)) \\ & & &= f_1(\bar{x}) \end{aligned}$$

The maps  $g$  and  $q$  are inverse homotopy equivalences as

$$gq = f_1 \simeq f_0 = 1 \text{ via } f_t \text{ and}$$

$$qg = f_1 \simeq \overline{f_0} = 1 \text{ via } \overline{f_t}.$$

Def:  $W \simeq Z \text{ rel } Y$

for  $(W, Y)$  and  $(Z, Y)$ , there are maps  $\varphi: W \rightarrow Z$  and  $\psi: Z \rightarrow W$  restricting to identity on  $Y$  s.t.  $\psi\varphi \simeq 1_W$  and  $\varphi\psi \simeq 1_Z$  via homotopies that restrict to the identity on  $Y$  at all times.

Proposition

If  $(X_1, A)$  is a CW pair and we have attaching maps  $f, g: A \hookrightarrow X_0$  that are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

Proof:

Let  $F: A \times I \rightarrow X_0$  is a homotopy from  $f$  to  $g$ , consider the space  $X_0 \sqcup_F (X_1 \times I)$ , which has both  $X_0 \sqcup_f X_1$  and  $X_0 \sqcup_g X_1$  as subspaces

We can deformation retract  $X_1 \times I$  onto  $X_1 \times \{0\} \cup A \times I$  which induces a def retraction of  $X_0 \sqcup_F (X_1 \times I)$  onto  $X_0 \sqcup_f X_1$

Similarly,  $X_0 \sqcup_f (X_1 \times I)$  def retracts onto  $X_0 \sqcup_g X_1$

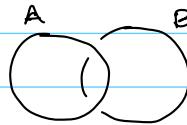
Both of them are identity on  $X_0$  so we get the homotopy equivalence

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

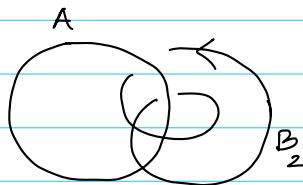
## Fundamental Group

### Intuition

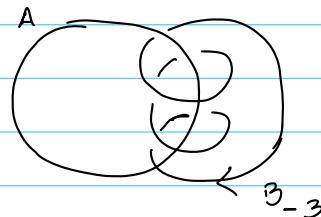
Two linked circles in  $\mathbb{R}^3$  :



Link B with A two times  
in the forward direction :



Link B with A three times  
in the backward direction :

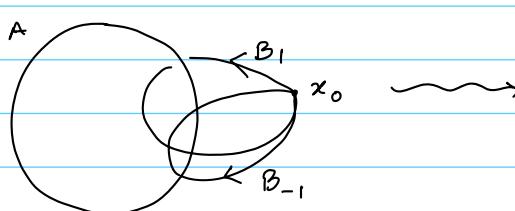
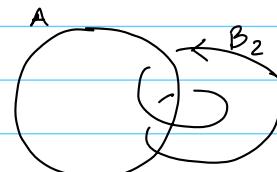
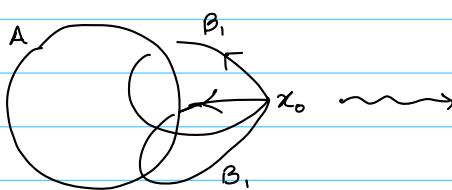


$B_2$  and  $B_{-3}$  are oriented circles/loops.

Two loops,  $B$  and  $B'$ , starting and ending at the same point  $x_0$  can be added to form a new loop that travels around  $B$  and  $B'$ .

$$\text{So, } B_1 + B_1 = B_2$$

$$B_1 + B_{-1} = B_0 \leftarrow \text{unlinked from A}$$



$$\text{More generally, } B_m + B_n = B_{m+n}$$

## Paths and Homotopy of paths

Def: Path in  $X$

A continuous map  $f: I \rightarrow X$  where  $I = [0, 1]$

→ Notice that any path is nullhomotopic.

Def: Homotopy

A family  $f_t: I \rightarrow X$  where  $t \in I$  s.t

(1)  $f_t(0) = x_0$  and  $f_t(1) = x_1 \forall t$

(2) The associated map  $f: I \times I \rightarrow X$   
is continuous

We say  $f_b \simeq f_1$ .

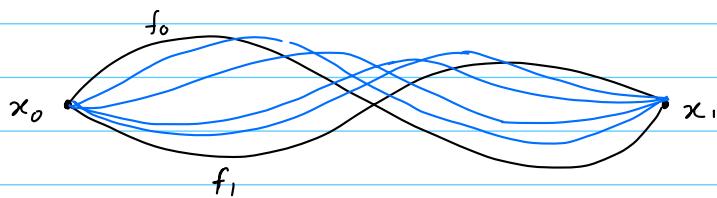
→  $f_0 \simeq f_1$  means homotopic rel.  $\partial I = \{0, 1\}$  as the endpoints are fixed.

Examples

(1) Linear homotopies:

Any 2 paths  $f_0$  and  $f_1$  in  $\mathbb{R}^n$  with endpoints  $x_0$  and  $x_1$ ,  
are homotopic by  $f_t(x) = (1-t)f_0(x) + tf_1(x)$

Here,  $F(x, t) = f_t(x) = (1-t)f_0(x) + tf_1(x)$  is continuous  
since  $f_0$  and  $f_1$  are continuous, and sum & and scalar  
multiplication preserve continuity.



Non-example

$$f_0, f_1 : I \rightarrow S^1$$

$$\left. \begin{array}{l} f_0(t) = 1 \\ f_1(t) = e^{2\pi i t} \end{array} \right\} \text{They are not path homotopic}$$

### Proposition

The relation of homotopy on paths with fixed endpoints

in any space is an equivalence relation.

We denote the equivalence class of  $f$  by  $[f]$  and is called the homotopy class of  $f$ .

### Proof:

Reflexivity:  $f \simeq f$  by homotopy  $f_t = f$

Symmetry: If  $f_0 \simeq f_1$  via  $f_t$ , then  $f_1 \simeq f_0$  via  $f_{1-t}$ .

Transitivity: Suppose  $f_0 \simeq f_1$  via  $f_t$ . and if  $f_1 = g_0$  with  $g_0 \simeq g_1$  via  $g_t$ , then the homotopy

$$h_t = \begin{cases} f_{2t}, & t \in [0, \frac{1}{2}] \\ g_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}$$

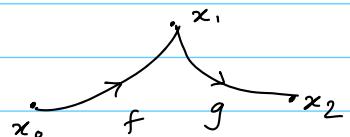
The associated function  $H(s, t) = h_t(s)$  is continuous.

A function on the union of 2 closed sets is continuous if it is continuous restricted to each of the 2 sets separately.

### Def: Product path

Given two paths  $f, g: I \rightarrow X$  s.t  $f(1) = g(0)$ , the product path  $f \cdot g$  first traverses  $f$  and then  $g$ :

$$f \cdot g(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$



This product path preserves homotopy classes:

if  $f_0 \simeq f_1$  and  $g_0 \simeq g_1$  via  $f_t$  and  $g_t$  homotopies respectively

and if  $f_0(1) = g_0(0)$  so that  $f \cdot g_0$  is well-defined

then  $f_t \cdot g_t$  provides the homotopy

$$f \cdot g_0 \simeq f \cdot g_1$$

Def : Loop

Paths  $f: I \rightarrow X$  s.t  $f(0) = f(1) = x_0 \in X$

$x_0 \rightarrow \text{basepoint}$

→ The set of all homotopy classes  $[f]$  of loops  $f: I \rightarrow X$  at the basepoint  $x_0 \in X$  is denoted  $\pi_1(X, x_0)$

Proposition :

$\pi_1(X, x_0)$  is a group w.r.t the product  
 $[f][g] = [f \cdot g]$

This group is called the fundamental group of  $X$  at basepoint  $x_0$ .

Proof :

Since the basepoint  $x_0 \in X$  is fixed, the product of any two paths,  $f$  and  $g$  in  $\pi_1(X, x_0)$  is defined.

Firstly, define reparametrisation of a path  $f$  to be a composition  $f\varphi$  where  $\varphi: I \rightarrow X$  is a continuous map r.t  $\varphi(0) = 0$  and  $\varphi(1) = 1$ .

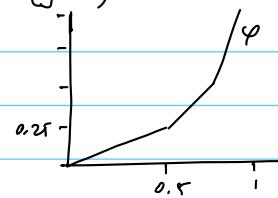
Reparametrisation preserves homotopy class of  $f$  since  $f\varphi \simeq f$  via homotopy  $f\varphi_t$  where  $\varphi_t(x) = (1-t)\varphi(x) + tx$  so  $\varphi_0(x) = \varphi(x)$  and  $\varphi_1(x) = x$

We often show that  $f$  is a reparametrisation of  $g$  to prove  $f \simeq g$ .

Given the paths  $f, g$  and  $h$  with  $f(1) = g(0)$  and  $g(1) = h(0)$ , then both  $(f \cdot g) \cdot h$  and  $f \cdot (g \cdot h)$  are defined.

Note  $(f \cdot g) \cdot h$  is a reparametrisation of  $f \cdot (g \cdot h)$  via  $f \cdot (g \cdot h) = (f \cdot g) \cdot h \varphi$  where  $\varphi$  is a continuous map s.t  $\varphi: [0, \frac{1}{2}] \rightarrow [0, \frac{1}{4}]$   $\varphi: [\frac{1}{2}, 1] \rightarrow [\frac{1}{4}, 1]$

$$\text{So, } (f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$$



Given a path  $f: I \hookrightarrow X$ , let  $c$  be the constant path at  $f(1)$  defined by  $c(s) = f(1)$ ,  $\forall s \in I$ . Then,  $f \cdot c$  is a reparametrisation of  $f$ :

$$f \cdot c(x) = \begin{cases} f(2x) & \text{for } x \in [0, \frac{1}{2}] \\ c(2x-1) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$\text{So, } f \cdot c = f \varphi \text{ where } \varphi: [0, \frac{1}{2}] \longrightarrow [0, 1]$$

$$\varphi: [\frac{1}{2}, 1] \longrightarrow \text{---}$$

$$\therefore f \cdot c \simeq f$$

Similarly  $c \cdot f \simeq f$  where  $c$  is constant path at  $f(0)$ .

Taking  $f$  to be a loop, the homotopy class of the constant path is a two-sided identity.

Now, let  $f$  be a path from  $x_0$  to  $x_1$ . Its inverse path is  $\bar{f}$  from  $x_1$  to  $x_0$  defined by  $\bar{f}(s) = f(1-s)$

Then,  $f \cdot \bar{f}$  is homotopic to a constant path via homotopy

$$h_t = f_t \cdot g_t$$

where  $f_t = f$  on  $[0, 1-t]$  and  $f_t = f(1-t)$  on  $[1-t, 1]$

$$\text{and } g_t = \bar{f}_t$$

Then,  $f_0 = f$  and  $f_1 = \text{constant path } c \text{ at } x_0$

So,  $h_t$  is a homotopy from  $f \cdot \bar{f}$  to  $c \cdot \bar{c}$

$$\text{as } h_0 = f_0 \cdot g_0 = \begin{cases} f & \text{for } x \in [0, \frac{1}{2}] \\ \bar{f} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$h_1 = f_1 \cdot g_1 = \begin{cases} c, & x \in [0, \frac{1}{2}] \\ \bar{c}, & x \in [\frac{1}{2}, 1] \end{cases}$$

$$\therefore f \cdot \bar{f} \simeq c \quad (\text{defining } c \cdot \bar{c} = c) \text{ where } c = x_0$$

Replacing  $f$  by  $\bar{f}$  gives  $\bar{f} \cdot f = c$

Take  $f$  to be the loop at  $x_0$ , then  $[\bar{f}]$  is a 2-sided inverse for  $[f]$  in  $\pi_1(X, x_0)$ .

Fundamental Group of  $X$  at  $x_0$ :  $\pi_1(X, x_0)$

$\pi_1(X, x_0) = \{ \text{loops home } x_0 \text{ to itself in } X \} / (\text{path homotopy})$

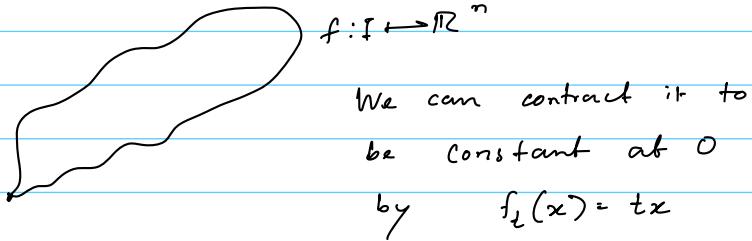
$$\rightarrow [f] \cdot [g] = [fg]$$

$$\rightarrow [f]^{-1} = [\bar{f}] \quad \text{where} \quad \bar{f}(t) = f(1-t)$$

$$\rightarrow [\text{constant}_{x_0}] = 1$$

### Examples

(i)  $\pi_1(\mathbb{R}^n, 0) = 1$



We say  $\pi_1(X) = 1$  if  $X$  is contractible

$$f_t(x) = r_t \circ f$$

homotopy from  $\text{id}_X$  to constant map

Properties

(1)

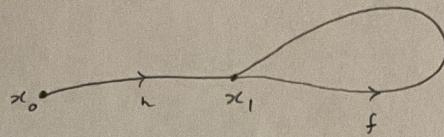
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### Change of basepoint

Let  $x_0$  and  $x_1$  lie in the same path-component of  $X$ .

Let  $h: I \rightarrow X$  be a path from  $x_0$  to  $x_1$ , with the inverse path  $\bar{h}(s) = h(1-s)$  from  $x_1$  to  $x_0$ .

Then, for each loop  $f$  based at  $x_1$ , define the loop  $h \cdot f \cdot \bar{h}$  based at  $x_0$ .



Alternatively, we can define a general  $n$ -fold product  $f_1, \dots, f_n$  in which the path  $f_i$  is traversed in  $[\frac{i-1}{n}, \frac{i}{n}]$ .

Then, define the change of basepoint map  $\beta_h: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  by  $\underline{\beta_h[f] = [h \cdot f \cdot \bar{h}]}$

Proposition: The map  $\beta_h: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  is an isomorphism.  
So,  $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Proof: Homomorphism as

$$\begin{aligned}\beta_h[f \cdot g] &= [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}] \\ &= \beta_h[f] \beta_h[g]\end{aligned}$$

This has the inverse  $\beta_{\bar{h}}$  as

$$\begin{aligned}\beta_h \beta_{\bar{h}}[f] &= \beta_h[\bar{h} \cdot f \cdot h] = [h \cdot \bar{h} \cdot f \cdot h \cdot \bar{h}] \\ &= [f]\end{aligned}$$

$$\text{Similarly, } \beta_{\bar{h}} \beta_h[f] = [f]$$

Def: Simply connected

A space is simply connected if it is path connected

and has trivial fundamental group.

i.e. the constant path

Proposition

A space  $X$  is simply connected iff there is a unique homotopy class of paths connecting any two points in  $X$ .

Proof :

$\Rightarrow$  : Need to show uniqueness.

Suppose

let  $f$  and  $g$  be 2 paths from  $x_0$  to  $x_1$ .

Then  $f \simeq f \cdot \bar{g} \cdot g \simeq g$  since the loops  $\bar{g} \cdot g$  and  $f \cdot \bar{g}$  are each homotopic to constant loops, given  $\pi_1(X) = 0$

$\Leftarrow$  : If there is only one homotopy class of paths loops at  $x_0$ , then all loops at  $x_0$  are homotopic to the constant loop  
 $\therefore \pi_1(X, x_0) = \pi_1(X) = 0$

If  $X$  is path connected, then  $\pi_1(X, x_0)$  is independent of  $x_0$ . We write it as  $\pi_1(X)$ .

### Def: Induced Homomorphism

Suppose,  $\varphi: X \rightarrow Y$  is a map taking basepoint  $x_0 \in X$  to the basepoint  $y_0 \in Y$

We say  $\varphi: (X, x_0) \mapsto (Y, y_0)$

Then,  $\varphi$  induces a homomorphism

$$\varphi_*: \pi_1(X, x_0) \mapsto \pi_1(Y, y_0)$$

defined by composing the loops  $f: I \rightarrow X$  based at  $x_0$  with  $\varphi$ :

$$\varphi_*[f] = [\varphi f]$$

→ Well-defined:

Homotopy  $f_t$  of loops at  $x_0$  yields a homotopy  $\varphi f_t$  of loops based at  $y_0$ .

$$\therefore \varphi_*[f_0] = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$$

→  $\varphi_*$  is a homomorphism:

$$\begin{aligned} \varphi(f \cdot g) &= \varphi f \cdot \varphi g && \text{both functions have values} \\ &&& \varphi f(2s), \quad 0 \leq s \leq \frac{1}{2} \\ &&& \varphi g(2s-1), \quad \frac{1}{2} \leq s \leq 1 \end{aligned}$$

### Properties of induced homomorphisms

$$(1) \quad (X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$$

$$(\varphi\varphi)_* = \varphi_*\varphi_*$$

Proof:

$$(\varphi\varphi)_f = \varphi(\varphi_f)$$

$$(2) \quad \mathbb{1}_* = \mathbb{1} \quad \text{which is saying } \mathbb{1}: X \rightarrow X \text{ induces } \mathbb{1}: \pi_1(X, x_0) \mapsto \pi_1(X, x_0)$$

(3) If  $\varphi$  is a homeomorphism with inverse  $\psi$

then  $\varphi_*$  is an isomorphism with inverse  $\psi_*$  since

$$\varphi_*\psi_* = (\varphi\psi)_* = \mathbb{1}_* = \mathbb{1} \quad \text{and similarly } \psi_*\varphi_* = \mathbb{1}$$

(4) Let  $\varphi, \psi: X \rightarrow Y$ .

If  $\varphi$  and  $\psi$  are homotopic, then  $\varphi_* = \psi_*$

Proof:

$$\varphi_*[f] = [\varphi f]$$

=  $[\psi f]$  (via homotopy of  $\varphi$  and  $\psi$ )

$$= \psi_*[f]$$

(5) Proposition:

If a space  $X$  retracts onto a subspace  $A$ , then the induced homomorphism  $i_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$  induced by the inclusion  $i: A \hookrightarrow X$  is injective. If  $A$  is a deformation retract of  $X$ , then  $i_*$  is an isomorphism.

Proof:

Suppose,  $X$  retracts onto  $A \subset X$  via  $r: X \rightarrow A$

Then  $r_i = \text{id}_A$

$$\text{So, } (ri)_* = r_* i_* = \text{id}$$

Suppose  $i_*(f) = \text{id}$  for some  $f \in \pi_1(A, x_0)$   
Therefore  $i_*$  is injective

$$\text{Then, } (r_* i_*)(f) = r_*(\text{id}) = \text{id}. \text{ But } r_* i_* = \text{id}$$

$$\text{so } f = \text{id}.$$

Now, suppose  $X$  def. retracts onto  $A$  via  $r_t: X \rightarrow X$

$$\text{so, } r_0 = \text{id}_X, r_t|_A = \text{id}_A \text{ and } r_t(X) \subset A$$

then, for any loop  $f: I \rightarrow X$  based at  $x_0 \in A$ ,

the composition  $r_t f$  gives a homotopy of  $f$  to a loop in  $A$ , so  $i_*$  is also surjective.

$$\hookrightarrow \text{as } r_t(X) \subseteq A$$

i.e. for any  $f: I \rightarrow X$ ,  
first def retract to  $f': I \rightarrow A$   
where  $f' = r_t f$ . Then  $i_*(f') = f' \in \pi_1(X, x_0)$   
and  $[f'] = [f]$  by  
the homotopy  $r_t$ .

### Def : Basepoint Preserving Homotopy

Consider a homotopy  $\varphi_t$  taking  $A \subset X$  to a subspace  $B \subset Y$  for all  $t$ , then we speak of maps of pairs

$$\varphi_t : (X, A) \rightarrow (Y, B)$$

A basepoint-preserving homotopy  $\varphi_t : (X, x_0) \rightarrow (Y, y_0)$  is the case where  $\varphi_t(x_0) = y_0 \quad \forall t$ .

(6) If  $\varphi_t : (X, x_0) \rightarrow (Y, y_0)$  is a basepoint preserving homotopy, then  $\varphi_{0*} = \varphi_{1*}$

$$\begin{aligned} \text{Proof : } \varphi_{0*}[f] &= [\varphi_0 f] \\ &= [\varphi, f] \quad (\text{via homotopy } \varphi_t f) \\ &= \varphi_{1*}[f] \end{aligned}$$

### Def : Homotopy Equivalence for spaces with basepoints

We say  $(X, x_0) \simeq (Y, y_0)$  if there are maps  $\varphi : (X, x_0) \rightarrow (Y, y_0)$  and  $\psi : (Y, y_0) \rightarrow (X, x_0)$  with homotopies  $\varphi \psi \simeq \text{id}_{(Y, y_0)}$  and  $\psi \varphi \simeq \text{id}_{(X, x_0)}$  through maps that fix the basepoint.

In this case, the induced maps on  $\pi_1$  satisfy

$$\varphi_* \psi_* = (\varphi \psi)_* = \text{id}_* = \text{id}$$

$$\psi_* \varphi_* = (\psi \varphi)_* = \text{id}_* = \text{id}$$

$\therefore \varphi_*$  and  $\psi_*$  are inverse isomorphisms

$$\therefore \pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

What if  $\varphi_t$  does not send  $x_0$  to a fixed  $y_0 \in Y$  for all  $t$ ? This means the basepoint in  $X$  is not always mapped to the same point by a homotopy.

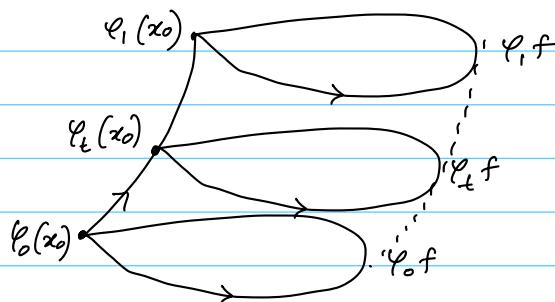
Lemma:

If  $\varphi_t : X \rightarrow Y$  is a homotopy and  $h$  is the path  $\varphi_t(x_0)$  formed by the images of a basepoint  $x_0 \in X$ , then the three maps in the diagram satisfy  $\varphi_{0*} = \beta_h \varphi_{1*}$

$$\begin{array}{ccc} & \varphi_{1*} & \rightarrow \pi_1(Y, \varphi_1(x_0)) \\ \pi_1(X, x_0) & \swarrow & \downarrow \beta_h \\ & \varphi_{0*} & \rightarrow \pi_1(Y, \varphi_0(x_0)) \end{array}$$

Proof:

Let  $h_t$  be the restriction of  $h$  to the interval  $[0, t]$  (with a reparametrization so that domain of  $h_t$  is  $[0, 1]$ ):  
So.  $h_t(s) = h(ts)$  where  $h : I \rightarrow Y$  with  $h(\tilde{t}) = \varphi_{\tilde{t}}(x_0)$



Then, if  $f$  is a loop in  $X$  at basepoint  $x_0$ , then the product  $h_t \cdot (\varphi_t f) \cdot \bar{h}$  gives a homotopy of loops at  $\varphi_0(x_0)$ .

Restricting this to  $t = 0$  and  $t = 1$ ,

$$\text{we see } \varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f]))$$

Theorem :

If  $\varphi: X \rightarrow Y$  is a homotopy equivalence, then the induced homomorphism  $\varphi_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, \varphi(x_0))$  is an isomorphism for  $\forall x_0 \in X$ .

Proof :

Let  $\varphi: X \rightarrow Y$  be a homotopy equivalence  $\Rightarrow$   
So, Let  $\psi: Y \rightarrow X$  be the homotopy inverse

$$\begin{aligned} \text{So, } \varphi \psi &\simeq \text{Id} \\ \psi \varphi &\simeq \text{Id} \end{aligned}$$

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi \varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi \psi \varphi(x_0))$$

Given  $\psi \varphi \simeq \text{Id}$ , then  $\psi_* \varphi_* = \text{Id}$  for some  $h$  by the previous lemma.  $\Rightarrow \psi_* \varphi_*$  is an isomorphism

Since  $\psi_* \varphi_*$  is an isomorphism

$\varphi_*$  is injective.

Similarly, with  $\psi_* \varphi_*$ , we conclude  $\psi_*$  is injective.

$\therefore \varphi_*, \psi_*$  are injections and  $\psi_* \varphi_*$  is an isomorphism. so  $\varphi_*$  is a surjection too.

$$(4) \quad \pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Proof :

A path  $I \rightarrow X \times Y$   
is a pair of paths  $(f: I \rightarrow X, g: I \rightarrow Y)$

## Fundamental Group of the Circle

Some preliminary tools:

- (1) Let  $w(s) = (\cos 2\pi s, \sin 2\pi s)$  for  $s \in I$  be a loop based at  $(1,0)$ .  
 Then,  $[w]^n = [w_n]$  where  $w_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$  for  $n \in \mathbb{Z}$ .

- (2) Compare paths in  $S^1$  with paths in  $\mathbb{R}$ :

→ Let  $p: \mathbb{R} \rightarrow S^1$  via  $p(s) = (\cos 2\pi s, \sin 2\pi s)$

Visualization: first, consider the helix  $s \mapsto (\cos 2\pi s, \sin 2\pi s, s)$

Then, project  $\mathbb{R}^3$  onto  $\mathbb{R}^2$  by  $(x,y,z) \mapsto (x,y)$

So, projecting the helix onto  $\mathbb{R}^2$  gives  $p$



→  $w_n(s) = p \tilde{w}_n(s)$  where  $\tilde{w}_n: I \rightarrow \mathbb{R}$  is the path  $\tilde{w}_n(s) = ns$

$\tilde{w}_n$  starts at 0 and ends at  $n$   
 $\tilde{w}_n$  is called the lift of  $w_n$ .

$p \tilde{w}_n(s)$  winds around the helix  $|n|$  times  $\rightarrow$  upwards  
 if  $n > 0$  and downwards if  $n < 0$ .

- (3) Def: Covering Space

Given a space  $X$ , a covering space of  $X$  consists of a

space  $\tilde{X}$  and a map  $p: \tilde{X} \rightarrow X$  satisfying:

- (a) for each  $x \in X$ ,  $\exists$  open neighbourhood  $U \ni x$  in  $X$  st

$p^{-1}(U) = \bigcup_{x \in U} V_x$  where each  $V_x$  is open and each  
 $V_x$  is mapped homeomorphically onto  $U$  by  $p$ .  
 is a union of disjoint open sets each of

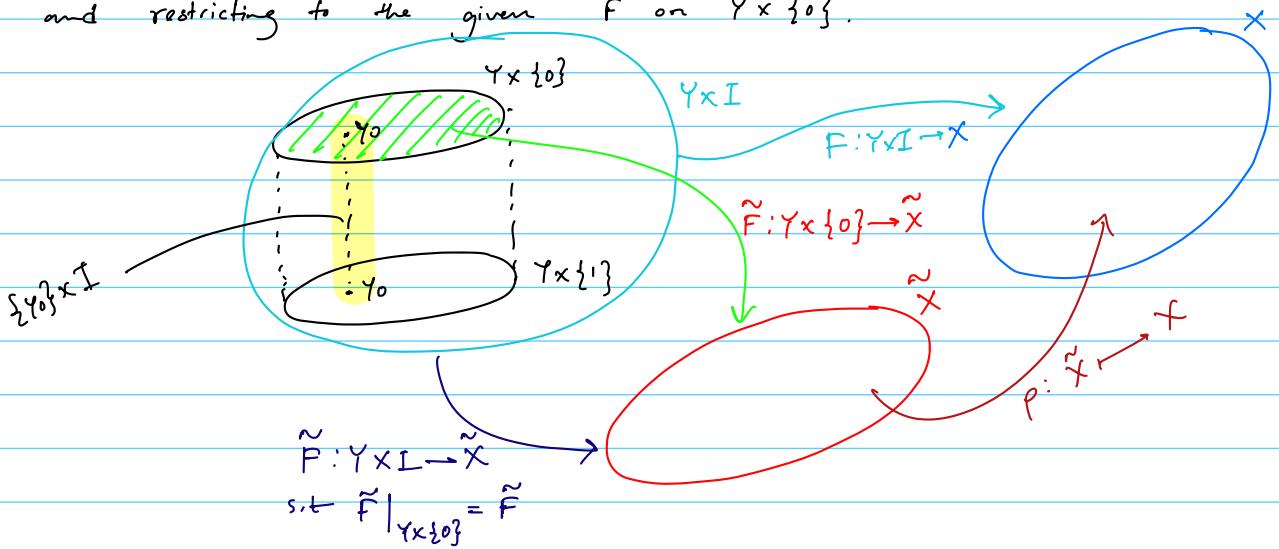
We say  $U$  is evenly covered.

Example:

(1)  $p: \mathbb{R} \rightarrow S'$ , an open arc in  $S'$  is evenly covered.

Lemma: Consider covering spaces  $p: \tilde{X} \rightarrow X$ .

Given a map  $f: Y \times I \rightarrow X$  and a map  $\tilde{f}: Y \times \{0\} \rightarrow \tilde{X}$  lifting  $f|_{Y \times \{0\}}$ , then there is a unique map  $\tilde{F}: Y \times I \rightarrow \tilde{X}$  lifting  $f$  and restricting to the given  $\tilde{f}$  on  $Y \times \{0\}$ .



Proof:

First, construct a lift  $\tilde{F}: N \times I \rightarrow \tilde{X}$  for some neighbourhood  $N$  of  $y_0 \in Y$ .

Given  $F$  is continuous,  $\forall (y_0, t) \in Y \times I$  has a product

neighbourhood  $N_t \times (a_t, b_t)$  s.t.  $F(N_t \times (a_t, b_t))$  is contained in an evenly covered neighbourhood of  $F(y_0, t)$ .

around  $F(y_0, t)$ ,  $\exists$  an evenly covered neighbourhood since  $p: \tilde{X} \rightarrow X$  is a covering space.

By continuity, we can always shrink  $N_t \times (a_t, b_t)$  so that  $F(N_t \times (a_t, b_t))$  is inside this evenly covered nbd.

By compactness of  $\{y_0\} \times I$ , finitely many such  $N_t \times (a_t, b_t)$  products cover  $\{y_0\} \times I$ . Thus, we can choose one neighbourhood  $N$  of  $\{y_0\}$  and a partition  $0 = t_0 < t_1 < \dots < t_m = 1$  of  $I$  s.t. for each  $i$ ,  $F(N \times [t_i, t_{i+1}])$  is contained in an evenly covered neighbourhood  $U_i$ .

Assume, inductively,  $\tilde{F}$  has been constructed on  $N \times [0, t_i]$  starting with our given  $\tilde{f}$  on  $N \times \{0\}$ . Thus,  $F(N \times [t_i, t_{i+1}]) \subset U_i$ , so since  $U_i$  is evenly covered,  $\exists$  open set  $\tilde{U}_i \subset \tilde{X}$  projecting homeomorphically onto  $U_i$  by  $p$  and containing  $\tilde{F}(y_0, t_i)$  because

$\tilde{F}$  is a lift of  $F$   
so  $p(\tilde{F}(y_0, t_i)) = F(y_0, t_i)$   
we know it is a lift  
or we have already  
constructed the lift  
on  $N \times [0, t_i]$ .

We can extend  $\tilde{F}$  on  
 $N \times [t_i, t_{i+1}]$  by  
composing  $p^{-1}: U_i \rightarrow \tilde{U}_i$   
(since  $p$  is a homeomorphism)  
with  $F$ . For this to  
be continuous,  $\tilde{F}$  must agree with  $\tilde{F}$  on  
 $N \times [0, t_i]$ , in particular at  
 $(y_0, t_i)$ .

Replace  $N$  by a small enough nbd of  $y_0$ , we can get that  
 $\tilde{F}(N \times \{t_i\})$  is contained in  $\tilde{U}_i$  by replacing  $N \times \{t_i\}$  by  
its intersection with  $(\tilde{F}|_{N \times \{t_i\}})^{-1}(\tilde{U}_i)$ . Then define  $\tilde{F}$  on  
 $N \times [t_i, t_{i+1}]$  to be the composition of  $F$   
with  $p^{-1}: U_i \rightarrow \tilde{U}_i$ .

Continuing, we get  $\tilde{F}: N \times I \rightarrow \tilde{X}$ , a lift, for  
some neighbourhood  $N$  of  $y_0$ .

Next we show uniqueness of this lift. We prove for when  
 $Y$  is a point. Since  $Y$  is a point, we suppress  
it from our notation.

Let  $\tilde{F}$  and  $\tilde{F}'$  be 2 lifts of  $F: I \rightarrow X$   
s.t.  $\tilde{F}(0) = \tilde{F}'(0)$ .

Again, we choose a partition  $\mathcal{D} = t_1 < t_2 < \dots < t_m = 1$   
of  $I$  s.t. for each  $i$ ,  $F([t_i, t_{i+1}])$  is contained in  
evenly covered nbd  $U_i$ .

Assume inductively that  $\tilde{F} = \tilde{F}'$  on  $[0, t_i]$ . As  $[t_i, t_{i+1}]$   
is connected, so is  $\tilde{F}([t_i, t_{i+1}]) \Rightarrow$  it must lie in single  
one of the disjoint open sets  $\tilde{U}_i$  projecting homeomorphically  
to  $U_i$ . By same logic,  $\tilde{F}'([t_i, t_{i+1}])$  lies in a  
single  $\tilde{U}_i$  and it must be the same one as  
 $\tilde{F}'(t_i) = \tilde{F}(t_i)$ . As  $p$  is injective on  $\tilde{U}_i$  and  
 $p \tilde{F} = p \tilde{F}'$ , we get  $\tilde{F} = \tilde{F}'$  on  $[t_i, t_{i+1}]$ . Continuing this way,  
 $\tilde{F} = \tilde{F}'$ .

Lastly, observe that since the lift  $\tilde{F}$  constructed on sets of the form  
 $N \times I$  is unique when restricted to each segment  $\{y\} \times I$ ,  
they must agree when two such sets  $N \times I$  overlap.

∴ we have a well-defined lift  $\tilde{F}$  on all of  $Y \times I$ .

$\tilde{F}$  is continuous as it is continuous on each  $N \times I$ , and unique as it is unique on each segment  $\{y\} \times I$ .

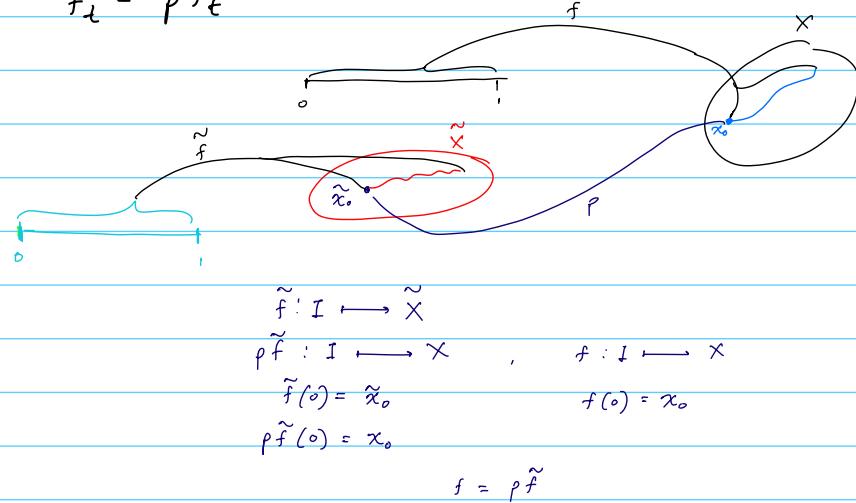
Lemma : Consider covering spaces  $p: \tilde{X} \rightarrow X$ .

(1) For each path  $f: I \rightarrow X$  starting at a point  $x_0 \in X$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$ ,  $\exists$  a unique lift  $\tilde{f}: I \rightarrow \tilde{X}$  starting at  $\tilde{x}_0$ .

Hence,  $f = p\tilde{f}$ .

(2) For each homotopy  $f_t: I \rightarrow X$  of paths starting at  $x_0$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$ ,  $\exists$  a unique lifted homotopy  $\tilde{f}_t: I \rightarrow \tilde{X}$  of paths starting at  $\tilde{x}_0$ .

Hence,  $f_t = p\tilde{f}_t$



Proof :

(1) follows from prev. lemma when  $Y$  is a point

(2) Let  $Y = I$ .

Then for the homotopy  $f_t: I \rightarrow X$ , we have a map  $F: I \times I \rightarrow X$  with  $F(s, t) = f_t(s)$ .

We get a unique lift  $\tilde{F}: I \times \{0\} \rightarrow \tilde{X}$  using part (1).

Then, by prev. lemma, we get a unique lift

$$\tilde{F}: I \times I \rightarrow \tilde{X}$$

The restrictions  $\tilde{F}|_{\{0\} \times I}$  and  $\tilde{F}|_{\{1\} \times I}$  are paths

lifting constant paths, so they must also be constant by uniqueness of part (1).

So,  $\tilde{f}_t(s) = \tilde{F}(s, t)$  is a homotopy of paths

and  $\tilde{f}_t$  lifts  $f_t$  as  $F = p\tilde{F}$

Theorem:  $\pi_1(S')$  is an infinite cyclic group generated by the homotopy class of the loop  $w(s) = (\cos 2\pi s, \sin 2\pi s)$  based at  $(1, 0)$ .

Proof: Let  $f: I \rightarrow S'$  be a loop at the basepoint  $x_0 = (1, 0)$  which is one element of the group  $\pi_1(S', x_0)$ .

Then,  $\underbrace{\exists}$  a lift  $\tilde{f}$  starting at 0 and must end at some integer  $n$  since  $p\tilde{f}(1) = f(1) = x_0$  and  $\underbrace{p^{-1}(x_0)}_{\text{recall } p(s) = (\cos(2\pi s), \sin(2\pi s))} = \mathbb{Z} \subset R$

Another path in  $R$  from 0 to  $n$  is  $\tilde{w}_n$  and  $\tilde{f} \simeq \tilde{w}_n$  via the linear homotopy  $(1-t)\tilde{f} + t\tilde{w}_n$ . Compose the homotopy with  $p$  gives the homotopy  $f \simeq w_n$  so  $[f] = [w_n]$ .

Next, we show that  $n$  is uniquely determined by  $[f]$ : Suppose  $f \simeq w_m$  and  $f \simeq w_n$ . Let  $f_t$  be a homotopy from  $w_m = f_0$  to  $w_n = f_1$ .

Then,  $\underbrace{f_t}$  lifts to a homotopy  $\tilde{f}_t$  of paths starting at 0 by previous lemma (2)

The uniqueness of  $\tilde{f}$  (by prev. lemma) implies that  $\tilde{f}_0 = \tilde{w}_m$  and  $\tilde{f}_1 = \tilde{w}_n$ . Since  $\tilde{f}_t$  is a homotopy of paths, the endpoint  $\tilde{f}_t(1)$  is independent of  $t$ . For  $t=0$ , the endpoint is  $m$  and for  $t=1$ , it is  $n$  so  $m=n$ .

The fact that this group is generated by  $w(s)$  is obvious from noting that  $[w]^n = [w_n]$