Algebraic Topology
Notes
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These notes are a combination of notes from Allen Hatcher's book "Algebraic Topology" and Prof. Ciprian Manolescu's lectures from MATH 215a at Stanford University and my own.

Introduction

Our goal is to develop algebraic invarianees associated with topological spaces.

We will look at

(1) Fundamental Group:

$$\pi_1(X) = \{\text{loops in } X\} / \text{homotopy}$$

(2) Homology Group:

$H_n(X)$, $n \in \mathbb{N}$ and abelian

Intuitively, they count "holes" in X

(3) Cohomolog Group:

$H^n(X) =$ Dual to $H_n(X)$

$\bigoplus_n H^n(X)$ is a ring!

Basic Constructions

Def : Homeomorphism

Let X and Y be topological spaces.

$f: X \to Y$ is a homeomorphism if f is bijective and both f and f^{-1} are continuous.

We say $X \cong Y$.

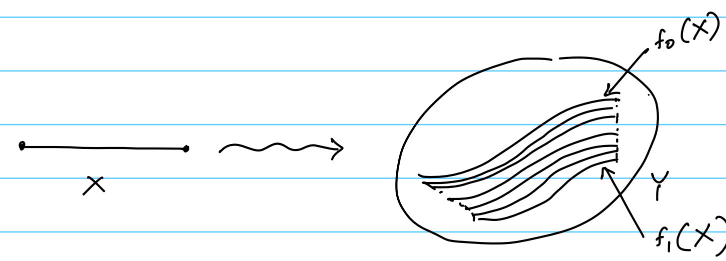
Def : Homotopy

A family of maps, $f_t : X \to Y$ where $t \in I = [0,1]$ s.t the associated map $F : X \times [0,1] \to Y$ given by $F(x,t) = f_t(x)$ is continuous.

Two maps $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy $F : X \times [0,1] \to Y$ s.t

$$F(x,0) = f_0(x) \quad \forall x \in X$$
$$F(x,1) = f_1(x)$$

We say $f_0 \simeq f_1$.



Def: Homotopy Equivalence

A map $f: X \to Y$ is a homotopy equivalence if $\exists\, g: Y \to X$ s.t $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

We say the spaces X and Y are homotopy equivalent and $X \simeq Y$.

$\to$ can prove easily that this is an equivalence relation.

Examples of homotopy equivalence

(1) $\mathbb{R}^n \simeq$ a point (even though $\mathbb{R}^n \not\cong$ a point)

$\mathbb{R}^n$ ↑ infinite, a point ↑ finite

Why?

$f : \mathbb{R}^n \longrightarrow \{0\}$

and take $g : \{0\} \longrightarrow \mathbb{R}^n$ by $g(0) = 0$

Then $f \circ g = id_{\{0\}}$ and $(g \circ f)(x) = 0 \quad \forall x \in \mathbb{R}^n$

Now $g \circ f \sim id_{\mathbb{R}^n}$ by $f_t(x) = tx$ where $f_0 = 0$ and $f_1 = id_{\mathbb{R}^n}$

(2) $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \simeq$ a point

$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \simeq$ a point

Def : Contractible

We say the space X is contractible if $X \simeq$ a point

↕

Equivalent definition : the identity map of X is null homotopic

i.e $id_X \simeq$ constant map

null homotopic ↓ homotopic to a constant map.

Def : Retractions

Let X be a space and let $A \subset X$.

Then, a retraction is a map $r : X \to X$ s.t

$r(X) = A$ and $r|_A = id_A$.

Def : Deformation Retraction

A deformation retraction of X onto a subspace A is

a family of maps $f_t : X \to X$, with $t \in I$ s.t

$f_0 = id_X$ and $f_1(X) = A$ and $f_t|_A = id_A$ for $\forall t \in I$.

The family f_t must also be continuous

→ an example of a homotopy from id_X to a retraction of X onto $A \subset X$.

→ in this case, $\boxed{A \simeq X}$ as $f_0 : A \hookrightarrow X$ by id_X

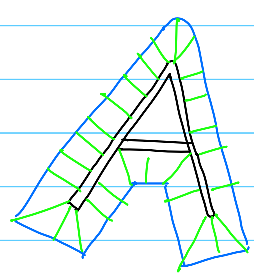
$f_1 : X \to A$ as above

then $f_0 \circ f_1 \simeq id_X$ $\quad (\text{since } f_0 \circ f_1 = f_1 \simeq f_0 = id_X)$

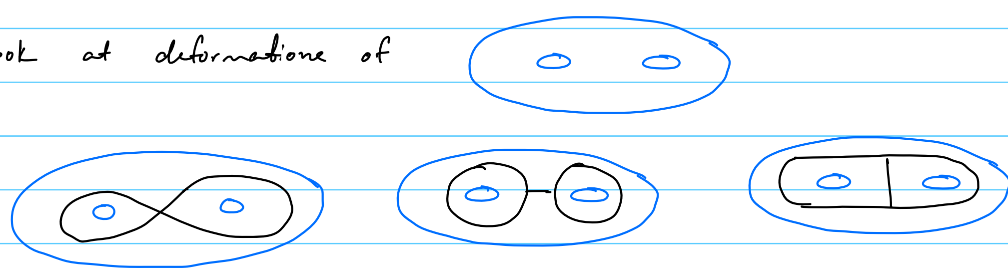
and $f_1 \circ f_0 = id_A$

Examples of deformation retraction :

(1)



(2) Look at deformations of



(3) $X = \mathbb{R}^2 - \{0\}$. $A = S^1$

Use $F(x,t) = (1-t)x + t\frac{x}{\|x\|}$.

$X \rightsquigarrow x$

Proposition:

If X def. retracts to a point $x \in X$, then for any $U \subset X$, $x \in U$. $\exists\, V \subset U$ with $x \in V$ s.t the inclusion map $V \hookrightarrow U$ is nullhomotopic.

$\longrightarrow$ homotopic to constant map

Def: Deformation Retraction in the weak sense:

Let $A \subset X$.

Then, this is the homotopy $f_t : X \mapsto X$ s.t $f_0 = id_X$ and $f_1(X) \subset A$ with $f_t(A) \subset A$, $\forall t \in I$.

Lemma:

If X deformation retracts to A in the weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Proof:

Let the weak def. ret be f_t.

Let $\iota : A \mapsto X$ by inclusion i.e $\iota(a) = a$, $\forall a \in A$.

Then, $(\iota \circ f_1)(x) = \iota(f_1(x)) = f_1(x)$, $\forall x \in X$

But $f_1 \simeq f_0 = id_X$

So, $\iota \circ f \simeq id_X$

Also, $(f_1 \circ \iota)(a) = f_1(a)$ ~~and~~ $\forall a \in A$

But $f_1|_A \simeq f_0|_A = id_X|_A = id_A$

$\Rightarrow f_1 \circ \iota \simeq id_A$

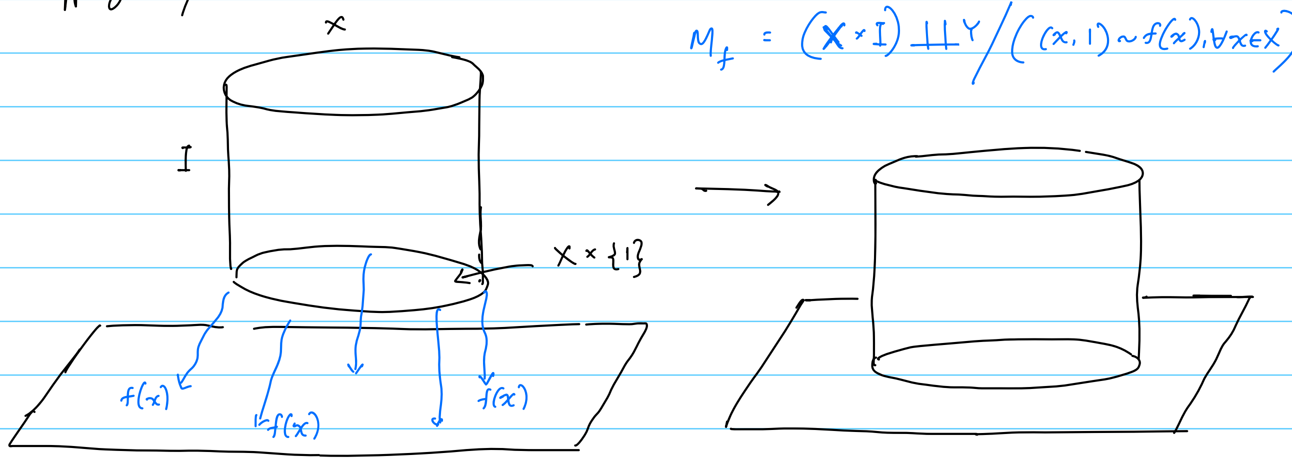
Def: Mapping Cylinder ("the structure through which the def. retraction occurs")

for a map $f: X \to Y$, the mapping cylinder M_f is the quotient space of the disjoint union $(X \times I) \sqcup Y$ obtained by the equivalence $(x, 1) \in X \times I \sim f(x) \in Y$

Make the endpoint of the deformation equivalent to the image of the map.

Mapping cylinders are continuous.

$M_f = (X \times I) \sqcup Y / ((x,1) \sim f(x), \forall x \in X)$



Def: Homotopy relative to A (homotopy rel. A)

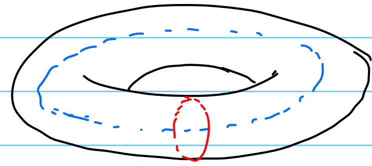
A homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t.

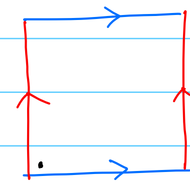
In other words, f_t is a homotopy and $f_t|_A$ is independent of t.

→ def. retraction of X onto A is a homotopy rel. A from id_X to a retraction of X onto $A \subset X$.

Cell Complexes

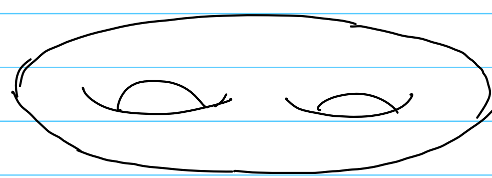
Examples :

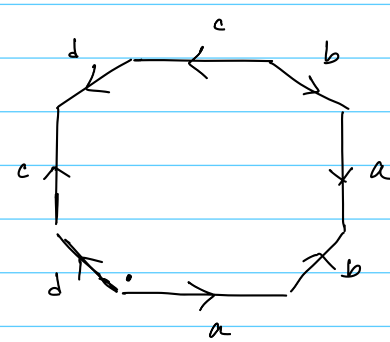




The torus $S^1 \times S^1$ can be constructed from the square

Generally, an orientable surface M_g of genus g can be constructed from a polygon of $4g$ sides, by identifying pairs of edges.





2 cell : interior of a polygon which is an open disk

1 cell : an open interval like $(0,1)$

3 cell : an open ball.

n-cell : open-disk

$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$

Def: Cell Complex (or CW complex)

A space constructed as follows:

(1) Start with discrete set X^0 → the points are 0-cells

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\varphi_\alpha : S^{n-1} \to X^{n-1}.$$

lives in $\mathbb{R}^n$ & is the boundary of D^n!

→ So, X^n is the quotient space of $X^{n-1} \sqcup_\alpha D^n_\alpha$ under the equivalence $x \sim \varphi_\alpha(x)$ $\forall x \in \partial D^n_\alpha$

(n-1)-skeleton n-disks

i.e. attach boundaries of the n-disk to the (n-1)-skeleton

$\therefore X^n = X^{n-1} \sqcup_\alpha e^n_\alpha$ where e^n_α is an open n-disk

(3) Either stop this induction at a finite stage and set $X = X^n$ for $n < \infty$

or continue indefinitely, setting

$$X = \bigcup_n X^n$$

↳ in this case, X has the weak topology:

$A \subset X$ is open iff $A \cap X^n$ is open in X^n for each n

Vocabulary:

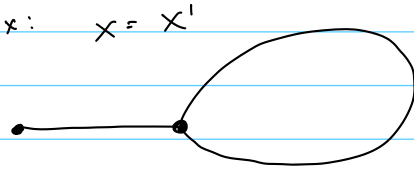
① X^n → n-skeleton

② Dimension of X → largest n s.t. an n-cell exists

Examples of Cell Complexes:

(1) 1-dimensional cell complex: $X = X^1$

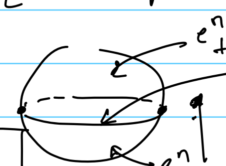
multigraphs



(2) The sphere S^n has a cell complex with two cells, e^0 and e^n, where e^n is attached by $\varphi : S^{n-1} \to e^0$

$\therefore$ S^n is being regarded as the quotient space $D^n / \partial D^n$

$S^n = e^0 \cup e^n$. Alternatively,



$S^n = S^{n-1} \cup e^n_+ \cup e^n_-$
$= e^0_+ \cup e^0_- \cup \cdots \cup e^n_+ \cup e^n_-$

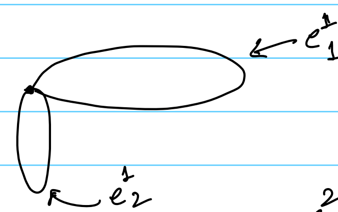
$\therefore S^\infty = \bigcup_n S^n$

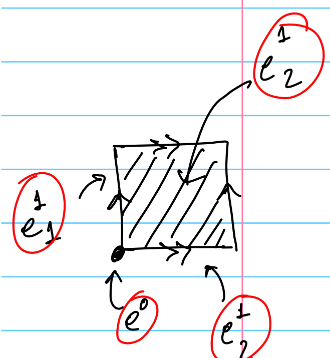
(3) Cell Complex of a torus:

Step 1: X^0 is just 1 point $\longrightarrow \bullet \leftarrow e^0$

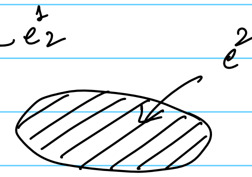
Step 2: Attach two 1-cells to this point

$X^1 =$





Step 3: Attach a disk



to X^1 by attaching its boundary to X^1.

(4) Real Projective Space, $\mathbb{R}P^n$

$(\mathbb{R}^{n+1} - \{0\}) / (v \sim \lambda v,\ \forall v \in \mathbb{R}^{n+1}.\ \lambda \neq 0)$

→ Restricting to vectors of length 1, $S^n / (v \sim -v)$

$\Rightarrow$ D^n with antipodal points of ∂D^n identified

To get this, think of

i.e. for the upper hemisphere's points, find where the line to south pole intersects with D^n

∂D^n with antipodal points equivalent is $\mathbb{R}P^{n-1}$

$\therefore$ $\mathbb{R}P^n$ can be formed from $\mathbb{R}P^{n-1}$ by attaching an n-cell. and the attaching map $\varphi : S^{n-1} \to \mathbb{R}P^{n-1}$

$\therefore$ $\mathbb{R}P^n$ has the cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$.

(5) <u>Complex Projective Space. $\mathbb{CP}^n$</u>

Space of all complex lines through the origin in $\mathbb{C}^{n+1}$

ie $\mathbb{CP}^n = (\mathbb{C}^{n+1} - \{0\}) / (v \sim \lambda v, \ \forall v \in \mathbb{C}^{n+1}, \lambda \neq 0)$

Equivalent to $S^{2n+1} / (v \sim \lambda v, \ |\lambda| = 1)$ $(S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1})$

Equivalent to $D^{2n} / (v \sim \lambda v, \ v \in \partial D^{2n})$

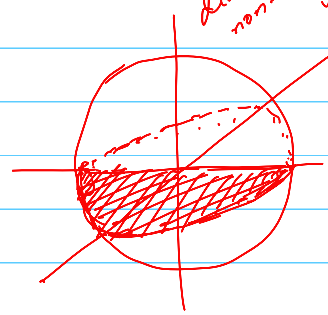
↳ <u>Why?</u>

$S^{2n+1} \subset \mathbb{C}^{n+1}$ → consider vectors in $\mathbb{C}^{n+1}$ whose last coordinate is ~~are~~ real and non-negative

in $\mathbb{R}^{2n+2}$, this kills one dimension another & makes dimension non-negative

These vectors are of the form $(w, \sqrt{1-w^2}) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$

They form the graph of the function $w \longmapsto \sqrt{1-w^2}$ with $|w| \leq 1$, $w \in \mathbb{C}^n$



Note : $w \in \mathbb{C}^n$ and $|w| \leq 1 \Rightarrow w \in D^{2n}$

This is a disk D^{2n}_+ bounded by the spheres S^{2n-1}.

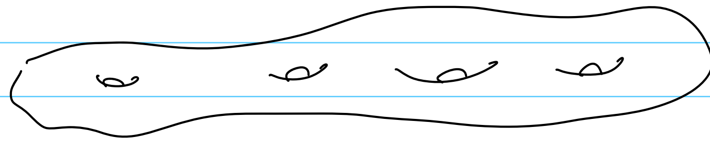
By adding another dimension and viewing them as $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$, we ~~view th~~ view them as vectors in $(D^{2n}_+, 0)$ bounded by $S^{2n-1} \subset S^{2n+1}$

Now each vector in S^{2n+1} is equivalent to a vector in D^{2n}_+ by identifying $v \sim \lambda v$. In particular, if the last coordinate is zero, we have $v \sim \lambda v$, $\forall v \in S^{2n-1}$.

∴ $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a cell e^{2n} using the attaching map $\varphi : S^{2n-1} \longrightarrow \mathbb{CP}^{n-1}$

∴ $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions

(6)



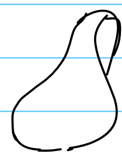
Orientable surface $\longrightarrow \Sigma_g \longrightarrow g$ holes

genus

Can be constructed from a $4g$ polygon

$\hookrightarrow$ Start with one e^0

(7)

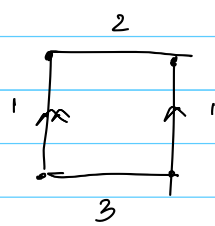


Non-orientable surface $\longrightarrow N_g$

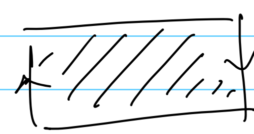
Eg: $N_2 \longrightarrow$ Klein bottle

$N_1 \longrightarrow \mathbb{RP}^2$

(8) Annulus:

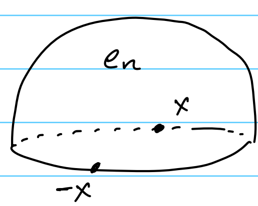


(9) Mobius band



(9) $\mathbb{RP}^n$ revisited

$$\mathbb{RP}^n = S^n / (x \sim -x, \forall x)$$



$$\Rightarrow \mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n$$

$$\therefore \mathbb{RP}^n = e^0 \cup e^1 \cup \cdots\cdots \cup e^n$$

Then, $\mathbb{RP}^\infty = e^0 \cup e^1 \cup e^2 \cup \cdots\cdots = \bigcup_n \mathbb{RP}^n$

(10) $\mathbb{CP}^n$ revisited

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - \{0\}\right) / \left(x \sim \lambda x,\ \lambda \in \mathbb{C}^*\right)$$

norm 1

$$\therefore\ z \sim \frac{z}{\|z\|} \Rightarrow \mathbb{CP}^n \cong S^{2n+1} / \left(z \sim \lambda z,\ \lambda \in S^1\right)$$

Divide everything by x_1 i.e last coordinate in $\mathbb{R}_{\geq 0}$

$$z = (\underbrace{z_0, \cdots, z_n}_{w}, \underbrace{z_{n+1}}_{\sqrt{1-\|w\|^2}})$$

with $\|w\| \leq 1$

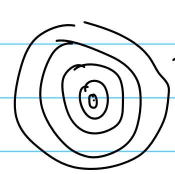
$$D^{2n}_+ = \text{graph}\left(w \longmapsto \sqrt{1-\|w\|^2}\right)$$

$$\therefore\ \mathbb{CP}^n = D^{2n}_+ / \left(w \sim \lambda w \text{ if } w \in S^{2n-1}\right)$$

$$= e^{2n} \cup \left(S^{2n-1}/(w \sim \lambda w)\right)$$

$$= \mathbb{CP}^{n-1} \cup e^{2n}$$

$$= e^0 \cup e^2 \cup \cdots \cup e^{2n}$$

(11)  $\leftarrow$ Circles of Radii $\frac{1}{n}$ $\longrightarrow$ NOT a cell complex

Properties of CW Complexes

(1) They are normal ($\therefore$ also Hausdorff)

(2) Any finite cell complex is compact

(3) A compact subspace of a cell cx is contained in a finite subcomplex

(4) Closure finiteness $\longrightarrow$ The closure of each cell $\bar{e}$ meets only finitely many cells.

(5) Locally contractible :

$\forall x \in X$, $U \ni x$ open, $\exists\ V \subset U$ with $x \in V$ s.t V is contractible

(6)

Recall :

Top manifolds $\longrightarrow$ 2^{ND} Countable, Hausdorff, locally Euclidean

Smooth manifolds $\longrightarrow$

Theorem : Every smooth manifold is homeomorphic to a cell complex.

Theorem : Every topological manifold is homotopy equivalent to a cell complex.

Theorem : Every top. manifold of dimension $\neq 4$ is homeomorphic to a cell complex (unknown in dim 4)

Def: Characteristic Map

Each cell e^n_α in a cell complex X has a characteristic map

$$\Phi_\alpha : D^n_\alpha \longmapsto X$$

which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α

→ Φ_α is the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \xrightarrow{q} X^n \hookrightarrow X$$

the quotient map that defines X^n

Example of characteristic map:

(1) Recall: S^n can be constructed by two cells: e^0 and e^n ← just one point

where e^n is attached to e^0 by

$$\varphi_\alpha : S^{n-1} \longrightarrow e^0$$

Then, the characteristic map of e^n is

$$\Phi_\alpha : D^n_\alpha \longrightarrow S^n$$ which collapses ∂D^n_α to e^0

Def : Subcomplex

A subcomplex of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X.

→ As A is closed, for each cell in A,

the image of its characteristic map
the image of its attaching map } contained in A

$\therefore$ A is a cell complex as well

Def : CW pair

A cell complex X and a subcomplex A forms a pair (X, A)

Example of subcomplex

→ Each skeleton, X^n, is a subcomplex.
→ in $\mathbb{R}P^n$ and $\mathbb{C}P^n$, the only subcomplexes are $\mathbb{R}P^k$ and $\mathbb{C}P^k$, $\forall k \leq n$

Properties of subcomplexes

(1) Closure of a collection of cells is a subcomplex.
(2) Any union and intersection of subcomplexes is a subcomplex.

Operations on Spaces

↳ Most of them preserves cell complexes ie if you start with CW complex, you get a CW complex.

I Products

$X, Y \longrightarrow$ cell complexes

$X \times Y \longrightarrow$ cell complex with the cells $\underbrace{e^m_\alpha}_{\text{cells of } X} \times \underbrace{e^n_\beta}_{\text{cells of } Y}$

II Quotients

$\longrightarrow A \subset X \rightsquigarrow X/A$

Given (X, A) a CW pair, (X: cell complex, A: cell subcomplex)

the quotient space X/A also has a cell complex structure:

→ the cells of X/A are the cells of $X-A$ and a new 0-cell which is the image of A in X/A.

→ for a cell e^n_α of $X-A$ attached by $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$, the attaching map for the corresponding cell in X/A is the composition $S^{n-1} \longrightarrow X^{n-1} \longrightarrow X^{n-1}/A^{n-1}$

Eg: (1) $D^n/S^{n-1} = S^n$

III Wedge Sum (for based spaces)

Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, the wedge sum $X \vee Y$ is the quotient of $X \amalg Y$ by identifying x_0 and y_0 to a single point

$$X \vee Y = X \amalg Y / (x_0 \sim y_0)$$

$\longrightarrow$ Examples: $S^1 \vee S^1 = \infty$

$\longrightarrow$ $\bigvee_\alpha X_\alpha$ for an arbitrary collection of spaces X_α: start with $\amalg_\alpha X_\alpha$ and then identify $x_\alpha \in X_\alpha$ to one point.

$\longrightarrow$ If X_α are cell complexes and the points x_α are 0-cells, then $\bigvee_\alpha X_\alpha$ is a cell complex because we obtain it from the cell complex $\amalg_\alpha X_\alpha$ and attach by collapsing a subcomplex to a point.

→ For a cell complex X, the quotient X^n / X^{n-1} is a wedge sum of n-spheres $\bigvee_\alpha S^n_\alpha$ with one sphere for each n-cell of X

□ <u>Smash Product</u> $X \wedge Y = (X \times Y) \big/ \big((x_0 \times Y) \cup (X \times y_0)\big)$

Inside the product space $X \times Y$, there are copies of X and Y : $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $y_0 \in Y$ and $x_0 \in X$.

These copies of X and Y intersect only at (x_0, y_0) so their union can be identifies with the wedge sum $X \vee Y$

ie $(X \times \{y_0\}) \cup (\{x_0\} \times Y) = X \vee Y$
$= (X \sqcup Y) / (x_0 \sim y_0)$

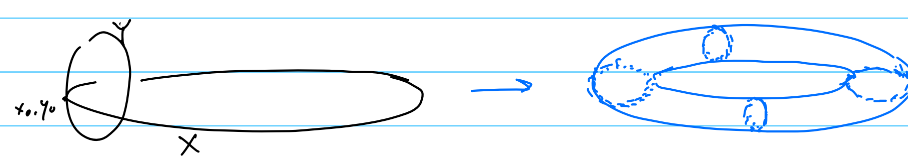
The smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$

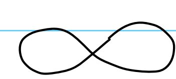
↳ i.e we are collapsing away the separate factors X and Y.

Eg: $S^1 \wedge S^1 = S^2$ ⟶ $S^1 = I / (0 \sim 1)$
$S^m \wedge S^n = S^{m+n}$

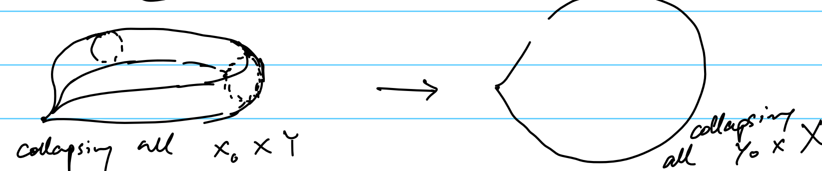
<u>Why?</u>

Firstly, $S^1 \times S^1$ results in a torus T^2



Secondly, $S^1 \vee S^1 =$ 

Then, quotienting :



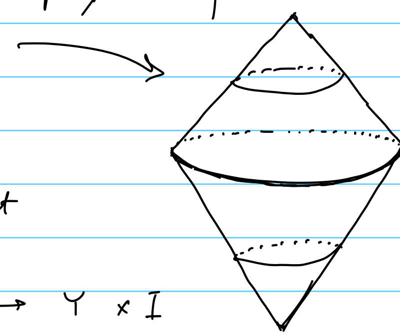
Suspension

For a space X, the suspension SX is the quotient of $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another.

Example

(i) $X = S^n$

$SX = S^{n+1}$ with the two suspension points at North and South of S^{n+1}



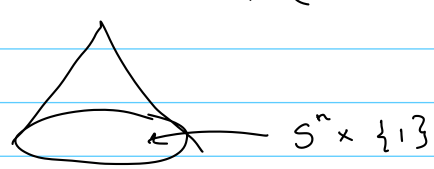
→ We can suspend maps too

$f: X \to Y \rightsquigarrow Sf : SX \to SY$ which is the quotient map of

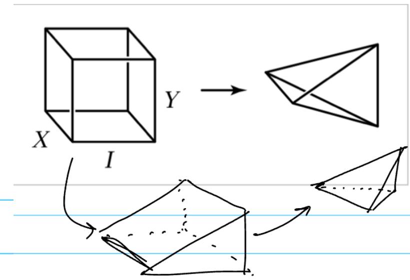
$$f \times \mathbb{1} : X \times I \to Y \times I$$

Cone:

$$CX = (X \times I) / (X \times \{0\})$$



→ If X is a CW complex, then so are SX and CX as quotients of $X \times I$ with its product cell structure with I given the standard cell structure of two 0-cells joined by one 1-cell.

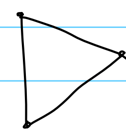


Join

Given X and Y, we can define the space of all line segments joining points in X to points in Y.

$$X * Y = (X \times Y \times I) \Big/ \left(\begin{array}{ll} (x, y_1, 0) \sim (x, y_2, 0) & \forall\, x, x_1, x_2 \in X \\ (x_1, y, 1) \sim (x_2, y, 1) & \forall\, y, y_1, y_2 \in Y \end{array} \right)$$

$\longrightarrow$ $pt * pt \longrightarrow$

$pt * pt * pt \longrightarrow$ 

$$\underbrace{pt * pt * \cdots\cdots * pt}_{n+1 \text{ points}} = \Delta^n \longrightarrow n\text{-simplex}$$

Def Reduced Suspension:

$X \to$ CW complex
$\{x_0\} \to$ base point

$$SX = (X \times I) / (X \times \{0\}) \cup (X \times \{1\})$$

$$\Sigma X = SX / (\{x_0\} \times I)$$

Thm

Criteria for Homotopy Equivalence

Recall:

Def: Homotopy Equivalence

A map $f: X \to Y$ is a homotopy equivalence if $\exists\, g: Y \to X$ s.t $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

We say the spaces X and Y are homotopy equivalent and $\boxed{X \simeq Y}$

$\to$ can prove easily that this is an equivalence relation.

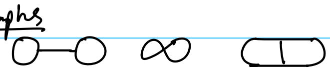
Collapsing Subspaces

Theorem:

If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \mapsto X/A$ is a homotopy equivalence.

Example

(1) Graphs

(a)  $\longrightarrow$ they are homotopy equivalent

A B C

$\to$ collapsing the middle edge of A and C produces B

(b) Let X be a graph with finitely many vertices and edges.

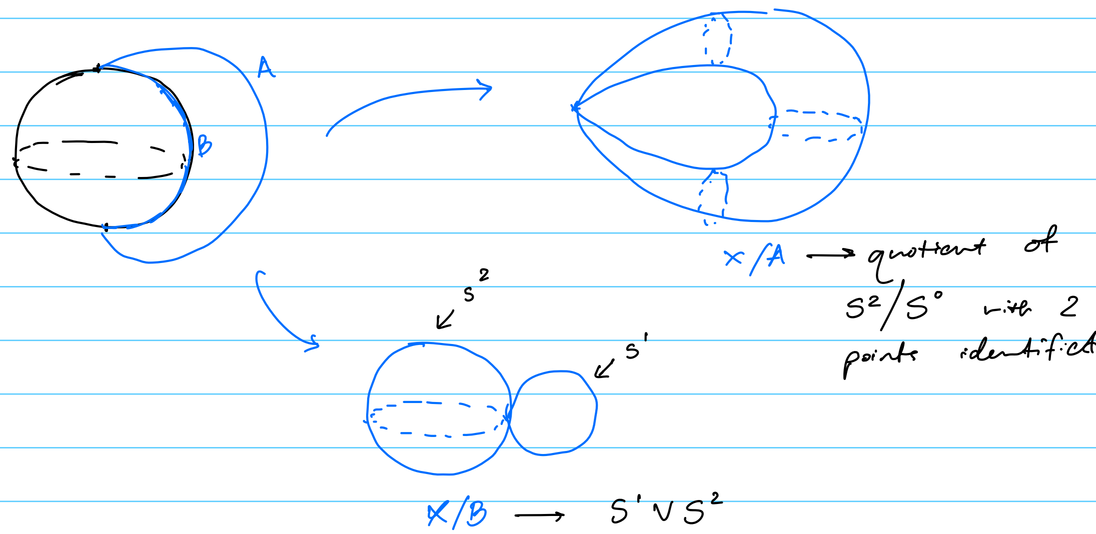
$\to$ if the two endpoints of any edge are distinct, we can collapse it to a pt.

$\downarrow$

Leads to a homotopy equivalent graph with one less edge.

Can Repeat until all edges are loops.

(2) $X \longrightarrow S^2$ but attach 2 ends of an arc A to N and S pole



Reduced Suspension

$$\Sigma X \simeq SX$$

Attaching Spaces

Start with space X_0 and another space X_1 which we will attach to X_0 by identifying points in a subspace $A \subset X_1$ with points of X_0.

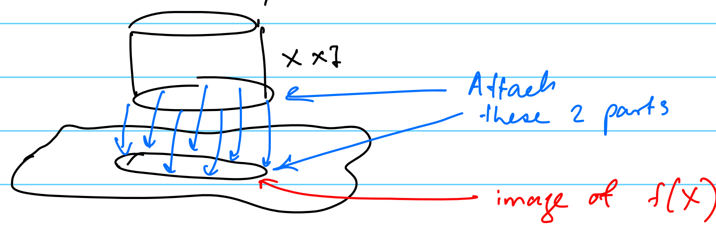
We do this using a map $f : A \longrightarrow X_0$ and then forming a quotient space of $X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A)$

We denote

$$X_0 \sqcup_f X_1 = X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A) \quad \text{where } f : A \longrightarrow X_0,\ A \subset X_1$$

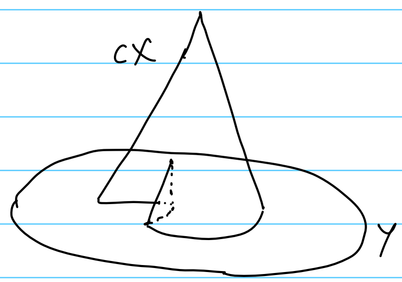
<u>Example</u> :

(1) Mapping cylinder of a map $f : X \mapsto Y$ is $M_f \to$ the space obtained from Y by attaching $X \times I$ along $X \times \{1\}$ via f.



(2) $\boxed{\text{Mapping Cone}} \longrightarrow$ $C_f = Y \sqcup_f CX$ where CX is the cone $(X \times I)/(X \times \{0\})$

and we attach this to Y along $X \times \{1\}$ via $(x,1) \sim f(x)$



Example : $X = S^{n-1}$

$C_f \longrightarrow$ attaching to Y the n-cell via $f : S^{n-1} \longmapsto Y$

<u>Proposition</u>

If (X_1, A) is a CW pair and the two attaching maps $f, g : A \mapsto X_0$ are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$$

Homotopy Extension Property

Intuition:

Consider the map $f_0 : X \mapsto Y$. Let $A \subset X$ and consider the homotopy on A $f_t : A \mapsto Y$ with $f_0 = f|_A$. We would like to extend this to a homotopy on X as a whole with f_0.

Def: Homotopy Extension

$A \subset X$

(X, A) has the homotopy extension property (h.e.p) if $\forall Y$, $\forall f_0 : X \longmapsto Y$, $\forall$ homotopy $G : A \times I \longmapsto Y$,

$$G(a,0) = f_0(a)$$

we can extend G to a homotopy $F : X \times I \longmapsto Y$

s.t $f(x,0) = f_0(x)$

$$\Updownarrow$$

$\overbrace{\phantom{X \times \{0\} \longmapsto Y}}^{f_0(x) = y}$

(X, A) has the h.e.p if every pair of maps $X \times \{0\} \longmapsto Y$ and $\underbrace{A \times I \longmapsto Y}_{f_t(a)}$ that agree on $\underbrace{A \times \{0\}}_{f_0 = f_0|_A}$ can be extended to a map $\underbrace{X \times I \longmapsto Y}_{f_t : X \longmapsto Y}$.

Lemma:

$$\begin{tikzcd} & X \arrow[dr, dotted] & \\ A \arrow[ur] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

$A \subset X$ top space.

$\forall Y$, any map $f : A \longmapsto Y$ extends to $X \longmapsto Y$ if and only if A is a retract of X

Proof:

$\Leftarrow$ Suppose A is a retract of X via $r : X \longmapsto A$ s.t $r|_A = id_A$

Then $(f \circ r) : X \longmapsto Y$ is our extension

$\Rightarrow$ Suppose, $\forall Y$ and any map $f : A \longmapsto Y$ extends to $X \longmapsto Y$.

i.e $f_t : X \longmapsto Y$ s.t $f_0|_A = f$

Then, let $Y = A$ and $f = id_A$ i.e $id_A : A \longmapsto A$ extends to $f_t : X \longmapsto A$ s.t $f_0|_A = id_A$ $\Rightarrow$ A is a retract of X

Lemma :

A pair (X, A) has the h.e.p if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof : By h.e.p, the identity map

$\Rightarrow$: $X \times \{0\} \cup A \times I \longmapsto X \times \{0\} \cup A \cup I$ extends to a map $X \times I \longmapsto X \times \{0\} \cup A \times I$

$\therefore$ $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

$\Leftarrow$ if A is closed : consider any two maps $X \times \{0\} \longmapsto Y$ and $A \times I \longmapsto Y$ that agree on $A \times \{0\}$. They combine to give a map $X \times \{0\} \cup A \times I \longmapsto Y$ which is continuous by continuity on the closed sets $X \times \{0\}$ and $A \times I$.

Compose this map $X \times \{0\} \cup A \times I \longmapsto Y$ with a retraction $X \times I \longmapsto X \times \{0\} \cup A \times I$ (we have this via hypothesis)

We get an extension $X \times I \longmapsto Y$

$\therefore$ (X, A) has the h.e.p.

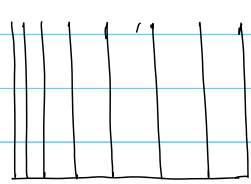
Properties

(i) $\left.\begin{array}{l} \text{H.e.p} \\ X - \text{hausdorff} \end{array}\right\} \Rightarrow A$ is closed in X

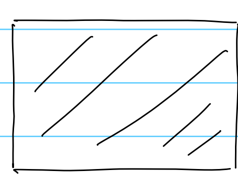
Non-example : (X, A) does not have h.e.p

(1) (I, A) where $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

There is no continuous retraction $I \times I \longmapsto I \times \{0\} \cup A \times I$ because of the structure of (I, A) near 0.



$I \times \{0\} \cup A \times I$



$I \times I$

Consider the ball ~~x_0~~ $B = B(x_0, \frac{1}{2})$

Then, $\exists \delta > 0$ s.t $r(B(x_0, \delta)) \subset B$

$\gamma \longrightarrow$ path in B from x_0 to x_1

but x_0 and x_1 are in diff components of $C \cap B$

path at $t=1$

$B(x_0, \delta)$

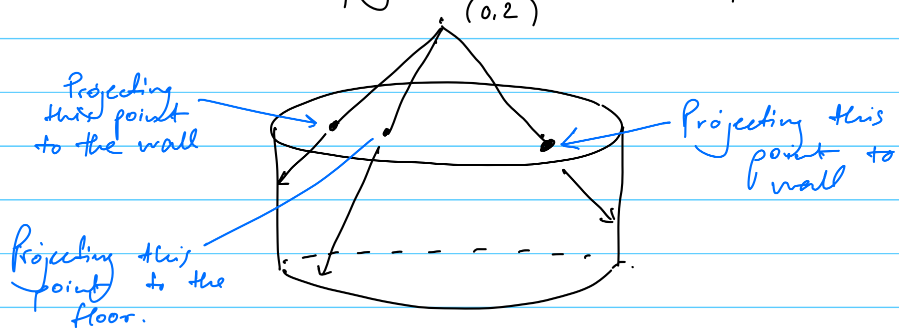
Proposition

If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the h.e.p.

Proof :

First, note that $\exists$ a retraction $r : D^n \times I \longrightarrow D^n \times \{0\} \cup \partial D^n \times I$

for ex - radial projection from the point $(0,2) \in D^n \times \mathbb{R}$



Now, set $r_t = tr + (1-t)\mathbb{1}$ is a deformation retraction of $D^n \times I$ onto $D^n \times \{0\} \cup \partial D^n \times I$.

Now, with this, we have a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ since $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$.

If we perform the def. ret. of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \times A^n) \times I$ during the t-interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, this infinite concatenation of homotopies is a def. ret. of $X \times I$ onto $X \times \{0\} \cup A \times I$.

Proposition

If the pair (X, A) satisfies h.e.p and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.

Proof :

Let $f_t : X \to X$ be the homotopy extending a contraction of A with $f_0 = \mathbb{1}$.

Now, $f_t(A) \subset A$ $\forall t$, so the composition

$$q \circ f_t : X \to X/A$$

sends A to a point and so factors as a composition

$$X \xrightarrow{q} \underbrace{X/A \longrightarrow X/A}$$

Denote this by $\bar{f}_t : X/A \longrightarrow X/A$)

So, $q f_t = \bar{f}_t q$

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{f}_t"'] & X/A \end{tikzcd}$$

When $t=1$, $f_1(A)$ equals to a point (since f_t is homotopy extension of the contraction of A), so f_1 induces a map

$g : X/A) \longrightarrow X$ with $gq = f_1$

$$\begin{tikzcd} X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{f}_1"'] \arrow[ur, "g"] & X/A \end{tikzcd}$$

So, $qg = \bar{f}_1$ since $qg(\bar{x}) = qgq(x)$

$= qf_1(x)$

$= \bar{f}_1 q(x)$

$= \bar{f}_1(\bar{x})$

The maps g and q are inverse homotopy equivalences as

$gq = f_1 \simeq f_0 = \mathbb{1}$ via f_t and
$qg = \bar{f}_1 \simeq \bar{f}_0 = \mathbb{1}$ via $\bar{f}_t$.

Def: $W \simeq Z$ rel Y

for (W,Y) and (Z,Y), there are maps $\varphi : W \mapsto Z$ and $\psi : Z \mapsto W$ restricting to identity on Y s.t. $\psi\varphi \simeq \mathbb{1}_W$ and $\varphi\psi \simeq \mathbb{1}_Z$ via homotopies that restrict to the identity on Y at all times.

Proposition

If (X_1, A) is a CW pair and we have attaching maps $f, g : A \mapsto X_0$ that are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

Proof:

Let $F : A \times I \mapsto X_0$ is a homotopy from f to g, consider the space $X_0 \sqcup_F (X_1 \times I)$ which have both $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ as subspaces

We can deformation retract $X_1 \times I$ onto $X_1 \times \{0\} \cup A \times I$ which induces a def retraction of

$$X_0 \sqcup_F (X_1 \times I) \text{ onto } X_0 \sqcup_f X_1$$

Similarly, $X_0 \sqcup_F (X_1 \times I)$ def retracts onto $X_0 \sqcup_g X_1$

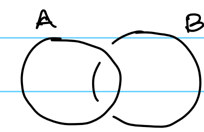
Both of them are identity on X_0 so we get the homotopy equivalence

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

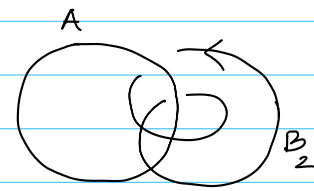
Fundamental Group

Intuition

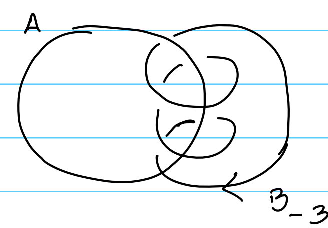
Two linked circles in $\mathbb{R}^3$:



Link B with A two times in the forward direction :



Link B with A three times in the backward direction :

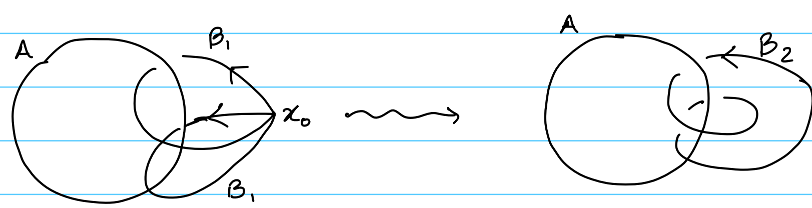


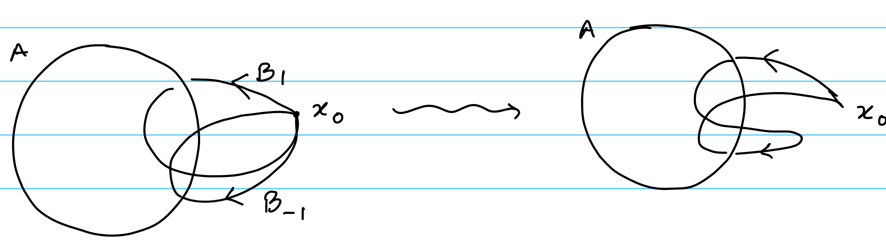
B_2 and B_{-3} are oriented circles/loops.

Two loops, B and B', starting and ending at the same point x_0 can be added to form a new loop that travels around B and B'.

So, $B_1 + B_1 = B_2$

$B_1 + B_{-1} = B_0 \leftarrow$ unlinked from A





More generally, $B_m + B_n = B_{m+n}$

Paths and Homotopy of paths

Def : Path in X

A continuous map $f : I \longmapsto X$ where $I = [0,1]$

Def : Homotopy

A family $f_t : I \longrightarrow X$ where $t \in I$ s.t

(1) $f_t(0) = x_0$ and $f_t(1) = x_1 \; \forall t$

(2) the associated map $F : I \times I \longmapsto X$ is continuous

We say $f_0 \simeq f_1$.

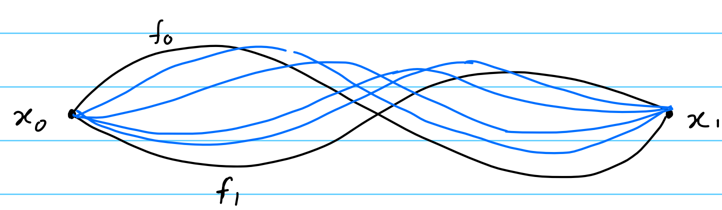
$\rightarrow$ $f_0 \simeq f_1$ means homotopic rel. $\partial I = \{0,1\}$ ie the endpoints are fixed.

Examples

(1) Linear homotopies in $\mathbb{R}^n$:

Any 2 paths f_0 and f_1 in $\mathbb{R}^n$ with endpoints ~~in~~ x_0 and x_1 are homotopic by $f_t(x) = (1-t) f_0(x) + t f_1(x)$

Here, $F(x,t) = f_t(x) = (1-t) f_0(x) + t f_1(x)$ is continuous since f_0 and f_1 are continuous. and sum ~~&~~ and scalar multiplication preserve continuity.



Non-example

$f_0, f_1 : I \longrightarrow S^1$

$\left.\begin{array}{l} f_0(t) = 1 \\ f_1(t) = e^{2\pi i t} \end{array}\right\}$ they are not path homotopic

Proposition

The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.
We denote the equivalence class of f by $[f]$ and is called the homotopy class of f.

Proof:

Reflexivity: $f \simeq f$ by homotopy $f_t = f$

Symmetry: If $f_0 \simeq f_1$ via f_t, then $f_1 \simeq f_0$ via f_{1-t}.

Transitivity: Suppose $f_0 \simeq f_1$ via f_t. and if $f_1 = g_0$ with $g_0 \simeq g_1$ via g_t, then the homotopy

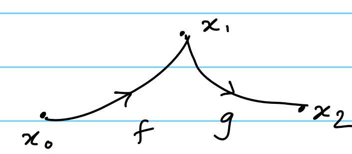
$$h_t = \begin{cases} f_{2t}, & t \in [0, \frac{1}{2}] \\ g_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}$$

The associated function $H(s,t) = h_t(s)$ is continuous.
A function on the union of 2 closed sets is continuous if it is continuous restricted to each of the 2 sets separately.

Def: Product Path

Given two paths $f, g : I \longmapsto X$ s.t $f(1) = g(0)$, the product path $f \cdot g$ first traverses f and then g:

$$f \cdot g(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$



This product path preserves homotopy classes:
if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via f_t and g_t homotopies respectively and if $f_0(1) = g_0(0)$ so that $f_0 g_0$ is well-defined then $f_t \cdot g_t$ provides the homotopy

$$f_0 g_0 \simeq f_1 g_1$$

Def : Loop

Paths $f : I \to X$ s.t $f(0) = f(1) = x_0 \in X$

$x_0 \longrightarrow$ basepoint

$\to$ The set of all homotopy classes $[f]$ of loops $f : I \to X$ at the basepoint $x_0 \in X$ is denoted $\pi_1(X, x_0)$

Proposition :

$\pi_1(X, x_0)$ is a group w.r.t the product
$$[f][g] = [f \cdot g]$$
This group is called the fundamental group of X at basepoint x_0.

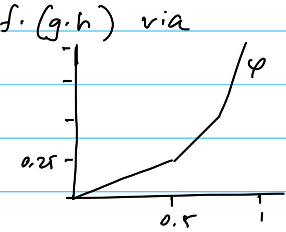
Proof :

Since the basepoint $x_0 \in X$ is fixed, the product of any two paths, f and g in $\pi_1(X, x_0)$ is defined.

Firstly, define reparametrisation of a path f to be a composition $f\varphi$ where $\varphi : I \to X$ is a continuous map s.t $\varphi(0) = 0$ and $\varphi(1) = 1$.
Reparametrisation preserves homotopy class of f since $f\varphi \simeq f$ via homotopy $f\varphi_t$ where $\varphi_t(x) = (1-t)\varphi(x) + tx$
so $\varphi_0(x) = \varphi(x)$ and $\varphi_1(x) = x$

We often show that f is a reparametrization of g to prove $f \simeq g$.

Given the paths f, g and h with $f(1) = g(0)$ and $g(1) = h(0)$, then both $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are defined.
Note $(f \cdot g) \cdot h$ is a reparametrisation of $f \cdot (g \cdot h)$ via
$f \cdot (g \cdot h) = (f \cdot g) \cdot h\,\varphi$ where φ is a continuous map s.t $\varphi : [0, \frac{1}{2}] \longmapsto [0, \frac{1}{4}]$
$\varphi : [\frac{1}{2}, 1] \longmapsto [\frac{1}{4}, 1]$



So, $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$

Given a path $f : I \mapsto X$, let c be the constant path at $f(1)$ defined by $c(s) = f(1)$, $\forall s \in I$.
Then, $f \cdot c$ is a reparametrisation of f:

$$f \cdot c(x) = \begin{cases} f(2x) & \text{for } x \in [0, \frac{1}{2}] \\ c(2x-1) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

So, $f \cdot c = f\varphi$ where $\varphi : [0, \frac{1}{2}] \longrightarrow [0, 1]$
$\varphi : [\frac{1}{2}, 1] \longrightarrow 1$

$\therefore f \cdot c \simeq f$
Similarly $c \cdot f \simeq f$ where c is constant path at $f(0)$.

Taking f to be a loop, the homotopy class of the constant path is a two-sided identity.

Now, let f be a path from x_0 to x_1. Its inverse path is $\bar{f}$ from x_1 to x_0. defined by $\bar{f}(s) = f(1-s)$
Then, $f \cdot \bar{f}$ is homotopic to a constant path via homotopy

$$h_t = f_t \cdot g_t$$

where $f_t = f$ on $[0, 1-t]$ and $f_t = f(1-t)$ on $[1-t, 1]$
and $g_t = \bar{f}_t$

Then, $f_0 = f$ and $f_1 =$ constant path c at x_0
So, h_t is a homotopy from $f \cdot \bar{f}$ to $c \cdot \bar{c}$
as

$$h_0 = f_0 \, g_0 = \begin{cases} f & \text{for } x \in [0, \frac{1}{2}] \\ \bar{f} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$h_1 = f_1 \cdot g_1 = \begin{cases} c, & x \in [0, \frac{1}{2}] \\ \bar{c}, & x \in [\frac{1}{2}, 1] \end{cases}$$

$\therefore f \cdot \bar{f} \simeq c$ (defining $c \cdot \bar{c} = c$) where $c = x_0$
Replacing f by $\bar{f}$ gives $\bar{f} \cdot f = c$
Take f to be the loop at x_0, then $[\bar{f}]$ is a 2-sided inverse for $[f]$ in $\pi_1(X, x_0)$.

Fundamental Group of X at x_0 : $\pi_1(X, x_0)$

$$\pi_1(X, x_0) = \{\text{loops from } x_0 \text{ to itself in } X\} / (\text{path homotopy})$$

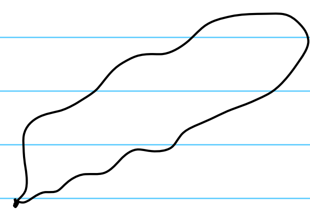
$\longrightarrow$ $[f]\cdot[g] = [fg]$

$\longrightarrow$ $[f]^{-1} = [\bar{f}]$ where $\bar{f}(t) = f(1-t)$

$\longrightarrow$ $[\text{constant}_{x_0}] = 1$

Examples

(1) $\pi_1(\mathbb{R}^n, 0) = 1$



$f : I \to \mathbb{R}^n$

We can contract it to be constant at 0 by $f_t(x) = tx$

We say $\pi_1(X) = 1$ if X is contractible

$f_t(x) = r_t \circ f$

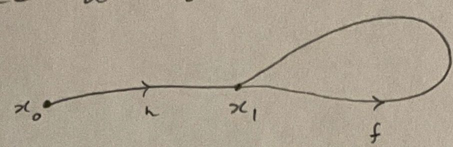
homotopy from id_X to constant map

Cta

Change of basepoint

Let x_0 and x_1 lie in the same path-component of X. Let $h : I \to X$ be a path from x_0 to x_1 with the inverse path $\bar{h}(s) = h(1-s)$. from x_1 to x_0.

Then, for each loop f based at x_1, define the loop $h \cdot f \cdot \bar{h}$ based at x_0.



Alternatively, we can define a general n-fold product $f_1 \cdot \dots \cdot f_n$ in which the path f_i is traversed in $\left[\frac{i-1}{n}, \frac{i}{n}\right]$.

Then define the change of basepoint map $\beta_h : \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $\beta_h[f] = [h \cdot f \cdot \bar{h}]$

Proposition: The map $\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ is an isomorphism.

So, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Proof: homomorphism as

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}]$$
$$= \beta_h[f]\,\beta_h[g]$$

This has the inverse $\beta_{\bar{h}}$ as

$$\beta_h \beta_{\bar{h}}[f] = \beta_h[\bar{h} \cdot f \cdot h] = [h \cdot \bar{h} \cdot f \cdot h \cdot \bar{h}]$$
$$= [f]$$

Similarly, $\beta_{\bar{h}} \beta_h [f] = [f]$

Def: Simply connected

A space is simply connected if it is $\boxed{\text{path connected}}$ and has $\boxed{\text{trivial fundamental group.}}$

ie the constant path

P

<u>Proposition</u>

A space X is <u>simply connected</u> iff there is a unique homotopy class of paths connecting any two points in X.

<u>Proof</u> :

$\Rightarrow$: Need to show uniqueness.

Suppose

Let f and g be 2 paths from x_0 to x_1

Then $f \simeq f \cdot \bar{g} \cdot g \simeq g$ since the loops $\bar{g} \cdot g$ and $f \cdot \bar{g}$ are each homotopic to constant loops, given $\pi_1(X) = 0$

$\Leftarrow$: If there is only one homotopy class of ~~paths~~ loops at x_0, then all loop at x_0 are homotopic to the constant loop

$\therefore \; \pi_1(X, x_0) = \pi_1(X) = 0$

If X is path connected, then $\pi_1(X, x_0)$ is independent of x_0. We write it as $\pi_1(X)$.

Induced Homomorphism

Def: Induced Homomorphism

Suppose, $\varphi : X \to Y$ is a map taking basepoint $x_0 \in X$ to the basepoint $y_0 \in Y$

We say $\varphi : (X, x_0) \longmapsto (Y, y_0)$

Then, φ induces a homomorphism

$$\varphi_* : \pi_1(X, x_0) \longmapsto \pi_1(Y, y_0)$$

defined by composing the loops $f : I \to X$ based at x_0 with φ :

$$\varphi_*([f]) = [\varphi f]$$

→ **Well-defined :**

Homotopy f_t of loops at x_0 yields a homotopy φf_t of loops based at y_0

$$\therefore \quad \varphi_*([f_0]) = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$$

→ **φ_* is a homomorphism :**

$$\varphi_*(f \cdot g) = \varphi(f \cdot g)$$
$$= \varphi f \cdot \varphi g$$
$$= \varphi_*(f) \cdot \varphi_*(g)$$

→ both functions have values

$\varphi f(2s)$, $0 \leq s \leq \frac{1}{2}$

$\varphi g(2s-1)$, $\frac{1}{2} \leq s \leq 1$

Properties of induced homomorphisms

(1) $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} Z, z_0$

$$(\varphi\psi)_* = \varphi_* \psi_* : \pi_1(X, x_0) \longmapsto \pi_1(Z, z_0)$$

Proof :

$$(\varphi\psi)f = \varphi(\psi f)$$

(2) $\mathbb{1}_* = \mathbb{1}$ which is saying $\mathbb{1} : X \to X$ induces $\mathbb{1} : \pi_1(X, x_0) \to \pi_1(X, x_0)$

(3) If φ is a homeomorphism with inverse φ^{-1} then φ_* is an isomorphism with inverse $(\varphi^{-1})_*$ since

$$\varphi_* \varphi^{-1}_* = (\varphi\varphi^{-1})_* = \mathbb{1}_* = \mathbb{1} \quad \text{and similarly} \quad \varphi^{-1}_* \varphi_* = \mathbb{1}$$

(4) Let $\varphi, \psi : X \longrightarrow Y$.
If φ and ψ are homotopic, then $\psi_* = \varphi_*$

Proof: $\varphi_*[f] = [\varphi f]$
$= [\psi f]$ (via homotopy of φ and ψ)
$= \psi_*[f]$

(5) Proposition:

If a space X retracts onto a subspace A, then the induced homomorphism $i_* : \pi_1(A, x_0) \longmapsto \pi_1(X, x_0)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. If A is a deformation retract of X, then i_* is an isomorphism

Proof:

Suppose, X retracts onto $A \subset X$. via $r : X \longrightarrow A$
Then $ri = id_A$
So, $(ri)_* = r_* i_* = \mathbb{1}$

Therefore i_* is injective

Suppose $i_*(f) = 1$ for some $f \in \pi_1(A, x_0)$
Then, $(r_* i_*)(f) = r_*(1) = 1$. But $r_* i_* = \mathbb{1}$
so $f = 1$.

Now, suppose X def. retracts onto A via $r_t : X \longmapsto X$
So, $r_0 = id_X$, $r_t|_A = id_A$ and $r_1(X) \subset A$
then, for any loop $f : I \longrightarrow X$ based at $x_0 \in A$, the composition $r_t f$ gives a homotopy of f to a loop in A, so i_* is also surjective.

as $r_1(X) \subseteq A$

i.e for any $f : I \to X$, first def retract to $f' : I \to A$
where $f' = r_1 of$. Then $i_*(f') = f' \in \pi_1(X, x_0)$
and $[f'] = [f]$ by the homotopy r_t.

Lemma :

If a space X is the union of a collection of path connected open sets A_α, each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

Proof :

Consider a loop $f: I \longrightarrow X$ at x_0.

Partition I into $0 = s_1 < s_2 < \dots < s_m = 1$ s.t each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α

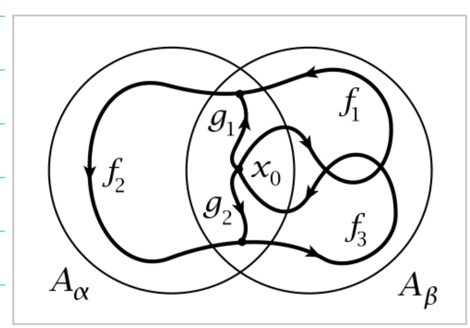
$\hookrightarrow$ Since f is continuous, each s in I has an open nbhd $V_s \subset I$ s.t f maps V_s to ~~A_α~~ some A_α. We can take $V_s \subset I$ s.t f maps $\overline{V_s}$ (closure of V_s) to a single V_α. The endpoints of this finite set of intervals will define the partition $0 = s_1 < s_1 \dots < s_m = 1$

We denote A_i to be the set containing $f([s_{i-1}, s_i])$ and we let f_i be the path obtained by restricting $f|_{[s_{i-1}, s_i]}$

Now, f is the composition $f_1 \cdots\cdots f_m$ with f_i, a path in A_i. Since $A_i \cap A_{i+1}$ is path connected, we can find a path $g_i \in A_i \cap A_{i+1}$ from x_0 to the point $f(s_i) \in A_i \cap A_{i+1}$. Then, the loop

$$(f_1 \cdot \overline{g_1}) \cdot (\overline{g_1} \cdot f_2 \cdot \overline{g_2}) \cdot \cdots\cdots (g_{m-1} \cdot f_m)$$

is homotopic to f and is a composition of loops that each lie in a single A_i.



Def: Basepoint Preserving Homotopy

Consider a homotopy φ_t taking $A \subset X$ to a subspace $B \subset Y$ for all t, then we speak of maps of pairs

$$\varphi_t : (X, A) \longmapsto (Y, B)$$

A basepoint-preserving homotopy $\varphi_t : (X, x_0) \longmapsto (Y, y_0)$ is the case where $\varphi_t(x_0) = y_0$ $\forall t$.

(6) If $\varphi_t : (X, x_0) \longmapsto (Y, y_0)$ is a basepoint preserving homotopy, then $\varphi_{0*} = \varphi_{1*}$

Proof:
$$\begin{aligned}\varphi_{0*}[f] &= [\varphi_0 f] \\ &= [\varphi_1 f] \quad \text{(via homotopy } \varphi_t f) \\ &= \varphi_{1*}[f]\end{aligned}$$

Def: Homotopy Equivalence for spaces with basepoints

We say $(X, x_0) \simeq (Y, y_0)$ if there are maps $\varphi : (X, x_0) \longmapsto (Y, y_0)$ and $\psi : (Y, y_0) \longmapsto (X, x_0)$ with homotopies $\varphi\psi \simeq \mathbb{1}_{(Y, y_0)}$ and $\psi\varphi \simeq \mathbb{1}_{(X, x_0)}$. through maps that fix the basepoint.

In this case, the induced maps on π_1 satisfy

$$\varphi_* \psi_* = (\varphi\psi)_* = \mathbb{1}_* = \mathbb{1}$$

$$\psi_* \varphi_* = (\psi\varphi)_* = \mathbb{1}_* = \mathbb{1}$$

$\therefore$ φ_* and ψ_* are ~~a~~ inverse isomorphisms

$\therefore$ $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$

What if φ_t does not ~~at~~ send x_0 to a fixed $y_0 \in Y$ for all t? This means the basepoint in X is not always mapped to the same point by a homotopy.

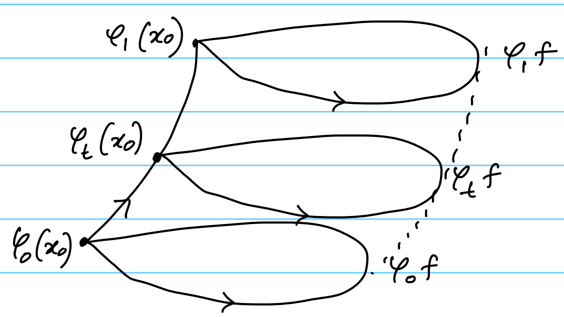
Lemma:

If $\varphi_t : X \to Y$ is a homotopy and h is the path $\varphi_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the three maps in the diagram satisfy $\varphi_{0*} = \beta_h \varphi_{1*}$

$$\begin{tikzcd}
 & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\
\pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[r, "\varphi_{0*}"'] & \pi_1(Y, \varphi_0(x_0))
\end{tikzcd}$$

Proof:

Let h_t be the restriction of h to the interval $[0,t]$ (with a reparametrization so that domain of h_t is $[0,1]$): So. $h_t(s) = h(ts)$ where $h : I \to Y$ with $h(\tilde{t}) = \varphi_{\tilde{t}}(x_0)$



Then, if f is a loop in X at basepoint x_0, then the product $h_t \cdot (\varphi_t f) \cdot \bar{h}$ gives a homotopy of loops at $\varphi_0(x_0)$.

Restricting this ~~to~~ to $t = 0$ and $t = 1$, we see $\varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f]))$

Theorem :

If $\varphi : X \longmapsto Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \longmapsto \pi_1(Y, \varphi(x_0))$ is an isomorphism for $\forall x_0 \in X$.

Proof :

Let $\varphi : X \longmapsto Y$ be a homotopy equivalence

So, Let $\psi : Y \longmapsto X$ be the homotopy inverse

So, $\varphi\psi \simeq \mathbb{1}$

$\psi\varphi \simeq \mathbb{1}$

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi\psi\varphi(x_0))$$

Given $\psi\varphi \simeq \mathbb{1}$, then $\psi_*\varphi_* = \beta_h$ for some h by the previous lemma. $\Rightarrow \psi_*\varphi_*$ is an isomorphism

Since $\psi_*\varphi_*$ is an isomorphism

φ_* is injective.

Similarly, with $\psi_*\varphi_*$, we conclude ψ_* is injective.

$\therefore$ φ_*, ψ_* are injections and $\psi_*\varphi_*$ is an isomorphism. so φ_* is a surjection too.

(4) $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$

Proof :

A path $I \longrightarrow X \times Y$

is a pair of paths $(f : I \to X,\ g : I \to Y)$

Fundamental Group of the Circle

Some preliminary tools:

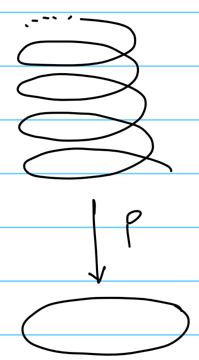
(1) Let $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ for $s \in I$ be a loop based at $(1,0)$.
Then, $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s)$ for $n \in \mathbb{Z}$.
by the definition of product path and the fact that product preserves homotopy.

(2) Compare paths in S^1 with paths in $\mathbb{R}$:
$\rightarrow$ Let $p: \mathbb{R} \mapsto S^1$ via $p(s) = (\cos 2\pi s, \sin 2\pi s)$

Visualization: first, consider the helix $s \longmapsto (\cos 2\pi s, \sin 2\pi s, s)$

Then, project $\mathbb{R}^3$ onto $\mathbb{R}^2$ by $(x,y,z) \longmapsto (x,y)$

So, projecting the helix onto $\mathbb{R}^2$ gives p



$\rightarrow$ $\boxed{\omega_n(s) = p\tilde{\omega}_n(s)}$ where $\underbrace{\tilde{\omega}_n : I \longmapsto \mathbb{R} \text{ is the path } \tilde{\omega}_n(s) = ns}_{\text{starts at } 0 \text{ and ends at } n}$

$\tilde{\omega}_n$ is called the lift of ω_n.

$p\tilde{\omega}_n(s)$ winds around the helix $|n|$ times $\rightarrow$ upwards if $n > 0$ and downwards if $n < 0$.

(3) Def: Covering Space

Given a space X, a covering space of X consists of a space $\tilde{X}$ and a map $p: \tilde{X} \longmapsto X$ satisfying:

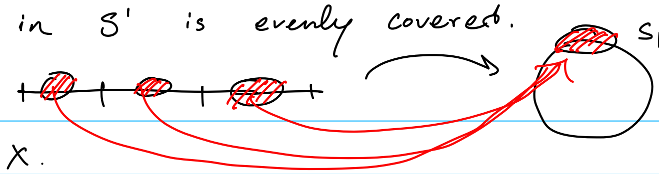
(a) For each $x \in X$, $\exists$ open neighbourhood $U \ni x$ in X s.t $p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$ where each V_α is open and each V_α is mapped homeomorphically onto U by p.

is a union of disjoint open sets (each of $p|_{V_\alpha} : V_\alpha \longrightarrow U$ is a homeomorphism

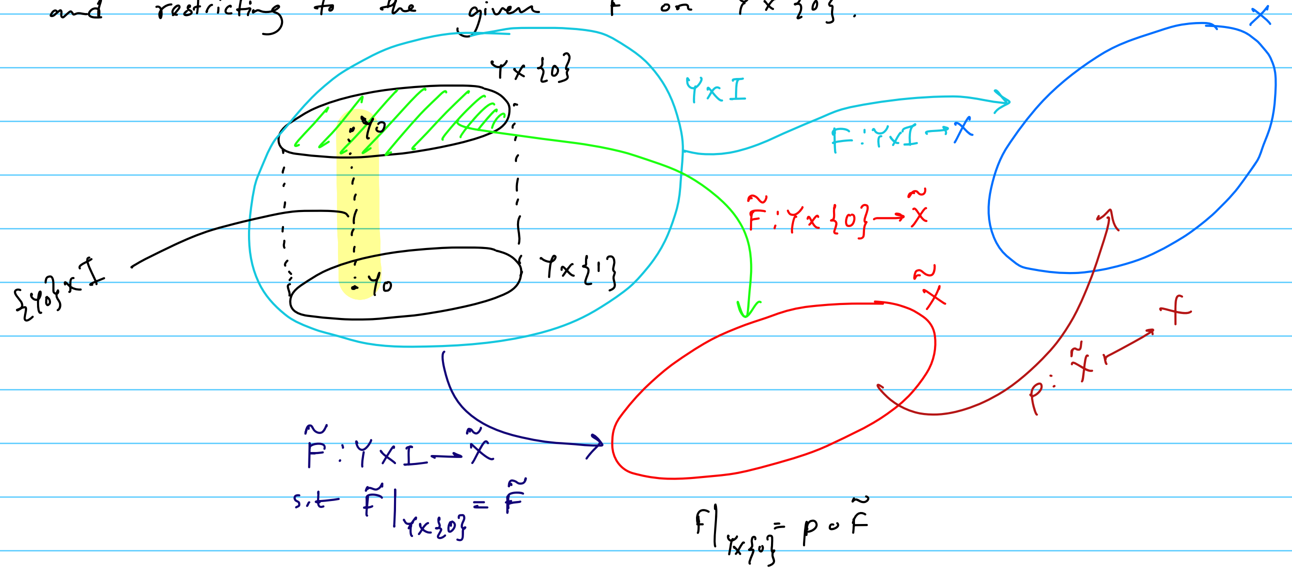
We say U is evenly covered.

Example:

(1) $p: \mathbb{R} \longmapsto S^1$, an open arc in S^1 is evenly covered.
Define it by $p(\theta) = e^{2\pi i \theta}$.



Lemma: Consider covering spaces $p: \tilde{X} \mapsto X$.
Given a map $F: Y \times I \to X$ and a map $\tilde{F}: Y \times \{0\} \longmapsto \tilde{X}$ lifting $F|_{Y\times\{0\}}$, then there is a unique map $\tilde{F}: Y \times I \longrightarrow \tilde{X}$ lifting F and restricting to the given $\tilde{F}$ on $Y \times \{0\}$.



Proof:

First, construct a lift $\tilde{F}: N \times I \longrightarrow \tilde{X}$ for some neighbourhood N of $y_0 \in Y$.

$\hookrightarrow$ Given F is continuous, $\forall\ (y_0, t) \in Y \times I$ has a product neighbourhood $N_t \times (a_t, b_t)$ s.t $F(N_t \times (a_t, b_t))$ is contained in an evenly covered neighborhood of $F(y_0, t)$.

$\longrightarrow$ around $F(y_0, t)$, $\exists$ an evenly covered neighbourhood, since $p: \tilde{X} \longmapsto X$ is a covering space.
By continuity, we can always shrink $N_t \times (a_t, b_t)$ so that $F(N_t \times (a_t, b_t))$ is inside this evenly covered nbd.

By compactness of $\{y_0\} \times I$, finitely many such $N_t \times (a_t, b_t)$ products cover $\{y_0\} \times I$. Thus, we can choose one neighbourhood N of $\{y_0\}$ and a partition $0 = t_0 < t_1 < \dots < t_m = 1$ of I s.t for each i, $F(N \times [t_i, t_{i+1}])$ is contained in an evenly covered neighbourhood U_i.

Assume, inductively, $\tilde{F}$ has been constructed on $N \times [0, t_i]$ starting with our given $\tilde{F}$ on $N \times \{0\}$. Thus, $F(N \times [t_i, t_{i+1}]) \subset U_i$, so since U_i is evenly covered, $\exists$ open set $\tilde{U}_i \subset \tilde{X}$ projecting homeomorphically onto U_i by p and containing $\tilde{F}(y_0, t_i)$ $\longrightarrow$ because

$\tilde{F}|_{N\times[0,t_i]}$ is a lift of $F|_{N\times[0,t_i]}$
so $p(\tilde{F}(y_0,t_i)) = F(y_0,t_i)$
– we know it is a lift as we have already constructed the lift on $N\times[0,t_i]$.

We can extend $\tilde{F}$ on $N\times[t_i,t_{i+1}]$ by composing $p^{-1}: U_i \to \tilde{U}_i$ (since p is a homeomorphism) with F. For this to be continuous, ~~t~~ this must agree with $\tilde{F}$ on $N\times[0,t_i]$, in particular at (y_0,t_i).

Replace N by a small enough nbd of y_0, we ~~can~~ get that $\tilde{F}(N\times\{t_i\})$ is contained in $\tilde{U}_i \Rightarrow$ by replacing $N\times\{t_i\}$ by its intersection with $\left(\tilde{F}|_{N\times\{t_i\}}\right)^{-1}(\tilde{U}_i)$. Then define $\tilde{F}$ on $N\times[t_i,t_{i+1}]$ to be the composition of F with $p^{-1}: U_i \to \tilde{U}_i$.

Continuing, we get $\tilde{F}: N\times I \to \tilde{X}$, a lift, for some neighbourhood N of y_0.

Next we show uniqueness of this lift. We prove for when Y is a point. ~~the~~ Since Y is a point, we suppress it from our notation.

Let $\tilde{F}$ and $\tilde{F}'$ be 2 lifts of $F: I\to X$ s.t $\tilde{F}(0) = \tilde{F}'(0)$.

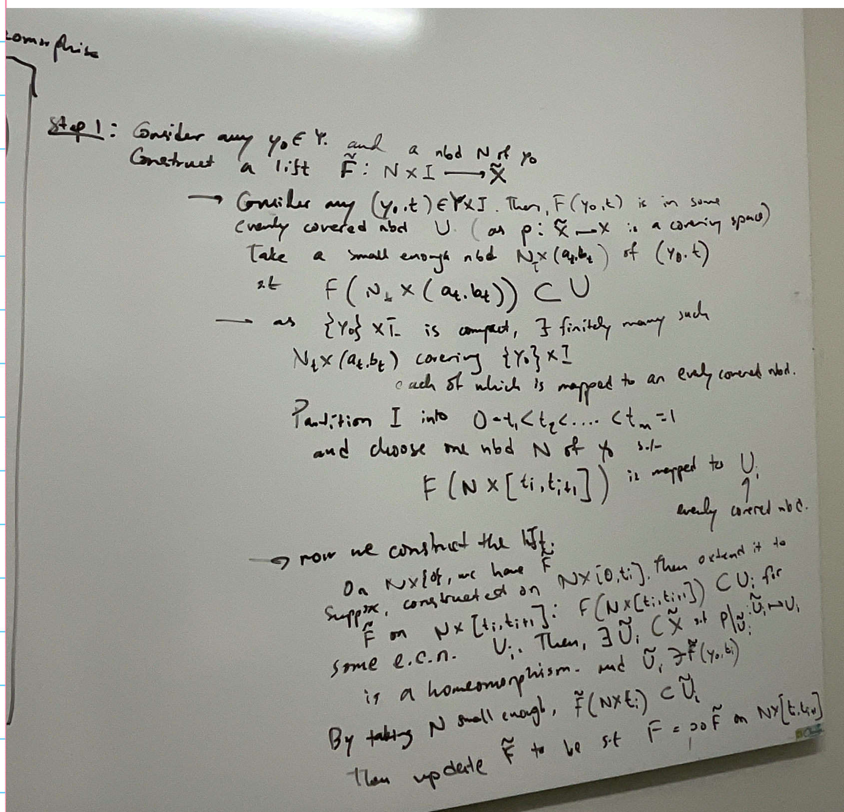
Again, we choose a partition $0 = t_1 < t_2 < \dots < t_m = 1$ of I s.t for each i, $F([t_i,t_{i+1}])$ is contained in evenly covered nbd U_i.

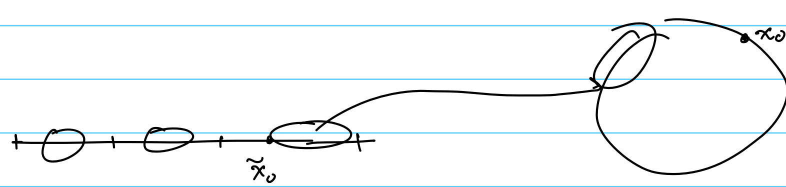
Assume inductively that $\tilde{F} = \tilde{F}'$ on $[0,t_i]$. As $[t_i,t_{i+1}]$ is connected, so is $\tilde{F}([t_i,t_{i+1}]) \Rightarrow$ it must lie in single one of the disjoint open sets $\tilde{U}_i$ projecting homeomorphically to U_i. By same logic, $\tilde{F}'([t_i,t_{i+1}])$ lies in a single $\tilde{U}_i$ and it must be the same one as $\tilde{F}'(t_i) = \tilde{F}(t_i)$. As p is injective on $\tilde{U}_i$ and $p\tilde{F} = p\tilde{F}'$, we get $\tilde{F} = \tilde{F}'$ on $[t_i,t_{i+1}]$. Continuing this way, $\tilde{F} = \tilde{F}'$.

Lastly, observe that since the lift $\tilde{F}$ constructed on sets of the form $N\times I$ is unique when restricted to each segment $\{y\}\times I$, they ~~a~~ must agree when two such sets $N\times I$ overlap.

$\therefore$ We have a well-defined lift $\tilde{F}$ on all of $Y \times I$.

$\tilde{F}$ is continuous as it is continuous on each $N \times I$. and unique as it is unique on each segment $\{y\} \times I$.





Path Lifting Property

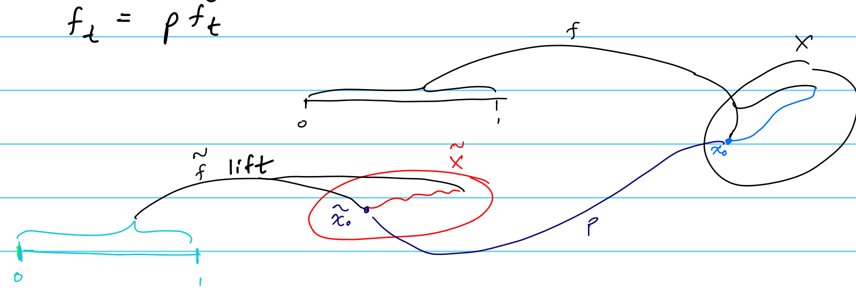
!! **Lemma**: Consider covering spaces $p: \tilde{X} \mapsto X$.

(1) For each path $f: I \to X$ s.t $f(0) = x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lift $\tilde{f}: I \to \tilde{X}$ starting at $\tilde{x}_0$.

Hence, $f = p\tilde{f}$.

(2) For each homotopy $f_t : I \to X$ of paths s.t $f_t(0) = x_0$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lifted homotopy $\tilde{f}_t : I \to \tilde{X}$ of paths starting at $\tilde{x}_0$.

Hence, $f_t = p\tilde{f}_t$



$\tilde{f}: I \longmapsto \tilde{X}$

$p\tilde{f} : I \longmapsto X$, $f: I \longmapsto X$

$\tilde{f}(0) = \tilde{x}_0$ $\quad f(0) = x_0$

$p\tilde{f}(0) = x_0$

$f = p\tilde{f}$

Proof:

(1) follows from prev. lemma when Y is a point

(2) Let $Y = I$.

Then for the homotopy $f_t : I \to X$, we have a map $F: I \times I \mapsto X$ with $F(s,t) = f_t(s)$.

We get a unique lift $\tilde{F}: I \times \{0\} \mapsto \tilde{X}$ using part (1).

Then, by prev. lemma, we get a unique lift $\tilde{F}: I \times I \to \tilde{X}$.

The restrictions $\tilde{F}|_{\{0\} \times I}$ and $\tilde{F}|_{\{1\} \times I}$ are paths lifting constant paths, so they must also be constant by uniqueness of part (1).

So, $\tilde{f}_t(s) = \tilde{F}(s,t)$ is a homotopy of paths and $\tilde{f}_t$ lifts f_t as $F = p\tilde{F}$

We set $\tilde{X}$ to be $\mathbb{R}$ here as $p: \mathbb{R} \to S^1$ is a covering space.

Theorem: $\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1,0)$. So, $\pi_1(S^1) \cong \mathbb{Z}$ as a group.

Proof: Let $f: I \longrightarrow S^1$ be a loop at the basepoint $x_0 = (1,0)$ which is one element of the group $\pi_1(S^1, x_0)$.

Then, $\underbrace{\exists \text{ a lift}}_{\text{by previous lemma (1)}}$ $\tilde{f}$ starting at 0 and must end at some integer n since $p\tilde{f}(1) = f(1) = x_0$ and $\underbrace{p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}}_{\text{recall } p(s) = e^{2\pi i s}}$.

Another path in $\mathbb{R}$ from 0 to n is $\underbrace{\tilde{\omega}_n}_{\text{recall } \tilde{\omega}_n(s) = ns}$ and $\tilde{f} \simeq \tilde{\omega}_n$ via the linear homotopy $(1-t)\tilde{f} + t\tilde{\omega}_n$. Compose the homotopy with p gives the homotopy $\tilde{f} \simeq \tilde{\omega}_n$ so $[f] = [\omega_n]$.

$\therefore$ for any loop f, $f = [\omega_n]$. ~~Ca~~ Is n fixed here? Yes.

Next, we show that n is uniquely determined by $[f]$: Suppose $f \simeq \omega_n$ and $f \simeq \omega_m$. Let f_t be a homotopy from $\omega_m = f_0$ ~~to~~ to $\omega_n = f_1$.

Then, $\underbrace{\text{this } f_t \text{ lifts}}_{\text{by previous lemma (2)}}$ to a homotopy $\tilde{f}_t$ of paths starting at 0.

The uniqueness of $\tilde{f}$ (by prev. lemma) implies that $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. Since $\tilde{f}_t$ is a homotopy of paths, the endpoint $\tilde{f}_t(1)$ is independent of t. For $t=0$, this endpoint is m and for $t=1$, it is n. So, $m = n$.

The fact that this group is generated by $\omega(s)$ is obvious from noting that $[\omega]^n = [\omega_n]$

Theorem: <u>Fundamental Theorem of Algebra</u>

Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof:

Consider an arbitrary polynomial $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n$.
Suppose, $p(z)$ has no roots in $\mathbb{C}$ (for contradiction)

Since $p(z)$ has no roots in $\mathbb{C}$, then $\forall r \in \mathbb{R}$,

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|} \quad \text{is a loop in } S^1 \subset \mathbb{C} \text{ based at } 1.$$

$$\hookrightarrow f_r(0) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1$$

$$f_r(1) = \frac{p(r\cos(2\pi) + ri\sin(2\pi))/p(r)}{(\dots)} = 1$$

Then, as r varies, f_r is a homotopy of loops with basepoint 1.
For $r = 0$, f_0 is the trivial loop constant at 1.

$\therefore [f_r] = 0 \quad \forall r \quad$ in $\pi_1(S^1)$

$\therefore p(z) \longleftrightarrow 0 \in \pi_1(S^1) \longrightarrow$ 1

Now, consider a large r s.t $r > |a_1| + \dots + |a_n|$ and $r > 1$
Then, for $\underline{|z| = r}$, $p(z)$ has no solution on $|z| = r$:

$$|z^n| > (|a_1| + \dots + |a_n|)|z^{n-1}| > |a_1 z^{n-1}| + \dots + |a_n| \geq |a_1 z^{n-1} + \dots + a_n|$$

$$\Rightarrow |z|^n > |a_1 z^{n-1} + \dots + a_n|$$

$\Rightarrow p_t(z) := z^n + t(a_1 z^{n-1} + \dots + a_n)$. $\forall t \in I$ has no root on the circle $|z| = r$ $\longrightarrow$ this is a deformation of our polynomial to z^n

Then, redefine $f_r(s) := \dfrac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}$

Let t go from 1 to 0, we find a homotopy from the loop f_r to $\omega_n(s) = e^{2\pi i n s}$.

But $[w_n] = [w]^n$ $\therefore$ $p(z) \longleftrightarrow n \in \pi_1(S^1) \longrightarrow$ ②

$\Rightarrow [w_n] = [f_r] = 0$ using ① and ②

$\therefore n = 0.$ $\longrightarrow$ contradiction as we assumed the degree was n.

Theorem: Browser Fixed Point Theorem in 2 dimensions

Every continuous map $h: D^2 \longmapsto D^2$ has a fixed point i.e $x \in D^2$ s.t $h(x) = x$.

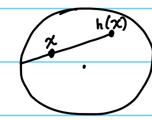
Proof:

Suppose, $\forall x \in D^2$, $h(x) \neq x$.

Define $r: D^2 \longmapsto S^1$ (where $\partial D^2 = S^1$)

to be the point where the line from $h(x)$ through x meets S^1:

$$r(x) = \frac{x - h(x)}{\| x - h(x) \|}$$



Clearly, r is continuous. Also, $r(x) = x$ $\forall x \in S^1$.

Thus, r is a retraction of D^2 onto S^1.

However, no such retraction exists.

$\hookrightarrow$ Let $f_0 \in \pi_1(S^1)$

In D^2, $f_0 \simeq$ constant loop by linear homotopy

$f_t(x) = (1-t) f_0(x) + t x_0$ $\leftarrow$ basepoint of f_0

Since $r = id$ on S^1, $r \circ f_t$ is a homotopy in S^1 from $r \circ f_0 = f_0$ to the constant loop at x_0, since r is a retraction of D^2 onto S^1.

But this contradicts the fact that $\pi_1(S^1)$ is non-zero.

Theorem: <u>Borsak-Ulam Theorem in 2 dimensions</u>

For every continuous map $f: S^2 \mapsto \mathbb{R}^2$, $\exists$ a pair of antipodal points x and $-x$ in S^2 s.t $f(x) = f(-x)$.

<u>OR</u>

<u>Weather Theorem</u>

At any moment, there exists a pair of antipodal points on Earth s.t they have

<u>Proof</u>:

Suppose not for $f: S^2 \longmapsto \mathbb{R}^2$

Define $g: S^2 \longmapsto S^1$ by $g(x) = \dfrac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$. Notice: $-g(x) = g(-x)$.

Let the loop η in $S^2 \subseteq \mathbb{R}^3$ be $\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)$ → circle the equator of S^2 once.

and let $h: I \longmapsto S^1$ be the composed loop $h = g \circ \eta$

Now, $g(-x) = -g(x) \Rightarrow h(s + \frac{1}{2}) = -h(s) \quad \forall s \in [0, \frac{1}{2}]$.

Now, the loop h can be lifted to $\tilde{h}: I \longrightarrow \mathbb{R}$.

Since $h(s+\frac{1}{2}) = -h(s)$, $\tilde{h}(s+\frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$ for some odd integer.

Now, q is independent of s: q depends on $s \in [0, \frac{1}{2}]$ continuously but can take on odd integer values $\Rightarrow$ it must be constant.

Also, $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$

$\therefore$ h represents q times the generator of $\pi_1(S^1)$

Since q is odd, h is not nullhomotopic

But $h = g \circ \eta : I \longrightarrow S^2 \longmapsto S^1$ and η is nullhomotopic in S^2 →

Shrink it along the surface.

$\therefore$ $g \circ \eta$ is null homotopic

$\therefore$ h is nullhomotopic ⟶ contradiction.

Free Product of Groups

First, we fix some notation:

(1) $G = \langle X | R \rangle$ is a group.

$X \longrightarrow$ set of generators

$R \longmapsto$ set of relations

Example 1: $G = \langle a, b \mid a^5 b^{-1} a b^3 = 1,\ b^7 a^9 = 1 \rangle$

$= \langle a, b \rangle \big/$ normal subgroup generated by $a^5 b^{-1} a b^3$, $b^7 a^9$

Example 2: $\mathbb{Z} = \langle g \rangle$

$\mathbb{Z}/n = \langle g \mid g^n \rangle$

(2) Product of groups:

Given a collection of groups G_α, $\alpha \in A$, the product is $\prod_{\alpha \in A} G_\alpha$ which can be regarded as functions $\alpha \longmapsto g_\alpha \in G_\alpha$.

↳ Suppose $(g_1, g_2, g_3, \ldots\ldots) \in \prod_{\alpha \in A} g_\alpha$

Then, this corresponds to a function f

s.t $f(\alpha) = g_\alpha \in G_\alpha$. So, $f(1) = g_1$, $f(2) = g_2$, ...

↳ $(g_1, g_2, \ldots\ldots) \cdot (h_1, h_2, \ldots\ldots) = f \cdot h$

as $f(1) \cdot h(1) = g_1 \cdot h_1$, $f(i) \cdot h(i) = g_i \cdot h_i$

(3) Free Product :

$\underset{\alpha}{*} G_\alpha$ consists of elements of the form $g_1 g_2 \cdots g_m$ for finite $m \geq 0$ s.t :

(1) each $g_i \in G_{\alpha_i}$

(2) $g_i \neq 1_{G_i}$

(3) g_i and g_{i+1} belong to different groups (i.e $\alpha_i \neq \alpha_{i+1}$)

→ words "$g_1 g_2 \cdots g_m$" satisfying these conditions are called reduced

→ unreduced words can be simplified to reduced ones by writing adjacent letters in the same G_{α_i} as a single letter and by cancelling trivial letters.

→ empty word = identity of $\underset{\alpha}{*} G_\alpha$.

→ Group operation : $(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$ and this should be simplified to reduced form i.e if $g_m h_1 \in G_\alpha$, then write $(g_m h_1)$ as one letter and if it is identity, we cancel it.

Ex : $(g_1 \cdots g_m)(g_m^{-1} \cdots g_1^{-1})$ = identity/empty word.

Associative :

Let W be the set of reduced words $g_1 \cdots g_m$ including empty word. For each $g \in G_\alpha$, we associate the function $L_g : W \longrightarrow W$ by multiplication on the left : $L_g(g_1 \cdots g_m) = g g_1 \cdots g_m$ (& simplify)

Property of this association $g \longmapsto L_g$ is that $L_{gg'} = L_g L_{g'}$ for $g, g' \in G_\alpha$ i.e $g(g'(g_1 \cdots g_m)) = (gg')(g_1 \cdots g_m)$ ⟶ this associativity follows from associativity in G_α.

Now $L_{gg'} = L_g L_{g'} \Rightarrow L_g$ is invertible with the inverse $L_{g^{-1}}$.

The association $g \longmapsto L_g$ is, thus, a homomorphism from G_α to the group $P(W)$ of all permutations of W. More generally, we can define :

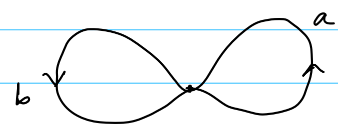
$L : W \longrightarrow P(W)$ by $L(g_1 \cdots g_m) = L_{g_1} \cdots L_{g_m}$ for each reduced word $g_1 \cdots g_m \in W$.

L is injective as the permutation $L(g_1 \cdots g_m)$ sends the empty word to $g_1 \cdots g_m$.

Now, the product operation in W corresponds under L to composition in $P(W)$ as $L_{gg'} = L_g L_{g'}$. Since composition in $P(W)$ is associative, the product in W is associative.

Eg:

(1) $\mathbb{Z} * \mathbb{Z}$:



Consider circles A and B at the basepoint x_0.

Suppose $\pi_1(A)$ is generated by a

$\pi_1(B)$ is generated by b

Then $\underbrace{a^5 b^2 a^{-3} b}$ is a loop in the $A \vee B$ described above

goes around A 5 times, around B 2 times, inverse around A 3 times, around B once

This is a word in $\mathbb{Z} * \mathbb{Z}$

Multiplication: $(b^4 a^5 b^2)(a^3 b^{-1} a) = b^4 a^5 b^2 a^3 b^{-1} a$

This is an example of a free group → the free product of any no. of copies of $\mathbb{Z}$ (can be infinite)

→ one generator for each $\mathbb{Z}$

→ the generators are called a basis for the free group

→ no. of basis elements = rank of the free group.

(2) $\mathbb{Z}_2 * \mathbb{Z}_2$ → not a free group

Here, $a^2 = b^2 =$ identity

$\mathbb{Z}_2 * \mathbb{Z}_2$ → alternating words like $a, b, ab, ba, aba, bab, \ldots$ and empty word.

Consider $\varphi: \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ which outputs the length of the word mod 2. Then φ is surjective, and its kernel is the set of words of even length. These words of even length form an infinite cyclic subgroup generated by ab or $(ba) = (ab)^{-1} \in \mathbb{Z}_2 * \mathbb{Z}_2$.

Called the infinite dihedral group.

Now, for a free product $\ast_\alpha G_\alpha$, each group G_α can be identified with a subgroup of $\ast_\alpha G_\alpha$ consisting of the empty word and the non-identity one letter words $g \in G_\alpha$.

→ ∴ the empty word is the common identity element of all the subgroup G_α (which are otherwise disjoint).

→ A consequence of associativity is that any product $g_1 \cdots g_m$ of elements $g_i \in G_\alpha$ has a <u>unique</u> reduced form.

<u>Proposition</u> :

for the free product $\ast_\alpha G_\alpha$, any collection of homomorphisms $\varphi_\alpha : G_\alpha \longmapsto H$ extends uniquely to a homomorphism

$$\varphi : \ast_\alpha G_\alpha \longleftrightarrow H$$

s.t $\varphi(g_1 \cdots g_n) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$

Example : for a free product $G * H$, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \longrightarrow G \times H$.

<u>Amalgamated Free Product</u>

$G_1, G_2, H \longrightarrow$ groups

$$\left.\begin{array}{l} f_1 : H \longrightarrow G \\ f_2 : H \longrightarrow G \end{array}\right\} \text{homomorphism}$$

Amalgamated free product : $G_1 *_H G_2 = G_1 * G_2 / f_1(h) = f_2(h) \ \forall h \in H$

<u>Ex:</u>

$$\mathbb{Z} * _{\mathbb{Z}} \mathbb{Z} = \langle g_1, g_2 \mid g_1^n = g_2^m \rangle$$

(with $\langle g_1 \rangle$ over the first $\mathbb{Z}$, $\langle g_2 \rangle$ over the second, and f_1, f_2 on the maps from the middle $\mathbb{Z}$)

$f_1(h) = g_1^n$

$f_2(h) = g_2^m$

Van Kampen's Theorem

Suppose, the space X can be decomposed as the union of a collection of path-connected, open subsets A_α, each of which contains the basepoint $x_0 \in X$.

Consider the inclusion $A_\alpha \hookrightarrow X$ which induces the homomorphisms

$$j_\alpha : \pi_1(A_\alpha) \longrightarrow \pi_1(X).$$

This can be extended to the homomorphism

$$\Phi : \ast_\alpha \, \pi_1(A_\alpha) \longmapsto \pi_1(X)$$

$$\text{s.t.} \quad \Phi(f_1 f_2 \cdots f_n) = j_{\alpha_1}(f_1) \cdots\cdots j_{\alpha_n}(f_n)$$

Consider the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ inducing $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \mapsto \pi_1(A_\alpha)$.

Then, $j_\alpha i_{\alpha\beta}(\omega) = j_\beta i_{\beta\alpha}(\omega)$ for any loop in $A_\alpha \cap A_\beta$.

and both of them are induced by the inclusion

$$A_\alpha \cap A_\beta \hookrightarrow X.$$

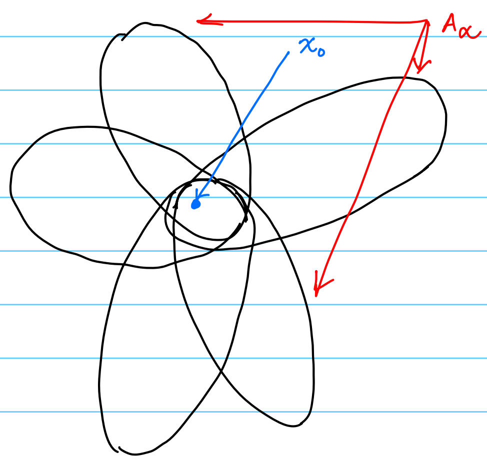
$\therefore$ kernel of Φ contains all elements of the form $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

$$\hookrightarrow \Phi\left(i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}\right)$$

$$= j_\alpha(i_{\alpha\beta}(\omega))\, j_\beta(i_{\beta\alpha}(\omega)^{-1})$$

$$= j_\alpha(i_{\alpha\beta}(\omega))\left(j_\beta(i_{\beta\alpha}(\omega))\right)^{-1}$$

$$= \text{constant loop.}$$



$$\begin{array}{c} A_\alpha \hookrightarrow X \\ \Downarrow \\ j_\alpha : \pi_1(A_\alpha) \longrightarrow \pi_1(X) \end{array} \qquad \begin{array}{c} A_\alpha \cap A_\beta \hookrightarrow A_\alpha \\ \Downarrow \\ i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \longmapsto \pi_1(A_\alpha) \end{array}$$

$$\Phi : \ast_\alpha \, \pi_1(A_\alpha) \longmapsto \pi_1(X)$$

by

$$\Phi(f_1 f_2 \cdots f_n) = j_{\alpha_1}(f_1) \cdots\cdots j_{\alpha_n}(f_n)$$

Seifert-van Kampen Theorem

(1) If X is the union of path connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then the homomorphism

$$\Phi : \ast_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$$

is surjective.

(2) In addition, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of Φ is the normal subgroup N generated by elements of the form $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$ and, hence, Φ induces an isomorphism

$$\pi_1(X) \approx \ast_\alpha \pi_1(A_\alpha)/N.$$

Proof:

(1) is true by the following:

> Lemma:
> If a space X is the union of a collection of path connected open sets A_α, each containing the basepoint $x_\alpha \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

(2). We need to prove that $\ker(\Phi)$ is N.

Def: Factorization of a loop

factorization of $[f] \in \pi_1(X)$ is a formal product $[f_1] \cdots [f_k]$

s.t (1) each $f_i \in A_\alpha$ for some α at basepoint x_α and $[f_i] \in \pi_1(A_\alpha)$

(2) the loop f is homotopic to $f_1 \cdots f_k$ in X.

The factorization of $[f]$ is a word in $\ast_\alpha \pi_1(A_\alpha)$, possibly unreduced that is mapped to $[f]$ by Φ.

Surjectivity of Φ is equivalent to saying that every $[f] \in \pi_1(X)$ has a factorization.

Def: Equivalent Factorizations

Two factorizations are equivalent if they are related by sequences of the following two moves or their inverses:

(move 1): combine adjacent terms $[f_i][f_{i+1}]$ into $[f_i \cdot f_{i+1}]$ if $[f_i]$, $[f_{i+1}]$ both belong to the same $\pi_1(A_\alpha)$

(move 2): regard $[f_i] \in \pi_1(A_\alpha)$ as lying in $\pi_1(A_\beta)$ instead if f_i is a loop in $A_\alpha \cap A_\beta$

move 1 does not change the element in $\ast_\alpha A_\alpha$ w.r.t the definition of factorization

move 2 does not change the image of this element in the quotient group $\boxed{Q := \ast_\alpha \pi_1(A_\alpha)/N}$

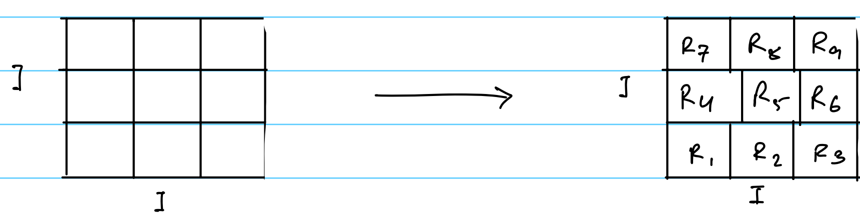
We want to prove that any two factorizations of f are equivalent. Then, we will have proven that $Q \longmapsto \pi_1(X)$ is injective $\Rightarrow$ N is the kernel of Φ $\Rightarrow$ $Q \simeq \pi_1(X)$.

Let $[f_1] \cdots [f_k]$ and $[f_1'] \cdots [f_\ell']$ be two factorizations of $[f]$. Then, the composed paths $f_1 \cdots f_k$ and $f_1' \cdots f_\ell'$ are homotopic via $F : I \times I \longrightarrow X$.

Now, $\exists$ partitions $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_1 < t_2 < \cdots < t_n = 1$ s.t each rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F into a single A_α called A_{ij} $\longrightarrow$ we get these partitions by covering $I \times I$ by finitely many rectangles $[a,b] \times [c,d]$ each mapping to a single A_α and then partitioning $I \times I$ by the union of all vertical and horizontal lines containing edges of these rectangles.

$\llcorner$ The s-partition subdivides these partitions to give the products $f_1 \cdots f_k$ and $f_1' \cdots f_\ell'$.

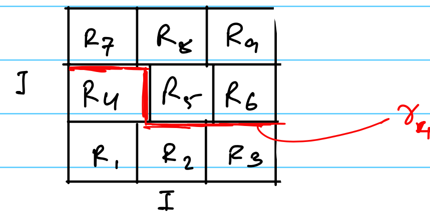
Now, F maps a nbhd of R_{ij} to A_{ij}, so we may perturb the vertical sides of the rectangles R_{ij} so that each point in $I \times I$ is in at most three R_{ij}'s :



We are perturbing only the middle rows (not the first and last — we are assuming there are at least three).
Label the rectangles $R_1, R_2, \ldots, R_{mn}$.

If γ is a path in $I \times I$ from the left to the right edge, then the restriction $F|_\gamma$ is a loop at the basepoint x_0 since F maps both the left and right edges of $I \times I$ to x_0.
Let γ_r be the path separating the first r rectangles

from the rest.



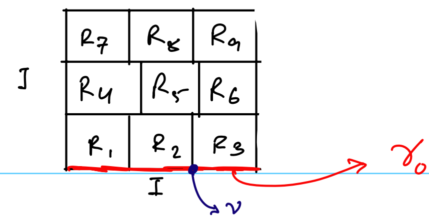
Then, γ_0 is the bottom edge of $I \times I$

γ_{mn} is the top edge.

We go from γ_r to γ_{r+1} by pushing across the rectangle R_{r+1}.

Now, consider the vertices of R_r. For each vertex v with $F(v) \neq x_0$, we choose a path g_v from x_0 to $F(v)$ that lies in the <u>intersection</u> of the two or three A_{ij}'s corresponding to the R_r's containing v.

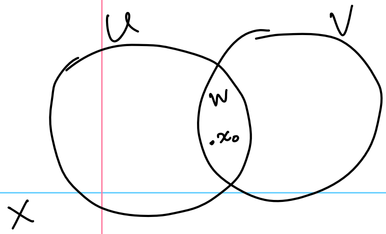
Then, we have a factorization of $[F|_{\gamma_r}]$ by inserting the appropriate paths $\bar{g}_v g_v$ into $F|_{\gamma_r}$ at successive vertices (similar to the way we did it in the proof of surjectivity). This factorization depends on our choices: consider the path between two successive vertices which can lie in 2 different A_{ij}'s since the path may be in 2 different R_{ij}'s. However, different choices of A_{ij}'s here gives equivalent factorizations. Also, the factorization for successive paths γ_r and γ_{r+1} are equivalent since pushing γ_r across R_{r+1} to γ_{r+1} changes $F|_{\gamma_r}$ to $F|_{\gamma_{r+1}}$ by a homotopy within the A_{ij} corresponding to R_{r+1} and we can choose this A_{ij} for all the segments of γ_r and γ_{r+1} in R_{r+1}.



We can arrange so that the factorization associated to γ_0 is equivalent to the factorization $[f_1] \cdots [f_k]$ by choosing the path g_v for each vertex v along the lower edge of $I \times I$ to lie not just in the two A_{ij}'s corresponding to the R_s's containing v but also in the A_α for the f_i containing v in its domain.

↳ in case v is the common endpoint of the domains of two ~~com~~ f_i's, $F(v) = x_0$, so there is no need to choose a g_v here.

Similarly, assume that the factorization associated to the final γ_{mn} is equivalent to $[f_1'] \cdots [f_\ell']$.
Since the factorization associated to all the γ_r's are equivalent, $[f_1] \cdots [f_k]$ and $[f_1'] \cdots [f_\ell']$ are equivalent.



Seifert-van Kampen Theorem

W has to be path connected.

Theorem: Let $X = U \cup V$ where $U, V \subseteq X$ open

Let $W = U \cap V$ be path connected. Pick $x_0 \in W$

Define the inclusions $i : W \hookrightarrow U$, $j : W \hookrightarrow V$ $\rightsquigarrow$ maps i_*, j_* on $\pi_1(-, x_0)$

Then,

$$\pi_1(X, x_0) = \pi_1(U, x_0) \underset{\pi_1(W,x_0)}{*} \pi_1(V, x_0) \quad \text{w.r.t } i_*, j_*$$

Proof:

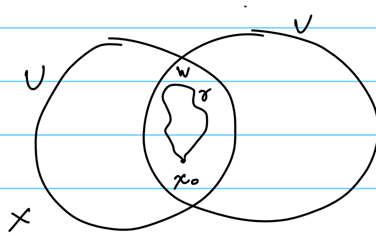
Some more notation: Let $k : U \hookrightarrow X$, $\ell : V \hookrightarrow X$ inclusions.

Then, $(k_*, \ell_*) : \pi_1(U, x_0) \times \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$

by $(k_*, \ell_*)([\gamma_1], [\varphi_2]) = [\gamma_1][\gamma_2]$

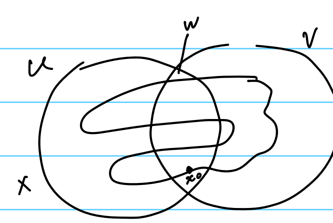
Check that the relations: $\gamma \in \pi_1(W)$

$$[i_*(\gamma)] \cdot [(j_*(\gamma)^{-1})] \longrightarrow [\gamma][\gamma]^{-1} = 1$$



$\rightsquigarrow$ homom: $\Phi : \pi_1(U, x_0) \underset{\pi_1(W,x_0)}{*} \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$

Claim: Φ is onto (i.e surjective)



Let $[\gamma] \in \pi_1(X, x_0)$

i.e $\gamma : I \longrightarrow X = U \cup V$

Then, $\exists\ 0 = s_0 < s_1 < s_2 < \dots < s_n = 1$ by compactness of I.

s.t $\gamma|_{[s_i, s_{i+1}]} \subset U$ or $\gamma|_{[s_i, s_{i+1}]} \subset V$ $\Rightarrow \gamma = \gamma_1 * \gamma_2 * \dots * \gamma_n$

and $\gamma(s_i) \in U \cap V$

Pick paths h_i from x_0 to $\gamma(s_i)$ $\longrightarrow$ $\gamma = (\gamma_1 \cdot h_1^{-1}) \cdot (h_1 \gamma_2 \cdot h_2^{-1}) \cdot (h_2 \gamma_3 h_3^{-1}) \dots \cdot (h_{n-1} \gamma_n)$

($W \longrightarrow$ path connected)

$\Rightarrow [\gamma] \in \operatorname{Im}(\Phi)$

Claim: Φ is injective

We look at the kernel:

An element in $\ker \Phi$ is $[f_1][f_2]\cdots[f_n]$ where each

$f_i : I \to U$ or V

s.t. $f_1 f_2 \cdots f_n \sim$ constant in $\pi_1(X, x_0)$

i.e. $[f_1 \cdots f_n] = 1$

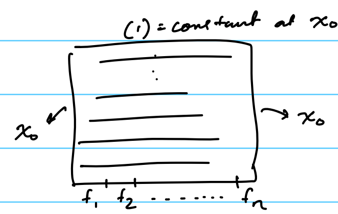
We want to show: this factorization of 1 can be related to the trivial factorization by moves: $[f_i] \cdot [f_j] \rightleftarrows [f_i \cdot f_j]$

if $f_i, f_j \to$ both in $\pi_1(U)$ or both in $\pi_1(V)$

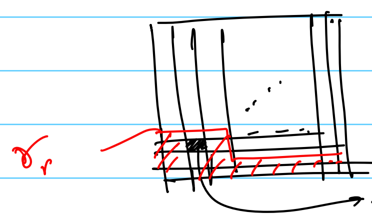
If $[f_i] \in \pi_1(W)$, regard it as either in $\pi_1(V)$ or $\pi_1(U)$

$\Rightarrow$ element in $\ker(\Phi)$ is $1 \in \pi_1(U) *_{\pi_1(W)} \pi_1(V)$.

$\exists$ homotopy $F: I \times I \to X$



$\rightsquigarrow$ split, by compactness of $I \times I$, into small rectangles whose images are in either U or V



each rectangle is mapped to either U or V

Let γ_r be the path separating the first r rectangles from the rest

γ_r ~~counts~~ comes factorized as a product of paths, each in U or V.