

Algebraic Geometry

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Contents

1	Introduction	6
1.1	Terminology	6
2	Hilbert Basis Theorem	8
3	Module-finite, Ring-finite, Field extensions	10
4	Hilbert's Nullstellensatz	14
5	Irreducible Components of Algebraic Sets	18
6	Zariski Topology	20
7	Coordinate Rings, Varieties, Morphisms	23
7.1	Coordinate rings as k -algebra homomorphisms	23
7.2	Pullback, relationship between $Map(V, W)$ and $Mor_k(O(W), O(V))$	24
7.3	Product of varieties	25
7.4	$MSpec(R)$ and abstract varieties	26
7.5	Localization	28
7.6	Coordinate Changes	30
7.7	Morphisms	30
8	Schemes	32
9	Dimensions	35
9.1	Krull Dimension	35

9.2	Noether Normalization	38
10	Sheaves	40
10.1	Presheaf, Germs, Stalks and Sheaves	40
10.2	Structure Sheaf	42
11	Local Rings and Valuations	45
11.1	Local Rings	45
12	Discrete Valuation Rings	49
13	Forms, product of Rings and operations on ideals	51
13.1	Forms	51
13.2	Direct Products of Rings	52
14	Affine plane curves and Intersection number	54
14.1	Preliminaries	54
14.2	Intersection Number	55
15	Projective Space	57
15.1	Construction of Projective Space	57
15.2	Projective Variety	59
15.3	Graded ring and importance of forms+homogenous ideals	60
15.4	Projective Nullstellensatz	61
15.5	Homogenous coordinate ring	61
16	Intersection Number in Projective Spaces, Bezout's Theorem	64

16.1	Affine and Projective Dimension Theorems	64
16.2	Numerical Polynomials, Hilbert Polynomials	65
16.3	Graded Modules	65
17	Projective Plane Curves	69
17.1	Linear System of Curves	70
17.2	Bezout's Theorem	71
17.3	Max Noether's Fundamental Theorem	71
18	Resolution of Singularities	73
19	Riemann-Roch Theorem	74
20	References	75
A	Category Theory	75
A.1	Basic terminology	75
A.2	Universal properties	78
B	Ring Theory Revision	79
B.1	Rings	79
B.2	Integral Domains and Subrings	80
B.3	Ideals	81
B.4	Maximal Ideals	83
B.5	Prime Ideals	83
B.6	Rings of Fractions and Fields of Fractions	84
B.7	Euclidean Domains and Discrete Valuation Rings	84

B.8	Principal Ideal Domains	85
B.9	Irreducible elements and Prime elements	86
B.10	Unique Factorization Domain	87
B.11	Polynomial Rings	87
B.12	Polynomial Rings over Fields	88
B.13	Irreducibility Criteria	89
C	Real Projective Space	92
D	Complex Projective Space	94

1 Introduction

Algebraic geometry is about solutions of polynomial equations and the geometric structures on the space of those solutions. We use the language and techniques from abstract algebra on these geometric objects.

Geometry becomes interesting when local properties reveal to us global properties. Algebra provides us a very powerful tool to do that.

1.1 Terminology

A field k is algebraically closed if any non-constant polynomial $f \in k[x]$ has at least one root/zero in k i.e if $f \in k[x]$, then $f(x) = \mu \prod (x - \lambda_i)^{e_i}$ where $\lambda_i \in k$ are the roots. The field \mathbb{R} is not algebraically closed as $f(x)x^2+1$ has no root in \mathbb{R} , whereas \mathbb{C} is algebraically closed.

The affine space of field k is denoted by \mathbb{A}_k^n which is the Cartesian n-product of k .

The true coordinate ring $O(\mathbb{A}^n)$ of functions on \mathbb{A}^n is the commutative ring $k[x_1, \dots, x_n]$ of polynomials with n variables.

Let $f \in k[x_1, \dots, x_n]$ be a polynomial. Then, $V(f)$ is the set of zeros of f and is called the hypersurface defined by f . If S is a set of polynomials from $k[x_1, \dots, x_n]$, then $V(S) := \{p \in \mathbb{A}_k^n \mid f(p) = 0, \forall f \in S\}$. One can check that $V(S) = \cap_{f \in S} V(f)$. When $S = \{f_1, \dots, f_r\}$, we write $V(S)$ as $V(f_1, \dots, f_r)$.

Example: Consider $k[x]$ which is a principal ideal domain. Therefore, every algebraic set can be written as the set of zeros of a single polynomial.

A subset $X \subseteq \mathbb{A}_k^n$ is called an affine algebraic set if $X = V(S)$ for some set S of polynomials in $k[x_1, \dots, x_n]$. Throughout these notes, we will use the term affine variety to mean the same thing as affine algebraic sets (although some texts refer to only *irreducible* algebraic sets as affine varieties). One can easily show that if I is the ideal in $k[x_1, \dots, x_n]$ generated by polynomials in S , then $V(S) = V(I)$. Suppose, $I = (f_1, \dots, f_n)$, then, $V(I) = \cap_{i=1}^n V(f_i)$. Some more properties:

(1) If $\{I_\alpha\}$ is a collection of ideals, then $V(\cup_\alpha I_\alpha) = \cap_\alpha V(I_\alpha)$. (2) $I \subset J \implies V(J) \subset V(I)$
(3) $V(fg) = V(f) \cup V(g)$ (4) Any finite subset of \mathbb{A}_k^n is an algebraic set (5) $V(A) = V((A))$ where (A) is the ideal generated by A .

The ideal generated by a set of functions $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ is the set $(f_1, \dots, f_m) :=$

$\{\sum_{i=1}^m g_i f_i : g_i \in k[x_1, \dots, x_n]\}$. For a subset $X \subseteq \mathbb{A}_k^n$, consider the ideal in $k[x_1, \dots, x_n]$ generated by polynomials that vanish on X . This ideal is called the vanishing ideal of X , denoted by $I(X)$. So,

$I(X) = \{f \in k[x_1, \dots, x_n] : f(a) = 0, \forall a \in X\}$. So, if $f, g \in I$, then $f + g \in I$ and for any $h \in k[x_1, \dots, x_n]$, $hf \in I$. Some more properties:

(1) $X \subset Y \implies I(Y) \subset I(X)$ (2) $I(\emptyset) = k[x_1, \dots, x_n]$, $I(\mathbb{A}^n) = \emptyset$, $I(\{a\}) = (x_1 - a_1, \dots, x_n - a_n)$.

We say f_1, \dots, f_m scheme-theoretically define the affine variety $X \subset \mathbb{A}^n$ if $I(X) = (f_1, \dots, f_m)$ i.e the ideal generated by f_1, \dots, f_m . Furthermore, the ideal I is said to set-theoretically define variety X if $X = V(I)$ if It can be easily shown that $V(I(X)) = X$. $V(-)$ and $I(-)$ allow us to switch between the geometric world and the algebraic world which is a key tool used in algebraic geometry. In particular, later on, we will see that using Hilbert's Nullstellensatz, there is no information lost after we make this switch.

We also define fractional fields. Let R be an integral domain. Its fractional field $K = \text{Frac}(R)$ is defined as the ring

$$K := \left\{ \frac{f}{g} : f, g \in R, g \neq 0 \right\}$$

.

A polynomial mapping/morphism $p : V \rightarrow W$, where $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ are varieties, is a mapping such that $(x_1, \dots, x_n) \rightarrow f(x_1, \dots, x_n) := (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$, $f_i \in k[x_1, \dots, x_n]$ and the image of the algebraic set V lies inside the algebraic set W . The mapping set $\text{Map}(V, W)$ is the set of all polynomial maps from V to W and in our case this is the set of all polynomial maps from V to W . We need polynomial mappings in order to investigate the relationships between varieties. Given X is an affine variety, an **automorphism** of X is a polynomial map $f : X \rightarrow X$ which is an isomorphism. $\text{Aut}(X)$ denotes the group of all automorphisms of X .

2 Hilbert Basis Theorem

First, we note that for $a := (a_1, \dots, a_n) \in \mathbb{A}_k^n$, $I(\{a\}) = (x_1 - a_1, \dots, x_n - a_n) \subset k[x_1, \dots, x_n]$. To see this, note that $(x_1 - a_1, \dots, x_n - a_n) \subset I(\{a\})$ which is straightforward. To see the other direction, suppose $f \in I(\{a\})$. Since $f \in k[x_1, \dots, x_n]$, we can write it as $f = \sum_{i_1, \dots, i_n \geq 0} a_{i_1 \dots i_n} x_1^{i_1} \dots x_n^{i_n}$. Since $f(a) = 0$, we can write this as $f(x) = \sum_{i_1, \dots, i_n \geq 0} b_{i_1 \dots i_n} (x_1 - a_1)^{i_1} \dots (x_n - a_n)^{i_n}$ and so $f(x) \in (x_1 - a_1, \dots, x_n - a_n)$.

Definition 1. A ring R is called Noetherian if every ideal in R is finitely generated.

Example: Fields and Principal Ideal Domains (PIDs) are Noetherian rings.

One can easily verify the following:

R is Noetherian if and only if every sequence of ideals $I_1 \subset I_2 \subset \dots$ stabilizes i.e there exists N such that $I_N = I_{N+1} = \dots$.

Proof. Forward direction: If every ideal is finitely generated then the ideal $\cup_i I_i$ is finitely generated and so the generating set of $\cup_i I_i$ must lie in some I_N . Conversely, suppose the sequence stabilizes but there exists an I that is not finitely generated. Then take a sequence of $f_i \in I$ such that $f_i \notin (f_1, \dots, f_{i-1})$ yields an increasing sequence of ideals i.e $(f_1) \subset (f_1, f_2) \subset (f_1, f_2, f_3) \subset \dots$ that does not stabilize - contradiction. \square

Theorem 1. (Hilbert Basis Theorem) If R is a Noetherian ring, then $R[x_1, \dots, x_n]$ is a Noetherian Ring.

Proof. We know $R[x_1, \dots, x_n] \cong R[x_1, \dots, x_{n-1}][x_n]$. So, if we can prove that R Noetherian implies $R[x]$ is Noetherian, by induction we will have proven that $R[x_1, \dots, x_n]$ is also Noetherian.

Suppose R is Noetherian. Let I be an ideal in $R[x]$. Let J denote the set of leading coefficients of polynomials in I . Then, given I is an ideal, J is an ideal in R . Since R is Noetherian, we can write that J is generated by the leading coefficients of $f_1, \dots, f_r \in I$. Suppose $N \in \mathbb{Z}$ such that N is greater than the degrees of all polynomials f_1, \dots, f_r . Then, for any $m \leq N$, we define J_m to be the ideal in R generated by the leading coefficients of all polynomials f in I such that $\deg(f) \leq m$. Once again, since J_m is an ideal in R , we can say that J_m is generated by the finite set of polynomials, $\{f_{mj}\}$, such that each polynomial's degree is less than or equal to m . Finally, define I' be the ideal generated by polynomials $\{f_{mj}\}$ and f_i .

We claim $I' = I$. Suppose not i.e suppose there exists elements in I that are not in I' . Let g be the minimal element such that $g \in I, g \notin I'$.

Case 1: $\deg(g) > N$. Then, there exists polynomials Q_i such that $\sum_i Q_i f_i$ has the same leading term as g . Therefore, $\deg(g - \sum_i Q_i f_i) < \deg(g)$. Clearly, $g - \sum_i Q_i f_i$ is in I' . But since g is the minimal element and $\deg(g - \sum_i Q_i f_i) < \deg(g)$, therefore $g - \sum_i Q_i f_i \in I'$, which implies $g \in I'$.

Case 2: $m := \deg(g) \leq N$. Then, there exists polynomials Q_j such that $\sum_j Q_j f_{mj}$ and g have the same leading term. Using a similar argument, we get that $g \in I'$. \square

This has the following interesting implication:

Theorem 2. An algebraic set is the intersection of a finite number of hypersurfaces.

Proof. Let $V(I)$ be an algebraic set. We prove that I is finitely generated since that implies $V(I) = V(f_1, \dots, f_r) = \cap_{i=1}^r V(f_i)$. Given k is a field, k is a Noetherian ring and by the Hilbert Basis Theorem, $k[x]$ is also Noetherian. Therefore, the ideal I in $k[x]$ is finitely generated. \square

Corollary 3. $k[x_1, \dots, x_n]$ is a Noetherian ring for any field k .

Proof. Follows from the Hilbert Basis Theorem. \square

We have some other useful corollaries:

- Any descending chain of subvarieties of \mathbb{A}^n must stabilize i.e if $V_1 \supset V_2 \supset V_3 \dots$, then there exists N such that $V_N = V_{N+1} = \dots$.

- There exists a finite subset $B \subset A$ such that $V(A) = V(B)$.

Exercise:

Define

$$R[[x]] = \{f(x) = \sum_{n=0}^{\infty} a_n x^n : a_n \in R\}.$$

Prove (1) Given $f \in R[[x]]$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and suppose there exists b_0 s.t $a_0 b_0 = 1$. Then, there exists $g \in R[[x]]$ s.t $fg = 1$. (2) Given R is Noetherian, $R[[x]]$ is also Noetherian. *Hint: Similar proof to Theorem 1, but use trailing coefficient (coefficient of the smallest power) instead of leading coefficient.*

3 Module-finite, Ring-finite, Field extensions

Definition 2. R -Module.

Let R be a ring. Let M be an abelian group $(M, +)$. Then, an R -module is M with multiplication $R \times M \rightarrow M$ such that for any $a, b \in R, m \in M, (a + b)m = am + bm, a(m + n) = am + an, (ab)m = a(bm), 1_R m = m$.

Definition 3. Submodule.

A submodule N is a subgroup of R -module, M , such that $an \in N$ for any $a \in R, n \in N$.

One can check that for any $m \in M, 0_R m = 0_M$ by noting that $0_R m = (x - x)m = xm - xm = 0_M$ for any $x \in R, m \in M$. Also, the submodule N of an R -module is an R -module itself.

Definition 4. Submodule generated by S .

Let $S := \{s_1, s_2, \dots\}$ be a set of elements of the R -module M . Then the submodule generated by S is $\{\sum_i r_i s_i \mid r_i \in R, s_i \in S\}$.

When S is finite, we denote the submodule generated by S as $\sum_i R s_i$.

Definition 5. Finiteness conditions of subrings of a ring.

Let S be a ring and let R be a subring of S .

(1) S is module-finite over R if S is finitely-generated as an R -module i.e $S = \sum_{i=1}^n R v_i$ where $v_1, \dots, v_n \in S$. More explicitly, $S = \{\sum_{i=1}^n r_i v_i \mid r_i \in R\}$, for $v_1, \dots, v_n \in S$ fixed.

(2) S is ring-finite over R if $S = R[v_1, \dots, v_n] = \{\sum_i a_i v_1^{i_1} \cdots v_n^{i_n} \mid a_i \in R\}$ where $v_1, \dots, v_n \in S$.

(3) S is a finitely-generated field extension of R if S and R are fields and $S = R(v_1, \dots, v_n)$ (the quotient field of $R[v_1, \dots, v_n]$) where $v_1, \dots, v_n \in S$.

(Recall: the definition of field extension. Firstly, given A is a field, then a subset $B \subseteq A$ is a subfield if it contains 1 and it is closed under addition and multiplication and taking the inverse of non-zero elements of B . Given B is a subfield of A , we call A a field extension of B .)

Properties:

1. If S is module-finite over R , then S is ring-finite over R . (This is straightforwardly seen from the definitions)
2. If $L = K(x)$, then L is a finitely-generated field extension of K but L is not ring-finite over K .

Proof. Using the definition, L is a finitely generated field extension of K and so $K(X)$ is a finitely-generated field extension of K . Now, suppose L is ring-finite over K . Then, $L = K[v_1, \dots, v_n]$ and so $K(x) = K[v_1, \dots, v_n]$, where $v_1, \dots, v_n \in k(x)$.

Then, there exists $\frac{s_i}{t_i} \in K(X)$ that generate L where $i = 1, \dots, n$. Define $p := 1/q$ where q is an irreducible polynomial that has a higher degree than all t_i 's. Then, as $p \in K(X)$, $p = \frac{h}{t_1^{e_1} \dots t_n^{e_n}}$. Since q has a higher degree than all the t_i 's, we see that p cannot be equal to $\frac{1}{q}$. \square

Definition 6. Integral elements

Let R be a subring of the ring S . Then, $v \in S$ is integral over R if there exists a monic polynomial $f = x^n + a_1x^{n-1} + \dots + a_n \in R[x]$ such that $f(v) = 0$ and $a_i \in R$. If R and S are fields, we say v is algebraic over R .

When all elements of S is integral over R , we say S is integral over R . When S and R are fields and S is integral over R , we call S an algebraic extension of R .

Theorem 4. Let R be a subring over an integral domain S and let $v \in S$. Then, the following are equivalent:

- (1) v is integral over R .
- (2) $R[v]$ is module-finite over R .
- (3) There exists a subring R' of S such that R' contains $R[v]$ and it is module-finite over R .

Proof. We see (2) implies (3) readily. Now, (1) implies (2): Suppose v is integral over R with the monic polynomial $f(x) = x^n + a_1x^{n-1} + \dots + a_n$. Then, $f(v) = 0 \implies v^n \in \sum_{i=0}^{n-1} Rv^i$. Therefore, for any integer m , $v^m \in \sum_{i=0}^{n-1} Rv^i$. This implies $R[v]$ is module-finite over R .

Lastly, (3) implies (1) as follows: Suppose R' is module-finite over R . Then, $R' = \sum R w_i$, where $w_i \in R'$. Then, $v w_i \in R[v] \subset R'$, so $v w_i = \sum_j a_{ij} w_j$ where $a_{ij} \in R$. Now, $v w_i - v w_i = 0$ implies $\sum_{j=1}^n \delta_{ij} v w_j - v w_i = 0$ which then implies $\sum_{j=1}^n (\delta_{ij} v - a_{ij}) w_j = 0$ (here $\delta_{ij} = 1 \{i = j\}$). Write this in matrix notation and consider these equations in the quotient field of S and note that (w_1, \dots, w_n) is a non-trivial solution to these equations (as we see, they give 0). Therefore, $\det(\delta_{ij} v - a_{ij}) = 0$ from which we get $v^n + a_1 v^{n-1} + \dots + a_n = 0$. Therefore, v is integral over R . \square

Corollary 5. The set of elements of S that are integral over R is a subring of R that contains R .

Proof. Suppose a, b are elements in S that are integral over R . Now, b is integral over R implies b is integral over $R[a]$ as $R \subset R[a]$. Therefore, by the previous theorem, $R[a, b]$ is

module-finite over R . Then by the previous theorem $a + b, a - b, ab \in R[a, b]$ and so they are all integral over R . \square

We will need one result from linear algebra: if $A = (r_{ij})$ is an $n \times n$ matrix over R and V is a column vector s.t $AV = 0$, then $\det(A)V = 0$. This is because $\det(A)V = \det(A)I_n V = \text{adj}(A)AV = 0$

We will require the following results:

Theorem 6. Suppose an integral domain S is ring-finite over R . Then, S is module-finite over R if and only if S is integral over R .

Proof. For the forward direction: suppose the generators of S are s_1, \dots, s_n (where we take $s_1 = 1$ because we enlarge the set of generators as we please as long as it's finite) so $S = \sum_{i=1}^n R s_i$. Then, for any $s \in S$, we can write s as $s = r_1 s_1 + \dots + r_n s_n$.

Now, $s s_i = \sum_{j=1}^n r_{ij} s_j$ because $s s_i \in S$ so can be written as a linear combination of s_i . Then, let I_n be the $n \times n$ identity matrix, V is the n dimensional column vectors where $V_i = s_i$ and $B = (r_{ij})$. Then, we can write these equations as $sIV = BV \implies (sI - B)V = 0$. Then, $\det(sI - B)V = 0$. However, $v_1 = s_1 = 1$, so $\det(sI - B) = 0$ which implies s is the root of a characteristic polynomial of B over R so s is integral over R .

Conversely, suppose S is integral over R and we are told that S is ring-finite over R i.e $S = R[s_1, \dots, s_n]$. Then, for each $s_i \in S$, we have a monic polynomial from which we can write, after rearranging $s_i^{k_i} = a_{1,i} s_i^{k_i-1} + \dots + a_{k_i-1,i} s_i + a_{k_i,i}$. Therefore, $s_i^{k_i}$ is in the submodule of S generated by $\{s_i, \dots, s_i^{k_i-1}\}$ i.e s_i^m is in this submodule for any m . Now, we know S is ring-finite over R with s_1, \dots, s_n as the generators. Now, the direct sum of the submodules (as we saw for each s_i) is also a finitely generated as an R -module and so S is itself module-finite over R . \square

Theorem 7. Let L be a field and let k be an algebraically closed subfield of L . Then an element of L that is algebraic over k is in k . Furthermore, an algebraically closed field has no module-finite field extension except itself.

Proof. Proof of the first part - suppose $p \in L$ that is algebraic over k . Therefore, $p^n + a_1 p^{n-1} + \dots + a_n = 0$ with $a_i \in k$. This is a polynomial in $k[x]$ with a root, so by definition of algebraic closure, $p \in k$.

Now, we prove the second part. Suppose L is module-finite over k . Then, by the previous theorem, L is integral over k . Then, by the first part $L = k$. \square

Lastly,

Theorem 8. Let k be a field. Let $L = k(x)$ be the field of rational functions over k . Then, (a) any element of L that is integral over $k[x]$ is also in $k[x]$. (b) There is no non-zero element $f \in k[x]$ such that $\forall z \in L, f^n z$ is integral over $k[x]$ for some $n > 0$.

Proof. (a) p is integral over $k[x]$ implies there exists the following polynomial $p^n + a_1 p^{n-1} + \dots = 0$. Now, since $p \in k(x)$, we may write it as $p = \frac{s}{t}$ where $s, t \in k[x], t \neq 0$. Then, we get $s^n + a_1 s^{n-1} t + \dots + a_n t^n = 0$. Rearranging, we get $s^n = -a_1 s^{n-1} t - \dots - a_n t^n$. Since t divides the right hand side, t divides s . This means, s/t is a polynomial in $k[x]$. Therefore, $p \in k[x]$.

(b) Suppose, not. Let f be such a function. Let $p(x) \in k[x]$ such that $p(x)$ does not divide f^m for any m . Set $z = \frac{1}{p}$, so $z \in L = k(x)$. Then, $f^n z = \frac{f^n}{p}$ is integral over $k[x]$. This means, there exists $a_i \in k[x]$ such that $(\frac{f^n}{p})^d + \sum_{i=1}^{d-1} a_i (\frac{f^n}{p})^i = 0$. From this, we get $f^{nd} = \sum_{i=1}^{d-1} a_i p^{d-i} f^{in}$. Since p divides the right hand side, we get that p divides f^{nd} which contradicts our definition of p . \square

4 Hilbert's Nullstellensatz

First, we prove the following:

Theorem 9. (Zariski) If a field L is ring-finite over a subfield k , then L is module finite (and, hence, algebraic) over k .

Note that L is module finite over k if and only if L is integral over k which means L is algebraic over k .

Proof. Suppose L is ring-finite over k . Then, $L = k[v_1, \dots, v_n]$ where $v_i \in L$. We proceed by induction.

Suppose $n = 1$. We have that k is a subfield of L and $L = k[v]$. Let $\psi : k[x] \rightarrow L$ be a homomorphism that takes x to v . Now $\ker(\psi) = (f)$ for some f since $k[x]$ is a principal ideal domain. Then, $k[x]/(f) \cong k[v]$ by the first isomorphism theorem. This implies (f) is prime (since $k[v]$ is an integral domain).

Now, if $f = 0$. Then $k[x] \cong k[v]$, so $L \cong k[x]$. However, by the second property following definition 5, this cannot be true. Therefore, $f \neq 0$.

Given $f \neq 0$, we can assume f is monic. Then, (f) prime implies f is irreducible and (f) is a maximal ideal (check Dummit and Foote). This means, $k[v] \cong k[x]/(f)$ is a field (check Dummit and Foote). Therefore, $k[v] = k(v)$. Since $f(v) = 0$, so v is algebraic over k and so, by theorem 4, $L = k[v]$ is module-finite over k . This concludes the proof for $n = 1$.

Now, for the inductive step, assume true for $n - 1$ i.e $k[v_1, \dots, v_{n-1}]$ is module-finite over k . Let $L = k_1[v_2, \dots, v_n]$ where $k_1 = k(v_1)$. Then, by the inductive hypothesis, $k_1[v_2, \dots, v_n]$ is module-finite over k_1 .

We show that v_1 is algebraic over k which would say $k[v_1]$ is module-finite over k concluding the proof. Suppose, v_1 is not algebraic over k . Then, using the inductive hypothesis, for each $i = 2, \dots, n$, we have an equation $v_i^{n_i} + a_{i1}v_i^{n_i-1} + \dots = 0$ where $a_{ij} \in k_1$.

Let $a \in k[v_1]$ such that a is a multiple of all the denominators of $a_{ij} \in k(v_1)$. We get $av_i^{n_i} + aa_{i1}(av_1)^{n_i-1} + \dots = 0$. Then, by corollary 5, for any $z \in L = k[v_1, \dots, v_n]$, there exists N such that $a^N z$ is integral over $k[v_1]$ (since the set of integral elements forms a subring). Since this holds for any $z \in L$, this also holds for any $z \in k(v_1)$. But by theorem 8, this is impossible. This gives us the contradiction. \square

Assume k is algebraically closed.

Theorem 10. (Nullstellensatz Version I) If I is a proper ideal in $k[x_1, \dots, x_n]$, then $V(I) \neq \emptyset$.

Proof. For any ideal I , there exists a maximal ideal J containing I (since we are assuming our ring has an identity $1 \neq 0$, see Dummit and Foote). So, for simplicity, we assume I is the maximal ideal itself since $V(J) \subset V(I)$. Then, $L = k[x_1, \dots, x_n]/I$ is a field (since I is maximal, see Dummit and Foote) and k is an algebraically closed subfield of L . Note that there is a ring-homomorphism from $k[x_1, \dots, x_n]$ onto L , which is the identity. This means, L is ring-finite over k . Then, by theorem 9, L is module-finite over k . Then, by theorem 7, $L = k$ i.e $k = k[x_1, \dots, x_n]/I$.

Now, since $k = L$, in particular this means $k \cong k[x_1, \dots, x_n]/I$. Suppose $x_i \in k[x_1, \dots, x_n]$ is mapped to a_i by the homomorphism ψ whose kernel is I . Then, $x_i - a_i$ is mapped to 0, so $x_i - a_i \in I$. Now, note that $(x_1 - a_1, \dots, x_n - a_n)$ is a maximal ideal as one can easily verify and it contains I , so $I = (x_1 - a_1, \dots, x_n - a_n)$. So, $(a_1, \dots, a_n) \in V(I)$. Therefore, $V(I) \neq \emptyset$. \square

The fact that every maximal ideal in the polynomial ring over n variables is of the form $(x_1 - a_1, \dots, x_n - a_n)$ is a very important thing to remember. In fact, we will often use the fact that points in affine varieties correspond to maximal ideals made rigorous in the following:

Lemma 11. There is a natural bijection between a point $a \in \mathbb{A}^n$ and k -algebra homomorphisms $k[x_1, \dots, x_n] \rightarrow k$. We say the point a corresponds to the maximal ideal defined by the kernel of this homomorphism.

Proof. Let $\phi : k[x_1, \dots, x_n] \rightarrow k$ be a k -algebra homomorphism defined by $\phi(x_i) = a_i$, so $x_i - a_i \in \ker(\phi)$. Given ϕ is surjective (since it's k -algebra homomorphism), $k[x_1, \dots, x_n]/\ker(\phi) \cong k$ so $\ker(\phi)$ is a maximal ideal. \square

We recall some definitions before moving to Hilbert's Nullstellensatz. The radical of an ideal I in R is $\sqrt{I} := \{a \in R : a^n \in I, \text{ for some } n \in \mathbb{Z}, n > 0\}$. It can be easily shown that \sqrt{I} is an ideal itself and $I \subseteq \sqrt{I}$. I is called a radical ideal if $I = \sqrt{I}$.

For any ideal I in $k[x_1, \dots, x_n]$, $V(I) = V(\sqrt{I})$. To see this, note that $I \subseteq \sqrt{I}$ implies $V(\sqrt{I}) \subseteq V(I)$. Conversely, let $v \in V(I)$ and let $f \in \sqrt{I}$. Then, $f^n \in I$ for some $n > 0$. This implies $f^n(v) = 0$ which implies $f(v) = 0$ as k has no zero divisor. Therefore, $v \in V(\sqrt{I})$.

Lastly, $\sqrt{I} \subset I(V(I))$. To see this, suppose $s \in \sqrt{I}$. Then, $s^n \in I$ for some n . Now, let $v \in V(I)$. Then, $s^n(v) = 0$ implies $s(v) = 0$, so $s \in I(V(I))$.

Now, we prove Hilbert's Nullstellensatz:

Theorem 12. (Hilbert's Nullstellensatz) Let I be an ideal in $k[x_1, \dots, x_n]$ where k is algebraically closed. Then, $I(V(I)) = \sqrt{I}$.

Proof. We already know $\sqrt{I} \subset I(V(I))$. So, we only need to prove the other direction. Let $I = (f_1, \dots, f_r)$ where $f_i \in k[x_1, \dots, x_n]$. Suppose, $G \in I(V(I))$. Define $J := (f_1, \dots, f_r, x_{n+1}G - 1) \subset k[x_1, \dots, x_n, x_{n+1}]$. Then, $V(J) \subset \mathbb{A}_k^n$ is \emptyset since G is 0 whenever all f_i are 0 and therefore, $x_{n+1}G - 1 \neq 0$ at those points.

Since $V(J) = \emptyset$, J is not a proper ideal by the previous theorem. Therefore, $J = k[x_1, \dots, x_{n+1}]$. So, $1 \in J$ (check Dummit and Foote; an ideal in R is all of R iff it contains a unit). So $1 = \sum_i a_i(x_1, \dots, x_{n+1})f_i + b(x_1, \dots, x_{n+1})(x_{n+1}G - 1)$.

In particular, if $x_{n+1} = \frac{1}{G}$, then, $1 = \sum_i a_i f_i + b(1 - 1) = \sum_i a_i f_i$. Therefore, $G^N = G^N \sum_i a_i f_i$, so $G^N \in I$. Therefore, $G \in \sqrt{I}$. Therefore, $I(V(I)) \subseteq \sqrt{I}$. \square

This has a series of interesting applications.

Corollary 13. If I is a radical ideal in $k[x_1, \dots, x_n]$, then $I(V(I)) = I$. Therefore, there is a one-to-one correspondence between radical ideals and algebraic sets.

Corollary 14. If I is a prime ideal, then $V(I)$ is irreducible. There is a one-to-one correspondence between prime ideals and irreducible algebraic sets. The maximal ideals correspond to points.

Corollary 15. Let F be a non-constant polynomial in $k[x_1, \dots, x_n]$ with the irreducible decomposition of F being $F = F_1^{n_1} F_2^{n_2} \dots F_r^{n_r}$. Then, $V(F) = V(F_1) \cup \dots \cup V(F_r)$ is the decomposition of $V(F)$ into irreducible components and $I(V(F)) = (F_1 \dots F_r)$. Therefore, there is a one-to-one correspondence between irreducible polynomials $F \in k[x_1, \dots, x_n]$ (up to multiplication by a non-zero element of k) and irreducible hypersurfaces in \mathbb{A}_k^n .

Corollary 16. Let I be an ideal in $k[x_1, \dots, x_n]$. Then, $V(I)$ is a finite set if and only if $k[x_1, \dots, x_n]/I$ is a finite dimensional vector space over k . If this occurs, then, the number of points in $V(I)$ is at most $\dim_k(k[x_1, \dots, x_n]/I)$.

Proof. Let $p_1, \dots, p_r \in V(I)$. Choose $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ such that $f_i(p_j) = 0$ if $i \neq j$ and $f_i(p_i) = 1$ and let \bar{f}_i be the residue class of f_i . Now, if $\sum_i \lambda_i \bar{f}_i = 0$ with $\lambda_i \in k$, then, $\sum_i \lambda_i f_i \in I$. Therefore, $\lambda_j = (\sum_i \lambda_i f_i)(p_j) = 0$. Therefore, \bar{f}_i are linearly independent over k . So $r \leq \dim_k(k[x_1, \dots, x_n]/I)$.

Conversely, suppose $V(I) = \{p_1, \dots, p_r\}$ and so is finite. Let $p_i = (a_{1i}, \dots, a_{ni})$ and define $f_j := \prod_{i=1}^r (x_j - a_{ij})$, $j = 1, \dots, n$. Then, $f_j \in I(V(I))$, so for all j , $f_j^N \in I$ for some large enough $N > 0$. Now, taking I -residues, $\bar{f}_j^N = 0$. By expanding f_j^N , we get that \bar{x}_j^{rN} is

a k -linear combination of $\bar{1}, \bar{x}_j, \dots, \bar{x}_j^{rN-1}$. So, for all s , \bar{x}_j^s is a k -linear combination of $\bar{1}, \bar{x}_j, \dots, \bar{x}_j^{rN-1}$. Therefore, the set $\{\bar{x}_1^{m_1}, \dots, \bar{x}_n^{m_n} : m_i < rN\}$ generates $k[x_1, \dots, x_n]/I$ as a vector space over k . \square

Definition 7. Reduced Rings. A ring R is called reduced if $f^N = 0 \in R$ implies $f = 0$.

Next, we find irreducible decompositions of algebraic sets of an affine space.

5 Irreducible Components of Algebraic Sets

So far, we have seen polynomials and the varieties defined over them. Now, we bring in topological invariants.

Definition 8. Irreducible decomposition of a set. Let $V \in \mathbb{A}_k^n$ be an algebraic set. Then, V is reducible if $V = V_1 \cup V_2$ where V_1, V_2 are non-empty, algebraic sets in \mathbb{A}_k^n i.e $V_i \neq V$ for $i = 1, 2$. If V is not irreducible, we call it reducible.

Theorem 17. The algebraic set V is irreducible if and only if $I(V)$ is prime.

Proof. Suppose, V is irreducible. Now, suppose for contradiction, $I(V)$ is not prime. Therefore, by definition of prime, there exists $f_1 f_2 \in I(V)$ such that $f_1 \notin I(V)$ and $f_2 \notin I(V)$. Now, $V = (V \cap V(f_1)) \cup (V \cap V(f_2))$ and $V \cap V(f_i) \subset V, V \cap V(f_i) \neq V$ to see this, note that for any $p \in V$ such that p is a zero of $f_1 f_2$, p has to be a root of either f_1 or f_2 since f_i belong to an integral domain, therefore, $p \in (V \cap V(f_1)) \cup (V \cap V(f_2))$ (the other direction is obvious). Then, $V = (V \cap V(f_1)) \cup (V \cap V(f_2))$ is decomposition of V which means V is not irreducible - contradiction.

Conversely, suppose $I(V)$ is prime. For contradiction, suppose V is reducible with $V = V_1 \cup V_2$, V_i non-empty. Then, consider $f_i \in I(V_i)$ such that $f_i \notin V$. Clearly, $f_1 f_2 \in I(V)$, so $I(V)$ is not prime - contradiction. \square

Corollary 18. The affine space \mathbb{A}_k^n is irreducible if k is infinite.

Theorem 19. Let A be a non-empty collection of ideals in a Noetherian ring R . Then, A has a maximal ideal i.e an ideal I such that $I \in A$ and no other ideal in A contains I .

Proof. Given our collection of ideals, A , choose an ideal $I_0 \in A$. Then, define $A_1 = \{I \in A : I_0 \subsetneq I\}$ and $I_1 \in A_1$, $A_2 = \{I \in A : I_1 \subsetneq I\}$ and $I_2 \in A_2$ and so on. Then, the statement in the theorem is equivalent to saying that there exists positive integer n such that A_n is empty since that would mean there exists no ideal containing I_{n-1} . Suppose this is not true. Then, with $I := \bigcup_{n=0}^{\infty} I_n$, since R is Noetherian, therefore there exists f_1, \dots, f_m that generates the ideal I where each $f_i \in I_n$ for n sufficiently large. But since the generates are all in I_n , $I = I_n$ and so $I_{n'} = I_n$ for any $n' > n$ (since $I = \bigcup_{n=0}^{\infty} I_n$ by definition) - contradiction. \square

We finally prove the main result. Note that this is pretty closely tied to the Hilbert Basis Theorem which says that every algebraic set is the intersection of a finite number of algebraic sets/hypersurfaces:

Theorem 20. Let V be an algebraic set in \mathbb{A}_k^n . Then, there exists unique, irreducible algebraic sets V_1, \dots, V_r such that $V = V_1 \cup V_2 \cdots \cup V_r$ and $V_i \subsetneq V_j$ for any $i \neq j$.

Proof. Proving this statement is equivalent to disproving that \mathcal{F} is non-empty where $\mathcal{F} := \{\text{algebraic set } V \in \mathbb{A}_k^n : V \text{ is not the union of finitely many irreducible algebraic sets}\}$.

Suppose, \mathcal{F} is not empty. Let $V \in \mathcal{F}$ such that V is the minimal member of \mathcal{F} i.e V cannot be written as the union of sets in \mathcal{F} .

Now, since $V \in \mathcal{F}$, V is reducible (if V is irreducible, then it is trivially the union of 1 irreducible subsets). Since V is reducible, $V = V_1 \cup V_2$ where $V_i \neq \emptyset$. Since V is the minimal member of \mathcal{F} , $V_i \notin \mathcal{F}$. Since $V_i \notin \mathcal{F}$, it is the union of finitely many irreducible algebraic sets, so let $V_i = V_{i1} \cup V_{i2} \cdots \cup V_{im_i}$. Then, $V = \cup_{i,j} V_{ij}$, so $V \notin \mathcal{F}$. So, we have shown that V can be written as $V = V_1 \cup \cdots \cup V_m$ where each V_i is irreducible. First, remove any V_i such that $V_i \subset V_j$. Now we prove uniqueness. Suppose $V = W_1 \cup \cdots \cup W_m$ be another such decomposition. Then, $V_i = \cup_j (W_j \cap V_i)$. Now, $W_j \cap V_i = V_i$ since otherwise we will have found a decomposition of the irreducible set V_i . Therefore, $V_i \subset W_{j(i)}$ for some $j(i)$. Similarly, by symmetry, $W_{j(i)} \subset V_k$ for some k . But then, $V_i \subset V_k$ implies $i = k$ and so $V_i = W_{j(i)}$. Continuing this for each $i \in \{1, \dots, m\}$, we get that the two decompositions are equal. \square

Furthermore, we use the following terms:

Definition 9. An ideal $I \subset k[x_1, \dots, x_n]$ set-theoretically defines a variety V if $V = V(I)$. An ideal $J \subset \mathbb{A}^n$ scheme-theoretically defines a variety V if $J = I(V)$.

Here's a pretty straightforward result:

Theorem 21. For an affine variety X , if f_1, \dots, f_m scheme-theoretically define X , then $V(I(X)) = X$

Two affine-varities can be isomorphic in the usual sense using the language of polynomial maps:

Definition 10. Isomorphic affine varieties. Two affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ are isomorphic if there exists polynomial maps $f : V \rightarrow W$ and $g : W \rightarrow V$ such that $f \circ g = g \circ f = i_d$.

Theorem 22. Let f and g be two polynomials in $k[x, y]$ with no common factors. Then, $V(f, g)$ is a finite set of points.

Proof. Check [1]. \square

6 Zariski Topology

We will require this following result:

Theorem 23. Let $Z \subset \mathbb{A}^n$ be an affine variety and let $x \in \mathbb{A}^n - Z$. Then, there exists $f \in k[x_1, \dots, x_n]$ such that $f(Z) = 0$ and $f(x) \neq 0$.

Proof. Suppose this is not true. Then, $f \in I(Z) \implies f \in I(Z \cup \{x\})$. Then, $I(Z) = I(Z \cup \{x\})$. Therefore, $Z = Z \cup \{x\}$ since $V(I(X)) = X$. This is contradiction since this implies $x \in Z$. \square

Now, we move on to define a topology on \mathbb{A}^n .

Definition 11. Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then, $Z \subseteq X$ is closed if $Z \subseteq X \subseteq \mathbb{A}^n$ is an affine variety i.e there exists $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ such that $Z = V(f_1, \dots, f_m) \subset X$.

This forms a topology. \emptyset is closed as $\emptyset = V(1)$. X itself is closed since $X = V(g_1, \dots, g_m)$ by definition (since it's an affine variety). Now, suppose $\{Z_i\}_{i \in A}$ are affine varieties. Then, $\bigcap_{i \in A} Z_i = V(\sum_i I(Z_i))$. Lastly, $V(f_1, \dots, f_m) \cup V(h_1, \dots, h_r) = V(\sum_{i,j} f_i h_j)$.

Given any affine variety has a unique irreducible, this gives us topological invariants. This allows us to move between worlds:

$$\{\text{Polynomials}\} \leftrightarrow \{\text{Varieties}\} \rightarrow \{\text{Topological Invariants}\}$$

Theorem 24. The pre-image of an affine variety under a polynomial map $p : V \rightarrow W$ is a variety. Therefore, in Zariski topology, polynomial maps/morphisms between varieties are continuous.

Proof. Let $V \subseteq \mathbb{A}_k^n$, $W \subseteq \mathbb{A}_k^m$ be affine varieties. Write p as $p = (p_1, \dots, p_m)$ where the image of each p_i is in k . Now, suppose $Z := V(g_1, \dots, g_m) \subseteq W$ is closed. We show $f^{-1}(Z)$ is closed. $f^{-1}(Z) = \{x = (x_1, \dots, x_n) \in V : (p_1(x), \dots, p_m(x)) \in Z\} = \{x \in V : g_j(f(x)) = 0, \forall j\} \implies f^{-1}(Z) \text{ is closed.}$ \square

For an example, consider the Zariski topology on \mathbb{A}_k^1 and let $V(f_1, \dots, f_m) \subset \mathbb{A}_k^1$. Now, given K is a field, $k[x]$ is a principal ideal domain so $(f_1, \dots, f_m) = (g)$ for some $g \in k[x]$. Then, the closed subset i.e variety of \mathbb{A}_k^1 is of the form $V(g) = \{x \in k : g(x) = 0\}$ which is finite since g is a polynomial of some degree. This means that the closed subsets of \mathbb{A}_k^1 are of the form \emptyset , \mathbb{A}_k^1 and finite subsets of \mathbb{A}_k^1 .

Definition 12. Coordinate Ring. Let $X \subset \mathbb{A}^n$ be an affine variety. The coordinate ring of functions on V is

$$O(X) := k[x_1, \dots, x_n]/I(X)$$

is the quotient ring of polynomials in n -variables. *Intuitively, two polynomials in $O(X)$ are equivalent as long as they have the same values at every point of V .*

Note that, for a point $a = (a_1, \dots, a_n) \in X$ and $f \in O(X)$, the value of $f(a) \in k$ is well-defined. This is because for any $f' \in I(X)$, $f'(a) = 0$, so the value $f(a)$ is independent of our choice of function from $I(V)$.

The coordinate ring $O(X)$ can be thought of as a ring of polynomials such that we only care about their values on X since we identify two polynomials that are equal on X to be the same.

We can always write, using first isomorphism theorem, $O(X) = k[x_1, \dots, x_n]/I(X) \cong k$ by sending each $f \in I(X)$ to 0 which means $(x_1, \dots, x_n) \in X$ (by properties of homomorphisms).

Definition 13. Reduced ring. A ring R is called reduced if, given $x \in R$ with $x^m = 0$ for some $m > 0$, we can conclude $x = 0$.

Lemma 25. The coordinate ring $O(X)$ is always reduced.

This is because if $f^m = 0$ in $O(X)$, then $f^m(x) = 0$ for all $x \in V$ which implies $f(x) = 0$ for all $x \in V$ and so $f = 0 \in O(X)$.

Lemma 26. $O(X)$ is a finitely generated k -algebra.

Definition 14. First, we define $V(f)_X := V(f) \cap X$ where $X \subset \mathbb{A}^n$ is an affine variety. Now, we define basic closed sets of X be sets of the form $V(f)_X$. Note that $V(\{f_i\}_{i \in I}) = \bigcap_i V(f_i)$. Any closed set in the Zariski topology is a union of basic closed sets for some set of functions. On the other hand, the basic open sets of X are of the form $D(f)_X := \{x \in X : f(x) \neq 0\}$ i.e $D(f)_X = X - V(f)$. Any open set in Zariski topology is the union of some basic open sets.

Note that, by Hilbert Basis Theorem, every closed subset of X is a finite intersection of basic closed sets. Similarly, every open set is a finite union of basic open sets.

There is a particularly **local nature of algebraic geometry** as evident by the following:

Corollary 27. Let $U \subseteq X$ be a basic open subset of an affine variety X . Then, for any $x \in U$, there exists a basic open subset $D(f) \subset X$ and $f \in k[x_1, \dots, x_n]$ such that $x \in D(f) \subseteq U$.

Proof. Let $Z = X - U$ be the closed subset of X i.e an affine variety. Then, Theorem 15 allows us to conclude the statement. \square

Some more useful corollaries:

Corollary 28. (Hausdorff property of Zariski topology) Let $x, y \in X$ such that $x \neq y$. There exists an open subset U_x containing x but not y .

This can be proven using the theorem at the start of this section.

Corollary 29. $D(1) = V, D(0) = \emptyset, D(fg) = D(f) \cap D(g)$.

Definition 15. Zariski Closure. For a subset X of V , the Zariski closure of X in V is the minimal closed subset of V that contains X which we denote by $\bar{X} \subseteq V$.

Proposition 30. S is irreducible if and only if $\bar{S} \subseteq V$ is irreducible.

7 Coordinate Rings, Varieties, Morphisms

7.1 Coordinate rings as k -algebra homomorphisms

First, we recall that given $V \subset \mathbb{A}^n$, $W \subset \mathbb{A}^m$ are varieties, $f : V \rightarrow W$ is a polynomial map if $f(x_1, \dots, x_n) = (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))$, $f_i \in k[x_1, \dots, x_n]$ and $f(V) \subset W$.

Furthermore, given the definition of coordinate ring and $I(\mathbb{A}_k^n) = 0$, $O(\mathbb{A}_k^n = k[x_1, \dots, x_n]$ is the true coordinate ring of \mathbb{A}_k^n .

One can easily prove that the coordinate ring $O(V)$ is Noetherian.

Definition 16. k -algebra. Let k be a field (i.e a commutative division ring). A ring R is a k -algebra if $k \subseteq Z(R) := \{x \in R : xy = yx, \forall y \in R\}$ and the identity of k is the same as the identity of R .

Note, $Z(R)$ is the center of the ring R .

Definition 17. Finitely generated k -algebra. A finitely generated k -algebra is a ring that is isomorphic to a quotient of a polynomial ring $k[x_1, \dots, x_n]/I$.

Equivalently, a ring R is a finitely-generated k -algebra if R is generated as a ring by k with some finite set r_1, \dots, r_n of elements of R i.e $k[r_1, \dots, r_n]$.

These definitions are equivalent. Suppose, R is a finitely generated k -algebra i.e $R = k[r_1, \dots, r_n]$. Then, by the first isomorphism theorem, $R \cong k[r_1, \dots, r_n]/I$. Conversely, suppose $R \cong k[r_1, \dots, r_n]/I$ i.e $\varphi : k[r_1, \dots, r_n]/I \rightarrow R$. Then, with $\pi : k[r_1, \dots, r_n] \rightarrow k[r_1, \dots, r_n]/I$, $f := \varphi \circ \pi : k[x_1, \dots, x_n] \rightarrow R$ is a surjective homomorphism. Since f is a homomorphism, $f(p(x_1, \dots, x_n)) = p(f(x_1), f(x_2), \dots, f(x_n))$ and so all elements of R is a polynomial in $f(x_1), \dots, f(x_n)$ with coefficients in R so they are generated by these n elements as a k -algebra.

Definition 18. $\text{Mor}_k(R, S)$. Let R and S be k -algebras. Then, $\psi : R \rightarrow S$ is a k -algebra homomorphism, $\psi \in \text{Mor}_k(R, S)$ if ψ is a ring homomorphism that is identity on k .

Note that if $\phi : R \rightarrow k$ is a k -algebra homomorphism, then ϕ is surjective.

Theorem 31. $O(X) \cong \text{Map}(X, \mathbb{A}^1)$. Here, $\text{Map}(X, \mathbb{A}^1)$ is a commutative k -algebra under addition and multiplication on \mathbb{A}^1 . Furthermore, $O(X)^m \cong \text{Map}(X, \mathbb{A}^m)$

Proof. Let $\varphi : O(X) \rightarrow \text{Map}(X, \mathbb{A}^1)$. Then, define $\varphi(f)(a) = f(a)$ for any $a \in X$. This is a homomorphism by design. To show surjectivity, by definition of $\text{Map}(X, \mathbb{A}^1)$, $f \in$

$\text{Map}(X, \mathbb{A}^1)$ implies $f(x) \in k[x_1, \dots, x_n]$ so $\bar{f} \in O(X)$ is mapped to f . To show injectivity, suppose $f \in O(X)$ is mapped to 0. Then, $f(x) = 0$ for all $x \in X$. This means, $f \in I(X)$ implying $f = 0$ in $O(X)$. \square

Corollary 32. Given X and Y are affine varieties, $X \cong Y$ implies $O(X) \cong O(Y)$.

With these results in mind, we note that a key idea in algebraic geometry is to characterize an affine variety X by the ring of functions $O(X) \cong \text{Map}(X, \mathbb{A}^1)$.

Furthermore, just as before, a point p in a variety X corresponds to a maximal ideal in $R = O(X)$, $m_p = \{f \in O(X) : f(p) = 0\} \subseteq O(X)$. This can be seen by considering the k -algebra homomorphism $\phi : O(X) \rightarrow k$ such that $\phi(x_i) = p_i$. Then, $m_p = \ker(\phi)$.

7.2 Pullback, relationship between $\text{Map}(V, W)$ and $\text{Mor}_k(O(W), O(V))$

Definition 19. Given $X \in \mathbb{A}^n, Y \in \mathbb{A}^m$ are affine varieties, $p \in \text{Map}(X, Y)$, define p^* to be the map $p^* : \text{Mor}_k(O(Y), O(X)), p^*(f) = f \circ p$.

Note that p is a map from X to Y whereas p^* is a morphism from $O(Y)$ to $O(X)$. In light of the previous theorem, we can also say $p^* : \text{Map}(Y, \mathbb{A}^1 \rightarrow \text{Map}(X, \mathbb{A}^1)$.

Next, we prove that there is a one-to-one correspondence between p and p^* :

Theorem 33. Let $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ be affine varieties. There exists a natural 1-1 correspondence between $\text{Map}(V, W)$ and $\text{Mor}_k(O(W), O(V))$.

Proof. Define p and p^* as in the definition of pullbacks. We claim that the map $p \rightarrow p^*$ is injective.

Let $s, s' \in \text{Map}(V, W)$ with $s = (f_1, \dots, f_m)$ and $s' = (f'_1, \dots, f'_m)$. We want to show that if $s^* = s'^*$ i.e $s^*(f) = s'^*(f)$ for all $f \in O(W)$, then $s = s'$. To see this, note that $f_i = x_i \circ s = s^*(x_1) = s'^*(x_i) = x_i \circ s' = f'_i$. Given $f_i = f'_i$ for all $i = 1, \dots, m$, therefore $s = s'$.

Now we claim that the map $p \rightarrow p^*$ is surjective. Let $\lambda \in \text{Mor}_k(O(W), O(V))$. We construct a map $s \in \text{Map}(V, W)$ such that $\lambda = s^*$.

Let $f_i \in k[x_1, \dots, x_n]$ such that $\lambda(y_i) = f_i$ for $i = 1, \dots, m$. Define $s : \mathbb{A}^n \rightarrow \mathbb{A}^m$ such that $s(a_1, \dots, a_n) = (f_1(a_1, \dots, a_n), \dots, f_m(a_1, \dots, a_n))$. Now, if $g \in I(W)$, then $g(f_1, \dots, f_m) = g(\lambda(y_1), \dots, \lambda(y_m)) = \lambda g(y_1, \dots, y_m) = 0$, where we got the last inequality by noting that $g \in I(W)$ so it is 0 in $O(W)$ and λ is a homomorphism so it must send 0s to 0s. Note that for any $g \in k[y_1, \dots, y_m]$, $\lambda(g) = g(f_1, \dots, f_m)$; to see this, write $g(y_1, \dots, y_m) = \sum_i c_i y_1^{i_1} \cdots y_m^{i_m}$,

so $\lambda(g(y_1, \dots, y_m)) = \lambda(\sum_i c_i y_1^{i_1} \cdots y_m^{i_m}) = \sum_i \lambda(c_i y_1^{i_1} \cdots y_m^{i_m}) = \lambda(c_i) \lambda(y_1^{i_1} \cdots y_m^{i_m}) = \sum_i c_i \lambda(y_1^{i_1} \cdots y_m^{i_m}) = g(f_1, \dots, f_m)$

This means, for any $a = (a_1, \dots, a_n) \in V$, $g(s(a)) = g(f_1(a), \dots, f_m(a)) = 0$. Therefore, all $g \in I(W)$ vanish on $s(a)$, $a \in V$. So, $s(a) \in W, \forall a \in V$. This means s restricted to V is a polynomial map i.e $s|_V \in \text{Map}(V, W)$.

Note that $\lambda = s^*$ on y_1, \dots, y_m because if $s = (f_1, \dots, f_m)$, then $s^*(y_i) = y_i \circ s = y_i \circ (f_1, \dots, f_m) = y_i \circ (\lambda(y_1), \dots, \lambda(y_m)) = \lambda(y_i)$. Since they agree on y_1, \dots, y_m , they agree on all of $O(W)$. \square

7.3 Product of varieties

Now, we can naturally discover the notion of tensor products by considering the product of varieties:

Theorem 34. Let k be any field. Let X, Y be two affine varieties. Then, the coordinate ring of $X \times Y$ is $O(X \times Y) = O(X) \otimes_k O(Y)$.

Proof. Let $X \subset \mathbb{A}^n$. Let $Y \subset \mathbb{A}^m$. Let $X = V(I_1), Y = V(I_2)$. Let $I_1 = I(X)$ and $I_2 = I(Y)$ (ignoring the radicals for ease of notation). We claim $I(X \times Y) = I(X) \otimes_k k[y_1, \dots, y_m] + k[x_1, \dots, x_n] \otimes I(Y)$. We prove this as follows:

Let $f \in I(X \times Y)$, $f = \sum_{i=1}^t f_i \otimes g_i$, where $f_i \in k[x_1, \dots, x_n], g_i \in k[y_1, \dots, y_m]$. Then, for any $(x, y) \in X \times Y$, $\sum_{i=1}^t f_i \otimes g_i(x, y) = \sum_{i=1}^t f_i(x)g_i(y) = 0$. We do induction on t . If $t = 1$, then, $f(x)g(y) = 0$ so either $f_1 \in I(X)$ or $g_1 \in I(Y)$. Now for general t , if $g_j(y) = 0$ for all $y \in Y$ and j , then, $f \in k[x_1, \dots, x_n] \otimes_k I(Y)$. Otherwise, we may find $y_0 \in Y$ and j such that $g_j(y_0) \neq 0$. Then, $f_j(x) = \frac{-\sum_{i \neq j} f_i(x)g_i(y_0)}{g_j(y_0)}, \forall x \in X$. Therefore, $f_j - \frac{-\sum_{i \neq j} f_i g_i(y_0)}{g_j(y_0)} \in I(X) \otimes_k k[y_1, \dots, y_m]$ (by inductive hypothesis) and so $f = \sum_{i \neq j} f_i g'_i + I(X) \otimes_k k[y_1, \dots, y_m]$. By inductive hypothesis, $\sum_{i \neq j} f_i g'_i \in I(X) \otimes_k k[y_1, \dots, y_m] + k[x_1, \dots, x_n] \otimes_k I(Y)$, so $f \in I(X) \otimes_k k[y_1, \dots, y_m] + k[x_1, \dots, x_n] \otimes_k I(Y)$. This completes the proof of the claim.

Now, we have $O(X \times Y) = k[x_1, \dots, x_n] \times k[y_1, \dots, y_m] / (I(X \times Y)) = k[x_1, \dots, x_n] \times k[y_1, \dots, y_m] / (I(X) \otimes_k k[y_1, \dots, y_m] + k[x_1, \dots, x_n] \otimes_k I(Y)) = O(X) \otimes_k O(Y)$. \square

Definition 20. Kernel ideal. For a k -algebra homomorphism $\phi : R \rightarrow S$, the ideal $\ker(\phi) = \{x \in R : \phi(x) = 0\}$ is called the kernel ideal.

Definition 21. m_a . Given $a = (a_1, \dots, a_n)$,

$$m_a = (x_1 - a_1, \dots, x_n - a_n)$$

.

One can check that m_a is the kernel of the k -algebra homomorphism $\psi : k[x_1, \dots, x_n] \rightarrow k$ such that $\psi(x_i) = a_i$.

Therefore, $m_a = I(\{a_1, \dots, a_n\})$.

Theorem 35. For $I \subset k[x_1, \dots, x_n]$, the vanishing ideal $I(V(I))$ is given by the intersection of all kernels of k -algebra homomorphisms $k[x_1, \dots, x_n] \rightarrow k$ such that I is mapped to 0 i.e

$$I(V(I)) = \bigcap_{a \in V(I)} m_a$$

Proof. This is pretty straightforward. Suppose $f \in I(V(I))$. Then, $f(a) = 0, \forall a \in V(I)$. Therefore, $f \in m_a, \forall a \in V(I)$. Therefore, $f \in \bigcap_{a \in V(I)} m_a$. On the other hand, suppose $f \in \bigcap_{a \in V(I)} m_a$. Then, $f(a) = 0$ for all $a \in V(I)$. So, $f \in I(V(I))$. \square

Corollary 36. A finitely generated k -algebra R is the coordinate ring of an affine variety if and only if for all $f \neq 0, f \in R$, there exists a k -algebra homomorphism $\phi : R \rightarrow k$ such that $\phi(f) \neq 0$.

Proof. Suppose R is the coordinate ring of an affine variety, i.e $R = O(X)$. Let $X := V(I)$. Then, $I(X) = I(V(I)) = \bigcap_{a \in V(I)} m_a$. Now, if $f \neq 0$ in $R = O(X)$, then $f \notin I(V(I))$ implying $f \notin m_a$ for some $a \in V(I)$. Therefore, there exists some k -algebra homomorphism $\psi : k[x_1, \dots, x_n] \rightarrow k$ such that $\psi(x_i) = a_i$ and $\psi(f) \neq 0$.

Conversely, suppose for any $f \neq 0, f \in R$, there exists a k -algebra homomorphism $\psi : R \rightarrow k$ such that $\psi(f) \neq 0$.

Now, suppose $f \neq 0$. Then, $f \notin m_a$ for some $a \in V := V(I)$. So, $f \in m_a$ for all $a \in V = V(I)$ implies $f = 0$. So, $f \in I(V(I))$ implies $f = 0$. Therefore, $I(V(I)) = \{0\}$ and we can write $R = O(V)$ i.e a coordinate ring of the affine variety V . \square

With these results, we can redefine coordinate ring: a finitely generated k -algebra R is a coordinate ring if for any $f \neq 0, f \in R$, there exists a k -algebra homomorphism $\psi : R \rightarrow k$ such that $\psi(f) \neq 0$.

Theorem 37. Let $p : V \rightarrow W$ be a morphism between 2 varieties where $p = (p_1, \dots, p_m)$. Let $w \in W$ be a point with the vanishing ideal $m_w = (x - w_1, \dots, x - w_m) \subset I(W)$, then the fiber $p^{-1}(w) \subset V$ is defined by $p^*(m_w) = (p_1(x) - w_1, \dots, p_m(x) - w_m) \subset O(V)$.

7.4 MSpec(R) and abstract varieties

We can now define three abstract structures that are ubiquitous in algebraic geometry:

Definition 22. Maximal space $\text{MSpec}(R)$. For R a coordinate ring, the maximal space $\text{MSpec}(R)$ of R is the set of k -algebra homomorphisms $p : R \rightarrow k$ which we call points. For $f \in R$, we say f vanishes at point p if $p(f) = 0$.

Let's make this more clear. Let $R = k[x_1, \dots, x_n]/I(X)$. Let $\varphi : R \rightarrow k$ be a k -algebra homomorphism where I is the kernel. Then, we can visualize φ as:

$$\begin{array}{ccc} k[x_1, \dots, x_n] & \longrightarrow & k[x_1, \dots, x_n]/I \\ & \searrow & \downarrow \varphi \in \text{MSpec}(k[x_1, \dots, x_n]/I) \\ & & k \end{array}$$

With this in mind, we note that we can associate an element of $\text{MSpec}(k[x_1, \dots, x_n]/I)$ with a point in $(a_1, \dots, a_n) \in X$ in k . This is because $\varphi(f_i) = 0, \forall f_i \in I$, so $x_i \rightarrow a_i \in k$ and since x_i generate all of the quotient ring, therefore, we can send all polynomials to k .

Therefore, we will often write:

$$\text{MSpec}(k[x_1, \dots, x_n]/(f_1, \dots, f_m)) = \{(a_1, \dots, a_n) \in \mathbb{A}^n : f_i(a_1, \dots, a_n) = 0, \forall i = 1, \dots, m\}$$

We can turn $\text{MSpec}(R)$ into a topological space by letting the basic closed sets be $V_{\text{MSpec}(R)}(f) = \{p \in \text{MSpec}(R) : p(f) = 0\}$.

Definition 23. Abstract affine variety and ring of functions. A pair (V, R) is an abstract affine variety if R is a coordinate ring and V is identified with the topological space $\text{MSpec}(R)$.

We often just write V instead of (V, R) . We call R the ring of functions on V .

Here's an intuition for why abstract affine varieties are required. We want to study polynomials in $R = O(X) = k[x_1, \dots, x_n]$. We can move to the geometric world by studying the affine variety of a polynomial in R . Given a polynomial, we can fully determine the set of zeros and therefore attain the variety. But given a just variety in the geometric world which is just a set of elements in \mathbb{A}^n , we cannot hope to recover $R = O(X)$. Therefore, in an almost silly manner, we remember the data by just letting the abstract affine variety be (V, R) .

Definition 24. Morphism of abstract varieties. For (V, R) and (W, S) abstract varieties,

the morphism of (V, R) and (W, S) is a k -algebra homomorphism $\psi^* : W \rightarrow V$ with the induced map $\psi : V = \text{MSpec}(R) \rightarrow W = \text{MSpec}(S)$.

7.5 Localization

Theorem 38. (Rabinowitch Trick) The solutions (a_1, \dots, a_n) to $f_1 = f_2 = \dots = f_n = 0$ and $f \neq 0$ are in bijection with the solutions $(a_1, \dots, a_n, a_{n+1})$ to $f_1 = f_2 = \dots = f_n = 0$ and $x_{n+1}f(x_1, \dots, x_n) - 1 = 0$.

Proof. The bijections take a_1, \dots, a_n to $a_1, \dots, a_n, \frac{1}{f(a_1, \dots, a_n)}$ and (reverse) a_1, \dots, a_n, a_{n+1} to a_1, \dots, a_n . \square

Now, we define something central to a lot of the techniques.

Definition 25. Localization of ring R at element $g \in R$. We define

$$R_g := R[x]/(xg - 1).$$

Because $xg - 1 = 0$ in R_g , we refer to x as g^{-1} (here g is a unit).

Since $x = g^{-1}$ in R_g , therefore, we will often write R_g as $R[g^{-1}]$. Localization is very useful; for example, one can notice that we had used localization in the proof of Hilbert's Nullstellensatz (although we used it as an arbitrary "trick" without really looking deep into the construction).

Here are some immediate properties of the localization of a ring at an element.

Lemma 39. For any ring R , we have:

- (a) Every element of R_g can be written as rg^{-i} for some $r \in R$ and $i \geq 0$.
- (b) $rg^{-i} = sg^{-i} \in R_g$ if and only if $g^N(r - s) = 0 \in R_g$ for some N .
- (c) A ring map $R_g \rightarrow S$ is the same as a ring map $R \rightarrow S$ such that $\text{Im}(g) \in S^\times$ i.e g is mapped to a unit in S . In other words,

$$\text{Mor}(R_g, S) = \{\varphi \in \text{Mor}(R, S) : \varphi(g) \in S^\times\}$$

- (d) The map $R \rightarrow R_g$ is an isomorphism exactly when $g \in R^\times$.

Proof. (a) Suppose $s \in R_g$. Then, $s = f(x)$. Given f is a polynomial, for i large enough, sg^i not have any x (since $x = g^{-1}$), so $sg^i = r \in R$. This implies the result.

(b) Suppose $g^N(r - s) = 0$. Then, since g is a unit, this implies $r - s = 0$, so $r = s$ and $rg^{-i} = sg^{-i}$. Conversely, suppose $rg^{-i} = sg^{-i}$. Then, $(r - s)g^{-i} = 0$. Since we are operating in R_g , this implies $(r - s)g^{-i} = (1 - xg)(a_0 + a_1x + \cdots + a_nx^n)$. Expanding the right hand side and comparing the coefficients of x^0, x^1, \dots, x^n , we see that $a_0 = r - s$, $a_1 = (r - s)g$ and $a_n = (r - s)g^n$. Now, comparing the coefficient of x^{n+1} , we get that $a_n g = 0 \implies (r - s)g^{n+1} = 0$. // (c) A ring map $\psi : R_g \rightarrow S$ is the same as a map $\phi : R \rightarrow S$ such that ϕ sends $(xg - 1)$ to 0. But then, $\phi(x)\phi(g) - \phi(1) = 0$, so $\phi(x) = \phi(g)^{-1} = \phi(g^{-1})$. Therefore, g has to be a unit. If g is not a unit, these maps cannot be equivalent.
(d) If g is a unit, then $R[x]/(xg - 1) = R[x](x - g^{-1}) = R$. Conversely, if g is not a unit, then R is not isomorphic to R_g . \square

Now, we get the following result:

Theorem 40. If $V \subset \mathbb{A}^n$ is a variety with coordinate ring R , then, $\{(v, f(v)^{-1}) : v \in V, f(v) \neq 0\} \subset \mathbb{A}_k^{n+1}$ is a variety with coordinate ring R_f and isomorphic to $D(f) \subseteq \mathbb{A}^n$.

Proof. First, we prove that for $R = O(V)$, R_f is a coordinate ring. Given R is the quotient of a polynomial ring, it is a finitely generated algebra. Now, using corollary 22, R_f is a coordinate ring if and only if for all $g \in R_f$, $g \neq 0$, there exists a k -algebra homomorphism $q : R_f \rightarrow k$ such that $q(g) \neq 0$. Let $g = hf^{-i}$ for $i \geq 0$, $h \in R$, $g \neq 0$ in R_f . Then, $hf \neq 0$, since otherwise $g = 0$. Now, we know there exists a k -algebra homomorphism $p : R := O(V) \rightarrow k$ such that $p(hf) = p(h)p(f) \neq 0$. As $p(f) \neq 0$, therefore, we get a map from $R_f \rightarrow k$ such that $p(g) \neq 0$. Therefore, R_f is a coordinate ring.

Furthermore, $MSpec(R_f) = D(f)$. To see this, suppose $V = (f_1, \dots, f_m)$, so $R_f = (k[x_1, \dots, x_n]/(f_1, \dots, f_m))_f = (k[x_1, \dots, x_n]/(f_1, \dots, f_m))[f^{-1}]$. Therefore, $MSpec(R_f)$ is associated with points such that (f_1, \dots, f_m) are 0 but $1/f(x_1, \dots, x_n)$ is not 0. This is the same as $D(f)_V$. Furthermore, $D(f)$ can be considered an affine variety too by considering it as $V(f_1, \dots, f_m, xf - 1)$. \square

Using the language of localization, we can also determine what ideals are radical which is important in light of Hilbert's Nullstellensatz:

Theorem 41. $I \subset R$ is radical if and only if R/I is reduced if and only if $(R/I)_f = 0$ implies $f = 0$

We define a few more important objects.

Definition 26. R is a Jacobson ring, if every radical ideal I is the intersection of these maximal ideals containing it.

Definition 27. A quasi-affine variety is an open subset of an affine variety.

As we close this section, we note that we now have a better understanding of what closed and open subsets are in Zariski topology:

Suppose k is an algebraically closed field and $V \subset \mathbb{A}^n$ is an irreducible affine variety. Then, let $R := O(V)$ be our coordinate ring. Given $I \subset R$ is an ideal, $V(I)$ is a variety, $V(I) \cap V$ is a closed subset and with $V(I)$, we can associate the coordinate ring $R/I(V(I))$.

On the other hand, let $g \in R$, then $D(g) \subset V$ is a basic open set and with it, we can associate the coordinate ring R_g in $k[x_1, \dots, x_n][1/g] = k[x_1, \dots, x_n, x_{n+1}]/(x_{n+1} - \frac{1}{f(x_1, \dots, x_n)}) = O(V(x_{n+1} - \frac{1}{f(x_1, \dots, x_n)}))$. Later on, we will see that we can study algebraic geometry of open subsets too using sheaves.

7.6 Coordinate Changes

Definition 28. Change of coordinates. Let $T = (T_1, \dots, T_m)$ be a morphism/polynomial map from \mathbb{A}^n to \mathbb{A}^m . Let f be a polynomial in $k[x_1, \dots, x_m]$. Then, define

$$f^T = \tilde{T}(f) = f(T_1, \dots, T_m).$$

Given $I \subset \mathbb{A}^m$ is an ideal and V is an algebraic set in \mathbb{A}^m , $I^T \subset \mathbb{A}^n$ will be the ideal in $k[x_1, \dots, x_n]$ generated by $\{f^T : f \in I\}$. V^T is the algebraic set $T^{-1}(V) = V(I^T)$ where $I = I(V)$. So if V is the hypersurface of f , then V^T is the hypersurface of f^T given f^T is not constant.

Definition 29. Affine Change of coordinates. An affine change of coordinates on \mathbb{A}^n is a polynomial map $T = (T_1, \dots, T_n) : \mathbb{A}^n \rightarrow \mathbb{A}^n$ where each T_i is a polynomial of degree 1 and T is one-to-one and onto. If $T_i = \sum_{ij} a_{ij}x_j + a_{i0}$, then $T = T'' \circ T'$ where T' is a linear map ($T'_i = \sum_{ij} a_{ij}$) and T'' is a translation ($T''_i = x_i + a_{i0}$). Translations are invertible. So T is one-to-one and onto if and only if T' is invertible.

If T and U are affine change of coordinates on \mathbb{A}^n , then so are $T \circ U$ and T^{-1} .

7.7 Morphisms

This is mostly from Hartshorne's text.

Let Y be a quasi-affine variety in \mathbb{A}^n i.e it is an open subset of an irreducible affine variety.

Definition 30. Regular at point p . A function $f : Y \rightarrow k$ is regular at $p \in Y$ if there exists an open neighbourhood U such that $p \in U \subseteq Y$ and $g, h \in A := k[x_1, \dots, x_n]$ such that h is not zero anywhere on U and $f = \frac{g}{h}$ on U .

Definition 31. Regular function. A function f is regular on Y if it regular at every point of Y .

Lemma 42. A regular function is continuous when k is identified with \mathbb{A}_k^1 in its Zariski topology.

Proof. We show that f^{-1} of a closed set is closed. Any closed set of \mathbb{A}_k^1 is a finite set of point. So we show $f^{-1}(a) = \{p \in Y : f(p) = a\}$ is closed for any $a \in k$. We know a subset $Z \subseteq Y$ is closed iff Y can be covered by open subsets U such that $Z \cap U$ is closed in U for each U (proving this is simple - consider $Y - Z$ which we can write as $(Y - Z) \cap (\cup U_i) = \cup (U_i - Z)$). Now, let U be an open set on which f can be represented as g/h s.t h is not 0 anywhere on U . Then, $f^{-1}(a) \cap U = \{p \in U : g(p)/h(p) = a\}$. Now, $g(p)/h(p) = a$ iff $(g - ah)(p) = 0$. So $f^{-1}(a) \cap U = V(g - ah) \cap U$ which is closed. Therefore, $f^{-1}(a)$ is closed in Y . \square

8 Schemes

(This chapter is from Scholze's lectures)

Let V be an algebraic set. Then, we can define the coordinate ring over V i.e $O(V)$. Now, consider $Map(V, k)$ which we can consider to be a ring with pointwise addition and multiplication in k . Then,

$$O(X) = im(k[x_1, \dots, x_n] \rightarrow Map(V, k)).$$

We also express $V(I)$ where I is an ideal by writing

$$V(I) = Hom_{k-alg}(k[x_1, \dots, x_n]/I, k)$$

Assume, for this section, that we are working with commutative rings with an identity and that all maps are ring homomorphisms.

Definition 32. $\text{Spec } A$. Let A be a ring. Then $\text{Spec } A$ is defined as the collection of all ring homomorphisms $A \rightarrow k$ where k is some field and we identify two maps $A \rightarrow k$ and $A \rightarrow k'$ to be equivalent if we have the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & k \\ & \searrow & \downarrow \\ & & k' \end{array}$$

Proposition 43. The map $\text{Spec } A \rightarrow \{ \text{prime ideals in } A \}$ defined by $(f : A \rightarrow k) \rightarrow \ker(f)$ is a well-defined bijection.

Proof. If $f : A \rightarrow k$ is a map where k is a field, then $\ker(f)$ is a proper ideal in A and if we have $xy \in \ker(f)$ then, $0 = f(xy) = f(x)f(y)$ so $x \in \ker(f)$ or $y \in \ker(f)$ given k is a field. Therefore, $\ker(f)$ is also prime. Now, we have the following diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & k \\ & \searrow f' & \downarrow \\ & & k' \end{array}$$

The map $k \rightarrow k'$ must be injective so $\ker(f) = \ker(f')$ and so our map is well-defined.

Next, we show bijection. We do so by constructing an inverse map. If $p \subseteq A$ is a prime ideal, then A/p is an integral domain, so we have the inclusion $A/p \hookrightarrow \text{Frac}(A/p) := k(p)$ from A/p into a field. This gives us the composition

$$f_p : A \rightarrow A/p \hookrightarrow k(p)$$

which is a map to a field with $\ker(f_p) = p$. This gives us surjectivity of f_p . For any map $f : A \rightarrow k$, we can factor the map f through $\ker(f) = p$ and then the map $A/p \hookrightarrow k$ can be factored through $\text{Frac}(A/p)$ which makes f equivalent to f_p . Then, we have two maps $f : A \rightarrow k$ and $f' : A \rightarrow k'$ with $\ker(f) = \ker(f')$ and so $f = f'$. \square

(Here, $\text{Frac}(A)$ of an integral domain is a field of elements a/b where b is not 0 with the mod relation $a/b \sim a'/b'$ if $ab' = a'b$. The addition and multiplication is defined in the usual way)

With this, we redefine/define the following:

Definition 33. We define $\text{Spec } A$ as the set of all prime ideals in A called the spectrum of the ring A . Furthermore, for any $x \in \text{Spec } A$, we let p_x be the prime ideal which is x , then

$$k(x) := k(p_x) = \text{Frac}(A/p_x)$$

called the residue field of at x . For $x \in \text{Spec } A$, we have a natural map $A \rightarrow k(x)$ which maps $g \rightarrow g(x)$. In other words, we want to consider any element $g \in A$ as a function on $\text{Spec } A$, whose value at $x \in \text{Spec } A$ is $g(x) \in k(x)$. In other words, for each $g \in A$, we can map $x \in \text{Spec } A$ to $g(x) \in k(x)$ which is the same as viewing g modulo p_x inside the field of fractions of A/p .

Definition 34. Given a ring A , we define $\text{Spec}_{\max} A$ as the subset of $\text{Spec } A$ consisting of all the maximal ideals of A .

Proposition 44. Let k be an algebraically closed field and let A be a finitely generated k -algebra so $A = k[x_1, \dots, x_n]/a$. Then,

$$k^n \supseteq V(a) = \text{Hom}_{k\text{-alg}}(A, k) \rightarrow \text{Spec}_{\max} A \subseteq \text{Spec}(A)$$

the map which sends $f : A \rightarrow k$ to $\ker(f)$ is a bijection.

Proof. Given $f : A \rightarrow k$, we know $\ker(f)$ is a maximal ideal and for any maximal ideal $m \subseteq A$, we saw in the proof of Nullstellensatz, that $k \cong A/m$. Therefore, for any maximal ideal $m \subseteq A$, we have a map of k -algebras $A \rightarrow A/m \cong k$. \square

Proposition 45. Given $f : A \rightarrow B$, then $\text{Spec } f$ defined by $q \rightarrow f^{-1}(q)$ is a well-defined map.

Proof. if $x, y \in A$ and $xy \in f^{-1}(q)$, then $f(xy) = f(x)f(y) \in q$. So, either $f(x) \in q$ or $f(y) \in q$ and so either x or y is in $f^{-1}(q)$. Therefore, $f^{-1}(q)$ is a prime ideal in A . \square

9 Dimensions

Most of this section is from Hartshorne's book and Zhiyu Zhang's lectures.

9.1 Krull Dimension

We need a notion of dimension on topological spaces which we can hope to use on Zariski topology. We have some expectations. We would want \mathbb{A}^n to be of dimension n . If X_1 and X_2 are closed, then $\dim(X_1 \cap X_2) = \dim(X_1) + \dim(X_2) - \dim(X)$. Dimension should be such that it can be understood locally through local rings or fractional fields. Given Y_1, Y_2 are closed subvarieties of X , we expect $\dim(Y_1 \cap Y_2) = \dim(Y_1) + \dim(Y_2) - \dim(X)$. For $f \in \mathcal{O}(X)$, then $\dim(V(f)) = \dim(X) - 1$ for general f .

For motivation, we look at linear algebra. For a vector space V , we can define $\dim(V) = \max\{k : \exists V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_k, \text{ linear subspaces of } V\}$.

We can use the same idea to define dimension for topological spaces. (recall: irreducible closed subsets of X are closed subsets $Y \subseteq X$ such that Y cannot be written as the union of two closed subsets Y_1, Y_2 such that one of these is empty.)

Definition 35. Krull dimension of a topological space. Given X is a topological space. Then,

$$\dim(X) = \max\{k : \exists X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k \subsetneq X, X_i \text{ irreducible closed subsets in } X\}.$$

We set $\dim(\emptyset) = 0$.

Example: $\dim(\mathbb{A}^1) = 1$ since the irreducible closed subsets are \emptyset , finite subsets, \mathbb{A}^1 i.e the maximal chains look like $\emptyset \subset \{p\} \subset \mathbb{A}^1$.

Definition 36. Equi-dimensional. A topological space X is equidimensional of dimension n , if any irreducible component of X has the same dimension n .

Example: Let $X := \{xy = 0\} \subseteq \mathbb{A}^2$. This has two irreducible components only - $\{x = 0\} \cong \mathbb{A}^1$ and $\{y = 0\} \cong \mathbb{A}^1$. Both the irreducible components are of dimension 1, so X is equi-dimensional of dimension 1.

With this definition of dimension, we get the following:

Proposition 46. If $X \subseteq Y$ are closed, then $\dim(X) \leq \dim(Y)$. If Y is irreducible and X is a proper closed subset of Y (i.e $X \neq Y$ is closed), then $\dim(X) < \dim(Y)$.

Proof. For the first part, any strictly increasing chain of non-empty irreducible closed subsets of X is also a strictly increasing chain of non-empty irreducible closed subsets of Y . For the second part, note that we are working with a ring with identity and so every proper ideal is contained in some maximal ideal (which is also prime). \square

Given we have defined Zariski topology over affine varieties, we can now use Krull Dimension as a notion of dimension over affine varieties.

Definition 37. Krull dimension of an affine variety. Using Zariski topology,

$$\dim(V) = \max\{k : X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_k \subsetneq V, X_i \text{ irreducible closed subsets of } V\}$$

Definition 38. Krull dimension of ring. Let R be any ring. The Krull dimension $\dim(R)$ is the maximal length of strict chains of prime ideals of R .

Proposition 47. If X is an affine variety, then $\dim(X) = 0$ if and only if X is finite.

Proof. Write X as the union of irreducible components : $X = \bigcup_{i=1}^n X_i$. Then suppose $\dim(X) = 0$. Then, choose $x \in X_i$. Now, given $\dim(X) = 0$, $X_i = \{x_i\}$ and so X is the union of finitely many points. Conversely, if X is finite, then the topology is discrete and so any subset is closed. The irreducible closed subsets are points so $\dim(X) = 0$. \square

We call 1 dimensional varieties algebraic curves. We call 2 dimensional varieties algebraic surfaces.

Proposition: Let X be an affine algebraic set and let $R = O(X)$. Then, $\dim(X) = \dim(R) = \dim(O(X))$.

Proof. If X is an affine algebraic set in \mathbb{A}^n , then the closed irreducible subsets of Y correspond to prime ideals of $k[x_1, \dots, x_n]$ containing $I(X)$. These correspond to prime ideals of $O(X)$. So, $\dim(X)$ is equal to the length of the longest chain of prime ideals of $O(X)$. \square

The following is from Hartshorne's text:

Theorem 48. Let Y be a quasi-affine variety (i.e $Y \subset V$ where V is an affine variety and Y is an open set in V), then $\dim(Y) = \dim(\bar{Y})$

Proposition 49. A Noetherian integral domain A is a UFD if and only if every prime ideal of height 1 is principal.

Theorem 50. Krull's Hauptidealsatz. Let A be a Noetherian ring. Let $f \in A$ be an element s.t f is neither a zero divisor nor a unit. Then, every minimal prime ideal p containing f has height 1.

Proposition 51. An affine variety Y in \mathbb{A}^n has dimension $n - 1$ if and only if it is the zero set $Z(f)$ of a single nonconstant irreducible polynomial in $A = k[x_1, \dots, x_n]$.

Definition 39. Catenary Rings. A ring R is catenary if for any prime ideals $p \subseteq q$ of R , any maximal chain $p = p_0 \subseteq p_1 \subseteq \dots \subseteq p_e = q$ has the same length $e = e(p, q)$.

Theorem 52. Let k be a field. Any finitely generated k -algebra R is catenary.

Note: This means $k[x_1, \dots, x_n]/I$ is catenary.

Example: $\dim \mathbb{A}^n = n$ since we have a strictly increasing chain $0 \subseteq \mathbb{A}^1 \subseteq \dots \subseteq \mathbb{A}^n$.

Corollary 53. If $V_i \subset \dots \subset V_j$ is a maximal chain of irreducible subvarieties of X , then $\dim V_{i+k} = \dim V_i + k$.

Now, we define the following:

Definition 40. Algebraically independent and algebraic over. Let L/k be a field extension (i.e k is a subfield of L). Then, $a_1, \dots, a_n \in L$ are **algebraically independent** over k if $f \in k[x_1, \dots, x_n]$ such that $f(a_1, \dots, a_n) = 0$ implies $f = 0$. Otherwise, a_1, \dots, a_n are algebraically dependent over k . (recall: $a \in L$ is **algebraic over** k if there exists a monic polynomial $f \in k[x]$ such that $f(a) = 0$)

Definition 41. Transcendence degree.

$$tr.deg(L/k) := \max\{r : \exists a_1, \dots, a_r \in L \text{ algebraically independent over } k\}.$$

To get a sense of this, note that $\dim(k(\mathbb{A}^n)) = n$. This is because the maximal set of algebraically independent elements of $k(\mathbb{A}^n)$ is of length n . Choose $x_1, \dots, x_n \in k(\mathbb{A}^n)$. Then, $p(x_1, \dots, x_n) = 0$ implies $p = 0$ (note that x_i are variables here, so $p(x_1) = 0$ would also imply p is 0). On top of that, if we chose any other $f \in k(\mathbb{A}^n)$, we could write it as a combination of x_i 's so $p(T_1, \dots, T_n, T_{n+1}) = f(T_1, \dots, T_n) - T_{n+1} = 0$ is a non-zero polynomial that vanishes at (x_1, \dots, x_n, f) .

With this, we have the following:

Theorem 54. Let V be an irreducible affine variety. Let $R = O(V)$.

$$\dim(R) = tr.deg(k(V)).$$

Note: here we are treating $k(V)$ as a field extension of k .

Sketch of proof: It can be easily seen that $tr.deg(k(V)) \geq \dim(R)$. This is because $tr.deg(\text{Frac}(R)) \geq tr.deg(R)$. Now, consider the number of algebraically independent elements of R . Suppose,

$x_1, \dots, x_r \in R/P$ (where P is a prime ideal) are algebraically independent. Then, for any $y \neq 0, y \in P$, we have $\{x_1, \dots, x_r, y\}$ is an algebraically independent set so the number of algebraically independent elements of R is greater than or equal to the number of prime ideals. Now, we need to show $\text{tr.deg}(k(V)) \leq \dim(R)$. Proving this requires Noether Normalization which we will do later.

Corollary 55.

$$\dim(\mathbb{A}^n) = \text{trdeg}(k(x_1, \dots, x_n)/k) = n.$$

Corollary 56. $f : X \rightarrow Y$ is a finite and surjective morphism between two affine varieties, then $\dim(X) = \dim(Y)$.

Recall: $f : X \rightarrow Y$ and $y \in Y$. Then, $f^{-1}(y)$ is the fiber of y .

Definition 42. Quasi-finite. Let $f : X \rightarrow Y$ be a polynomial map between two affine varieties. Then, f is quasi-finite if the number of elements in the fiber over $y \in Y$ is finite for any $y \in Y$.

Theorem 57. Krull's Principle Ideal Theorem. If the affine variety V is irreducible and $0 \neq f \in O(V)$ is not a unit, then all irreducible components V_i of $V(f)$ have dimension $\dim(V) - 1$.

9.2 Noether Normalization

Lemma 58. Let $f \in k[x_1, \dots, x_n]$ where $n \geq 2$ be a non-zero polynomial over an infinite field k . Then, there are elements $\lambda, a_1, \dots, a_{n-1} \in k$ such that $\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) \in k[y_1, \dots, y_n]$ is monic in y_n .

Proof. Let f_d be the homogenous part of f of the highest degree. We have $f_d(\lambda x_1, \dots, \lambda x_n) = \lambda^d f_d(x_1, \dots, x_n)$ where d is the degree of f . Since k is infinite, we can always find a_1, \dots, a_{n-1} such that $f_d(a_1, \dots, a_{n-1}, 1) \neq 0$. Then, let $y_j = x_j - a_j x_n$ for $j = 1, 2, \dots, n-1$ and $y_n = x_n$ and $\lambda = f_d(a_1, \dots, a_{n-1}, 1)^{-1}$. Then, $\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) = \lambda f_d(a_1, \dots, a_{n-1}, 1) y_n^d + \text{lower order}$ (in degree of y_n terms). This is a monic polynomial in y_n . \square

Theorem 59. (Noether Normalization). Let R be a finitely generated algebra over a infinite field k with generators $x_1, \dots, x_n \in R$. Then, there exists an injective k -algebra homomorphism $\phi : k[t_1, \dots, t_r] \rightarrow R$ from a polynomial ring to R such that R is integral over $k[t_1, \dots, t_r]$.

Proof. Proceed by induction. For $n = 1$, let $t_1 = x_1$. Now suppose $n > 1$. If x_1, \dots, x_n are algebraically independent, we can choose $t_i = x_i$ and the result follows. Now suppose

there exists some algebraic dependence between the generators x_1, \dots, x_n i.e there exists a non-zero polynomial f over k such that $f(x_1, \dots, x_n) = 0$. Let f_d be the homogenous part of the highest degree of f . Then, by the previous lemma, we can find $\lambda, a_1, \dots, a_{n-1}$ such that $\lambda f(y_1 + a_1 y_n, \dots, y_{n-1} + a_{n-1} y_n, y_n) \in k[y_1, \dots, y_n]$ is monic in y_n . The new coordinates are then given by $y_1 = x_1 - a_1 x_n, \dots, y_{n-1} = x_{n-1} - a_{n-1} x_n, y_n = x_n$ and so $\lambda f(x_1, \dots, x_n) = 0$. Furthermore, y_n is integral over $k[y_1, \dots, y_{n-1}]$ by the previous lemma. By inductive hypothesis, there exists an injective algebra homomorphism $\phi : k[t_1, \dots, t_r] \rightarrow k[y_1, \dots, y_{n-1}]$ s.t $k[y_1, \dots, y_{n-1}]$ is integral over $k[t_1, \dots, t_r]$. But y_n is integral over $k[y_1, \dots, y_{n-1}]$ and so y_n is integral over $k[t_1, \dots, t_r]$. \square

Example 1: Let $R = k[x_1, x_2]/(x_1 x_2 - 1) \cong k[x, \frac{1}{x}]$. Then, R is not integral over $k[x]$ - if $1/x$ were integral over $k[x]$ with the polynomial $(\frac{1}{x})^n + a_1(x)(\frac{1}{x})^{n-1} + \dots + a_n(x) = 0$ with $a_i(x) \in k[x]$. After multiplying both sides by x^{n-1} , we can see that $\frac{1}{x} \in k[x]$ which is impossible. Let $x_1 := t_1 + t_2, x_2 := t_2$, then, $R = k[t_1, t_2]/(t_2^2 + t_1 t_2 - 1)$ and there is injective map $\phi : k[t_1] \rightarrow R$ with R integral over $k[t_1]$.

10 Sheaves

10.1 Presheaf, Germs, Stalks and Sheaves

So far, we have studied morphisms from an affine variety X to another affine variety Y . Now consider any *open* subset U of X . Now, given a polynomial map $f \in \text{Map}(X, Y)$, we would like to localize f i.e $f|_U \in \text{Map}(U, Y)$ where U is an open subset of the affine variety X . The problem is that U is not affine. However, we know that U is locally covered by affine varieties, by the local nature of algebraic geometry introduced in the section on Zariski topology. With this, we can define $\text{Map}(U, \mathbb{A}^1)$ as:

$$\{p_{D(f)} \in \bigcup_{D(f) \subseteq U, f \in O(X)} \text{Map}(D(f), \mathbb{A}^1) : \forall D(f), D(g) \subseteq U, p_{D(f)}|_{D(f) \cap D(g)} = p_{D(g)}|_{D(f) \cap D(g)}\}$$

Recall that $D(f)$ is isomorphic to $\{(v, f(v)^{-1}) : v \in X, f(v) \neq 0\}$ and has the coordinate ring $R_f = k[x_1, \dots, x_n][1/f]$. Note that if $U = D(f)$ is an affine variety, then our definition agrees with what we already know. We saw in the section on coordinate rings that $O(X) \cong \text{Map}(X, \mathbb{A}^1)$, so $O(U) \cong \text{Map}(U, \mathbb{A}^1)$.

We define sheaves in a similar way. While this is abstract, there are a few examples to keep in mind. In differential topology, for any manifold X , we define an atlas and then we define smooth functions on X using the atlas. Considering $X = \mathbb{R}^n$, here is a more straightforward example - consider smooth functions on \mathbb{R}^n . Then, the sheaf of smooth functions on X is the data of all smooth functions on open subsets of X . Let U be an open subset of X - then the ring of smooth functions on U is denoted by $\mathcal{O}(U)$. Given $V \subset U$, we can restrict smooth functions on U to V by $\text{res}_{U,V} : \mathcal{O}(U) \rightarrow \mathcal{O}(V)$. These restrictions commute i.e if we have $W \subset V \subset U$, we could first restrict a function in $\mathcal{O}(U)$ to V and then restrict that to W but this would be equivalent to directly restriction the function in $\mathcal{O}(U)$ to W .

Now, if we want to do algebraic geometry on a general (possibly open subset) U , we need to remember all the coordinate rings $\text{Map}(D(f), \mathbb{A}^1)$.

Although sheaves can defined using any category, we will define it using sets or rings.

First, we define the germ of a smooth function:

Definition 43. Germ of a smooth function at $p \in X$. Germs are objects of the form $(f, \text{open set } U)$ such that $p \in U, f \in \mathcal{O}(U)$ with the equivalence $(f, U) \sim (g, V)$ if there exists an open set $W \subset U, W \subset V$ and $p \in W$ such that $f|_W = g|_W$ i.e $\text{res}_{U,W}f = \text{res}_{V,W}g$.

Therefore, two germs are equivalent as long as they agree on some open neighbourhood of p (even though they might disagree elsewhere).

The set of germs is called the **stalk** at p , denoted by \mathcal{O}_p .

The stalk is a ring. One can add two germs and get another germ in the stalk: if f is defined on U and g is defined on V , then $f + g$ is defined on $U \cap V$. Also, $f + g$ is well-defined: if \tilde{f} has the same germ as f (i.e f and \tilde{f} agree on some open neighbourhood W of p) and \tilde{g} has the same germ as g (i.e g and \tilde{g} agree on some open neighbourhood W' of p), then $\tilde{f} + \tilde{g}$ agrees with $f + g$ on $U \cap V \cap W \cap W'$.

Furthermore, for $p \in U$, there is a natural map $\mathcal{O}(U) \rightarrow \mathcal{O}_p$.

Lastly, \mathcal{O}_p is a ring itself. Let $m_p \subset \mathcal{O}_p$ be the set of germs that vanish at p . These germs form an ideal since m_p is closed under addition and multiplying any element in m_p by any function, the rest is also in m_p . This is also a maximal ideal since the quotient ring is a field.

Definition 44. Presheaf. A presheaf of sets, \mathcal{F} , on a topological space X is the following data:

(1) For each open set U in X , we have $\mathcal{F}(U)$ (alternative notations include $\mathcal{F}(U) = H^0(U, \mathcal{F}) = \Gamma(U, \mathcal{F})$ where $\mathcal{F}(U)$ is called the **sections or functions over U** . When U is omitted, we assume we are talking about X i.e \mathcal{F} contains sections/functions over X which are often called the global sections.

(2) For each inclusion $U \subseteq V$, we have the **restriction** map:

$$res_{V,U} : \mathcal{F}(V) \rightarrow \mathcal{F}(U).$$

Given $f \in \mathcal{F}(V)$, we write $f|_U = res_{V,U}(f)$.

We also require two more conditions:

(3) The map $res_{U,U}$ is the identity (as one would expect) $res_{U,U} = id_{\mathcal{F}(U)}$

(4) If $U \subseteq V \subseteq W$ are inclusions of open sets, then restrictions commute as the following diagram shows:

$$\begin{array}{ccc}
\mathcal{F}(U) & \xrightarrow{\text{res}_{U,V}} & \mathcal{F}(V) \\
& \searrow \text{res}_{U,W} & \downarrow \text{res}_{V,W} \\
& & \mathcal{F}(W)
\end{array}$$

With the notion of a presheaf, we can now generalize our notion of germs and stalks:

Germs and stalks of presheaf: Let \mathcal{F} be our presheaf. Then, each germ at p is a section over some open set containing p such that two sections are equivalent if they agree on some smaller open neighbourhood of p . Then, the stalk of a presheaf \mathcal{F} at a point p is the set of germs at p and is denoted by \mathcal{F}_p i.e

$$\mathcal{F}_p := \{(f, \text{open set } U) : p \in U, f \in \mathcal{F}(U)\}$$

with the relation $(f, U) \sim (g, V)$ if $\exists W \subset U, W \subset V, p \in W$ such that $\text{res}_{U,W}f = \text{res}_{V,W}g$.

Now we finally define sheaf:

Definition 45. Sheaf. A presheaf \mathcal{F} is a sheaf if it satisfies two more axioms:

Identity/injectivity axiom. For any open set U , if $\{U_i\}_{i \in I}$ is an open cover of U and $f_1, f_2 \in \mathcal{F}(U)$ with $f_1|_{U_i} = f_2|_{U_i}$ for all $i \in I$, then $f_1 = f_2$.

Glueability axiom. If $\{U_i\}_{i \in I}$ is an open cover of U , then given $f_i \in \mathcal{F}(U_i)$ for all i such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists some $f \in \mathcal{F}(U)$ such that $\text{res}_{U, U_i}f = f_i$ for all i .

10.2 Structure Sheaf

Sheaf of functions \mathcal{O} on $X = MSpec(R)$. (Recall: given R is a k -algebra (and integral domain), $X = MSpec(R)$ is the set of all homomorphisms from R to k .) So, X is an affine variety. The sheaf of functions \mathcal{O} on $X = MSpec(R)$ has sections $\mathcal{O}(U) \subset \text{Frac}(R)$ where $f \in \mathcal{O}(U)$ if for any $p \in U$, we can write $f = \frac{g}{h}$, $g, h \in R$ and $h(p) \neq 0$.

Proposition 60. If R is an integral domain, then $\mathcal{O}(X = MSpec(R)) = R$.

Proof. If $f \in R$, then $f = f/1 \in \mathcal{O}(X)$ and we know $1 \in R$ does not vanish on any point of X . So $R \subset \mathcal{O}(X)$. Now, conversely, suppose $f \in \mathcal{O}(X) \subset \text{Frac}(R)$. We want to show

$f \in R$. Define the ideal of denominators of f to be $I := \{r \in R : rf \in R\} \subset R$. Now given $f \in \mathcal{O}(X)$, near any p , there exists some $h \in I$ where $h(p) \neq 0$ and $f = \frac{g}{h}$. This implies $V(I) = \emptyset$. By Hilbert's Nullstellensatz, this means $I = R$ and so $I = (1)$, implying $f = f/1 \in R$. \square

Proposition 61. Let $S = k[x_1, \dots, x_n]/I(V)$. Recall that we can write $MSpec(S)$ as $V \subset \mathbb{A}^n$. Then, consider any open set $U \subset MSpec(S)$ (or $U \subset V$) where $U = D(f_1, \dots, f_n)$. Then, $U = \cup_i D(f_i)$. Furthermore, $D(f) \cup D(g) = D(fg)$.

The proof of this is straightforward.

Proposition 62. Let R be an integral domain with $z \in R$. Then,

$$\mathcal{O}(D(z)) = R_z = \left\{ \frac{1}{z^n} f : n \in \mathbb{Z}_{\geq 0}, f \in R \right\}$$

Proof. Suppose $f \in \text{Frac}(R)$ with $f \in \mathcal{O}(D(z))$. We want to show that $f \in R_z$. Let $I \subset R$ be the ideal of denominators of f i.e elements $r \in R$ with $rf \in R$. Then, near any point $p \notin V(z)$, we can write $f = \frac{g}{h}$, $h(p) \neq 0$. Therefore, $p \notin V(I)$. Thus, $V(I) \subset V(z)$. We know $z \in I(V(I)) \implies z \in \sqrt{I}$ (by Nullstellensatz) $\implies z^k \in I, k \in \mathbb{Z}^+$. \square

Now, we go back to the question of studying algebraic geometry on not just closed subsets (i.e affine varieties) but also open subsets. We come to an important sheaf:

Proposition 63. Let k be an algebraically closed field and let R be a finitely generated k -algebra with $f \in R$. Then, the sheaf over $D(f)$ is given by

$$\mathcal{O}(D(f)) = R_f = R[x]/(xf - 1) = \left\{ \frac{a}{f^n} : n \in \mathbb{Z}^{\geq 0}, a \in R \right\}.$$

Proof. We assume k is algebraically closed so that we can use Hilbert's Nullstellensatz. Let R be the coordinate ring of $D(f)$ and let $U_i := D(f_i), i = 1, \dots, m$ be a collection of open sets covering $D(f)$. Then, $V(f_1, \dots, f_m) = \emptyset$ in $D(f)$ so by Hilbert's Nullstellensatz, $(f_1, \dots, f_m) = 1$. Now, we check the two axioms of sheaves:

(1) injectivity: we need to show that the map $R \rightarrow \prod_i R_{f_i}$ is injective. Suppose $x \in R$ is 0 in every R_{f_i} . Then, $x = \frac{a}{f_i^N} = 0$, so $f_i^N x = 0$ (since $f_i \neq 0$ in U_i) for all i (given N is large enough). But we know f_i are non-zero, so $x = 0$.

(2) Glueability: Suppose we have $x_i \in R_{f_i}$ such that $x_i = x_j \in R_{f_i f_j}$ (caution: x_i is not a variable, it's a section/function over U_i). We want to show there exists $f \in R$ such that $f|_{U_i} = x_i$. To see this, we first write $x_i = \frac{s_i}{f_i^{n_i}}$. Now, if we take N large enough, we can

write all $x_i = \frac{s_i}{f_i^N}$. Then, $x_i = x_j$ implies $(s_i f_j^N - s_j f_i^N) f_i^N f_j^N = 0$. Replace s_i with s_i with $s_i f_i^N$ and f_i with f_i^{2N} , we may assume that $s_i f_j - s_j f_i = 0$ in R for all i, j . Now, $f \in \sqrt{(f)} = \sqrt{(x_1, \dots, x_m)}$ (since we already argued that $(x_1, \dots, x_m) = 1$ so it generates (f)), therefore, we can write $f^N = c_1 x_1 + \dots + c_m x_m$ where $c_i \in R$. Let $a = c_1 s_1 + \dots + c_m s_m$. Then, $a f_i = \sum_j c_j s_j x_i = \sum_j c_j x_j s_i = f^N s_i$. So, $\frac{a}{f^N} = \frac{s_i}{f_i} = \frac{s_i f_i^N}{f_i^{2N}} = \frac{s_i}{f_i^N} = x_i$. \square

Now, we put this all under a suitable dictionary:

Definition 46. $\text{Spec}(R)$. Let R be any ring. Denote by $\text{Spec}(R)$ the set of all prime ideals of R . For any subset $T \subset R$, define $V_{\text{prime}}(T) = \{p \in \text{Spec}(R) : \forall f \in T, f \in p\}$ i.e the set primes of R that contain T . Similarly, $D_{\text{prime}}(T)$ is the set of primes of R not containing T .

$\text{Spec}(R)$ is equipped with its Zariski topology: a subset $Z \subseteq \text{Spec}(R)$ is closed if $Z = V(I)$ for some $I \subset R$

With the last proposition, we have:

Theorem 64. The presheaf of rings \mathcal{O} on $\text{Spec}(R)$ sending $D(f)$ to $\mathcal{O}(D(f)) = R_f$ is a sheaf called the structure sheaf of $\text{Spec}(R)$.

Definition 47. Ringed space. Ring spectrum. A ringed space (X, \mathcal{O}_X) is a topological space with a sheaf of rings \mathcal{O}_X (called structure sheaves). The ring spectrum for any ring R is the ringed space $(\text{Spec}(R), \mathcal{O})$.

Definition 48. Prevariety. A prevariety over k is a ringed space (X, \mathcal{O}_X) over k such that the pair (X, \mathcal{O}_X) is locally isomorphic to ring spectrum of affine varieties.

11 Local Rings and Valuations

11.1 Local Rings

We first provide a series of definitions:

Definition 49. Rational Function. Let V be an affine variety and let $O(V)$ be its coordinate ring. A rational function f on V is an element in $k(V) = \text{Frac}(O(V)) = \text{Frac}(k[x_1, \dots, x_n]/I(V))$ i.e. $f \in k(V)$ can be expressed as $f = g/h$ where $h \neq 0$. A rational f is **defined at a point** $p \in V$ if $f = \frac{g_p}{h_p}$ with $g_p, h_p \in R$ and $h(p) \neq 0$.

Definition 50. Birational equivalence. We say two varieties X and Y are birational or birationally isomorphic if there exists rational maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g = \text{id}$, $g \circ f = \text{id}$. If X and Y are irreducible, this is equivalent to $k(X) \cong k(Y)$.

Definition 51. Ideal of denominators. For $f \in k(V)$, we have the ideal of denominators $\text{Den}_R(f) = \{r \in R : rf \subseteq R\} \subseteq R$. Here $R = O(V)$.

Theorem 65. For an open subset $U \subseteq X$, the ring of polynomial functions/regular functions $\mathcal{O}(U)$ on U can be identified with the subring of $\text{Frac}(\mathbb{O}(X))$ consisting of all rational functions defined on all points of U .

Definition 52. Localization $S^{-1}R$. Let R be a ring. A subset $S \subset R$ is a multiplicative subset of R if $1 \in S$ and $g, h \in S \implies gh \in S$. Define the localization $S^{-1}R$ of R at S to be the set $\{a/s : a \in R, s \in S\}$ under the equivalence relation $a_1/s_1 = a_2/s_2 \leftrightarrow \exists s \in S, s(a_1s_2 - a_2s_1) = 0$.

Note that this localization is, at least on face value, defined differently from the definition of localization before where we had R_f . Fortunately, they are not different:

Example: If $S = \{f^n\}_{n \geq 0}$, then $S^{-1}R = R_f$.

Example: Let q be a prime ideal of R . Then, $S = R - q$ gives $R_q := (R - q)^{-1}R$ which is the localization of R at q .

Definition 53. Local ring. A ring R is called a local ring if it has a unique maximal ideal.

We now look at two examples of local rings: (1) R_q and (2) $\mathcal{O}_p(V)$

Theorem 66. Let R be a ring and q be a prime ideal. Then, $R_q(R - q)^{-1}R$ is a local ring with maximal ideal qR_q .

Proof. Recall : $R_q = (R - q)^{-1}R$. First, we claim qR_q is a maximal ideal. Note, $qR_q = q(R - q)^{-1}R = (R - q)^{-1}qR = (R - q)^{-1}q$. This is an ideal because $(R - q)^{-1}qR = (R - q)^{-1}q$ (similarly for left ideal). Furthermore, $qR_q \neq R_q$. To do so, we will show that $1 = 1/1 \notin qR_q$. Suppose, $1/1 = r/s$ where $r \in q, s \in R - q$. Then, $\exists t \in R - q$ s.t $(s - r)t = 0$ (using the equivalence relation in localizations), so $rt = st$. This is impossible as $rt \in q$ and $st \notin q$ (since neither s nor t are in q but q is prime).

Suppose $r/s \in R_q$ such that $r/s \notin qR_q$. Then, $r \notin q$ since if it were, then $r/s = (r/1)(1/s) \in qR_q$.

Now, $s/r \in R_q$ and so r/s is invertible. Therefore, if an ideal contains any element that is not in qR_q , it contains a unit and therefore, the ideal becomes all of R i.e not a maximal ideal. \square

We know germs already.

Definition 54. Local ring $\mathcal{O}_p(V)$. For $p \in V$, we define the local ring $\mathcal{O}_p(V)$ of V to be the germs at p (see previous chapter).

$$\mathcal{O}_p := \{(f, U) : p \in U \text{ open subset of } V, f \in \mathcal{O}(U) \text{ a regular function}\}$$

with the equivalence relation: $(f, U) \sim (g, V)$ if $\exists W \subset U, W \subset V, p \in W$ such that $\text{res}_{U,W}f = \text{res}_{V,W}g$.

(Recall: V an affine variety, $U \subset V$ open, we have $\mathcal{O}(U) \subset \text{Frac}(R)$ where $f \in \mathcal{O}(U)$ if for any $x \in U$, we can write $f = \frac{g}{h}, g, h \in R$ and $h(x) \neq 0$.)

Basically, $\mathcal{O}_p(V)$ is the set of rational functions on V that are defined at p .

The set of points $z \in V$ where a rational function f is not defined is called the *pole set* of f .

Furthermore, define

$$m_p = (t - p)\mathcal{O}_p$$

and define

$$M_p = (\text{Vanishing ideal of } p \in V) \subset \mathcal{O}(V)$$

Example: Consider $p \in \mathbb{A}^1$. Then, $\mathcal{O}_p = \{f(t)/g(t) : g(p) \neq 0\} \subset k(t)$. Then, we identify $\mathcal{O}_p^\times = \{f(t)/g(t) : f(p) \neq 0, g(p) \neq 0\}$ and the complement $\mathcal{O}_p \setminus \mathcal{O}_p^\times = \{f(t)/g(t) : f(p) = 0, g(p) \neq 0\} = (t - p)\mathcal{O}_p$. Any proper ideal $I \subset \mathcal{O}_p$ does not contain a unit so it must be contained in $(t - p)\mathcal{O}_p$ and so $(t - p)\mathcal{O}_p$ is the unique maximal ideal $m_p \subset \mathcal{O}_p$.

We have the following immediate properties:

Proposition 67. (1) The pole set of a rational function is an algebraic subset of V .
(2) $O(V) = \cap_{p \in V} \mathcal{O}_p(V)$.

Proof. (1) Suppose $V \subseteq \mathbb{A}^n$. For $G \in k[x_1, \dots, x_n]$, let \bar{G} be the residue of G in $O(V)$. Now, let $f \in k(V)$ and let $J_f = \{G \in k[x_1, \dots, x_n] : \bar{G}f \in O(V)\}$. Then, J_f is an ideal of $k[x_1, \dots, x_n]$ containing $I(V)$ and the points of $V(J_f)$ are the points where f is not defined.
(2) Suppose $f \in \cap_{p \in V} \mathcal{O}_p(V)$, then $V(J_f) = \emptyset$ (since if f is defined everywhere in V , then $\bar{G} \neq 0 \implies G$ is not 0 on V . Then, by Nullstellensatz, $1 \in J_f$, so $1 \cdot f = f \in O(V)$). \square

Lemma 68. The following conditions on a ring R are equivalent:

- (1) The set of non-units in R forms an ideal.
- (2) R has a unique maximal ideal that contains every proper ideal of R .

Proof. If $m = \{\text{non-units of } R\}$. Then, every proper ideal of R is contained in m (since proper ideal does not contain units). \square

Proposition 69. $\text{Spec}(R_q) = \{p \subseteq R_q : p \subseteq q\}$

Theorem 70. The following are true for any point $p \in V$:

- (1) \mathcal{O}_p is a Noetherian local ring with unique maximal ideal $m_p \subset \mathcal{O}_p$ of pairs (U, f) with $f(p) = 0$ and m_p is generated by the image of the vanishing ideal $M_p \subset \mathcal{O}(V)$ of p .
- (2) For any k -algebra, a homomorphism $\mathcal{O}_p \rightarrow R$ is the same as a homomorphism $\mathcal{O}(V) \rightarrow R$ with all elements in $\mathcal{O}(V) \setminus M_p$ sent to units i.e \mathcal{O}_p is the localization $S^{-1}\mathcal{O}(V)$ for the multiplicatively closed set $S = \mathcal{O}(V) \setminus M_p$. (Note: these are backslashes i.e complements, not quotients)
- (3) If V_1, \dots, V_i are the irreducible components of V containing p , then the map $\mathcal{O}(V) \rightarrow \mathcal{O}_p$ factors $\mathcal{O}(V) \rightarrow \mathcal{O}(V_1 \cup \dots \cup V_i) \rightarrow \mathcal{O}_p$.
- (4) $\mathcal{O}(V)/M_p^i = \mathcal{O}_p/m_p^i$ for all i with $M_p^j/M_p^i = m_p^j/m_p^i$ and $T_p V := (M_p/M_p^2)^\vee = (m_p/m_p^2)^\vee$.

Proof. (1) Consider a germ in \mathcal{O}_p , (f, U) , that does not vanish at p when restricted $(U \cap D(f), f|_{U \cap D(f)})$. Then, $(U \cap D(f), f|_{U \cap D(f)})$ has an inverse $(U \cap D(f), f|_{U \cap D(f)}^{-1})$ and ideals that contain points not in m_p (i.e contains regular functions that do not vanish at p) become all of \mathcal{O}_p . Therefore, m_p is the unique maximal ideal.

Now, suppose $(D(g), \frac{f}{g})$ is a germ in m_p . Then this is equivalent up to a unit to $(D(g), f)$ where $f \in \mathcal{O}(V)$. Therefore, m_p is generated by M_p .

Now \mathcal{O}_p is Noetherian. Suppose $(f_1/g_1) \subset (f_1/g_1, f_2/g_2) \subset \dots \subset \mathcal{O}_p$ is an increasing sequence of ideals. This is equivalent to the sequence $(f_1) \subset (f_1, f_2) \subset \dots \subset \mathcal{O}_p$ - this sequence stabilizes because $\mathcal{O}(V)$ is Noetherian.

(2) Follows from definition.

(3) There is a map $\mathcal{O}(V) \rightarrow \mathcal{O}_p/m_p^i$ with M_p^i in the kernel. The other way, note that $\mathcal{O}(V) \setminus M_p$ are all units in $\mathcal{O}(V) \setminus M_p^i$ so we have a map $\mathcal{O}_p \rightarrow \mathcal{O}(V)/M_p^i$ and it is easy to see that m_p^i lies in the kernel. \square

12 Discrete Valuation Rings

We will now see a very useful application of local rings. We will require the following:

Proposition 71. Let R be an integral domain that is *not* a field. Then, the following are equivalent:

- (1) R is Noetherian and local (i.e has a unique maximal ideal), and the maximal ideal is principal.
- (2) There is an irreducible element $t \in R$ s.t every non-zero $z \in R$ may be written uniquely in the form $z = ut^n$ where u is a unit in R , and n is a non-negative integer.

Proof. (1) implies (2): let m be a maximal ideal and let t be a generator for m . Suppose $ut^n = vt^m$ for $n \geq m$ and u, v are units. Then, $ut^{n-m} = v$ is a unit, so $n = m$ (since t is irreducible) and $u = v$. Thus the expression $z = ut^n$ is unique. Now we show there exists such an expression. Assume that z is not a unit. Now, every proper ideal is contained in a maximal ideal, so $z = z_1 t$ for some $z_1 \in R$. If z_1 is a unit, we are done. Otherwise, write $z_1 = z_2 t$. Continuing this way, we get an infinite sequence z_1, z_2, \dots with $z_i = z_{i+1} t$. Given R is Noetherian, the chain of ideals $(z_1) \subset (z_2) \subset \dots$ must have a maximal member so $(z_n) = (z_{n+1})$ for some n . Then, $z_{n+1} = vz_n$ and so $z_n = vtz_n$ and $vt = 1$. But t is not a unit.

(2) $m = (t)$ is the set of non-units. Then, the only ideals in R are the principal ideals (t^n) where n is a non-negative integer so R is a PID. PIDs are Noetherian. \square

Definition 55. Discrete valuation ring. A ring satisfying the conditions of the previous proposition is called a discrete valuation ring (DVR). Therefore, if ring R is a DVR, then:

- (1) R is Noetherian and has a unique maximal ideal which is principal
- (2) There exists an irreducible element $t \in R$ s.t for any non-zero $z \in R$, we can write it uniquely in the form $z = ut^n$ where $u \in R$ is a unit and $n \in \mathbb{Z}_{\geq 0}$

An element t as in the second condition is called a uniformizing parameter for R . Any other uniformizing parameter is of the form ut where u is a unit in R .

Definition 56. Order function on DVR. Let K be the quotient field of R which is a DVR. Then, when t is fixed, any non-zero element $z \in K$ has a unique expression $z = ut^n$ (u is a unit in R and $n \in \mathbb{Z}$). The exponent n is called the order of z and is written $n = \text{ord}(z)$ with $\text{ord}(0) = \infty$. Then, $R = \{z \in K : \text{ord}(z) \geq 0\}$ and $m = \{z \in K : \text{ord}(z) \geq 0\}$ is the maximal ideal in R .

Now, we motivate why discrete valuation ring/order of vanishing is important

Consider the curve $y = x^2$. we want to assign an "order of vanishing" at $(1, 1)$ for every function in $k[x, y]/(y - x^2)$. We define the order of vanishing at $(1, 1)$ to be the **largest i**

such that $f \in (x - 1, y - 1)^i$. Here's the intuition - f is the sum of products of i linear functions vanishing at 1 and each linear function vanishes to order at least 1.

Let us compute some orders of vanishing:

(1) Order of vanishing of y at $(1, 1)$: $y(1, 1) = 1 \neq 0$, so $y \notin (x - 1, y - 1)$ so the order of vanishing is 0.

(2) $y - x^2 = 0 \in (x - 1, y - 1)^i$ for all i , so the order is ∞ .

(3) $y - 1 \in (x - 1, y - 1)$ so the order is ≥ 1 although it is unclear if the order is exactly 1.

(4) $2x - y - 1 = -(x - 1)^2 \in (x - 1, y - 1)^2$ so the order is ≥ 2 although it is unclear if $2x - y - 1 \in (x - 1, y - 1)^3$.

(5) $(2x - y - 1)^2 = -(x - 1)^4 \in (x - 1, y - 1)^4$ so the order is ≥ 4 although it is unclear if $(2x - y - 1)^2 \in (x - 1, y - 1)^5$.

We then generalize discrete valuation:

Definition 57. Discrete valuation. A discrete valuation of a field K is a function $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ such that

- (1) $v(x) = \infty$ exactly when $x = 0$,
- (2) v is surjective,
- (3) $v(fg) = v(f) + v(g)$,
- (4) $v(f + g) \geq \min(v(f), v(g))$ and if $v(f) \neq v(g)$, this is an equality.

If L is a subfield of K , then $v(L^\times) = 0$.

We let \mathcal{O}_v be the elements of K where $v(f) \geq 0$.

$$\mathcal{O}_v := \{x \in K : v(x) \geq 0\}.$$

We call $v(x)$ the **order of vanishing** of x .

Lastly, we define the maximal ideal

$$m_v := \{x \in K : v(x) \geq 1\}$$

For $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ a discrete valuation of field K , $\mathcal{O}_v/m_v \cong k$.

13 Forms, product of Rings and operations on ideals

These sections are from Foulton [1]. Writing these down here for completeness.

13.1 Forms

(Treating forms to be the same as homogenous polynomials)

Let R be an integral domain. If $f \in R[x_1, \dots, x_{n+1}]$ is a form, we define $f_* \in R[x_1, \dots, x_n]$ by setting $f_* = f(x_1, \dots, x_n, 1)$. Conversely, for any polynomial $f \in R[x_1, \dots, x_n]$ of degree d , write $f = f_0 + \dots + f_d$ where f_i is a form of degree i and define $f^* \in R[x_1, \dots, x_n, x_{n+1}]$ by setting

$$\begin{aligned} f^* &:= x_{n+1}^d f_0 + x_{n+1}^{d-1} f_1 + \dots + f_d \\ &= x_{n+1}^d f(x_1/x_{n+1}, \dots, x_n/x_{n+1}) \end{aligned}$$

and so, f^* is a form of degree d .

So, f_* gives us a way to make a form into a polynomial in lower dimensions while f^* gives us a way to transform a polynomial into a form in higher dimensions.

Proposition 72. (1) $(fg)_* = f_*g_*$, and $(fg)^* = f^*g^*$

(2) if $f \neq 0$ and r is the highest power of x_{n+1} that divides f , then $x_{n+1}^r(f_*)^* = f$. Also, $(f^*)_* = f$

(3) $(f+g)_* = f_* + g_*$, and $x_{n+1}^t(f+g)^* = x_{n+1}^r f^* + x_{n+1}^s g^*$ where $r = \deg(g)$, $s = \deg(f)$ and $t = r + s - \deg(f+g)$

Proof. Proving just the first:

(1)

$$\begin{aligned} (fg)_* &= (fg)(x_1, \dots, x_n, 1) \\ &= f(x_1, \dots, x_n, 1)g(x_1, \dots, x_n, 1) \\ &= f_*g_* \end{aligned}$$

$$\begin{aligned} (fg)^* &= x_{n+1}^d (fg)(x_1/x_{n+1}, \dots, x_n/x_{n+1}) \\ &= x_{n+1}^{d_f} f(x_1/x_{n+1}, \dots, x_n/x_{n+1}) x_{n+1}^{d_g} g(x_1/x_{n+1}, \dots, x_n/x_{n+1}) \\ &= f^*g^* \end{aligned}$$

where $d_f + d_g = d$, d_f degree of f .

(2) Suppose $f \in R[x_1, \dots, x_{n+1}]$ is a form i.e $f = f_d$. Suppose r is the highest power such that x_{n+1}^r that divides f so $f(x_1, \dots, x_{n+1}) = \sum_{i=1}^r x_{n+1}^i g_{d-i}(x_1, \dots, x_n, x_{n+1})$. So $f(x_1, \dots, x_n, 1) = \sum_{i=1}^r g_{d-i}(x_1, \dots, x_n, 1)$.

$$\begin{aligned} x_{n+1}^r (f_*)^* &= x_{n+1}^r (f(x_1, \dots, x_n, 1))^* \\ &= x_{n+1}^r (x_{n+1}^{d-r} f(x_1/x_{n+1}, \dots, x_n/x_{n+1}, 1/x_{n+1})) \\ &= x_{n+1}^r f(x_1/x_{n+1}, \dots, x_n/x_{n+1}, 1/x_{n+1}) \\ &= f(x_1, \dots, x_{n+1}) \end{aligned}$$

The rest can also be proven straightforwardly. \square

Corollary 73. Up to power of x_{n+1} , factoring a form $f \in R[x_1, \dots, x_{n+1}]$ is the same as factoring $f_* \in R[x_1, \dots, x_n]$. In particular, if $f \in k[x, y]$ is a form and k is algebraically closed then f factors into a product of linear factors.

Proof. Use previous proposition's part 1 and 2 to prove the first part. For the second, let y be st y does not divide g , then let $f = y^r g$. Then, $f_* = g_* = \epsilon \prod (x - \lambda_i)$ since k is algebraically closed, so $f = \epsilon y^r \prod (x - \lambda_i y)$ \square

13.2 Direct Products of Rings

The cartesian product of rings is a ring with the operations $(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 + b_1, \dots, a_n + b_n)$ and $(a_1, \dots, a_n)(b_1, \dots, b_n) = (a_1 b_1, \dots, a_n b_n)$. This ring is called the direct product of R_1, \dots, R_n , denoted by $\prod_{i=1}^n R_i$.

Let $\pi_j : \prod_{i=1}^n R_i \rightarrow R_j$ which takes a_1, \dots, a_n to a_j be the natural projection map which can easily be shown to be a ring homomorphism.

Given any ring R and ring homomorphisms $\psi_i : R \rightarrow R_i$ for $i = 1, \dots, n$, there is a unique ring homomorphism $\psi : R \rightarrow \prod_{i=1}^n R_i$ such that $\pi_i \circ \psi = \psi_i$. In particular, if a field k is a subring of each R_i , k may be regarded as a subring of $\prod_{i=1}^n R_i$.

Proposition 74. Let I be an ideal in $k[x_1, \dots, x_n]$ where k is algebraically closed. Suppose $V(I) = \{p_1, \dots, p_N\}$ is finite. Let $\mathcal{O}_i := \mathcal{O}_{p_i}(\mathbb{A}^n)$. Then, there is a natural isomorphism of $k[x_1, \dots, x_n]/I$ with $\prod_{i=1}^N \mathcal{O}_i/I\mathcal{O}_i$.

The proof of this can be found in [1].

Corollary 75.

$$\dim_k(k[x_1, \dots, x_n]/I) = \sum_{i=1}^N \dim_k(\mathcal{O}_i/I\mathcal{O}_i).$$

Corollary 76. If $V(I) = \{p\}$, then $k[x_1, \dots, x_n]/I$ is isomorphic to $\mathcal{O}_p(\mathbb{A}^n)/I\mathcal{O}_p(\mathbb{A}^n)$.

14 Affine plane curves and Intersection number

14.1 Preliminaries

Definition 58. Affine plane curve. An affine plane curve is the set of points in \mathbb{A}^2 at which $f \in k[x, y]$ vanishes. Therefore, the affine plane curve $X := \{(x, y) : f(x, y) = 0\}$.

Alternatively, we say that an affine plane curve is an equivalence class of non-constant polynomials under the equivalence relation of polynomials in $k[x, y]$: $f \sim g$ iff $f = \lambda g$ where $\lambda \in k$.

Intuition: This definition of affine plane curve using equivalence relation makes sense because if the affine plane curve is defined by $X = \{(x, y) : x^2 + y^2 = 0\}$. Then, the polynomial $x^2 + y^2$ is equivalent to $2(x^2 + y^2)$ and $5(x^2 + y^2)$ and so on - which is justified since all these polynomials have the same 0s. Furthermore, since $k[x_1, x_2]$ is a UFD, two polynomials are equivalent is equivalent to this being satisfied.

The degree of a curve is the degree of any representative of the curve. A curve of degree one is a line i.e $ax + by + c = 0$ or simply $ax + by + c$.

Definition 59. Components of an affine plane curve. If $f = \prod f_i^{e_i}$ where f_i are irreducible factors of f , we say that f_i are the components of f and e_i is the multiplicity of the f_i . If $e_i = 1$, f_i is called a **simple component**. Otherwise, it is a **multiple component**.

If f is irreducible, then (f) is prime and so $V(f)$ is irreducible variety in \mathbb{A}^2 . We will often write $O(f)$ to mean $O(V(f))$. Similarly, write $k(f)$ instead of $k(V(f))$ and $\mathcal{O}_p(f)$ instead of $\mathcal{O}_p(V(f))$.

Definition 60. Simple point, tangent line, non-singular curve. Let f be a curve and let $p = (a, b) \in f$. The point p is called a simple point of f if either derivative $\frac{d}{dx}f(p) \neq 0$ or $\frac{d}{dy}f(p) \neq 0$. In this case, the line $\frac{d}{dx}f(p)(x - a) + \frac{d}{dy}f(p)(y - b) = 0$ is called the tangent line to f at p . A point that is not simple is called multiple (or singular). A curve with only simple points is called a non-singular curve.

Definition 61. Multiplicity. Let f be any curve and $p = (0, 0)$. Write $f = f_m + f_{m+1} + \dots + f_n$, where f_i is a form of degree i and $f_m \neq 0$. We define m to be the multiplicity of f at $p = (0, 0)$ and write $m = m_p(f)$.

Couple of immediate properties (for $p = (0, 0)$):

Lemma 77. Let $p = (0, 0)$. Then,

(1) $p \in f$ if and only if $m_p(f) > 0$.

(2) p is a simple point on f if and only if $m_p(f) = 1$ and in this case f_1 is the tangent line to f at p . If $m = 2$, p is called a double point. If $m = 3$, p is called a triple point and so on.

Proof: Suppose $p = (0, 0) \in f$. Then, if we write $f = f_m + f_{m+1} + \dots + f_n$ and $f_m \neq 0$, then since $f = 0$ at p , each f_i must disappear and so $m > 0$ (if $m = 0$, then $f_m(p) \neq 0$ since $f_m \neq 0$). Converse is straightforward.

Given $f_m \in k[x, y]$ is a form, write $f_m = \prod_i g_i^{r_i}$ where each g_i is a distinct line. The lines g_i are called the tangent lines to f at $p = (0, 0)$ and r_i is called the multiplicity; g_i is called simple tangent line if $r_i = 1$. If f has m distinct simple tangents at p , we say p is an ordinary multiple point of f . An ordinary double point is called a node.

Let $f = \prod_i f_i^{e_i}$ be the factorization of f into irreducible components. Then, $m_p(f) = \sum e_i m_p(f_i)$ and if L is a tangent line to f_i with multiplicity r_i .

Proposition 78. A point p is a simple point of f if and only if p belongs to just one component f_i of f where f_i is a simple component of f and p is a simple point of f_i .

Define multiplicity at an arbitrary point:

Let $P = (a, b) \neq (0, 0)$. Let T be the translation that takes $(0, 0)$ to P i.e $T(x, y) = (x + a, y + b)$. Then $f^T = f(x + a, y + b)$.

Definition 62. Multiplicity at any point p . We define $m_p(f)$ to be $m_{(0,0)}(f^T)$ i.e write $f^T = g_m + g_{m+1} + \dots$ where g_i is a form and $g_m \neq 0$ and let $m = m_p(f)$.

Definition 63. Tangent Lines. If $g_m = \prod_i L_i^{r_i}$ where $L_i = \alpha_i x + \beta_i y$, then the lines $\alpha_i(x - a) + \beta_i(y - b)$ are defined to be the tangent lines to f at p and r_i is the multiplicity of the tangent.

Therefore, p is a simple point on f if and only if $m_p(f) = 1$.

Theorem 79. Let f be an irreducible plane curve and let $p \in f$. Then, p is a simple point of f if and only if $\mathcal{O}_p(f) = \mathcal{O}_p(V(f))$ is a discrete valuation ring.

14.2 Intersection Number

Let f and g be plane curves with $p \in \mathbb{A}^2$. Our goal is to define what the intersection number of f and g is at p , denoted by $I(p, f \cap g)$.

Desired properties of intersection number:

We say that f and g intersect properly at p if f and g have no common component that passes through p .

- (1) $I(p, f \cap g)$ is a nonnegative integer for any f, g and p such that f and g intersect properly at p . Also, $I(p, f \cap g) = \infty$ if f and g do not intersect properly at p .
- (2) $I(p, f \cap g) = 0$ if and only if $p \notin f \cap g$. $I(p, f \cap g)$ depends only on the components of f and g that pass through p . $I(p, f \cap g) = 0$ if f or g is a non-zero constant.
- (3) If T is an affine change of coordinates on \mathbb{A}^2 and $T(q) = p$, then $I(p, f \cap g) = I(q, f^T \cap g^T)$.
- (4) $I(p, f \cap g) = I(p, g \cap f)$

Two curves, f and g intersect transversally at p if p is a simple point on both f and on g and if the tangent line to f at p is different from the tangent line of g at p .

So,

- (5) $I(p, f \cap g) \geq m_p(f)m_p(g)$, with equality occurring if and only if f and g have no tangent lines in common at p .
- (6) If $f = \prod f_i^{r_i}$ and $g = \prod g_j^{s_j}$, then $I(p, f \cap g) = \sum_{i,j} r_i s_j I(p, f_i \cap g_j)$.
- (7) $I(p, f \cap g) = I(p, f \cap (g + Af))$ for any $A \in k[x, y]$.

Theorem 80. There is a unique intersection number $I(p, f \cap g)$ defined for all plane curves f, g and all points $p \in \mathbb{A}^2$ satisfying the properties (1) - (7). It is given by

$$I(p, f \cap g) = \dim_k(\mathcal{O}_p(\mathbb{A}^2)/(f, g))$$

15 Projective Space

15.1 Construction of Projective Space

Consider two curves: $y^2 = x^2 + 1$ and $y^2 = \alpha x$ where $\alpha \in k$. These two curves intersect at two points when $\alpha \neq \pm 1$. When $\alpha = \pm 1$, the curves do not intersect however the curve is asymptotic to the line. Our goal is to enlarge the xy plane to allow such curves to intersect *at infinity*.

Identify each $(x, y) \in \mathbb{A}^2$ with $(x, y, 1) \in \mathbb{A}^3$. Then, every point $(x, y, 1)$ determines a line that passes through itself and $(0, 0, 0)$. Note that every line through the origin in \mathbb{A}^3 corresponds to exactly one such point, unless the line is on the plane $z = 0$. We say that the lines through $(0, 0, 0)$ in $z = 0$ plane correspond to points at infinity.

Definition 64. Projective n -space over k , \mathbb{P}_k^n . This is defined to be the set of all lines through $(0, 0, \dots, 0) \in \mathbb{A}_k^{n+1}$. Any point $(x) = (x_1, \dots, x_n, x_{n+1}) \neq (0, \dots, 0)$ determines a unique such line i.e $\{(\lambda x_1, \dots, \lambda x_n) : \lambda \in k^\times\}$. Two such points are equivalent i.e determine the same line if and only if there exists a non-zero $\lambda \in k$ such that $y_i = \lambda x_i$ for $i = 1, \dots, n, n+1$. Therefore, \mathbb{P}^n is the set of equivalence classes of points in $\mathbb{A}^{n+1} \setminus \{(0, \dots, 0)\}$. Elements of \mathbb{P}^n are called points. If $p \in \mathbb{P}^n$ is determined by some $(x, \dots, x_{n+1}) \in \mathbb{A}^{n+1}$, then we say (x_1, \dots, x_{n+1}) are **homogenous coordinates** for p . We write $p = [x_1 : \dots : x_{n+1}]$ to indicate that (x_1, \dots, x_{n+1}) are homogenous coordinates for $p \in \mathbb{P}^n$. While x_i itself is not well-defined (because of the equivalence), it is well-defined to say whether x_i is 0 or not; if $x_i \neq 0$, then $\frac{x_j}{x_i}$ are well-defined

Definition 65. U_i . We let $U_i = \{[x_1 : \dots : x_{n+1}] \in \mathbb{P}^n : x_i \neq 0\}$. Then, for any $p \in U_i$, p can be uniquely written as $p = [x_1 : \dots : x_{i-1} : 1 : x_{i+1} : \dots : x_{n+1}]$ which allows us to view U_i as an affine n -space. Visualize U_i as taking a sphere and cutting the space perpendicular to the i -th axis. The coordinates $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n+1})$ are called the **non-homogenous coordinates** for p with respect to U_i .

If we define $\varphi_i : \mathbb{A}^n \rightarrow U_i$ by $\varphi_i(a_1, \dots, a_n) = [a_1 : \dots : a_{i-1} : 1 : a_i : \dots : a_n]$, then φ_i sets up a one-to-one correspondence between points of \mathbb{A}^n and the points of U_i . Note,

$$\mathbb{P}^n = \bigcup_i^{n+1} U_i$$

so \mathbb{P}^n is covered by $n+1$ sets each of which looks like affine n -space.

Definition 66. Hyperplane at infinity.

$$H_\infty = \mathbb{P}^n \setminus U_{n+1} = \{[x_1 : \dots : x_{n+1}] : x_{n+1} = 0\}$$

Given there is a natural correspondence between $[x_1 : \dots : x_n : 0]$ and $[x_1 : \dots : x_n]$, H_∞ can be identified with \mathbb{P}^{n-1} and so $\mathbb{P}^n = U_{n+1} \cup H_\infty$

We will often use the following:

$$\mathbb{P}^n = H_\infty \cup U_{n+1}$$

Examples:

(1) \mathbb{P}_k^0 is a pont.

(2) $\mathbb{P}_k^1 = \{[x : 1] : x \in k\} \cup \{[1 : 0]\}$. \mathbb{P}_k^1 is the affine line plus one point at infinity. \mathbb{P}_k^1 is the projective line over k .

(3) $\mathbb{P}_k^2 = \{[x : y : 1] : (x, y) \in \mathbb{A}^2\} \cup \{[x : y : 0] : [x : y] \in \mathbb{P}^1\}$. Here H_∞ is called the line at infinity. \mathbb{P}_k^2 is called the projective plane over k .

(4) Consider the line $y = mx + b$ in \mathbb{A}^2 . Identify \mathbb{A}^2 with $U_3 \subset \mathbb{P}^2$, then the points on the line correspond to the points $y = mx + bz$ and $z \neq 0$. The set $\{[x : y : z] \in \mathbb{P}^2 : y = mx + bz\} \cap H_\infty = \{[1 : m : 0]\}$ so all lines with the same slope, when extended in this way, pass through the same point at infinity.

(5) Consider the curve $y^2 = x^2 + 1$. The corresponding set in \mathbb{P}^2 is given by the equation $y^2 = x^2 + z^2$ where $z \neq 0$. Then, $\{[x : y : z] \in \mathbb{P}^2 : y^2 = x^2 + z^2\}$ intersects H_∞ in the two points $[1 : 1 : 0]$ and $[1 : -1 : 0]$. These are the points where the lines $y = x$ and $y = -x$ intersect the curve.

Lemma 81. $\mathbb{P}^n = \mathbb{P}^{n-1} \cup \mathbb{A}^n$

Proof.

$$\begin{aligned} \mathbb{P}^n &= H_\infty \cup U_{n+1} \\ &= \mathbb{A}^n \cup U_{n+1} \\ &= \mathbb{A}^n \cup \mathbb{P}^{n-1} \end{aligned}$$

□

Proposition 82. Let L_1 and L_2 be two projective lines in \mathbb{P}^2 . Then, $L_1 \cap L_2 \neq \emptyset$; they intersect at a point if $L_1 \neq L_2$ or $L_1 \cap L_2 = L_1 = L_2$. However, in \mathbb{A}^2 , two lines may have no intersection.

Proof. Let the 2 lines in \mathbb{A}^2 be $ax + by + c = 0$ and $a'x + b'y + c' = 0$. Now, if these lines are not parallel, they already intersect in \mathbb{A}^2 . They would also intersect in \mathbb{P}^2 : replace x and y with $\frac{x}{z}, \frac{y}{z}$ to write $ax + by + cz = 0, a'x + b'y + c'z = 0$, subtract the two equations to get $x = \frac{c-c'}{\frac{a}{b} - \frac{a'}{b'}}z$ where the denominator is not 0 since the two lines were not parallel. Now, suppose the two lines were parallel. Then, we can rewrite the lines in \mathbb{P}^2 as $ax + by + cz = 0$ and $ax + by + c'z = 0$ where $c \neq c'$. Take $z = 0$ and we now get $y = -\frac{a}{b}x$. So the two lines intersect $[-b, a, 0]$. Note that if $c = c'$, then the two lines are the same. □

15.2 Projective Variety

Definition 67. Zero of a polynomial in \mathbb{P}^n . A point $p \in \mathbb{P}^n$ is a zero of a polynomial $f \in k[x_1, \dots, x_{n+1}]$ if $f(x_1, \dots, x_{n+1}) = 0$ for every choice of homogenous coordinates (x_1, \dots, x_{n+1}) for p . We say $f(p) = 0$. If f vanishes at one representative, then it vanishes at every representative of p .

Proposition 83. (1) Any polynomial $f \in k[x_1, \dots, x_{n+1}]$ may be written as $f = f_0 + \dots + f_r$ for $r = \deg(f)$ where f_i is a homogenous polynomial of degree i .
(2) If $f(p) = 0$, then $f_i(p) = 0$ for any set of homogenous coordinates for p .

Proof. (1) follows from just writing f as a sum of monomials of degree i . We prove (2) now: given $f_i(p)$ is a homogenous polynomial, $f_i(tp) = t^i f_i(p)$. Then, $\psi(t) = f(tp) = \sum_i t^i f_i(p)$. Given $f(tp) = 0$ for all $t \neq 0$ i.e infinitely many t , then $\psi(t) \in k[t]$ has infinitely many 0s, so $\psi(t)$ is the zero polynomial i.e $f_i(p) = 0$ for all i . \square

Definition 68. Projective varieties. For any set S of polynomials in $k[x_1, \dots, x_{n+1}]$, we let

$$V(S) = \{p \in \mathbb{P}^n : p \text{ is a zero of each } f \in S\}.$$

Vanishing ideal. If I is the ideal generated by S , then $V(I) = V(S)$. If $I = (f^{(1)}, \dots, f^{(r)})$ where $f^{(i)} = \sum_j f_j^{(i)}$ where $f_j^{(i)}$ is a form of degree j , then $V(I) = V(\{f_j^{(i)}\}_{i,j})$ (see the first proposition of this section) is the set of zeros of a finite number of forms. Such sets are called projective varieties.

Definition 69. Vanishing ideal. Let $X \subset \mathbb{P}^n$, we let

$$I(X) = \{f \in k[x_1, \dots, x_{n+1}] : f \text{ is zero on every } p \text{ in } X\}.$$

An ideal $I \in k[x_1, \dots, x_{n+1}]$ is homogenous if for every $f = \sum_{i=0}^m f_i \in I$ where f_i is a form of degree i , we also have $f_i \in I$. For any $X \subset \mathbb{P}^n$, $I(X)$ is a homogenous ideal.

Proposition 84. An ideal $I \subset k[x_1, \dots, x_{n+1}]$ is homogenous if and only if it is generated by a finite set of forms.

Proof. Suppose $I = (f^{(1)}, \dots, f^{(r)})$ is homogenous, then I is generated by $\{f_j^{(i)}\}_{i,j}$. Conversely, suppose $S = \{f^{(\alpha)}\}$ be a set of forms generating an ideal I with $\deg(f^{(\alpha)}) = d_\alpha$ and suppose $f = f_m + \dots + f_r \in I$ and $\deg(f_i) = i$. We need to show that $f_m \in I$ because then $f - f_m \in I$ and then by induction each f_i would be in I . Let $f = \sum_\alpha a^{(\alpha)} f^{(\alpha)} \in I$. Comparing terms of the same degree, we can see that $f_m = \sum_\alpha a_{m-d_\alpha}^{(\alpha)} f^{(\alpha)}$, so $f_m \in I$. \square

One can easily prove the following:

Lemma 85. The sum, product, intersection and radical of homogenous ideals are all homogenous.

Similar to the affine case, $V \subset \mathbb{P}^n$ is irreducible if and only if $I(V)$ is prime. Any algebraic variety can be written uniquely as a union of projective varieties, its irreducible components.

Notation: We use V_p, I_p for projective operations and V_a, I_a for affine ones.

Definition 70. Cone over V . If V is an algebraic variety in \mathbb{P}^n , we define

$$C(V) = \{(x_1, \dots, x_{n+1}) \in \mathbb{A}^{n+1} : [x_1 : \dots : x_{n+1}] \in V \text{ or } (x_1, \dots, x_{n+1}) = (0, \dots, 0)\}.$$

If $V = \emptyset$, then $I_a(C(V)) = I_p(V)$. If I is a homogenous ideal in $k[x_1, \dots, x_{n+1}]$ s.t $V_p(I) \neq \emptyset$, then $C(V_p(I)) = V_a(I)$

15.3 Graded ring and importance of forms+homogenous ideals

Most of this is from Hartshorne. We introduce the algebraic constructions on which the previous subsection was built.

Definition 71. Graded ring. A graded ring S is a ring with a decomposition $S = \bigoplus_{d \geq 0} S_d$ of S into a direct sum of abelian groups S_d such that for any $d, e \geq 0$, $S_d \cdot S_e \subseteq S_{d+e}$. An element of S_d is called a homogenous element of degree d . Therefore, any element of S can be written uniquely as a finite sum of homogenous elements.

Therefore, $S := k[x_0, \dots, x_n]$ is a graded ring. We let S_d be the set of all forms of degree d .

Definition 72. Homogenous ideal. S is a graded ring. An ideal $I \subseteq S$ is a homogenous ideal if $I = \bigoplus_{d \geq 0} (I \cap S_d)$. Alternatively: if I is a homogenous ideal, then for any $a \in I$ s.t $a = \sum_d a_d$ where $a_d \in S_d$, we have that $a_d \in I$ as well.

Importance of forms and homogenous ideals: if $f \in S = k[x_0, \dots, x_n]$ is a polynomial, it does not define a function on \mathbb{P}^n because of the non-uniqueness of the homogenous coordinates. However, if f is a form/homogenous polynomial of degree d , then, $f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$. This means the property of f being 0 or not *only* depends on the equivalence class of (a_0, \dots, a_n) . This allows us to write f as a function from \mathbb{P}^n to $\{0, 1\}$ by letting $f(p) = 0$ if $f(a_0, \dots, a_n) = 0$ and $f(p) = 1$ if $f(a_0, \dots, a_n) \neq 0$.

This, in turn, allows us to define $V(f) = \{p \in \mathbb{P}^n : f(p) = 0\}$.

Now, we can define the Zariski topology on \mathbb{P}^n since we have a notion of what a projective variety is.

Open cover of projective spaces: Let $f \in S := k[x_0, \dots, x_n]$ be a linear homogenous polynomial. Then, the zero set of f is called a **hyperplane**. We define

$$H_i := \{(x_0, \dots, x_n) \in \mathbb{P}^n : x_i = 0\}$$

and

$$U_i := \mathbb{P}^n - H_i.$$

Then, \mathbb{P}^n is covered by the open sets U_i - this is because if $P = (a_0, \dots, a_n) \in \mathbb{P}^n$ is a point, then at least one a_i is not 0 and so the point is in U_i .

15.4 Projective Nullstellensatz

Theorem 86. Projective Nullstellensatz. Let I be a homogenous ideal in $k[x_1, \dots, x_{n+1}]$. Then,

- (1) $V_p(I) = \emptyset$ if and only if there exists an integer N such that I contains all forms of degree $\geq N$.
- (2) If $V_p(I) \neq \emptyset$, then $I_p(V_p(I)) = \sqrt{(I)}$.

Proof. For the first part: The following conditions are equivalent: (1) $V_p(I) = \emptyset$, (2) $V_a(I) \subset \{(0, \dots, 0)\}$, (3) $\sqrt{(I)} = I_a(V_a(I)) \supset (x_1, \dots, x_{n+1})$ (by taking $I(-)$ of both sides in (2)) and (4) $(x_1, \dots, x_{n+1})^N \subset I$.

For the second part: $I_p(V_p(I)) = I_a(C(V_p(I))) = I_a(V_a(I)) = \sqrt{I}$. □

15.5 Homogenous coordinate ring

Definition 73. Coordinate ring. Let $I(V)$ be a **homogenous, prime** ideal and we call the residue ring $O_h(V) := k[x_1, \dots, x_{n+1}]/I(V)$, which is an integral domain, is called the homogenous coordinate ring.

More generally, let I be any homogenous ideal in $k[x_1, \dots, x_{n+1}]$ and let $k[x_1, \dots, x_{n+1}]/I$ be a coordinate ring.

Proposition 87. Every element $f \in k[x_1, \dots, x_{n+1}]/I$ (where I is homogenous) may be uniquely written as $f = f_0 + \dots + f_r$ where f_i is a form of degree i .

Proof. We already saw how to find one such decomposition. Suppose $f \in O_h(V)$ with $f = \sum_i f_i$. Suppose, we have another decomposition $f = \sum_i g_i$. Then, $f - \sum_i g_i = \sum_i (f_i - g_i) \in I$ and given I is homogenous, each $f_i - g_i \in I$ \square

Definition 74. Homogenous function field. Let $k_h(V)$ be the quotient field of $O_h(V)$. This is called the homogenous function field of V . No elements of $O_h(V)$ except the constants determine functions on V (since we are in the projective space so $tx = x \in \mathbb{P}^n$ for nonzero t). However, if $f, g \in O_h(V)$ are both **forms** of the same degree d , then $\frac{f}{g}$ does define a function where g is not zero because then $\frac{f(\lambda s)}{g(\lambda s)} = \frac{\lambda^d f(x)}{\lambda^d g(x)} = \frac{f(x)}{g(x)}$ so $\frac{f}{g}$ is independent of our choice of homogenous coordinates in \mathbb{P}^n .

Definition 75. Function field. The function field of V , written $k(V)$, is defined to be

$$\{z \in k_h(V) : \text{for some forms of same degree } f, g \in O_h(V), z = \frac{f}{g}\}.$$

It is not difficult to verify that $k(V)$ is a subfield of $k_h(V)$. So, $k \subset k(V) \subset k_h(V)$ but $O_h(V) \subsetneq k(V)$ and elements of $k(V)$ are called rational functions on V .

Similarly, we can also define local rings and so on.

Let $p \in V$ and $z \in k(V)$. Then, z is defined at p if z can be written as $z = \frac{f}{g}$ where f, g are forms of the same degree and $g(p) \neq 0$. Then,

$$\mathcal{O}_p(V) = \{z \in k(V) : z \text{ is defined at } p\}.$$

We see that $\mathcal{O}_p(V)$ is a subring of $k(V)$. It is a local ring, called the local ring of V at p , with the unique maximal ideal

$$m_p(V) = \{z \in k(V) : z = \frac{f}{g}, g(p) \neq 0, f(p) = 0\}$$

and the value $z(p)$ of a function $z \in \mathcal{O}_p(V)$ is well-defined.

Definition 76. Homogenization. Given a polynomial $f(x_1, \dots, x_n)$ of degree d , the homogenization with respect to x_i is the polynomial $f(x_1/x_i, \dots, x_n/x_i)x_i^d$. The dehomogenization of a polynomial $g(x_1, \dots, x_n, z)$ is $g(x_1, \dots, x_n, 1)$.

Definition 77. We define the following map $\varphi_i : U_i \rightarrow \mathbb{A}^n$ as follows: if $p = (a_0, \dots, a_n) \in U_i$, then

$$\varphi_i(p) = \left(\frac{a_0}{a_i}, \dots, \frac{a_{i-1}}{a_i}, 1, \frac{a_{i+1}}{a_i}, \dots, \frac{a_n}{a_i} \right)$$

Proposition 88. The map φ_i as defined above is a homeomorphism of U_i with its induced topology to \mathbb{A}^n with its Zariski topology.

Proof. We know φ_i is bijective. We now show that the closed subsets are sent to closed subsets by the function. Without loss of generality, suppose $i = 0$. We simply write $U := U_0$, $\varphi := \varphi_i$.

Suppose $A = k[y_1, \dots, y_n]$. First define $\alpha : S^h \rightarrow A$ where S^h is the set of homogenous elements of S . Also define $\beta : A \rightarrow S^h$. Given $f \in S^h$, we set $\alpha(f) = f(1, y_1, \dots, y_n)$ and given $g \in A$ of degree r , then $\beta(g) = x_0^r g(x_1/x_0, \dots, x_n/x_0)$ is a homogenous polynomial of degree r in the x_i .

Let $Y \subseteq U$ be a closed subset. Let \bar{Y} be its closure in \mathbb{P}^n . This is an algebraic set so $\bar{Y} = V(T)$ for some subset $T \subseteq S^h$. Let $T' = \alpha(T)$. Then, one can check that $\varphi(Y) = V(T')$. Conversely, let W be a closed subset of \mathbb{P}^n . Then, $W = V(T')$ for some subset T' of A . One can check that $\varphi^{-1}(W) = V(\beta(T')) \cap U$. Therefore, φ and φ^{-1} preserve closed sets. \square

Corollary 89. If Y is a projective (respectively, quasi-projective) variety, then Y is covered by the open sets $Y \cap U_i$ for $i = 0, \dots, n$ which are homeomorphic to affine (respectively, quasi-affine) varieties through the map φ_i defined above.

Example: $x^2 + y^2 + 1$ can be homogenized to get $z^2(x^2/x^2 + y^2/x^2 + 1) = x^2 + y^2 + z^2$. On the other hand, the dehomogenization of $x^2 + y^2 + z^2$ is $x^2 + y^2 + 1$.

Definition 78. Rational map between projective varieties. Let $X \subseteq \mathbb{P}^n$ and let $Y \subseteq \mathbb{P}^m$ be projective varieties. A rational map $f : X \rightarrow Y$ is a collection of homogenous polynomials of same degree :

$$[x_0, x_1, \dots, x_n] \rightarrow [f_0(x), f_1(x), \dots, f_m(x)],$$

such that $f(U) \subseteq Y$ where U is an open subset of X where f_i has no common zeros. We say two tuples $[f_i]$ and $[f'_i]$ are equal if they agree on an open, dense subset of X . We say f is regular at $p \in X$ (or defined at $p \in X$) if we can find such f_i with no common zero at p , so $f(p) \in Y$ is well-defined. A regular map $f : X \rightarrow Y$ is a rational map that is regular at every point of X .

16 Intersection Number in Projective Spaces, Bezout's Theorem

Most (if not all) of this is from Hartshorne's.

16.1 Affine and Projective Dimension Theorems

In linear algebra, we have seen the following: given $U, V \subseteq W$ s.t $\dim(U) = r$, $\dim(V) = s$, $\dim(W) = n$, we know $\dim(U \cap V) \geq r + s - n$. We will try to prove something similar for U, V irreducible algebraic sets.

Proposition 90. Affine Dimension Theorem. Let Y and Z be irreducible varieties of dimensions r, s in \mathbb{A}^n . Then, every irreducible component W of $Y \cap Z$ has dimension $\geq r + s - n$.

Proof. Suppose Z is a hypersurface defined by $f = 0$. Now, if $Y \subseteq Z$ then we are done: $Y \subseteq Z$ implies $Y = Z$ and there is no irreducible component other than Z itself. Now suppose $Y \not\subseteq Z$. We prove that each irreducible component $W \subset Y \cap Z$ has dimension $r - 1$ (because $\dim(Z) = n - 1$). Let $O(Y)$ be the coordinate ring of Y . Then, the irreducible components of $Y \cap Z$ directly correspond to minimal prime ideals p of the principal ideal $(f) \subset O(Y)$. From Krull dimension, each such p has height one and so $O(Y)/p$ has dimension $r - 1$. Using the fact that $\dim(V) = \dim(O(V))$, we conclude.

Now, for the general case: consider the product $Y \times Z \subseteq \mathbb{A}^{2n}$ - this is an irreducible variety of dimension $r + s$. Let Δ be the diagonal $\{p \times p : p \in \mathbb{A}^{2n}\} \subseteq \mathbb{A}^{2n}$. Then, $\mathbb{A}^n \cong \Delta$ are isomorphic (use the map $p \rightarrow p \times p$). Then, $Y \cap Z$ corresponds to $(Y \times Z) \cap \Delta$. Given Δ has dimension n (since \mathbb{A}^n has dimension n) and since $r + s - n = (r + s) + n - 2n$, then our problem can be rephased into proving the result for the two varieties $Y \times Z$ and Δ in \mathbb{A}^{2n} . We know Δ is the intersection of exactly n hypersurfaces namely $x_1 - y_1 = 0, x_2 - y_2 = 0, \dots, x_n - y_n = 0$ where $(x_1, \dots, x_n, y_1, \dots, y_n)$ are the coordinates of \mathbb{A}^{2n} . Apply the special case above here to conclude. \square

Theorem 91. Projective Dimension Theorem. Let Y and Z be two irreducible varieties of dimension r and s in \mathbb{P}^n . Then, every irreducible component of $Y \cap Z$ has dimension $\geq r + s - n$. Furthermore, if $r + s - n \geq 0$, then $Y \cap Z$ is nonempty.

Proof. First part follows from the affine dimension theorem since \mathbb{P}^n is covered by affine n -spaces. \square

16.2 Numerical Polynomials, Hilbert Polynomials

Definition 79. Numerical polynomial. A numerical polynomial is a polynomial $p(z) \in \mathbb{Q}[z]$ s.t $p(n) \in \mathbb{Z}$ for all $n \geq 0, n \in \mathbb{Z}$.

For an easy example, we have $f \in \mathbb{Z}[x]$. But to get a more general understanding of these polynomials, we have the following:

Proposition 92. (a) If $p \in \mathbb{Q}[z]$ is a numerical polynomial, then there are integers c_0, c_1, \dots, c_r s.t $p(z) = c_0 \binom{z}{r} + c_1 \binom{z}{r-1} + \dots + c_r$ where $\binom{z}{r} = \frac{1}{r!}(z)(z-1)\dots(z-r+1)$ is the familiar binomial coefficient. With this, $p(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

(b) If $f : \mathbb{Z} \rightarrow \mathbb{Z}$ is any function and if there exists a numerical polynomial $Q(z)$ s.t the difference function $\Delta(f) = f(n+1) - f(n)$ is equal to $Q(n)$ for all $n \gg 0$, then there exists a numerical polynomial $p(z)$ s.t $f(n) = p(n)$ for all $n \gg 0$.

Proof. (a) Prove by induction. When $\deg(p) = 0$, the statement is already proven. Now, $\binom{z}{r} = \frac{z^r}{r!} + \dots$, we can express $p \in \mathbb{Q}[z]$ of degree r in the form above given $c_0, \dots, c_r \in \mathbb{Q}$ (we still need to show that they are integers). Now, for any polynomial p , we define the difference polynomial Δp by $\Delta p(z) = p(z+1) - p(z)$. Since $\Delta \binom{z}{r} = \binom{z}{r-1}$ (i.e. $\binom{z+1}{r} - \binom{z}{r} = \binom{z}{r-1}$), we have that $\Delta p = c_0 \binom{z}{r-1} + c_1 \binom{z}{r-2} + \dots + c_{r-1}$. However, this is the expression of our numerical polynomial of degree $r-1$. By induction $c_0, \dots, c_{r-1} \in \mathbb{Z}$ - which then implies $c_r \in \mathbb{Z}$ since $p(n) \in \mathbb{Z}$ for $n \gg 0$.

(b) Let $q = c_0 \binom{z}{r} + \dots + c_r$ where $c_0, \dots, c_r \in \mathbb{Z}$. Now, let $p = c_0 \binom{z}{r+1} + \dots + c_r \binom{z}{1}$. Then, $\Delta p = q$. So, $\Delta(f - p)(n) = 0$ for all $n \gg 0$. So, $(f - p)(n) = c_{r+1}$ a constant for all $n \gg 0$. So, $f(n) = p(n) + c_{r+1}$ for all $n \gg 0$. \square

16.3 Graded Modules

Definition 80. Graded S -module. A graded S -module is an S -module M (where S is a graded ring) with the decomposition $M = \bigoplus_{d \in \mathbb{Z}} M_d$ s.t $S_d \cdot M_e \subseteq M_{d+e}$.

Definition 81. Twisted Module. Let M be a graded S -module. Let $l \in \mathbb{Z}$. Then, $M(l)$ is the twisted module defined $M(l)_d = M_{d+l}$ i.e shifting M to the right by l places.

Definition 82. Annihilator of a graded module. Let M be a graded S -module. Define

$$\text{Ann } M := \{s \in S : s \cdot M = 0\}$$

which is a homogenous ideal in S .

Recall: For a graded ring S with decomposition $S = \bigoplus_{d \geq 0} S_d$, an ideal I is called homogenous if $I = \bigoplus_{d \geq 0} (I \cap S_d)$. I find the alternative phrasing more easy to remember: if I is a homogenous ideal, then for any $a \in I$ s.t $a = \sum_d a_d$ where $a_d \in S_d$, we have that $a_d \in I$ as well.

Definition 83. Minimal primes of M . Let M be a graded S -module. Then, the minimal primes of M are the minimal prime ideals containing $\text{Ann } M$.

Proposition 93. Let M be a finitely generated graded module over a Noetherian graded ring S . Then, there exists a filtration $0 = M^0 \subseteq M^1 \subseteq M^2 \subseteq \dots \subseteq M^r = M$ by graded submodules s.t for each i we have

$$M^i/M^{i-1} \cong (S/p_i)(l_i)$$

where p_i is a homogenous prime ideal of S and $l_i \in \mathbb{Z}$. This filtration need not be unique but we have the following two properties:

- (a) if p is a homogenous prime ideal of S , then $p \supseteq \text{Ann } M \leftrightarrow p \supseteq p_i$ for some i . In particular, the minimal elements of the set $\{p_1, \dots, p_r\}$ are just the minimal primes of M i.e the primes which are minimal containing $\text{Ann } M$;
- (b) for each minimal prime of M , the number of times which p occurs in the set $\{p_1, \dots, p_r\}$ is equal to the length of M_p over the local ring S_p (and hence is independent of the filtration).

The proof can be found in Hartshorne's; I am writing out the part that I found non-obvious for reference.

Proof. Prove existence first. Consider first the set of graded submodules of M which admits such a filtration. The zero module does so the set is non-empty. Now, M is a Noetherian module so there is a maximal such submodule $M' \subseteq M$. Now, consider $M'' := M/M'$ - we want to show this is isomorphic to some $(S/p_i)(l_i)$.

If $M'' = 0$, we are done. If not, we consider the set of ideals $\mathcal{A} = \{I_m = \text{Ann}(m) : m \in M'' \text{ is a homogenous element, } m \neq 0\}$. Then, each I_m is a homogenous ideal and $I_m \neq S$. Given S is a Noetherian ring, we can find an element $m \in M''$, $m \neq 0$ s.t I_m is a maximal element of the set \mathcal{A} . Furthermore, I_m is a prime ideal. Let $a, b \in S$. Suppose $ab \in I_m$ but $b \notin I_m$. We can split a, b into homogenous components so we can assume that a and b are homogenous elements. Consider $bm \in M''$. Since $b \notin I_m$, we know $bm \neq 0$. Furthermore, $I_m \subseteq I_{bm}$, so by maximality of I_m , we know $I_m = I_{bm}$. But $ab \in I_m$ so $abm = 0$, so $a \in I_{bm} = I_m$. Therefore, I_m is a homogenous prime ideal of S .

Let $p := I_m := \text{Ann}(m)$ and let $m \in M'' := M/M'$ be of degree l . Then, the module $N \subseteq M''$ generated by m is isomorphic to $(S/p)(-l)$. Let $N' \subseteq M$ be the inverse image of N in M . Then, $M' \subseteq N'$ and $N'/M' \sim (S/p) \cong (S/p)(-l)$. This suggests that N' also has the filtration of the type required. This contradicts the maximality of M' . We conclude that M' was equal to M . \square

Before we proceed, we generalize the concept of localization to modules as well.

Definition 84. Localization of a module. Let A be a ring with S a multiplicative subset i.e $1 \in S$ and S is closed under multiplication. Let M be an A -module. Then, define an equivalence relation on $M \times S$ by saying $(m, s) \sim (n, t)$ iff $(mt - ns)u = 0$ for some $u \in S$. Let $S^{-1}M$ be the set of all such equivalence classes and the equivalence class of (m, s) is denoted by m/s . This makes $S^{-1}M$ an $S^{-1}A$ -module.

Definition 85. If p is a minimal prime of a graded S -module M , we define the multiplicity of M at p , denoted by $\mu_p(M)$, to be the length of M_p over S_p .

Recall: Here, $M_p = (S - p)^{-1}M$ where p is a prime ideal of S and this becomes a S_p -module.

Definition 86. Hilbert function of M . Let M be a graded module over the polynomial ring $S = k[x_0, \dots, x_n]$. Then, the Hilbert function is

$$\varphi_M(l) = \dim_k M_l$$

for each $l \in \mathbb{Z}$.

(recall the RHS is the largest length of strictly containing submodules of M_l)

Theorem 94. (Hilbert-Serre) Let M be a finitely generated graded $S = k[x_0, \dots, x_n]$ -module. Then, there exists a unique polynomial $p_M(z) \in Q[z]$ s.t $\varphi_M(l) = p_M(l)$ for all $l \gg 0$. Furthermore, $\deg p_M(Z) = \dim V(\text{Ann} M)$ where V is the variety/zero set in \mathbb{P}^n of a homogenous ideal.

Definition 87. The polynomial p_M of the theorem is called the Hilbert polynomial of M .

Notation: Write $S(Y)$ (instead of $O(Y)$) to make it clear that we are considering the homogenous coordinate ring of Y .

Definition 88. If $Y \subseteq \mathbb{P}^n$ is an algebraic set of dimension r , we define the Hilbert polynomial of Y to be the Hilbert polynomial P_Y of its homogenous coordinate ring $S(Y)$, which is a polynomial of degree r . We define the degree of Y to be $r!$ times the leading coefficient of P_Y .

Proposition 95. (a) If $Y \subseteq \mathbb{P}^n$ and $Y \neq \emptyset$, then the degree of Y is a positive integer. (b) Let $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 have the same dimension r and where $\dim(Y_1 \cap Y_2) < r$. Then, $\deg Y = \deg Y_1 + \deg Y_2$. (c) $\deg \mathbb{P}^n = 1$. (d) If $H \subseteq \mathbb{P}^n$ is a hypersurface whose ideal is generated by a homogenous polynomial of degree d , then $\deg H = d$.

Now, we finally come back to intersection.

Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r . Let H be a hypersurface not containing Y . Then, $Y \cap H = Z_1 \cup \dots \cup Z_s$ where Z_j are the irreducible varieties of dimension $r - 1$.

Let p_j be the homogenous prime ideal of Z_j . Then, we define the intersection multiplicity of Y and H . Define the intersection multiplicity of Y and H along Z_j to be $i(Z_j, Y \cap H) = \mu_{p_j}(S/(I_Y + I_H))$. Here I_Y and I_H are the homogenous ideals of Y and H . The module $M = S/(I_Y + I_H)$ has the annihilator $I_Y + I_H$ and $V(I_Y + I_H) = Y \cap H$, so p_j is a minimal prime of M and μ is the multiplicity defined above.

Theorem 96. Let Y be an irreducible variety of dimension ≥ 1 in \mathbb{P}^n and let H be a hypersurface not containing Y . Let Z_1, \dots, Z_s be the irreducible components of $Y \cap H$. Then,

$$\sum_{j=1}^s i(Z_j, Y \cap H) \cdot \deg(Z_j) = \deg(Y) \deg(H)$$

Theorem 97. Bezout's Theorem. Let Y and Z be distinct curves in \mathbb{P}^2 with degrees d and e respectively. let $Y \cap Z = \{p_1, \dots, p_s\}$. Then,

$$\sum_j i(p_j, Y \cap Z) = de$$

17 Projective Plane Curves

Definition 89. Projective Plane Curve. A projective plane curve is a hypersurface in \mathbb{P}^2 but we allow multiple components: two non-constant forms $f, g \in k[x, y, z]$ are equivalent if there exists a nonzero $\lambda \in k$ such that $f = \lambda g$. This makes sense because if f is a form of degree d , then $f(tx) = t^d f(x)$ and in projective space $tx \equiv x$ so $t^d f(x) \equiv f(x)$.

So, a projective plane curve is an equivalence class of forms. This also makes sense. Suppose $f \in k[x_0, \dots, x_n]$ is a projective plane curve. Then, $f(x) = 0 \implies f(tx) = 0$. But $f(tx) = \sum_{i=1}^d t^i f_i(x) \equiv f(x)$ forcing us to write the equivalence $t^i f(x) \equiv f(x)$.

The **degree of a curve** is the degree of a defining form. Curves of degree 1, 2, 3 and 4 are called lines, conics, cubics and quartics, respectively.

Notation: Once again, we will write $\mathcal{O}_p(f)$ to mean $\mathcal{O}_p(V(f))$ for an irreducible f .

When $p = [x : y : 1]$, then $\mathcal{O}_p(f)$ is canonically isomorphic to $\mathcal{O}_{(x,y)}(f_*)$

Multiplicity of projective curve - if f is a projective plane curve and $p \in U_i$ where $i = 1, 2$ or 3 , then, we can dehomogenize f with respect to x_i and define the multiplicity of f at p , $m_p(f)$, to be $m_p(f_*)$. The multiplicity is independent of the choice of U_i and invariant under projective change of coordinates.

We will require the following:

Proposition 98. Let $P = [x : y : z] \in \mathbb{P}^2$. Then, $\{(a, b, c) \in \mathbb{A}^3 : ax + by + cz = 0\}$ is a hyperplane in \mathbb{A}^3 . Furthermore, for any finite set of points, p_1, \dots, p_n , in \mathbb{P}^2 , there is a line not passing through any of them.

Proof. The first part is straightforward. Write the set as $V(T_1x + T_2y + T_3z)$ which makes this a hyperplane. Now we prove the second part. Then, the proposition is saying, there exists $a, b, c \in k$ such that $p_i \cdot (a, b, c) \neq 0$ (these a, b and c determine the direction vector of the line). Let $p_i = (x_i, y_i, z_i)$. Suppose for contradiction this is false. Then, $p(x, y, z) = \prod_{i=1}^n (T_1x_i + T_2y_i + T_3z_i)$ vanishes at every point. Since k is infinite, this means $p(x, y, z) = 0$ so $(T_1x_i + T_2y_i + T_3z_i)$ is 0 for some i which means $p_i = 0$ which cannot be the case in \mathbb{P}^2 . \square

Using this proposition, we know that if $p_1, \dots, p_n \in \mathbb{P}^2$ are a finite set of points. Then, we can always find a line L that does not pass through any of the points. If f is a curve of degree d , then define

$$f_* := \frac{f}{L^d} \in k(\mathbb{P}^2).$$

If instead of L , we chose the line L' , then $f/L'^d = (L/L')^d f_*$ and L/L' is a unit in each $\mathcal{O}_{p_i}(\mathbb{P}^2)$.

If p is a simple point on f i.e $m_p(f) = 1$ and f is irreducible, then $\mathcal{O}_p(f)$ is a discrete valuation ring (DVR). We let ord_p^f be the corresponding order function on $k(f)$. If g is a form in $k[x, y, z]$ and $g_* \in \mathcal{O}_p(\mathbb{P}^2)$ is as defined above and \bar{g}_* is the residue of g_* in $\mathcal{O}_p(f)$, we define $ord_p^f(g) = ord_p^f(\bar{g}_*)$. Equivalently, $ord_p^f(g)$ is the order of p at g/h where h is any form of degree $= \deg(g)$ such that $h(p) \neq 0$.

Definition 90. Intersection number of projective plane curves. Let f and g be projective plane curves and let $p \in \mathbb{P}^2$. Then,

$$I(p, f \cap g) := \dim_k(\mathcal{O}_p(\mathbb{P}^2)/(f_*, g_*)).$$

This is independent of the way f_*, g_* are formed. This definition satisfies all of the desired properties of intersection number except for (3), we have T a projective change of coordinates and for (7), A is a form with $\deg(A) = \deg(g) - \deg(f)$.

Definition 91. Tangent to a curve. Define a line L to be a tangent to a curve f at p if $I(p, f \cap L) > m_p(f)$

17.1 Linear System of Curves

Let's work in projective space. Let M_1, \dots, M_N be a fixed ordering of a set of monomials in x, y, z of degree d where $N = \frac{1}{2}(d+1)(d+2)$. Given a curve f of degree d is the same thing as choosing $a_1, \dots, a_N \in k$, not all zero, and letting $f = \sum_i a_i M_i$ except that (a_1, \dots, a_N) and $(\lambda a_1, \dots, \lambda a_N)$ determine the same curve. So, each curve f of degree d corresponds to a unique point in $\mathbb{P}^{N-1} = \mathbb{P}^{d(d+3)/2}$ and, conversely, each point in $\mathbb{P}^{d(d+3)/2}$ represents a unique curve.

Therefore, we often identify the curve f of degree d with a point of $\mathbb{P}^{d(d+3)/2}$. This is why we often say

Example: (1) For $d = 1$, each line $ax + by + cz$ corresponds to the points $[a : b : c] \in \mathbb{P}^2$.
(2) For $d = 2$, the conic $ax^2 + bxy + cxy + dy^2 + eyz + fz^2$ corresponds to the points $[a : b : c : d : e : f] \in \mathbb{P}^5$. The conics form a \mathbb{P}^5 . Similarly, the cubics form a \mathbb{P}^9 and the quartics form \mathbb{P}^{14}

Lemma 99. (1) Let $p \in \mathbb{P}^2$ be a fixed point. The set of curves of degree d that contain p forms a hyperplane in $\mathbb{P}^{d(d+3)/2}$.

(2) If $T : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is a projective change of coordinates, then the map $f \rightarrow f^T$ from $\{\text{curves of degree } d\}$ to $\{\text{curves of degree } d\}$ is a projective change of coordinates on $\mathbb{P}^{d(d+3)/2}$.

Proof. (1) If $p = [x : y : z]$, then the curve corresponding to $(a_1, \dots, a_N) \in \mathbb{P}^{d(d+3)/2}$ passes through p if and only if $\sum a_i M_i(x, y, z) = 0$. Since not all M_i are 0, the $[a_1 : \dots : a_N]$ satisfying this equation form a hyperplane. \square

17.2 Bezout's Theorem

Theorem 100. Let f and g be projective plane curves of degree m and n respectively. Assume f and g have no common component. Then, $\sum_p I(p, f \cap g) = mn$. In other words, the two curves intersect in mn points, counting multiplicity.

Corollary 101. If f and g have no common component, then $\sum_p m_p(f)m_p(g) \leq \deg(f)\deg(g)$

Corollary 102. If f and g meet in mn distinct points, $m = \deg(f)$, $n = \deg(g)$, then these points are all simple points on f and g .

Corollary 103. If two curves of degree m and n have more than mn points in common, then they have a common component.

Theorem 104. If f is an irreducible curve of degree n , then $\sum \frac{m_p(m_p-1)}{2} \leq \frac{(n-1)(n-2)}{2}$.

17.3 Max Noether's Fundamental Theorem

Definition 92. Zero cycle. A zero cycle on \mathbb{P}^2 is a formal sum

$$\sum_{p \in \mathbb{P}^2} n_p p$$

where $n_p \in \mathbb{Z}$ and all but finitely many n_p 's are zero. The set of all zero-cycles on \mathbb{P}^2 forms an abelian group. The degree of a zero cycle $\sum n_p p$ is defined to be

$$\sum n_p.$$

The zero cycle is positive if each $n_p \geq 0$.

We can also order zero cycles in some sense. We say $\sum_p n_p p \geq \sum_p m_p p$ i.e $\sum_p n_p p \geq \sum_p m_p p$ if each $n_p \geq m_p$.

Definition 93. Intersection cycle. Let f and g be projective plane curves of degree m, n such that they have no common components. The intersection cycle $f \cdot g$ is defined by

$$f \cdot g = \sum_{p \in \mathbb{P}^2} I(p, f \cap g) p.$$

By Bezout's theorem, $f \cdot g$ is a positive zero-cycle of degree mn .

- Lemma 105.** (1) $f \cdot g = g \cdot f$,
(2) $f \cdot gh = f \cdot g + f \cdot h$
(3) $f \cdot (g + af) = f \cdot g$ given a is a form and $\deg(a) = \deg(g) - \deg(f)$.

Motivating the theorem: Suppose f, g and h are curves and $h \cdot f \geq g \cdot g$ i.e h intersect f in a bigger cycle than g does. We want to find conditions s.t there exists a curve b so that $b \cdot f = h \cdot f - g \cdot f$.

To find such a b , we can find form a, b st $h = af + bg$. Then, $h \cdot f = (af + bg) \cdot f = bg \cdot f = b \cdot f + g \cdot f$.

Definition 94. Noether's conditions. Let $p \in \mathbb{P}^2$. Let f and g be curves with no component through p . Let h be another curve. We say that Noether's conditions are satisfied at p with respect to f, g and h if $H_* \in (f_*, g_*) \subset \mathcal{O}_p(\mathbb{P}^2)$ i.e if there are $a, b \in \mathcal{O}_p(\mathbb{P}^2)$ s.t $h_* = af_* + bg_*$.

Theorem 106. Max Noether's Fundamental Theorem. Let f, g and h be projective plane curves. Assume f and g have no common components. Then, there exists an equation $h = af + bg$ (with a, g forms of degree $\deg(h) - \deg(f)$, $\deg(h) - \deg(g)$ respectively) if and only if Noether's conditions are satisfied at every $p \in f \cap g$.

Proof is in Fulton's text.

Corollary 107. Let f, g and h be plane curves and $p \in f \cap g$. Then, Noether's conditions are satisfied at p if any of the following are true:

- (1) f and g meet transversally at p and $p \in H$.
- (2) p is a simple point on f and $i(p, h \cap g) \geq i(p, g \cap f)$
- (3) f and g have distinct tangents at p and $m_p(h) \geq m_p(f) + m_p(g) - 1$.

18 Resolution of Singularities

First, we recall some preliminaries:

Recalling rational maps: Rational maps (recall) and their domains. A rational map from X to Y (where X and Y are varieties) is an equivalence class of morphisms $f_i : U_i \rightarrow Y$ (U_i is an open subvariety of X) where f_i and f_j are equivalent if they agree on $U_1 \cap U_2$ (U_1 and U_2 are open, dense subsets of X). The domain of a rational map is the union of all open subvarieties $U_\alpha \subset X$ s.t some $f_\alpha : U_\alpha \rightarrow Y$ belongs to the equivalence class of the rational map. Given U is the domain of the rational map, the map $f : U \rightarrow Y$ defined by $f|_{U_\alpha} = f_\alpha$ is a morphism belonging to the equivalence class of the map. Every equivalent morphism is a restriction of f . Therefore, **a rational map from X to Y may be defined as a morphism from an open subvariety U of X to Y such that f cannot be extended to a morphism from any larger open subset of X to Y .**

Definition 95. Dominating rational map. A rational map from X to Y (where X and Y are irreducible varieties) is dominating if $f(U)$ is dense in Y where U is the domain of the map and $f : U \rightarrow Y$ is any morphism representing the map. If A and B are local rings, and A is a subring of B , we say B dominates A if the maximal ideal of B contains the maximal ideal of A .

Recalling simple points and non-singular curves: Point p on curve C is a simple point if $\mathcal{O}_p(C)$ is a discrete valuation ring (DVR). This agrees with our result from Theorem 79 (sequence might change - the result is in the chapter "Affine Plane Curves and Intersection Number"). Let $ord_p^C =: ord_p$ be the order function on $k(C)$ defined by $\mathcal{O}_p(C)$. A curve C is non-singular if all the points on it are simple.

Now, let K be a field and $k \subseteq K$. The local ring A is a local ring of K if A is a subring of K , K is the quotient field of A and A contains k . For example, let V be a variety, let $p \in V$. Then, $\mathcal{O}_p(V)$ is a local ring of $k(V)$. A DVR of K is a local ring of K .

19 Riemann-Roch Theorem

Let C be an irreducible projective curve and let $f : X \rightarrow C$ be a birational morphism from X (non-singular) onto C with $K = k(C) = k(X)$ the function field.

Definition 96. Divisors. A divisor on X is a formal sum $D = \sum_{p \in X} n_p p$ where $n_p \in \mathbb{Z}$ and $n_p = 0$ for all but finitely many $p \in X$. The divisors on X form an abelian group. The degree of the divisor $\deg(\sum_{p \in X} n_p p) = \sum_{p \in X} n_p$. So, $\deg(D + D') = \deg(D) + \deg(D')$. A divisor $D = \sum_{p \in X} n_p p$ is **effective/positive** if each $n_p \geq 0$. We say $\sum_{p \in X} n_p p \geq \sum_{p \in X} m_p p$ if each $n_p \geq m_p$.

Notation: Let $f : X \rightarrow C$ and let $Q \in X$ and $p = f(Q) \in C$. Let C be a plane curve. For any plane curve G , form $G_* \in \mathcal{O}_p(\mathbb{P}^2)$. Now, let g be the image of G_* in $\mathcal{O}_p(C) \subset k(C) = k(X)$. We define $\text{ord}_Q(G)$ to be $\text{ord}_Q(g)$. Firstly, as shown in [1],

$$\sum_{p \in X} \text{ord}_p(G) = \sum_{q \in C} I(q, C \cap G).$$

Secondly, using Bezout's theorem, $\text{div}(G)$ is a divisor of degree mn where m is the degree of G .

Let C be a plane curve of degree n and let G be a plane curve that does not contain C as a component. We define the divisor of G , $\text{div}(G)$, to be $\sum_{p \in X} \text{ord}_p(G)p$.

For any nonzero $z \in K$, define $\text{div}(z) := \sum_{p \in X} \text{ord}_p(z)p$. Since z has a finite number of poles and zeroes, $\text{div}(z)$ is a well-defined divisor. Let $(z)_0 = \sum_{\text{ord}_p(z) > 0} \text{ord}_p(z)p$ be the divisors of zeroes of z and $(z)_\infty = \sum_{\text{ord}_p(z) < 0} -\text{ord}_p(z)p$ be the divisor of poles of z , then $\text{div}(z) = (z)_0 - (z)_\infty$. Furthermore, $\text{div}(zz') = \text{div}(z) + \text{div}(z')$ and $\text{div}(z^{-1}) = -\text{div}(z)$.

Proposition 108. For any nonzero $z \in K$, $\text{div}(z)$ is a divisor of degree zero. A rational function has the same number of zeroes as poles.

Corollary 109. Let $0 \neq z \in K$. Then, the following are equivalent: (i) $\text{div}(z) \geq 0$ (ii) $z \in k$ and (iii) $\text{div}(z) = 0$.

Corollary 110. Let $z, z' \in K$ both be nonzero. Then, $\text{div}(z) = \text{div}(z')$ if and only if $z' = \lambda z$ for some $\lambda \in k$.

Definition 97. Equivalence of divisors. Two divisors D and D' are linearly equivalent if $D' = D + \text{div}(z)$ for some $z \in K$. We write $D' \equiv D$.

Suppose C is a plane curve with only ordinary multiple points. For each $Q \in X$, let $r_Q = m_{f(Q)}(C)$. Define the divisor $E = \sum_{Q \in X} (r_Q - 1)Q$. This E is effective; its degree is $\sum m_P(C)(m_P(C) - 1)$. Any plane curve G such that $\text{div}(G) \geq E$ is called an adjoint of C if and only if $m_P(G) \geq m_P(C) - 1$ for every (multiple) point $P \in C$. If C is nonsingular, every curve is an adjoint.

20 References

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A Category Theory

These introductory category theory notes are all taken, almost verbatim, from Ravi Vakil's textbook [5] - which I find to be arguably one of the best texts in any field of mathematics today.

A.1 Basic terminology

These notes are directly taken from (5).

Definition: Category. A **category** consists of a collection of **objects** and for each pair of objects, a set of **morphisms** or **arrows** between them (which are often called **maps**). The collection of objects of a category, \mathcal{C} , is denoted as $\text{obj}(\mathcal{C})$ but we will often denote this also by \mathcal{C} . If $A, B \in \mathcal{C}$, the set of morphisms from A to B are denoted by $\text{Mor}(A, B)$ where a morphism is often written as $f : A \rightarrow B$. A is the **source** of f whereas B is the **target** of f .

Morphisms compose as expected; $\text{Mor}(B, C) \times \text{Mor}(A, B) \rightarrow \text{Mor}(A, C)$. Composition is associative, i.e $(f \circ g) \circ h = f \circ (g \circ h)$.

For each object $A \in \mathcal{C}$, there exists an **identity morphism** $\text{id}_A : A \rightarrow A$ such that $f \circ \text{id}_A = f$

and $\text{id}_A \circ f = f$. Identity morphism is unique.

Definition: Isomorphism. An isomorphism between two objects is a morphism $f : A \rightarrow B$ such that there exists a unique morphism $g : B \rightarrow A$ such that $f \circ g = \text{id}_B$ and $g \circ f = \text{id}_A$.

Definition: Automorphism. The set of invertible elements of $\text{Mor}(A, A)$ forms a group called the **automorphism group of A** .

Examples:

- (1) Category of sets. The objects are sets and the morphisms are maps of sets.
- (2) Another good example is the category Vec_k of vector spaces over a given field k . The objects are k -vector spaces, and the morphisms are linear transformations.
- (3) Category of Abelian groups. The objects are Abelian groups and the morphisms are the group homomorphisms. This category is denoted as Ab .
- (4) Category of modules over a ring. If A is a ring, then the A -modules form a category $\text{Mod}(A)$.
- (5) Category of rings. Objects are rings and morphisms are ring homomorphisms.

Definition: Subcategory. A subcategory \mathcal{A} of a category \mathcal{C} includes some of the objects and morphisms of \mathcal{C} such that the objects of \mathcal{A} include the sources and targets of morphisms of \mathcal{A} and the morphisms of \mathcal{A} include the identity morphisms of the objects in \mathcal{A} and are preserved by composition.

Now, we define functors.

Definition: Covariant functor from category \mathcal{A} to category \mathcal{B} , denoted by $F : \mathcal{A} \rightarrow \mathcal{B}$. This is a map of objects $F : \text{obj}(\mathcal{A}) \rightarrow \text{obj}(\mathcal{B})$ and for each $A_1, A_2 \in \mathcal{A}$ and morphism $m : A_1 \rightarrow A_2$, a morphism $F(m) : F(A_1) \rightarrow F(A_2)$. We require F preserves identity morphisms i.e $F(\text{id}_A) = \text{id}_{F(A)}, \forall A \in \mathcal{A}$. F must also preserve composition i.e $F(m_1 \circ m_2) = F(m_1) \circ F(m_2)$.

To emphasize, a covariant functor has two "functions". One is mapping objects in one category to objects in another one i.e $F : \text{obj}(\mathcal{A}) \rightarrow \text{obj}(\mathcal{B})$. The other is mapping morphisms in one category to morphisms in another one by taking the "shadow" of the morphism i.e $F(\cdot) : \text{Mor}(\text{obj}(\mathcal{A}), \text{obj}(\mathcal{A})) \rightarrow \text{Mor}(\text{obj}(\mathcal{B}), \text{obj}(\mathcal{B}))$.

Example of covariant functor: Trivial example is the identity cofunctor $\text{id} : \mathcal{A} \rightarrow \mathcal{B}$

Forgetful Functor: Consider the functor from the category of vectors space over k i.e Vec_k to the category of sets by sending each vector space to its underlying set. Furthermore, F

sends each morphism m in Vec_k to the underlying map of sets. This is a forgetful functor because it forgets additional structure.

Definition: Faithfull and full covariant functors. A covariant functor F from category \mathcal{A} to \mathcal{B} is faithful if for any $A, A' \in \mathcal{A}$, the map $\text{Mor}_{\mathcal{A}}(A, A') \rightarrow \text{Mor}_{\mathcal{B}}(F(A), F(A'))$ is injective. It is full if it is surjective. A functor that is both full and faithful is called **fully faithful**.

Defintion: Contravariant functor from category \mathcal{A} to category \mathcal{B} , denoted by $F : \mathcal{A} \rightarrow \mathcal{B}$. This is a map of objects $F : \text{obj}(\mathcal{A}) \rightarrow \text{obj}(\mathcal{B})$ and for each $A_1, A_2 \in \mathcal{A}$ and morphism $m : A_1 \rightarrow A_2$, a morphism $F(m) : F(A_2) \rightarrow F(A_1)$. We require F preserves identity morphisms i.e $F(\text{id})_A = \text{id}_{F(A)}$, $\forall A \in \mathcal{A}$. Therefore, $F(m_1 \circ m_2) = F(m_2) \circ F(m_1)$.

Example: Consider the category Vec_k of vector spaces over field k . Then, we can take the duals to define a contravariant functor. Let $f : V \rightarrow W$ be a linear transformation. Then, the dual transformation is $f^* : W^* \rightarrow V^*$.

Example: Here is a pretty straightforward example of covariant and contravariant functors. Consider a category \mathcal{C} and the category of morphisms between sets \mathcal{S} (note that in this category, each object is a morphism and we have morphisms between morphisms. Then, let $A \in \text{obj}(\mathcal{C})$. We first define a covariant functor :

$h^A : \text{obj}\mathcal{C} \rightarrow (\text{some morphism between sets in } \mathcal{S})$ and in particular $h^A(B \in \text{obj}(\mathcal{C})) \in \text{Mor}(A, B)$. Furthermore, given $f \in \text{Mor}(B_1, B_2)$ in category \mathcal{C} , h^A will send f to a morphism from $\text{Mor}(A, B_1)$ to $\text{Mor}(A, B_2)$ and in particular $h^A(f \in \text{Mor}(B_1, B_2))(g \in \text{Mor}(A, B_1)) = (f \circ g) \in \text{Mor}(A, B_2)$.

Now, we define a contravariant functor. With the same set up i.e $A \in \text{obj}(\mathcal{C})$. Let $B \in \text{obj}(\mathcal{C})$, we have $h_A(B) \in \text{Mor}(B, A)$ (note that the direction has been reversed). Then, $h_A : \text{Mor}(B_1, B_2) \rightarrow (\text{Mor}(B_2, A) \rightarrow \text{Mor}(B_1, A))$ by the following: $h_A(f \in \text{Mor}(B_1, B_2))(g \in \text{Mor}(B_2, A)) = g \circ f \in \text{Mor}(B_1, A)$

Now, we introduce the concept of a **natural transformation** of covariant functors. Let F be a covariant functor. Then, we consider the transformation $F \rightarrow G$ as follows: consider the morphism $m_A : F(A) \rightarrow G(A)$ for each $A \in \mathcal{A}$ such that $f : A \rightarrow A'$ in \mathcal{A} . This has the following diagram:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(A') \\ m_A \downarrow & & \downarrow m_{A'} \\ G(A) & \xrightarrow{G(f)} & G(A') \end{array}$$

A **natural isomorphism** of functors is a natural transformation such that each m_A is an isomorphism. One can analogously define natural transformation of contravariant functors.

Next, we introduce the notion of equivalence of functors. The data of functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $F' : \mathcal{B} \rightarrow \mathcal{A}$ such that $F \circ F'$ is naturally isomorphic to $\text{id}_{\mathcal{B}}$ and $F' \circ F$ is naturally isomorphic to $\text{id}_{\mathcal{A}}$ is the **equivalence of categories**. Two categories are "essentially the same" when there is an equivalence of categories between them.

A.2 Universal properties

Definition: Initial, final and zero objects. An object of a category \mathcal{C} is initial if it has only one map to every object. An object is final if it has only one map from every object. An object is zero if it is both initial and final.

Lemma 111. Any two initial objects are uniquely isomorphic. Any two final objects are uniquely isomorphic.

The proof follows from the definition. This also shows that initial and final objects are unique up to isomorphism. The fact an object is an initial or final or zero object is a universal property.

Here are some more examples:

Localization of rings and modules. First, recall: a **multiplicative subset** S of a ring A is a subset that is closed under multiplication and contains 1. Then, define the ring $S^{-1}A$ whose elements are of the form a/s such that $a \in A, s \in S, s \neq 0$ and $a_1/s_1 = a_2/s_2$ if and only if $\exists s \in S, s(s_2a_1 - s_1a_2) = 0$. Lastly, $a_1/s_1 + a_2/s_2 := (s_2a_1 + s_1a_2)/(s_1s_2)$ and $a_1/s_1 \times a_2/s_2 = (a_1a_2)/(s_1s_2)$. Note, if $0 \in S$, then $S^{-1}A$ is the 0-ring. Also, we have the map $A \rightarrow S^{-1}A$ by sending each $a \rightarrow a/1$.

Now, we look at a few important multiplicative subsets.

The first is $S = \{1, f, f^2, \dots\}$ where $f \in A$. This is denoted by $A_f := S^{-1}A$. One can prove that this is isomorphic to $A_f \cong A[t]/(tf - 1)$. To prove this, we showed in the section on coordinate rings that all elements in $A[t]/(tf - 1)$ are of the form af^{-i} for some $a \in A, i \geq 0$. On the other hand, all elements in A_f can be trivially sent to the same element in $A[t]/(tf - 1)$.

The second important multiplicative subset is $S = A \setminus p$ where p is a prime ideal. We denote this by $A_p := S^{-1}A$ i.e we divide by elements that are *not* in p .

The third is constructed as follows: given A is an integral domain, $S := A \setminus \{0\}$. Then, $k(A) := S^{-1}A$ is called the fractional field of A .

B Ring Theory Revision

All the material here is from Dummit and Foote's "Abstract Algebra". Detailed proofs of the theorems can be found in the text.

B.1 Rings

Definition: Rings. A ring R is a set with binary operations \times and $+$ such that

- (1) $(R, +)$ is an abelian group (i.e has identity, inverses and associativity).
- (2) \times is associative i.e $(a \times b) \times c = a \times (b \times c)$
- (3) distributive laws hold in R i.e $\forall a, b, c \in R$, we have $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b + c) = a \cdot b + a \cdot c$.

Note: Rings that are commutative under multiplication are called **commutative rings**.

Example: Ring without identity The set of even integers $2\mathbb{Z}$ since 1 is not even.

Example: Ring of functions. For X a non-empty set and A any ring, the set of functions $f : X \rightarrow A$ forms a ring R with operations $(f + g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$. R is commutative if and only if A is commutative. R has identity 1 if and only if A has 1.

Example: Some other easy rings. $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all commutative rings. $\mathbb{Z}/n\mathbb{Z}$ is a commutative ring with identity 1.

Example: Trivial and Zero ring. Any abelian group is a trivial ring with the operation $x \cdot y = 0$ for any $x, y \in R$.

Definition: Division Ring. A ring R with identity $1 \neq 0$ such that every $x \in R$ has a multiplicative inverse $x^{-1} \in R$ with $xx^{-1} = x^{-1}x = 1$ is a division ring.

Definition: Field. A field is a commutative division ring.

Proposition: Immediate properties of rings For any ring R :

- (1) $0x = x0 = 0, \forall x \in R$
- (2) $(-x)y = x(-y) = -(xy), \forall x, y \in R$
- (3) $(-x)(-y) = xy, \forall x, y \in R$

(4) if $\exists 1 \in R$, then 1 is unique and $-x = (-1)x, \forall x \in R$.

Proposition: A finite division ring is a field.

Definition: Zero divisor. Let R be a ring. Let $x \neq 0$. Then, x is a zero divisor if $\exists y \in R, y \neq 0$ such that $xy = 0$ or $yx = 0$.

Definition: Unit. Let R be a ring with identity 1. Then, $x \in R$ is called a unit if there exists $y \in R$ such that $xy = yx = 1$.

R^\times is the set of units in ring R . (R^\times, \times) is a group under multiplication called the group of units.

Example of zero divisor: Let $x \neq 0$ be an integer and suppose x is relatively prime to $n \in \mathbb{Z}$. Then, x is a zero divisor in $\mathbb{Z}/n\mathbb{Z}$.

Lemma: If $x \in R$ is a zero divisor then x is not a unit. If $x \in R$ is a unit, then x is not a zero divisor.

Corollary: Fields have no zero divisors.

Example: zero divisor. Let $x \neq 0, x \in \mathbb{Z}$ and suppose x is relatively prime to $n \in \mathbb{Z}$. Then, \bar{x} is a zero divisor in $\mathbb{Z}/n\mathbb{Z}$.

B.2 Integral Domains and Subrings

Definition: Integral Domain. A commutative ring with identity $1 \neq 0$ such that it has no zero divisor.

Proposition: Cancellation laws hold in integral domains. Let $a, b, c \in R$ such that a is not a zero divisor. If $ab = ac$, then either $a = 0$ or $b = c$. In other words, if a, b, c are elements in an integral domain, then, $ab = ac \implies a = 0$ or $b = c$.

Proposition: Any finite integral domain is a field.

Proof. Let R be a finite integral domain. Let $a \in R$ s.t. $a \neq 0$. We find a multiplicative inverse for a . Consider the map $\varphi(x) = ax$ for all $x \in R$. This map is injective by the previous proposition. Since R is finite and the map is injective, this map must also be surjective. Thus, there exists x such that $ax = 1$. \square

Definition: Subring. A subring of the ring R is a subgroup of R that is closed under multiplication i.e $S \neq \emptyset$ is closed under addition, for each $x \in S$, there exists an additive

inverse in S , $0 \in S$ and S is closed under multiplication.

Definition: Polynomial Rings. Let R be a commutative ring with identity 1. Let x be an indeterminate. Then, $R[x]$ is the ring of polynomials $\sum_{i=1}^n a_i x^i$, $n \geq 0$, $a_i \in R$. If $a_n \neq 0$, degree of the polynomial is n . Monic polynomials are those with $a_n = 1$. $R \subset R[x]$ is the set of constant polynomials. $R[x]$ is itself a commutative ring with identity (where 1 is the same identity as in R).

Note: if S is a subring of R , then $S[x]$ is a subring of $R[x]$.

Proposition: immediate properties of polynomial rings. Let R be an integral domain.

Let $p(x), q(x)$ be non-zero elements of $R[x]$. Then,

- (1) $\text{degree } p(x)q(x) = \text{degree } p(x) + \text{degree } q(x)$.
- (2) the units of $R[x]$ are the same as the units of R
- (3) $R[x]$ is an integral domain.

Definition: Ring homomorphisms. Let R and S be rings. A ring homomorphism $f : R \rightarrow S$ is a map such that $f(x + y) = f(x) + f(y)$, $f(xy) = f(x)f(y)$, $\forall x, y \in R$. A bijective ring homomorphism is called an **isomorphism** and we say $R \cong S$.

Lemma: Let $f : R \rightarrow S$ be a ring homomorphism. Then, $\text{Im}(f)$ be a subring of S and $\text{ker}(f)$ is a subring of R .

Examples of subrings: \mathbb{Z} is a subring of \mathbb{Q} which is a subring of \mathbb{R} . $2\mathbb{Z}$ and $n\mathbb{Z}$ are subrings of \mathbb{Z} .

B.3 Ideals

Definition: Ideals. Let R be a ring, let $r \in R$ and let I be a subset of R . Then, $rI := \{rx : x \in I\}$. $Ir := \{xr : x \in I\}$. A subset I of R is a left ideal of R if I is a subring of R and $rI \subseteq I, \forall r \in R$. A subset I of R is a right ideal of R if I is a subring of R and $Ir \subseteq I, \forall r \in R$. If I is both a left and right ideal, it is called an ideal of R .

Definition (Quotient ring of R by an ideal). Let I be an ideal of the ring R . Then, R/I is the quotient ring of R by I . The elements of this quotient ring are of the form $r + I$ for $r \in R$ and $r + I = s + I$ if $r - s \in I$.

Proposition: Quotient ring is a ring. Let R be a ring and let I be an ideal of R . Then the additive quotient group R/I is a ring under the binary operations $(r + I) + (s + I) = (r + s) + I$, $(r + I)(s + I) = (rs + I)$, $\forall r, s \in R$. Conversely, if I is any subgroup of R such that these two operations are well-defined, then I is an ideal of R .

Proposition: Isomorphism Theorems for Rings.

- (1) (First Isomorphism Theorem for Rings) If $\psi : R \rightarrow S$ is a ring homomorphism, then $\ker(\psi)$ is an ideal of R , $\text{Im}(\psi)$ is a subring of S and $R/\ker(\psi) \cong \psi(R)$.
- (2) If I is an ideal of R , then the map $R \rightarrow R/I$ defined by $r \rightarrow r + I$ is a surjective ring homomorphism with kernel I . This is the natural projection of R onto R/I . Every ideal is the kernel of a ring homomorphism and vice-versa.
- (3) (Second Isomorphism Theorem for Rings) Let A be a subring and B be an ideal of the ring R . Then, $A + B = \{a + b | a \in A, b \in B\}$ is a subring of R and $A \cap B$ is an ideal of A and $(A + B)/B \cong A/(A \cap B)$.
- (4) (Third Isomorphism Theorem for Rings) Let I and J be ideals of the ring R with $I \subseteq J$. Then, J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$.
- (5) (Lattice/Fourth Isomorphism Theorem for Rings) Let I be an ideal of R . The correspondence $A \leftrightarrow A/I$ is an inclusion-preserving bijection between the set of subrings of R that contain I and the set of subrings of R/I (in other words, if $A \subseteq B$ and both contain I , then $A/I \subseteq B/I$ + if $J \subseteq K$ are ideals containing I , then $J/I \subseteq K/I$). Also, A (a subring containing I) is an ideal of R if and only if A/I is an ideal of R/I .

Definition: Proper ideal. An ideal I is proper if $I \neq R$.

Example: R and $\{0\}$ are ideals of R . $n\mathbb{Z}$ is an ideal of \mathbb{Z} for any $n \in \mathbb{Z}$.

Definition (Ideals generated by a set). Let R be a ring with identity 1. Let A be a subset of R . Let (A) be the smallest ideal of R containing A . Then,

- (1) (A) is the smallest ideal of R containing A , called the ideal generated by A :

$$(A) = \bigcap_{(I \text{ is an ideal, } A \subseteq I)} I$$

(2) Define $RA = \{\sum_i r_i a_i : r_i \in R, a_i \in A, n \in \mathbb{Z}^+\}$. Define RA and $RA R$ similarly. We say RA is the left ideal generated by A , AR is the right ideal generated by A and $RA R$ is the ideal generated by A . **If R is commutative**, $RA = AR = RA R = (A)$.

(3) Principle ideals are ideals generated by a single element.

(4) A finitely general ideal is an ideal generated by a finite set.

Proposition (conditions for proper ideal and fields): Let I be an ideal of R , where R is a ring with identity 1. (1) $I = R$ if and only if I contains a unit. (2) If R is commutative, then R is a field if and only if its only ideals are the zero ideal $\{0\}$ and R .

Corollary: If R is a field, then any non-zero ring homomorphism from R into another ring is an injection.

B.4 Maximal Ideals

Definition: Maximal Ideals Let R be a ring with identity $1 \in R$. An ideal M in R is called a maximal ideal if $M \neq R$ and the only ideals containing M are M and R .

Proposition: In a ring with identity 1, every proper ideal is contained in a maximal ideal.

Sketch of proof: Suppose I is a proper ideal. Let S be the set of proper ideals containing I (S is clearly non-empty and has partial order by inclusion). Let C be a chain in S and let J be the union of all ideals in C . Show that J is an ideal - $0 \in J$ and elements are closed under subtraction and left/ring multiplication by elements of R . Then, show that J is a proper ideal since otherwise $1 \in J$ and therefore, 1 is in at least one of the ideals in C making that ideal not proper. Then, each chain has an upper bound in S . Use Zorn's lemma to conclude S has a maximal element which is our maximal proper ideal containing I .

Proposition: Let R be a commutative ring with identity 1. The ideal M is a maximal ideal if and only if the quotient ring R/M is a field.

Sketch of proof: ideal M is maximal iff there are no ideals I st $M \subset I \subset R$. By lattice isomorphism, ideals of R containing M correspond bijectively with the ideals of R/M , so M is maximal if and only if the only ideals of R/M are 0 and R/M . But by a proposition above, R/M is a field iff the only ideals are 0 and R/M .

B.5 Prime Ideals

Definition: Prime ideal. Suppose R is a commutative ring with identity 1. An ideal P is called a prime ideal if $P \neq R$ and whenever $xy \in P$, we have $x \in P$ and/or $y \in P$.

Proposition: Suppose R is commutative with identity $1 \neq 0$. Then, the ideal P is a prime ideal in R if and only if the quotient ring R/P is an integral domain.

Proof: P is a prime ideal if and only if $\bar{R} \neq \bar{0}$ (since $P \neq R$) and $\bar{a}\bar{b} = \bar{0}$ implies either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ which is if and only if R/P is an integral domain.

Proposition: Assume R is commutative. Every maximal ideal of R is a prime ideal.

Proof: M is maximal implies R/M is a field and a field is an integral domain so M must be prime.

Example: The ideal (x) is a prime ideal in $\mathbb{Z}[x]$ since $\mathbb{Z}[x]/(x) \cong \mathbb{Z}$.

B.6 Rings of Fractions and Fields of Fractions

Let R be a commutative ring. We want to show that R is a subring of a larger ring Q in which every non-zero element of R that is not a zero divisor is a unit in Q .

First, we define the equivalence relation $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Define addition and multiplication of fractions as is normally done with rational numbers.

Definition: Let R be a commutative ring and let D be a non-empty subset of R such that $0 \notin D$, D contains no zero divisors and D is closed under multiplication. Then, there exists a commutative ring Q , denoted $Q = D^{-1}R$, with $1 \in Q$ such that Q contains R as a subring and every element of D is a unit in Q . This ring Q has the following properties:

- (1) every element of Q is of the form rd^{-1} for some $r \in R$ and $d \in D$.
- (2) If $D = R - \{0\}$, then Q is a field. We call Q the field of fractions or the quotient field of R .
- (3) The ring Q is the smallest ring containing R in which all elements of D are units. In other words, let S be any commutative ring with identity and let $\varphi : R \rightarrow S$ be any injective ring homomorphism s.t. $\varphi(d)$ is a unit in S for all $d \in D$. Then, there is an injective ring homomorphism $\phi : Q \rightarrow S$ s.t. $\phi|_R = \varphi$.

B.7 Euclidean Domains and Discrete Valuation Rings

Definition (Norm/Positive Norm). Any function $N : R \rightarrow \mathbb{Z}^+ \cup \{0\}$ with $N(0) = 0$ is called a norm on the integral domain R . If $N(a) > 0$ for all $a \neq 0$, then N is called a positive norm.

Definition (Euclidean Domains). An integral domain R is called a Euclidean Domain (or possess a Division Algorithm) if there exists a norm N on R such that for any two elements $a, b \in R$ with $b \neq 0$, there exists elements $q, r \in R$ s.t.

$$a = bq + r$$

with $r = 0$ or $N(r) < N(b)$. We call q the quotient and r the remainder.

Examples of Euclidean domains:

- (1) All fields are trivial examples of Euclidean Domains.
- (2) \mathbb{Z} is a Euclidean Domain with $N(a) = |a|$.
- (3) If F is a field, then the polynomial ring $F[x]$ is a Euclidean domain with norm $N(p(x)) = \deg(p(x))$.

Proposition: Every ideal in a Euclidean domain is a principal ideal. So, Euclidean domains are principal ideal domains.

Proof. If $I = 0$, we are done. Let $d \in I$ s.t. $N(d) \leq N(x)$ for all $x \in I$. Then, $(d) \subseteq I$. Consider any $a \in I$, where $a = qd + r$ (here, $r = 0$ or $N(r) \leq N(d)$). Then, $r = a - qd \in I$. By minimality of norm of d , $r = 0$, so $a = qd \in (d)$. So $I \subseteq (d)$. \square

Examples:

(1) Every ideal in \mathbb{Z} is a principal ideal.

Definition (Discrete Valuation): Let K be a field. A discrete valuation on K is a function

$$v : K^\times \rightarrow \mathbb{Z}$$

such that (1) $v(ab) = v(a) + v(b)$ (2) v is surjective and (3) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K^\times$ with $x + y \neq 0$.

Definition (Valuation Ring): The valuation ring of v is

$$\{x \in K^\times \mid v(x) \geq 0\} \cup \{0\}.$$

Definition(Discrete Valuation Ring). An integral domain R is a discrete valuation ring if there exists a valuation v on its field of fractions such that R is the valuation ring of v .

Note: A discrete valuation ring is a Euclidean Domain with the norm $N(0) = 0$ and $N = v$ on non-zero elements.

B.8 Principal Ideal Domains

Definition: Principal Ideal Domain (PID). A PID is an integral domain in which every ideal is principal.

Example: \mathbb{Z} is a PID.

Proposition: Let R be a PID and let $a, b \in R$ such that $a \neq 0, b \neq 0$. Let d be a generator for the principal ideal generated by a and b . Then,

- (1) d is the greatest common divisor of a and b .
- (2) d can be written as an R -linear combination of a and b i.e. there exists $x, y \in R$ such that $d = ax + by$ and
- (3) d is unique up to multiplication by a unit of R .

Proposition: Every non-zero prime ideal in a PID is a maximal ideal.

Proof. Let (p) be a prime ideal in a PID. Let $I = (m)$ contain (p) . We want to show $I = (p)$ or $I = R$. Since $(p) \subset (m) = I$, $p = rm$ for some $r \in R$. Now, (p) is prime, so either $r \in (p)$ or $m \in (p)$. If $m \in (p)$, then $(p) = (m) = 1$. If $r \in (p)$, write $r = ps$ so $p = rm = psm$. So $sm = 1$ so m is a unit and $I = R$. \square

Corollary: If R is any commutative ring such that the polynomial ring $R[x]$ is a PID or a Euclidean domain, then R is necessarily a field.

Proof. Suppose, $R[x]$ is a PID. Now, R is a subring of $R[x]$, then R must be an integral domain. The ideal (x) is a non-zero prime ideal in $R[x]$ as $R[x]/(x) \cong R$. Then, (x) is a maximal ideal by previous proposition, so R is a field. \square

B.9 Irreducible elements and Prime elements

Definition: Irreducible, prime and associate. Let R be an integral domain.

- (1) Let $x \in R$ such that x is not a unit. Then x is irreducible in R if $x = ab$ where $a, b \in R$ implies either a or b is a unit in R .
- (2) A non-zero element $x \in R$ is called a prime in R if the ideal (x) generated by x is a prime ideal. Equivalently, $x \neq 0$ is a prime if it is not a unit and whenever x divides $ab \in R$, either x divides a or x divides b .
- (3) Two elements x and y of R are associate if $x = uy$ for some unit $u \in R$.

Proposition: In an integral domain, a prime element is always irreducible.

Proof. Suppose (p) is a non-zero prime ideal and $p = ab$. Then, $ab \in (p)$ implies either $a \in (p)$ or $b \in (p)$. Assume the former WLOG. Then, $a = pr$ for some $r \in R$. Now, $p = ab = prb$ so $rb = 1$ so b is a unit. Thus, p is irreducible. \square

Proposition: prime = irreducible in PID. In a PID, a non-zero element x is a prime if and only if it is irreducible.

Proof. We already know prime implies irreducible, so we show the converse. Suppose M is an ideal containing (p) . Then, $M = (m)$ is a principal ideal and since $p \in (m)$, $p = rm$ for some r . Since p is irreducible, either r or m is a unit. So, either $(p) = (m)$ or $(m) = (1)$. Thus, the only ideals containing (p) are (p) or (1) . So (p) is a maximal ideal. We know maximal ideals are prime ideals, so p is a prime. \square

B.10 Unique Factorization Domain

Definition: Unique factorization domain (UFD) A unique factorization domain is an integral domain R in which every non-zero $x \in R$ which is not a unit has the following properties:

- (1) x is a finite product of irreducible p_i (not necessarily distinct) of R ; $x = p_1 \cdots p_r$
- (2) The decomposition is unique up to associates i.e $x = q_1 \cdots q_m$ is another decomposition, then $m = r$ and after renumbering p_i is associate to q_i for all i .

Example: A field F is a UFD since every element is a unit.

Example: When R is a UFD, $R[x]$ is also a UFD.

Proposition: prime = irreducible in UFD. In a UFD, a non-zero element x is a prime if and only if it is irreducible.

Proof. Let R be a UFD. We need to show irreducible implies prime. Suppose $p \in R$ is irreducible and suppose $p|ab$ for some $a, b \in R$. Then, $ab = pc$ for some c . Write a and b as their irreducible decomposition. Then p must be associate to some irreducible either in the decomposition of a or b - assume the former WLOG. Then, $a = (up)p_2 \cdots p_n$ where u is a unit. Then, p divides a and so p is a prime. \square

Proposition: Let $a, b \in R$ be non-zero and let R be a UFD. Suppose, $a = up_1^{e_1} \cdots p_n^{e_n}$ and $b = vp_1^{f_1} \cdots p_n^{f_n}$ are their prime factorizations for a and b with u, v units. Here the primes p_i are distinct and $e_i, f_i \geq 0$ for all i . Then, $d = p_1^{\min e_1, f_1} \cdots p_n^{\min e_n, f_n}$ is the greatest common divisor of a and b .

Proposition: Every Principal Ideal Domain is a Unique Factorization Domain. In particular, every Euclidean Domain is a Unique Factorization Domain.

B.11 Polynomial Rings

Definition (Polynomial Ring). $R[x]$ consists of formal sums $a_n x^n + \cdots + a_1 x + a_0$, where $a_i \in R$ and $n \geq 0$. If $a_n \neq 0$, then the degree of the polynomial is n . A monic polynomial is one where $a_n = 1$.

We already saw the following before:

Proposition 1: Let R be an integral domain. Then:

- (a) $\deg(p(x)q(x)) = \deg(p(x)) + \deg(q(x))$ given $p(x), q(x)$ are non-zero.

- (b) The units of $R[x]$ are the units of R .
(c) $R[x]$ is an integral domain.

Proposition 2: Quotient of polynomial ring. Let I be an ideal of the ring R . Then, $R[x]/I[x] \cong (R/I)[x]$. In particular, if I is a prime ideal of R , then $I[x]$ is a prime ideal of $R[x]$.

Proof. Consider map $\varphi : R[x] \rightarrow (R/I)[x]$ by taking each coefficient mod I ; this is easily seen to be a ring homomorphism. Then, $\ker(\varphi) = I[x]$ proves first part. For the second, since I is prime, R/I is integral domain (by previous proposition) so $(R/I)[x]$ is an integral domain and so $I[x]$ is a prime ideal of $R[x]$. \square

Note: It is *not* true that if I is a maximal ideal of R , then $I[x]$ is a maximal ideal of $R[x]$. However, if I is maximal in R , then the ideal of $R[x]$ generated by I and x is maximal in $R[x]$.

Definition: Polynomial ring of more than one variables. The polynomial ring in the variables x_1, \dots, x_n with coefficients in R denoted by $R[x_1, \dots, x_n]$ is defined inductively by $R[x_1, \dots, x_n] = R[x_1, \dots, x_{n-1}][x_n]$.

A polynomial in a polynomial ring of more than one variable is a finite sum of elements of the form $ax_1^{d_1} \cdots x_n^{d_n}$ where $a \in R, d_i \geq 0$ which are called the monomial terms. For a monomial term, if $a = 1$, we call it a monic term. A monomial term of this form is of degree $d = d_1 + \cdots + d_n$ and the n -tuple (d_1, \dots, d_n) is the multidegree of the term.

If f is a non-zero polynomial in n variables, the sum of all monomial terms in f of degree k is called the homogenous component of f of degree k . If f has degree d , then f may be written uniquely as the sum $f_0 + \cdots + f_d$ where f_k is the homogenous component of f of degree k .

B.12 Polynomial Rings over Fields

Theorem 3: Polynomial rings that are Euclidean Domains. Let F be a field. The polynomial ring $F[x]$ is a Euclidean Domain. If $a(x), b(x)$ are two polynomials in $F[x]$ with $b(x) \neq 0$, then there are unique polynomials $q(x), r(x) \in F[x]$ such that $a(x) = q(x)b(x) + r(x)$ with $r(x) = 0$ or $\deg r(x) < \deg b(x)$.

Sketch of proof: Use induction. Let $\deg(a(x)) = n, \deg(b(x)) = m$. If $a(x) = 0$, then $q(x) = r(x) = 0$. If $n < m$, let $q(x) = 0, r(x) = a(x)$. So let $n \geq m$. Construct $q(x)$ - if $a(x) = \sum_{i=0}^n a_i x^i, b(x) = \sum_{i=0}^m b_i x^i$. Define $a'(x) = a(x) - \frac{a_n}{b_m} x^{n-m} b(x)$ designed to subtract leading term from $a(x)$. By inductive hypothesis, $a'(x) = q'(x)b(x) + r(x)$. With $q(x) = q'(x) + \frac{a_n}{b_m} x^{n-m}$.

To prove uniqueness, assume there is another decomposition with $q_1(x), r_1(x)$ and leverage the fact that degree of $f(x)g(x)$ is the sum of degree $f(x)$ and degree $g(x)$.

Corollary 4: If F is a field, then $F[x]$ is a Principal Ideal Domain (PID) and a Unique Factorization Domain (UFD).

Now we look at polynomial rings that UFDs.

Proposition 5: Gauss's Lemma. Let R be a UFD with a field of fractions F and let $p(x) \in R[x]$. If $p(x)$ is reducible in $F[x]$, then $p(x)$ is reducible in $R[x]$. More precisely, if $p(x) = A(x)B(x)$ for some non-constant polynomials $A(x), B(x) \in F[x]$, then there are non-zero elements $r, s \in F$ such that $rA(x) = a(x)$ and $sB(x) = b(x)$ both in $R[x]$ and $p(x) = a(x)b(x)$ is a factorization in $R[x]$.

Sketch of proof for R a field: Let $p(x) = A(x)B(x)$ where on RHS, coefficients are in F . Multiply both sides by a common denominator for all coefficients to get $dp(x) = a'(x)b'(x)$ where on RHS we have elements in $R[x]$, $d \neq 0 \in R$. If d is unit, we are done with $a(x) = d^{-1}a'(x)$ and $b(x) = b'(x)$. Check Dummit and Foote for the proof in the case where d is not a unit.

Corollary: Let R be a UFD. Let F be its field of fractions and let $p(x) \in R[x]$. Suppose the greatest common divisor of the coefficients of $p(x)$ is 1. Then $p(x)$ is irreducible in $R[x]$ if and only if it is irreducible in $F[x]$. In particular, if $p(x)$ is a monic polynomial that is irreducible in $R[x]$, then $p(x)$ is irreducible in $F[x]$.

Proof: By Gauss's Lemma, if $p(x)$ is reducible in $F[x]$, then it is reducible in $R[x]$. Conversely, suppose the gcd of coefficients of $p(x)$ is 1. If p is reducible with $p(x) = a(x)b(x)$, then neither $a(x)$ nor $b(x)$ are constant polynomials - this factorization also shows $p(x)$ is reducible in $F[x]$.

Theorem: R is a UFD if and only if $R[x]$ is a UFD.

Corollary: If R is a UFD, then a polynomial ring in an arbitrary number of variables with coefficients in R is also a UFD.

B.13 Irreducibility Criteria

Now we look at **irreducible criteria** of polynomials.

We will require the following throughout the AG notes:

Proposition: Let R be a field. The prime ideals of $R[y]$ are the zero ideal (0) , the ideals $(f(y))$ where f is irreducible.

Proof: Given R is a field, $R[x]$ is a PID. If (f) is a prime ideal in $R[x]$, then f is prime which means f is irreducible.

Proposition: Let F be a field and let $p(x) \in F[x]$. Then $p(x)$ has a factor of degree one if and only if $p(x)$ has a root in F i.e there is an $\alpha \in F$ with $p(\alpha) = 0$.

Proof: If $p(x)$ has a factor of degree 1, then since F is a field, we may assume the factor is of the form $(x - a)$, $a \in F$. Then, $p(a) = 0$. Conversely, if $p(a) = 0$, then by division algorithm in $F[x]$ - theorem 3 in this section - $p(x) = q(x)(x - a) + r$ where r is constant and since $p(a) = 0$, $r = 0$ so $(x - a)$ is a factor.

Proposition: A polynomial of degree two or three over a field F is reducible if and only if has a root in F .

Proposition: Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial of degree n with integer coefficients. If $r/s \in \mathbb{Q}$ is in its lowest term (i.e r and s are relatively prime integers) and r/s is a root of $p(x)$, then r divides the constant term and s divides the leading coefficient of $p(x)$ i.e $r|a_0$ and $s|a_n$. In particular, if $p(x)$ is a monic polynomial with integer coefficients and $p(d) \neq 0$ for all integers d dividing the constant term of $p(x)$, then $p(x)$ has no roots in \mathbb{Q} .

The following are very important results:

Proposition: Let I be a proper ideal in the integer domain R and let $p(x)$ be a nonconstant monic polynomial in $R[x]$. If the image of $p(x)$ in $(R/I)[x]$ cannot be factored in $(R/I)[x]$ into two polynomials of smaller degree, then $p(x)$ is irreducible in $R[x]$.

Proof: Suppose $p(x)$ cannot be factored in $(R/I)[x]$ but $p(x)$ is reducible in $R[x]$ so $p(x) = a(x)b(x)$ where both $a(x)$ and $b(x)$ are monic, nonconstant in $R[x]$. But then reducing the coefficients modulo I gives a factorization in $(R/I)[x]$ - contradiction.

Proposition: Eisenstein's Criterion. Let P be a prime ideal of the integral domain R and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial in $R[x]$ ($n \geq 1$). Suppose $a_{n-1}, a_n, \dots, a_1, a_0$ are all elements of P and suppose a_0 is not an element of P^2 . Then $f(x)$ is irreducible in $R[x]$.

Proposition: The maximal ideals in $F[x]$ are the ideals $(f(x))$ generated by irreducible polynomials in $F[x]$. In particular, $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.

Proposition: Let $g(x)$ be a nonconstant element of $F[x]$ and let $g(x) = f_1(x)^{n_1}f_2(x)^{n_2} \cdots f_k(x)^{n_k}$ be its factorization into irreducibles, where the $f_i(x)$ are distinct. Then, we have the following isomorphism of things:

$$F[x]/(g(x)) \cong F[x]/(f_1(x)^{n_1}) \times F[x]/(f_2(x)^{n_2}) \times \cdots \times F[x]/(f_k(x)^{n_k})$$

Proposition: If the polynomial $f(x)$ has roots $a_1, \dots, a_k \in F$ (not necessarily distinct), then $f(x)$ has $(x - a_1) \cdots (x - a_k)$ as a factor. In particular, a polynomial of degree n in one variable over a field F has at most n roots in R , even counted with multiplicity.

C Real Projective Space

We define the real projective space as follows: for $n \geq 1$, define $\mathbb{RP}^n = S^n / \sim$ with the equivalence relation $x \sim y$ if and only if $x = y$ or $x = -y$. It can also be seen as the space attained by quotienting $\mathbb{R}^{n+1} \setminus \{0\}$ under the equivalence relation $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{R}$ and $\lambda \neq 0$.

Therefore, \mathbb{RP}^n identifies each direction through the origin of the n -dimensional sphere as unique points. We can also see that \mathbb{RP}^n contains "a copy of \mathbb{R}^n ". For example, if we consider \mathbb{RP}^n , then for each point (x, y, z) , we can consider the ratio $x : y : z$ and then write it as $(x/z) : (y/z)$. This can be done unless $z = 0$. Given every line through the centre meets $z = 1$ plane (unless the line is on the plane $z = 0$), we can then consider every line as points on the plane $z = 1$.

An interesting observation is the appearance of the mobius band inside \mathbb{RP}^2 . To see this, let D be a closed disk of radius 1 in \mathbb{R}^2 i.e $D := \{x \in \mathbb{R}^2 : |x| \leq 1\}$. Then, it is clear that $D \setminus \sim$ is homeomorphic to \mathbb{RP}^2 (via projection). Now, let $r \in (0, 1)$. We first cut out the disk D_r of radius r from inside D to get an annulus A . Now, $A \setminus \sim$ is homeomorphic to the mobius band.

Theorem 112. The real projective space, \mathbb{RP}^n , is a compact, n -dimensional manifold.

Proof. First, we show that \mathbb{RP}^n is compact. Note that S^n is compact. Consider the quotient map $p : S^n \rightarrow S^n / \sim$. Note that this mapping is continuous. To see this, let I be the identity function on S^n / \sim . Then, $(I \circ p)(x) = p(x)$. Now, given I is continuous, then $I \circ p$ is also continuous $\implies p$ is continuous. Since $p : S^n \rightarrow S^n / \sim$ is continuous and S^n is compact, therefore, \mathbb{RP}^n is compact.

Next, we show \mathbb{RP}^n is Hausdorff. Consider any $[x], [y]$ in \mathbb{RP}^n such that $[x] \neq [y]$. This means $x \neq y, x \neq -y$ in S^n . Now, in S^n , consider the following open sets - U_x which contains x , U_{-x} which contains $-x$, U_y which contains y and U_{-y} which contains $-y$. Given S^n is Hausdorff, we can let these sets be pairwise disjoint. Furthermore, $p(U_x), p(U_{-x}), p(U_y), p(U_{-y})$ are all open since. Furthermore, $p(U_x) \cup p(U_{-x})$ contains x and is open. We claim $(p(U_x) \cup p(U_{-x})) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$. This is because $p(U_x) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$ and $p(U_{-x}) \cap (p(U_y) \cup p(U_{-y})) = \emptyset$. For the first part, suppose $[z] \in p(U_x) \cap (p(U_y) \cup p(U_{-y})) \implies \exists a \in U_x$ such that $p(a) = z$ and $\exists b \in U_y$ such that $p(b) = z$ and $\exists c \in U_{-y}$ such that $p(c) = z$. Now $p(a) = p(b)$ implies $a = b$ or $a = -b$. If $a = b$, then $U_x \cap U_y \neq \emptyset$. So $a = -b$. By similar logic $a = -b' \implies -b = -b' \implies U_y \cap U_{-y} \neq \emptyset$ which is also a contradiction.

Next, we know \mathbb{RP}^n is second countable since S^n is second countable.

Now, we show \mathbb{RP}^n is locally Euclidean and has dimension n . Let $[x] \in \mathbb{RP}^n$. Without loss of generality, suppose $x_k \neq 0$ (if it is, then we can always rotate the space to ensure it is not 0). Then consider the following function $\pi([(x_0, x_1, \dots, x_n)]) = \left(\frac{x_1 x_k}{|x_k|}, \dots, \frac{x_n x_k}{|x_k|} \right)$. This function is bijective from the set $A_k := \{[x] \in \mathbb{RP}^n | x_k \neq 0\}$ to $\mathbb{D}^n := \{x \in \mathbb{R}^n | |x| < 1\}$. Its inverse is given by $\pi^{-1}((x_1, \dots, x_n)) = (x_1, \dots, x_{k-1}, \sqrt{1 - |x|^2}, x_k, \dots, x_n)$. Note that $\mathbb{RP}^n = \cup_{k=1, \dots, n+1} A_k$. Therefore π maps \mathbb{RP}^n to all of \mathbb{R}^n . Given π is continuous, \mathbb{RP}^n is locally Euclidean with dimension n . \square

Theorem 113. The real projective space, \mathbb{RP}^n , is a smooth manifold.

Proof. We denote points in the real projective space as $[x_0 : x_1 : \dots : x_n]$ which represents the equivalence class of $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} \setminus \{0\}$. We will construct an atlas on \mathbb{RP}^n with $n + 1$ charts (U_i, ϕ_i) for $i = 0, \dots, n$. Define $U_i = \{[x_0 : x_1 : \dots : x_n] | x_i \neq 0\}$. Then, as before, $\mathbb{RP}^n = \cup_i U_i$. Now, we define the homeomorphism:

$$\phi_i : U_i \rightarrow \mathbb{R}^n$$

such that $\phi_i([x_0 : \dots : x_n]) = \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right)$. We prove, now, that the transition functions, i.e $\phi_j \circ \phi_i^{-1}$, are smooth. Let $\phi_i([x_0 : \dots : x_n]) = \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i} \right) =: (y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_n)$ and let

$\phi_j([x_0 : \cdots : x_n]) = (\frac{x_0}{x_j}, \dots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \dots, \frac{x_n}{x_j}) = (\frac{y_0}{y_j}, \dots, \frac{1}{y_j}, \dots, \frac{y_n}{y_j})$. Therefore, our transition function becomes:

$$(\phi_j \circ \phi_i^{-1})(y_0, \dots, y_n) = \left(\frac{y_0}{y_j}, \dots, \frac{1}{y_j}, \dots, \frac{y_n}{y_j} \right)$$

which is smooth. This gives us an atlas and therefore, implicitly defines a smooth structure on \mathbb{RP}^n . \square

D Complex Projective Space

Let $X = \mathbb{C}^{n+1} \setminus \{0\}$. Now, define the following equivalence class on X : $x \sim y$ if and only if $x = \lambda y$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Then, the complex projective space is defined as $\mathbb{CP}^n = X / \sim$.

Note that \mathbb{C}^{n+1} is isomorphic to \mathbb{R}^{2n+2} . Therefore, if $p \in \mathbb{C}^{n+1}$ with $(p_1 + ip_2, p_3 + ip_4, \dots, p_{2n+1} + ip_{2n+2})$, then we can write p in \mathbb{R}^{2n+2} as $p = (p_1, p_2, \dots, p_{2n+2})$. Now, suppose $p \sim p'$ with $p = \lambda p'$ where $\lambda = \lambda_1 + i\lambda_2$. Then, in \mathbb{R}^{2n+2} , after expanding and simplifying, we see that $(p_1\lambda_1 - p_2\lambda_2, p_2\lambda_1 + p_1\lambda_2, \dots) = (p'_1, p'_2, \dots)$. This tells us that, for the first two coordinates, we have the following relation:

$$\begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} p'_1 \\ p'_2 \end{bmatrix}. \quad (1)$$

Similarly, for the third and fourth coordinates, we also have the similar relation. Therefore, the equivalence class of p in \mathbb{R}^{2n+2} can be written as the set consisting of

$$\left[\begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \quad \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \quad \cdots \quad \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \quad \begin{bmatrix} \lambda_1 & -\lambda_2 \\ \lambda_2 & \lambda_1 \end{bmatrix} \right] \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_{2n+1} \\ p_{2n+2} \end{bmatrix} \quad (2)$$

for any $\lambda_1, \lambda_2 \in \mathbb{R}$.

The complex projective space is the space of all complex lines through the origin or the set of all one-dimensional subspace. Furthermore, if we consider λ to have unit length, we can also write \mathbb{CP}^n as $S^{2n+1}/U(1)$ where $U(1)$ is the unitary group.

Theorem 114. The complex projective space, \mathbb{CP}^n , is a compact $2n$ -dimensional manifold.

Proof. Let $\pi : X \rightarrow X/\sim$.

First, we show that this space is compact. Consider the sphere S^{2n+1} . This sphere is compact. The function that maps points on the sphere to X/\sim is continuous (by the same reasoning as provided in the previous proof). Furthermore, the function is surjective, since for any equivalence class in X/\sim , there exists a point that is on the surface of the sphere. Therefore, the image space of this function, X/\sim is compact too.

Next, we show that this space is Hausdorff. Consider any $x \in S^{2n+1}$. Then, its equivalence class is $[x] = \{y \in X | y = \lambda x, \forall \lambda \in \mathbb{C} \setminus \{0\}\} =: O_x$. Given the function $(\lambda, x) \rightarrow \lambda x$ is continuous and since S^{2n+1} is compact, therefore, $[x]$ is compact. Given $[x] \neq [y]$, this means $O_x \cap O_y = \emptyset$ with both O_x, O_y being compact. On the other hand, since S^{2n+1} is Hausdorff, there exists open sets $O_x \subset U_x$ and $O_y \subset U_y$ with $y \in U_y, x \in U_x$ and $U_x \cap U_y = \emptyset$.

Now, \bar{U}_y is closed (given this is the closure of U_y) $\implies \pi(\bar{U}_y)$ is closed. On the other hand, define $U'_x := \mathbb{CP}^n \setminus \pi(\bar{U}_y)$, which must be open in \mathbb{CP}^n . Furthermore, define $U'_y := \pi(U_y)$, which must be open. Clearly, $U'_x \cap U'_y = \emptyset$. All that's left to show is $[x] \in U'_x$ and $[y] \in U'_y$. Let us show the first one. S^{2n+1} is compact and Hausdorff, while \bar{U}_x is closed, so \bar{U}_x is compact. On the other hand, O_x is compact. Given both O_x and \bar{U}_y are compact, we find two disjoint open sets U and W such that $O_x \subset U$ and $\bar{U}_y \subset W$ and $U \cap W = \emptyset$. Therefore, $O_x \cap W = \emptyset$. Now, $[x] \in O_x$ implies $[x] \notin \pi(W)$. Therefore, $[x] \notin \pi(\bar{U}_y)$. This means, $[x] \in \mathbb{CP}^n \setminus \pi(\bar{U}_y)$. Therefore, $[x] \in U'_x$. Similarly, $[y] \in U'_y$.

Now, we show that this space is locally Euclidean. Consider the following function:

$$[x_1, \dots, x_{n+1}] \rightarrow \frac{1}{x_k} [x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n].$$

Then, this is a function from the set $A_k := \{[x] \in \mathbb{CP}^n | x_k \neq 0\}$ to \mathbb{C}^n . The inverse of this function is:

$$(x_1, \dots, x_n) \rightarrow [x_1, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n].$$

Therefore, this is a homeomorphism. Now, $\mathbb{CP}^n = \cup_{k=1, \dots, n+1} A_k$ is open and so our functions maps all of the space too all of \mathbb{C}^n which is isomorphic to \mathbb{R}^{2n} . Therefore, \mathbb{CP}^n is locally Euclidean with dimension n .

Given, $\mathbb{CP}^n = \cup_{k=1, \dots, n+1} A_k$, the space is a finite union of second countable spaces, so \mathbb{CP}^n is second countable. \square

Theorem 115. The complex projective space, \mathbb{CP}^n , is a smooth manifold.

Proof. This can be proven with the same atlas as we defined for the real projective space. \square