

# Introduction to Whitney's Theorems for Embeddings

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## The Smooth Partition of Unity

Let  $X$  be a smooth manifold with open cover  $\{U_\alpha\}_{\alpha \in I}$ . A smooth partition of unity subordinate to this open cover is a sequence of smooth functions  $\{\theta_i : X \rightarrow \mathbb{R}\}_{i=1,2,\dots}$  such that:

- (a)  $0 \leq \theta_i(x) \leq 1$  for any  $x \in X$ .
- (b) For any  $x \in X$ , there exists a neighbourhood  $V_x$  such that  $\theta_i(y) = 0$  for any  $y \in V_x$  holds for at most finitely many  $i$ .
- (c) For any  $i$ ,  $\text{supp}(\theta_i) := \overline{\theta_i^{-1}(\mathbb{R} \setminus \{0\})} \subset U_\alpha$  for some  $\alpha \in I$ .
- (d) For any  $x \in X$ ,  $\sum_{i=1}^{\infty} \theta_i(x) = 1$ .

It can be proven that every open cover of a smooth manifold admits a smooth partition of unity subordinate to that cover. Additionally, if  $\{U_i\}_{i=1,\dots,N}$  is a finite open cover, we can take  $\{\theta_i\}_{i=1,\dots,n}$  such that  $\text{supp}(\theta_i) \subset U_i$  for each  $i$  and  $\theta_i = 0$  for  $i > n$  in our original infinite set of smooth functions.

## The Bump Function

We want to show the following: given  $X$  is a smooth manifold with  $(U, \phi)$  smooth chart and  $p \in U$ , then there exists a smooth bump function  $\beta : X \rightarrow \mathbb{R}$  and open neighbourhoods  $p \in W \subseteq V \subseteq U$  and  $\overline{V} \subseteq U$  such that  $\beta(x) = 1$  for  $x \in W$ ,  $\beta(x) = 0$  for  $x \notin V$  and  $\beta(x) \in [0, 1]$  for  $x \in X$ .

We first construct  $f_1(x)$  to be

$$f_1(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad (1)$$

Now, let

$$f_2(x) = \frac{f_1(2-x)}{f_1(2-x) - f_1(x-1)}.$$

Note that  $f_2(x) = 0$  for any  $x \geq 2$ ,  $f_2(x) = 1$  for any  $x \leq 1$  and  $f_2(x) \in [0, 1]$  for any  $x \in [1, 2]$ .

Now, suppose  $X$  is a smooth manifold. Then, for any  $p \in X$ , there exists a chart  $(U, \phi)$ . WLOG, suppose  $\phi(p) = 0$ . Also suppose we have open neighbourhoods such that  $W \subseteq V \subseteq U$  with  $x \in W$  and  $\bar{V} \subseteq U$ . Now, select  $\epsilon > 0$  such that  $B_{3\epsilon}(0)$  is inside  $\tilde{U}$  (which is the image of  $U$  under  $\phi$ ). Then, if  $W = \phi^{-1}(B_\epsilon(0))$  and  $V = \phi^{-1}(B_{2\epsilon}(0))$ , our bump function is defined to be

$$\beta(x) = \begin{cases} h(\frac{\|\phi(x)\|}{\epsilon}) & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$$

With this function,  $\overline{B_{2\epsilon}(0)} \subset B_{3\epsilon}(0) \subseteq \tilde{U}$  which implies  $\tilde{V} \subseteq U$ .

Note that  $W \subseteq V \subseteq U$  with  $x \in W$  and  $\bar{V} \subseteq U$ . We can easily check that  $\beta(x) = 1$  for  $x \in W$ ,  $\beta(x) = 0$  for  $x \notin V$  and  $\beta(x) \in [0, 1]$  for  $x \in X$ .

Now, we move on to the first important result.

**Theorem 1.** *Let  $X$  be a compact, smooth manifold of dimension  $m$ . Then, there exists  $N \geq m$  and a smooth embedding  $f : X \rightarrow \mathbb{R}^N$ .*

*Proof.* Pick any  $x \in X$  with the smooth chart  $(U_x, g_x)$  near it. Then, there exists  $\epsilon_x > 0$  such that  $B_{\epsilon_x}(g_x(x)) \subset \tilde{U}_x$ . Define  $W_x := g_x^{-1}(B_{\epsilon_x/2}(g_x(x)))$  and  $V_x := g_x^{-1}(B_{\epsilon_x}(g_x(x)))$  - both of these are subsets of  $\tilde{U}_x$ . Now,  $\{W_x\}_{x \in X}$  is a covering of  $X$  and since  $X$  is compact, there is a finite subcover given by  $W_1, \dots, W_n$  where  $W_i = W_{x_i}$ . For each  $W_i$ , let  $V_i$  and  $g_i$  be the corresponding  $V_{x_i}$  and  $g_{x_i}$ .

Now, we use the following bump function:

$$\phi : X \rightarrow [0, 1] \text{ such that } \phi_i(x) = \begin{cases} 1, & \text{if } x \in W_i \\ 0, & \text{if } x \in X - V_i \\ 0 \leq \phi_i(x) \leq 1 & \text{otherwise} \end{cases}$$

$$\text{Then, we define } h_i(x) = \begin{cases} \phi_i(x)g_i(x), & \text{on } V_i \\ 0 & \text{outside } V_i \end{cases}$$

Using these two functions, we define  $f : X \rightarrow \mathbb{R}^N$  where  $N = n(1 + m)$  to be  $f(x) = (\phi_1(x), \dots, \phi_n(x), h_1(x), \dots, h_n(x))$ . This map is smooth. Furthermore, we claim that  $f$  is injective. This is because, if  $f(x) = f(y)$ , then  $\phi_i(x) = \phi_i(y)$  and  $h_i(x) = h_i(y)$  for each  $i$ . Given  $x \in W_j$  for some  $j$ ,  $\phi_j(x) = 1$  and so  $\phi_j(y) = 1$  implying  $y \in W_j$ . Therefore,  $g_j(x) = \phi_j(x)g_j(x) = h_j(x) = h_j(y) = \phi_j(y)g_j(y) = g_j(y)$ . Given  $g$  is a homeomorphism,  $x = y$ , showing that  $f$  is injective.

Given  $X$  is compact and  $f$  is injective and continuous, therefore  $f$  is a topological embedding too. All that is left is to show that  $f$  is an immersion.

Given  $x \in X$ ,  $x \in W_i$  for some  $i$ . Now, for any  $y \in W_i$ , given  $\phi_i(y) = 1$  and  $h_i(y) = g_i(y)$ , therefore,  $f(y) = (1, \dots, 1, g_1(y), \dots, g_n(y))$ . Now consider the chart  $(W_i, g_i)$  where  $g_i$  is restricted to  $W_i$ . In this chart,  $g_i$  looks like the identity, so its derivative also looks like the identity which implies that  $Df_y$  has a non-zero  $m \times m$  minor. Therefore,  $Df_x$  is injective, implying  $f$  is a smoother immersion which tells us that  $f$  is a smooth embedding.  $\square$

**Lemma 2.** *Let  $X$  be a smooth manifold of dimension  $n$ . Then, there exists a smooth, proper function from  $X$  to  $\mathbb{R}$ .*

*Proof.* For any open set of  $X$ , we can get a compact closure by mapping the open set to the euclidean space using the chart function, then taking the closed ball around it and then mapping it back to  $X$ . Let  $\{U_\alpha\}_{\alpha \in I}$  be an open cover of  $X$  made up of subsets of  $X$  with compact closure i.e  $\overline{U_\alpha}$  is compact for each  $\alpha$ .

Let  $\{\theta_i\}$  be a subordinate partition of unity s.t  $\text{supp}(\theta_i) \subset U_{\alpha_i}$  for  $i = 1, 2, \dots$ . Now we define the following smooth function:  $\rho : X \rightarrow \mathbb{R}$  to be  $\rho = \sum_{i=1}^{\infty} i\theta_i$ . Given (b) in our definition of partition of unity,  $\rho(x)$  is finite.

We claim  $\rho$  is a proper map. Suppose  $K \subseteq \mathbb{R}$  is compact. We want to show that  $\rho^{-1}(K)$  is compact.

Since  $K$  is compact, it is closed and bounded, meaning there exists some  $j > 0$  such that  $K \subset [-j, j]$ . Then,  $\rho^{-1}(K)$  is also closed (since  $\rho$  is continuous) and is contained in the set  $\{x \in X | \rho(x) \leq j\}$ . We claim that if  $\rho(x) \leq j$ , then at least one of the function

$\theta_1, \dots, \theta_j$  must take  $x$  to a non-zero value. If not, then:

$$\begin{aligned}
\rho(x) &= \sum_{i=1}^{\infty} i\theta_i(x) \\
&= \sum_{i=j+1}^{\infty} i\theta_i(x) \\
&\geq \sum_{i=j+1}^{\infty} (j+1)\theta_i(x) \\
&= (j+1) \sum_{i=1}^{\infty} \theta_i(x) \\
&= (j+1)
\end{aligned}$$

This means,  $\rho(x) \geq j+1$  which is a contradiction.

With this, we can now write  $\rho^{-1}(K) \subseteq \{x \in X | \rho(x) \leq j\} \subseteq \cup_{i=1}^j \{x \in X | \theta_i(x) \neq 0\} \subseteq \cup_{i=1}^j U_{\alpha_i} \subseteq \cup_{i=1}^j \overline{U_{\alpha_i}}$ . Since  $\cup_{i=1}^j \overline{U_{\alpha_i}}$  is compact, we see that  $\rho^{-1}(K)$  is a closed subset of a compact set, so it is compact.  $\square$

**Theorem 3.** *Let  $X$  be a smooth manifold of dimension  $n$ . Then, there exists  $N \geq m$  and a proper, smooth embedding  $f : X \rightarrow \mathbb{R}^n$ .*

*Proof.* By Theorem 1, we have a smooth embedding  $g : X \rightarrow \mathbb{R}^p$  and by Lemma 2, we have a proper, smooth function  $\rho : X \rightarrow \mathbb{R}$ . Now, with  $N := p+1$ , define  $f : X \rightarrow \mathbb{R}^N$  such that  $f(x) = (g(x), \rho(x))$ . This is a smooth embedding -  $f$  is clearly smooth and since  $g$  is a smooth embedding, therefore, the derivative of  $f$  at any  $x$  is injective and  $f$  is a topological embedding.

We now claim  $f$  is proper. Suppose  $K \subset \mathbb{R}^{p+1}$  is compact, which implies it is closed and bounded - therefore,  $K \subset \mathbb{R}^p \times [-j, j]$  for some  $j > 0$ . Then,  $f^{-1}(K) \subseteq \rho^{-1}([-j, j])$ . Note that since  $\rho$  is compact,  $\rho^{-1}([-j, j])$  is compact, so  $f^{-1}(K)$  is a closed subset of a compact set which means it is compact.  $\square$

**Whitney's Theorem** While Whitney proved the following theorem for to embed  $X$  in  $\mathbb{R}^{2n}$ , we will prove it for  $2n+1$  instead because it is significantly simpler.

**Theorem 4.** *Let  $X$  be a smooth,  $n$ -dimensional manifold. Then,  $X$  admits a proper, smooth embedding into  $\mathbb{R}^{2n+1}$ .*

We will prove this by coming up with a proper, smooth immersion  $f : X \rightarrow \mathbb{R}^{2n+1}$  which automatically allows us to deduce that  $f$  is a smooth embedding and  $f(X)$  is therefore a smooth submanifold.

*Proof.* First, we construct  $f : X \rightarrow \mathbb{R}^{2n+1}$  to be an injective immersion:

By Theorem 1, we can find an injective immersion  $f : X \rightarrow \mathbb{R}^N$ . Now, consider any  $a \in \mathbb{R}^{2N}$ . Let  $H_a$  be the hyperplane that is orthogonal to  $a$  and let  $\pi_a : \mathbb{R}^N \rightarrow H_a$  be the orthogonal projection i.e  $\pi_a(x) = x - (x \cdot a)a$ . Note that  $\frac{\partial \pi_a(x)_i}{\partial x_j} = \delta_{ij} - (a_i a_j)$ , which means  $D(\pi_a)v = \pi_a$ . We claim that  $\pi_a \circ f : X \rightarrow H_a \cong \mathbb{R}^{N-1}$  is our injective immersion for almost all  $a$  in  $\mathbb{R}^N$ .

To prove this, construct  $h : X \times X \times \mathbb{R} \rightarrow \mathbb{R}^N$  s.t.  $h(x, y, t) = t(f(x) - f(y))$  and  $g : TX \rightarrow \mathbb{R}^N$  s.t.  $g(x, v) = Df_x(v) =: D_v f_x$  with  $x \in X$ ,  $v \in T_x X$ . Note that  $g$  is a function from a  $2n$  dimensional space to  $N$  and  $h$  is a function from  $2n+1$  dimensional space to  $N$ .

Now, by Sard's theorem, the set of critical values of  $g$  and  $h$  have measure zero and therefore their union is also measure zero. Therefore, we can select an arbitrary  $a \in \mathbb{R}^N$  such that  $a$  is a regular value for both  $h$  and  $g$ . By the definition of regular values,  $Dg(x', v')$  and  $Dh_{x', y', t'}$  are both surjective where the derivatives are taken at  $g^{-1}(a)$  and  $h^{-1}(a)$ . However, since domain of  $g$  and  $h$  are of dimensions  $2n+1 < N$  and  $2n < N$  respectively, therefore, the derivatives cannot be surjective. This means,  $a \notin \text{Im}(g)$  and  $a \notin \text{Im}(h)$ .

Now, we show that  $\pi_a \circ f$  is injective. Suppose  $(\pi_a \circ f)(x) = (\pi_a \circ f)(y)$ . Then,  $(\pi_a)(f(x) - f(y)) = 0$ . Given  $\pi_a$  is the projection map, this means,  $f(x) - f(y) = ta$  for some  $t$ . Furthermore,  $t = 0$  because if  $t \neq 0$ , then  $h(x, y, \frac{1}{t}) = \frac{1}{t}(f(x) - f(y)) = \frac{1}{t}ta = a \in \text{Im}(h)$ . Given  $t = 0$ , therefore  $f(x) = f(y)$  and since  $f$  is injective, therefore,  $x = y$ .

Next, we show  $\pi_a \circ f$  is an immersion i.e we show that  $D(\pi_a \circ f)$  is injective. Suppose not. Then, there exists  $v \neq 0$  such that  $D(\pi_a \circ f)_x(v) = 0$ . Then,

$$\begin{aligned} D(\pi_a \circ f)_x(v) &= 0 \\ D(\pi_a)_{f(x)}(Df_x(v)) &= 0 \\ \pi_a \circ Df_x(v) &= 0 \\ Df_x(v) &= ta \end{aligned}$$

for some  $t$ . Given  $f$  is an immersion, its derivative is injective and so  $t \neq 0$ . This means  $Df_x(\frac{v}{t}) = \frac{1}{t}(ta) = a \in \text{Im}(g)$  which is a contraction, so  $\pi_a \circ f$  is an immersion.

So far, we have shown that  $\pi_a \circ f$  is an injective immersion from  $X$  to  $\mathbb{R}^{N-1}$  for  $N > 2n + 1$ . Continuing this way and composing our immersions, we will get an immersion from  $X$  to  $\mathbb{R}^{2n+1}$ .

Next, we will make  $f$  a proper map.

Note that  $\mathbb{R}^{2n+1} \cong B^{2n+1} = B_1(0)$  by some diffeomorphism  $s$ . consider  $s \circ f : X \rightarrow B_1(0)$ . For simplicity in our notation, we will refer to  $s \circ f$  as just  $f$ . Since the image of  $f$  is in  $B_1(0)$ , therefore,  $\|f(x)\| < 1$  for any  $x \in X$ . Furthermore, by Lemma 2, there exists  $\rho : X \rightarrow \mathbb{R}$  that is smooth and proper.

Define  $F : X \rightarrow \mathbb{R}^{2n+2}$  s.t.  $F(x) = (f(x), \rho(x))$ . Then, consider the map  $\pi_a \circ F : X \rightarrow H_a \cong \mathbb{R}^{2n+2}$  for some  $a$  such that the map is an injective immersion as we showed before and  $\|a\| = 1$ . Then,  $a \in S^{2n+1}$ . Furthermore, suppose  $a \neq (0, \dots, 0, \pm 1)$  which we can assume given Sard's Theorem tells us almost all points are regular.

We claim  $\pi_a \circ F$  is a proper map.

$(\pi_a \circ F)(x) = \pi_a(f(x), \rho(x)) = F(x) - (F(x) \cdot a)a$ . Write  $a$  as  $a = (v, \alpha)$  where  $\alpha \in \mathbb{R}$ . Then,  $F(x) \cdot a = f(x) \cdot v + \rho(x) \cdot \alpha$  and therefore, the last coordinate of  $(\pi_a \circ F)(x)$  is  $\rho(x) - (f(x) \cdot v + \rho(x) \cdot \alpha)\alpha = \rho(x)(1 - \alpha^2) - \alpha f(x) \cdot v$ .

Now, suppose  $K \subset \mathbb{R}^{2n+1}$  is compact. We claim  $C := (\pi_a \circ f)^{-1}(K)$  is also compact. We know that  $K$  compact means  $K$  is closed and bounded. Since our function is smooth,  $C$  is also closed.

For any  $x \in C$  s.t.  $(\pi_a \circ F)(x) \in K$ , the last coordinate is  $\rho(x)(1 - \alpha^2) - \alpha f(x) \cdot v$ . Since  $K$  is bounded, this coordinate is also bounded. note that since  $|f(x)| < 1$  and  $\alpha, v$  are constants,  $-\alpha f(x) \cdot v$  is bounded. Therefore,  $\rho(x)(1 - \alpha^2)$  is bounded. Furthermore, since  $\alpha^2 \neq 1$  (given the last coordinate of  $a$  is neither  $+1$  nor  $-1$ ), so  $\rho(x)$  is bounded.

This means,  $\rho(C)$  is bounded. Then,  $\overline{\rho(C)}$  is closed and bounded and therefore, compact. Given  $\rho$  is proper,  $\rho^{-1}(\overline{\rho(C)})$  is compact. Now,  $C \subseteq \rho^{-1}(\overline{\rho(C)})$  is a closed subset, so  $C$  is compact. Therefore,  $\pi_a \circ F$  is a proper, injective immersion which implies  $\pi_a \circ F$  is a smooth, proper embedding.  $\square$

## Whitney Immersion Theorem

**Theorem 5.** *Every  $n$ -dimensional, smooth manifold can be immersed in  $\mathbb{R}^{2n}$ .*

*Proof.* Suppose,  $X$  is a smooth manifold of dimension  $n$ . By Whitney's Theorem, we can immerse this into  $\mathbb{R}^{2n+1}$ . Suppose the immersion is  $f$  and suppose it takes  $X$  to  $M \subset \mathbb{R}^{2n+1}$ . Now, we define  $g : TX \rightarrow \mathbb{R}^{2n+1}$  by  $g(x, v) = D_v f_x$ . Given  $f$  is an immersion, it is smooth and therefore, by Sard's Theorem, we know that almost all values of  $g$  are regular. Therefore, we can choose  $a \in \mathbb{R}^{2n+1}$  such that  $a$  is a regular value. However, note that  $g$ 's domain is  $TX$  is  $2n$  dimensional which is less than  $2n+1$ . This means,  $Dg_{(x', v')}$  (where  $(x', v')$  is in the preimage of  $a$  under  $g$ ) cannot be surjective. Therefore,  $a$  is not in the image of  $g$  i.e  $a \notin \text{Im}(g)$ .

Now, with  $a$  as a regular value of  $g$ , we claim  $\pi_a \circ f$  is a smooth immersion from  $X$  to  $\mathbb{R}^{2n}$ . To show this, we will show that  $D(\pi_a \circ f)_x$  is injective.

Suppose, there existed a non-zero  $v \in \mathbb{R}^n$  such that  $D(\pi_a \circ f)_x(v) = 0$ . Now,  $D(\pi_a)_{f(x)}(Df_x(v)) = \pi_a \circ Df_x(v)$ . Given this is equal to 0, therefore,  $Df_x(v) = ta$  for some  $t$ . Now, given  $f$  is an immersion,  $t \neq 0$ . But then,  $Df_x(\frac{v}{t}) = \frac{1}{t}(ta) = a \in \text{Im}(g)$ . This is a contradiction. Therefore,  $D(\pi_a \circ f)_x$  is injective. Furthermore,  $\pi_a \circ f$  is smooth. This gives us our immersion.  $\square$

## Exhaustion Function on a topological space $M$

Let  $M$  be a topological space. An exhaustion function  $f : M \rightarrow \mathbb{R}$  is a continuous function such that  $f^{-1}((-\infty, c])$  is compact in  $M$  for each  $c \in \mathbb{R}$ .

Turns out we can construct such a function for any smooth manifold  $M$  as shown below:

**Lemma 6.** *Every smooth manifold admits a smooth, positive exhaustion function.*

*Proof.* Given  $M$  is a smooth manifold, we can build a countable open cover of  $M$ . Let that be  $\{V_j\}_{j=1}^\infty$ . Furthermore, let  $\{\psi_j\}$  be the smooth partition of unity subordinate to this over cover. Now, we construct the following function:

$$f(p) = \sum_{j=1}^{\infty} j\psi_j(p).$$

Note that this is well-defined and smooth since for any neighbourhood that  $p$  is in, there exists only finitely many  $\psi_j$  that give non-zero terms. Furthermore,  $f$  is positive

since

$$f(p) = \sum_j j\psi_j(p) \geq \sum_j \psi_j(p) = 1.$$

Now we claim  $f$  is an exhaustion function. To show this, we will prove that for any  $c \in \mathbb{R}$ ,  $f^{-1}((-\infty, c])$  is compact.

Choose any arbitrary  $c \in \mathbb{R}$ . Let  $N > c$  be a positive integer.

Now, suppose  $p \notin \cup_{j=1}^N \overline{V_j}$ . Then,  $\psi_j(p) = 0$  using the definition of partition of unity for any  $j \in [1, N]$ . This means,  $f(p) = \sum_{j=N+1}^{\infty} j\psi_j(p) \geq \sum_{j=N+1}^{\infty} N\psi_j(p) = N \sum_{j=1}^{\infty} \psi_j(p) = N > c$ .

Therefore, if  $p \notin \cup_{j=1}^N \overline{V_j}$ , then  $f(p) > c$ . So, if  $f(p) \leq c$ , then  $p \in \cup_{j=1}^N \overline{V_j}$ . Therefore,  $f^{-1}((-\infty, c])$  is a closed subset of a compact set  $\cup_{j=1}^N \overline{V_j}$ , which means it is compact.  $\square$

Now, we can prove Whitney's Embedding Theorem for the non-compact manifolds.

**Theorem 7.** *Every non-compact smooth manifold  $X$  of dimension  $n$  admits a proper smooth embedding into  $\mathbb{R}^{2n+1}$ .*

*Proof.* Let  $f : X \rightarrow \mathbb{R}$  be a smooth exhaustion function on  $X$ . Then, by Sard's Theorem, for each non-negative integer  $i$ , there exists regular values  $a_i, b_i$  of  $f$  such that  $i < a_i < b_i < i + 1$ .

Now, we define the following sets:  $D_i, E_i \subset X$  such that  $D_0 = f^{-1}((-\infty, 1])$ ,  $E_0 = f^{-1}((-\infty, a_1])$ ,  $D_i = f^{-1}([i, i + 1])$  and  $E_i = f^{-1}([b_{i-1}, a_{i+1}])$  for  $i \geq 1$ . See figure 1.

Now, given  $f$  is a smooth exhaustion, each  $E_i$  is compact. Furthermore, one can show that each  $E_i$  is a submanifold with a boundary. Therefore, we can embed it into  $\mathbb{R}^{2n+1}$  by Theorem 4. Let  $\psi_i : E_i \rightarrow \mathbb{R}^{2n+1}$

Now,  $D_i \subset \text{Int}(E_i)$ . Then,  $X = \cup_i D_i$  with  $E_i \cap E_j = \emptyset$  unless  $j = i - 1, i$  or  $i + 1$ .

For each  $i$ , let  $\rho_i : X \rightarrow \mathbb{R}$  be a smooth bump function such that  $\rho_i = 1$  on an open neighbourhood of  $D_i$  and  $\text{supp}(\rho_i) \subset \text{Int}(E_i)$ .

Now, we define

$$F : X \rightarrow \mathbb{R}^{2n+1} \times \mathbb{R}^{2n+1} \times \mathbb{R} \text{ by } F(p) = (\sum_{i \text{ even}} \rho_i(p)\psi_i(p), \sum_{i \text{ odd}} \rho_i(p)\psi_i(p), f(p)).$$

(a)  $F$  is smooth and well-defined since for each  $p$ , there is only one term in each summation that is non-zero.



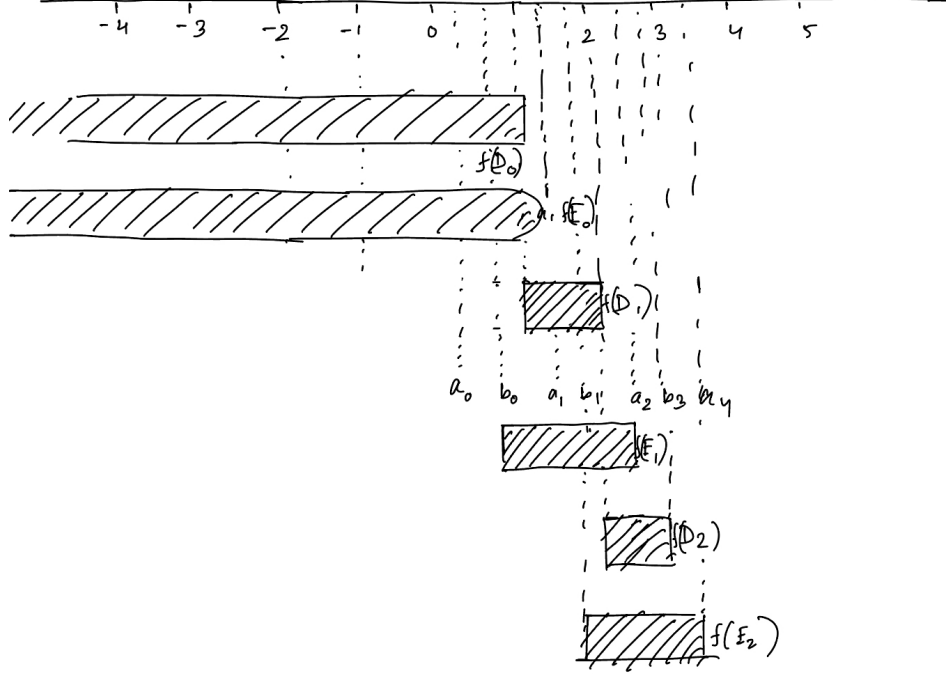


Figure 1: A visualization of  $D_i$  and  $E_i$  in Theorem 7

(b)  $F$  is proper as  $f$  is.

Furthermore  $F$  is injective since  $F(x) = F(y)$  implies  $\rho(x) = \rho(y)$  and using a similar argument as in theorem 1, we can show that  $x = y$ .  $F$  is an immersion too. Consider any  $p \in X$  and let  $j$  such that  $p \in D_j$ . Then,  $\rho_j = 1$  on a neighbourhood of  $p$  since  $p \in D_j$  and  $D_j$  has an open neighbourhood on which  $\rho_j$  is 1. Suppose  $j$  is odd. Then, for any  $q$  in this neighbourhood,  $F(q) = (\psi_j(q), \dots)$ . Then,  $dF_q$  is injective since  $\psi_j$  is an immersion.

Having found an immersion into Euclidean space, we can then use projection like we did with the compact case (using  $\pi_a$ ) to find an immersion into  $\mathbb{R}^{2n+1}$   $\square$