

Algebraic Topology

Notes

Jabayer Ibn Hamid

These notes are a combination of notes from Allen Hatcher's book "Algebraic Topology" and Prof. Ciprian Manolescu's lectures from MATH 215a at Stanford University, and my own.

## Introduction

Our goal is to develop algebraic invariants associated with topological spaces.

We will look at

(1) Fundamental Group:

$$\pi_1(X) = \{\text{loops in } X\} / \text{homotopy}$$

(2) Homology Group:

$$H_n(X), n \in \mathbb{N} \text{ and abelian}$$

Intuitively, they count "holes" in  $X$

(3) Cohomology Group:

$$H^n(X) = \text{Dual to } H_n(X)$$

$\oplus H^n(X)$  is a ring!

## Basic Constructions

### Def : Homeomorphism

Let  $X$  and  $Y$  be topological spaces.

$f: X \rightarrow Y$  is a homeomorphism if  $f$  is bijective and both  $f$  and  $f^{-1}$  are continuous.

We say  $\underline{X \cong Y}$ .

### Def : Homotopy

A family of maps,  $f_t: X \rightarrow Y$  where  $t \in I = [0, 1]$  s.t

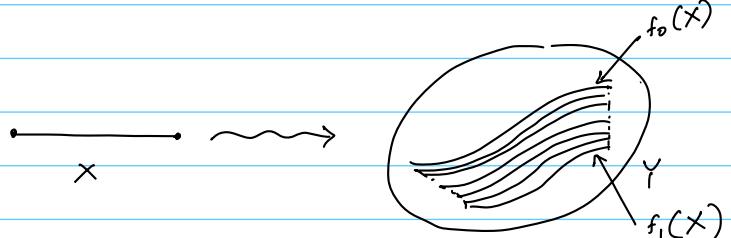
-the associated map  $F: X \times [0, 1] \rightarrow Y$  given by  $F(x, t) = f_t(x)$  is continuous.

Two maps  $f_0, f_1: X \rightarrow Y$  are homotopic if there exists a homotopy  $F: X \times [0, 1] \rightarrow Y$  s.t

$$f(x, 0) = f_0(x) \quad \forall x \in X$$

$$f(x, 1) = f_1(x)$$

we say  $\underline{f_0 \simeq f_1}$ .



### Def : Homotopy Equivalence

A map  $f: X \rightarrow Y$  is a homotopy equivalence if  $\exists g: Y \rightarrow X$  s.t  $f \circ g \simeq \text{id}_Y$  and  $g \circ f \simeq \text{id}_X$

We say the spaces  $X$  and  $Y$  are homotopy equivalent and  $\underline{X \simeq Y}$ .

→ can prove easily that this is an equivalence relation.

## Examples of homotopy equivalence

(1)  $\mathbb{R}^n \simeq$  a point (even though  $\mathbb{R}^n \not\simeq$  a point)  
infinite finite

Why?

$$f: \mathbb{R}^n \rightarrow \{0\}$$

and take  $g: \{0\} \rightarrow \mathbb{R}^n$  by  $g(0) = 0$

Then  $f \circ g = \text{id}_{\{0\}}$  and  $(g \circ f)(x) = 0 \quad \forall x \in \mathbb{R}^n$

Now  $g \circ f \sim \text{id}_{\mathbb{R}^n}$  by  $f_t(x) = tx$  where  $f_0 = 0$  and  $f_1 = \text{id}_{\mathbb{R}^n}$

(2)  $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \simeq$  a point

$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \simeq$  a point

Def : Contractible

We say the space  $X$  is contractible if  $X \simeq$  a point



Equivalent definition : the identity map of  $X$  is nullhomotopic

i.e.  $\text{id}_X \simeq$  constant map

homotopic to a  
constant map.

### Def : Retractions

Let  $X$  be a space and let  $A \subset X$ .

then, a retraction is a map  $r: X \rightarrow X$  s.t  
 $r(X) = A$  and  $r|_A = id_A$ .

### Def : Deformation Retraction

A deformation retraction of  $X$  onto a subspace  $A$  is

a family of maps  $f_t: X \rightarrow X$ , with  $t \in I$  s.t

$f_0 = id_X$  and  $f_1(X) = A$  and  $f_t|_A = id_A$  for  $\forall t \in I$ .

The family  $f_t$  must also be continuous

$\rightarrow$  an example of a homotopy from  $id_X$  to a retraction of  $X$  onto  $A \subset X$ .

$\rightarrow$  in this case,  $\boxed{A \cong X}$  as  $f_0: A \hookrightarrow X$  by  $id_X$

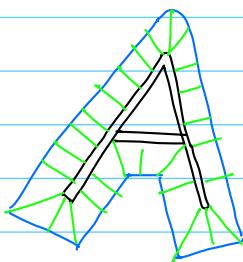
$f_1: X \rightarrow A$  as above

then  $f_0 \circ f_1 \simeq id_X$  (since  $f_0 \circ f_1 = f_1 \simeq f_0 = id_X$ )

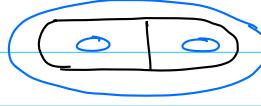
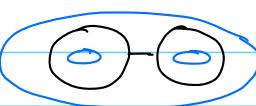
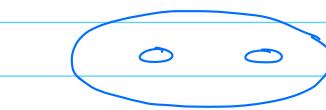
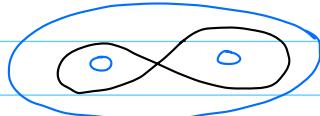
and  $f_1 \circ f_0 = id_A$

### Examples of deformation retraction:

(1)



(2) Look at deformations of



(3)  $X = \mathbb{R}^2 - \{0\}$ .  $A = S^1$

$$(i.e. f(x,t) = (1-t)x + t \frac{x}{\|x\|})$$



Proposition:

If  $X$  def. retracts to a point  $x \in X$ , then for any  $U \subset X$ ,  $x \in U$ .

$\exists V \subset U$  with  $x \in V$  s.t. the inclusion map  $V \hookrightarrow U$  is nullhomotopic.

homotopic to constant map

'Def: Deformation Retraction in the weak sense:

Let  $A \subset X$ .

Then, this is the homotopy  $f_t : X \rightarrow X$  s.t  $f_0 = \text{id}_X$   
and  $f_t(A) \subset A$  with  $f_t(A) \subset A$ ,  $\forall t \in I$ .

Lemma:

If  $X$  deformation retracts to  $A$  in the weak sense, then  
the inclusion  $A \hookrightarrow X$  is a homotopy equivalence.

Proof:

Let the weak def. ret be  $f_t$ .

Let  $i : A \hookrightarrow X$  by inclusion i.e.  $i(a) = a$ ,  $\forall a \in A$ .

Then,  $(i \circ f_t)(x) = i(f_t(x)) = f_t(x)$ .  $\forall x \in X$

But  $f_t \simeq f_0 = \text{id}_X$

So,  $i \circ f \simeq \text{id}_X$

Also,  $(f_1 \circ i)(a) = f_1(i(a)) = f_1(a)$   $\forall a \in A$

But  $f_1|_A \simeq f_0|_A = \text{id}_X|_A = \text{id}_A$

$\Rightarrow f_1 \circ i \simeq \text{id}_A$

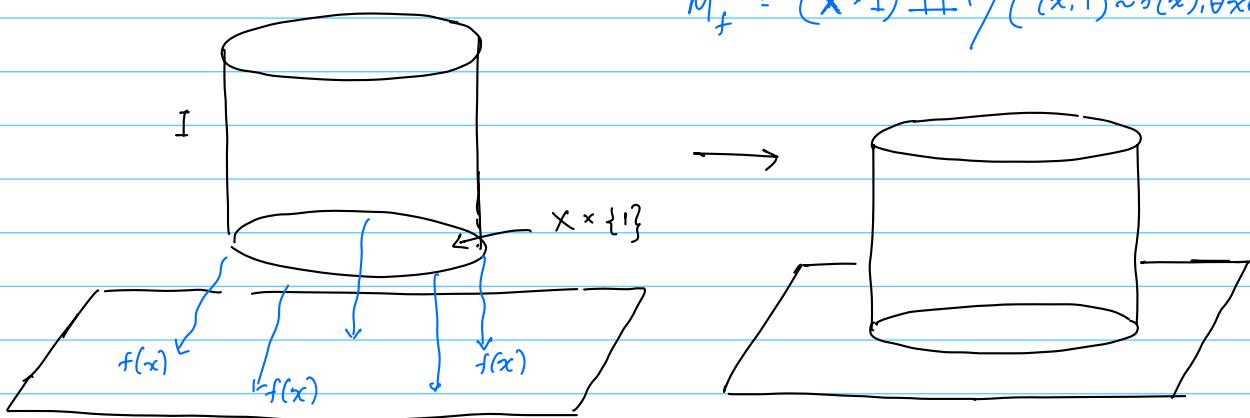
Def: Mapping Cylinder ("the structure through which the def. retraction occurs")

for a map  $f: X \rightarrow Y$ , the mapping cylinder  $M_f$  is the quotient space of the disjoint union  $(X \times I) \coprod Y$  obtained by the equivalence  $(x, 1) \in X \times I \sim f(x) \in Y$

↑      ↗  
Make the endpoint of the deformation  
equivalent to the image of the map.

Mapping cylinders are continuous.

$$M_f = (X \times I) \coprod Y / ((x, 1) \sim f(x), \forall x \in X)$$



Def: Homotopy relative to A (homotopy rel. A)

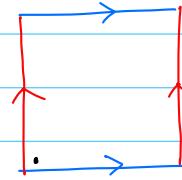
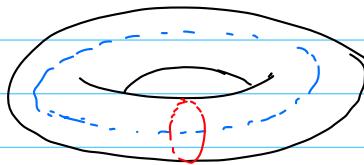
A homotopy  $f_t: X \rightarrow Y$  whose restriction to a subspace  $A \subset X$  is independent of  $t$ .

In other words,  $f_t$  is a homotopy and  $f_t|_A$  is independent of  $t$ .

→ def. retraction of  $X$  onto  $A$  is a homotopy rel. A from  $\text{id}_X$  to a retraction of  $X$  onto  $A \subset X$ .

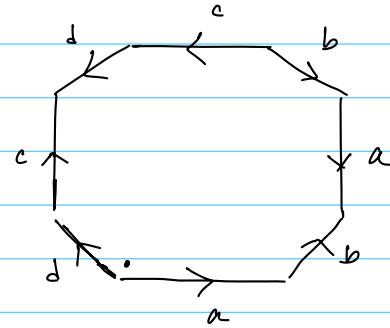
## Cell Complexes

Examples :



The torus  $S^1 \times S^1$  can be constructed from the square

Generally, an orientable surface  $M_g$  of genus  $g$  can be constructed from a polygon of  $4g$  sides by identifying pairs of edges.



2 cell: interior of a polygon which is an open disk

1 cell: an open interval like  $(0, 1)$

3 cell: an open ball.

$n$ -cell: open-disk

$$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$$

Def : Cell Complex (or CW complex)

A space constructed as follows:

- (1) Start with discrete set  $X^0 \rightarrow$  the points are D-cells
- (2) Inductively, form the n-skeleton  $X^n$  from  $X^{n-1}$  by attaching n-cells  $e_\alpha^n$  via maps

$\varphi_\alpha : S^{n-1} \rightarrow X^{n-1}$ .

So,  $X^n$  is the quotient space of  $X^{n-1} \coprod_\alpha D_\alpha^n$  under the equivalence  $x \sim \varphi_\alpha(x) \forall x \in \partial D_\alpha^n$

(n-1)-skeleton      n-disks

i.e attach boundaries of the n-disk to the (n-1)-skeleton

$$\therefore X^n = X^{n-1} \coprod_\alpha e_\alpha^n \text{ where } e_\alpha^n \text{ is an open n-disk}$$

(3) Either stop this induction at a finite stage and set  $X = X^n$  for  $n < \infty$

or continue indefinitely, setting

$$X = \bigcup_n X^n$$

in this case,  $X$  has the weak topology:

$A \subset X$  is open iff  $A \cap X^n$  is open in  $X^n$  for each  $n$

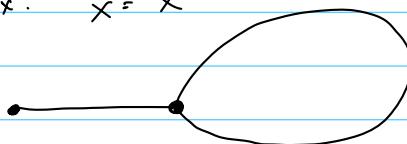
Vocabulary :

①  $X^n \rightarrow$  n-skeleton

② Dimension of  $X \rightarrow$  largest  $n$  s.t. an n-cell exists

### Examples of Cell Complexes:

(1) 1-dimensional cell complex:  $X = X^1$   
 (multigraphs)



(2) The sphere  $S^n$  has a cell complex with two cells,  $e^0$  and  $e^n$ , where  $e^n$  is attached by  $\varphi: S^{n-1} \rightarrow e^0$ .

$\therefore S^n$  is being regarded as the quotient space

$$D^n / \partial D^n$$

$$S^n = e^0 \cup e^n.$$

Alternatively,

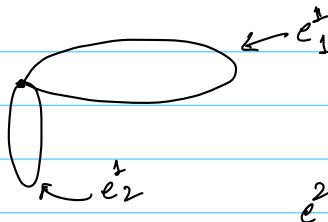
$$\begin{aligned} S^n &= S^{n-1} \cup e_+^n \cup e_-^n \\ &= e_+^0 \cup e_-^0 \cup \dots \cup e_+^n \cup e_-^n \end{aligned}$$

(3) Cell Complex of a torus:

Step 1:  $X^0$  is just a point  $\rightarrow \bullet \leftarrow e^0$

Step 2: Attach two 1-cells to this point

$$X^1 =$$



$$\therefore S^\infty = \bigcup_n S^n$$

Step 3: Attach a disk to  $X^1$  by attaching its boundary to  $X^1$ .

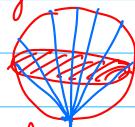
(4) Real Projective Space,  $\mathbb{R}P^n$

$$(\mathbb{R}^{n+1} - \{0\}) / (\nu \sim \lambda \nu, \forall \nu \in \mathbb{R}^{n+1}, \lambda \neq 0)$$

$\rightarrow$  Restricting to vectors of length 1,  $S^n / (\nu \sim -\nu)$

$\Rightarrow D^n$  with antipodal points of  $\partial D^n$  identified

To get this, think of



$\partial D^n$  with antipodal points equivalent is  $\mathbb{R}P^{n-1}$

$\therefore \mathbb{R}P^n$  can be formed from  $\mathbb{R}P^{n-1}$  by attaching an  $n$ -cell. and the attaching map  $\varphi: S^{n-1} \rightarrow \mathbb{R}P^{n-1}$

$\therefore \mathbb{R}P^n$  has the cell complex structure  $e^0 \cup e^1 \cup \dots \cup e^n$  with one cell  $e^i$  in each dimension  $i \leq n$ .

i.e. for the upper hemisphere's points, find where the line to south pole intersects with  $D^n$

## (5) Complex Projective Space. $\mathbb{C}P^n$

Space of all complex lines through the origin in  $\mathbb{C}^{n+1}$

$$\text{i.e. } \mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\}) / (v \sim \lambda v, \forall v \in \mathbb{C}^{n+1}, \lambda \neq 0)$$

Equivalent to  $S^{2n+1} / (v \sim \lambda v, |\lambda|=1)$  ( $S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1}$ )

Equivalent to  $D^{2n} / (v \sim \lambda v, v \in \partial D^{2n})$

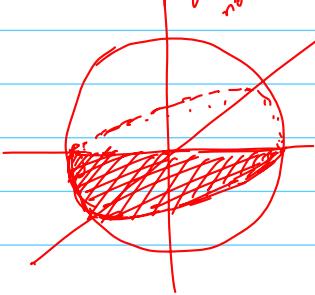
↳ Why?

$S^{2n+1} \subset \mathbb{C}^{n+1}$  → consider vectors in  $\mathbb{C}^{n+1}$  whose last coordinate is ~~one~~ real.

and non-negative

These vectors are of the form  $(w, \sqrt{1-w^2}) \in \mathbb{C}^n \times \mathbb{C}$  with  $|w| \leq 1$

They form the graph of the function  $w \mapsto \sqrt{1-w^2}$   
with  $|w| \leq 1, w \in \mathbb{C}^n$



Note:  $w \in \mathbb{C}^n$  and  $|w| \leq 1 \Rightarrow w \in D^{2n}$

This is a disk  $D^{2n}_+$  bounded by the spheres  $S^{2n-1}$ .

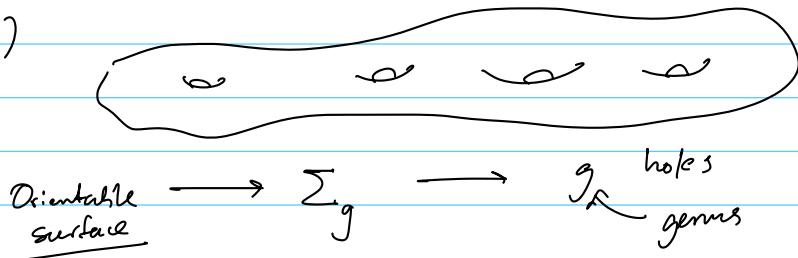
By adding another dimension and viewing them as  $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$ , we ~~view~~ them as vectors in  $(D^{2n}_+, 0)$  bounded by  $S^{2n-1} \subset S^{2n+1}$

Now, each vector in  $S^{2n+1}$  is equivalent to a vector in  $D^{2n}_+$  by identifying  $v \sim \lambda v$ . In particular, if the last coordinate is zero, we have  $v \sim \lambda v, \forall v \in S^{2n-1}$ .

$\therefore \mathbb{C}P^n$  is obtained from  $\mathbb{C}P^{n-1}$  by attaching a cell  $e^{2n}$  using the attaching map  $\varphi : S^{2n-1} \rightarrow \mathbb{C}P^{n-1}$

$\therefore \mathbb{C}P^n = e^0 \cup e^2 \cup \dots \cup e^{2n}$  with cells only in even dimensions

(6)

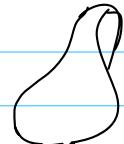


$\Sigma_g$   $\xrightarrow{\text{holes}}$   
g genus

Can be constructed from a  $4g$  polygon

Start with one  $e^o$

(7)

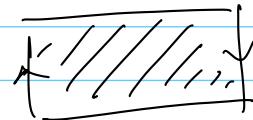
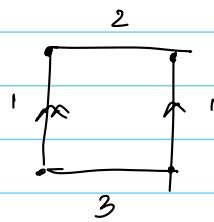


Non-orientable  
surface  $\longrightarrow N_g$

E.g.:  $N_2 \longrightarrow$  Klein bottle  
 $N_1 \longrightarrow RP^2$

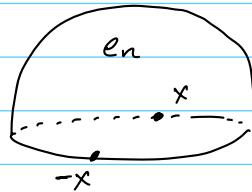
(8) Annulus:

(9) Möbius band



(a)  $RP^n$  revisited

$$RP^n = S^n / (x \sim -x, \forall x)$$



$$\Rightarrow RP^n = RP^{n-1} \cup e^n$$

$$\therefore RP^n = e^0 \cup e^1 \cup \dots \cup e^n$$

$$\text{Then, } RP^\infty = e^0 \cup e^1 \cup e^2 \cup \dots = \bigcup_n RP^n$$

(i)  $\mathbb{C}\mathbb{P}^n$  revisited

$$\mathbb{C}\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / (x \sim \lambda x, \lambda \in \mathbb{C}^*)$$

$$\therefore z \sim \frac{z}{\|z\|} \Rightarrow \mathbb{C}\mathbb{P}^n \cong S^{2n+1} / (z \sim \lambda z, \lambda \in S^1)$$

Divide everything by  $x_1$ , i.e. last coordinate in  $\mathbb{R}_{\geq 0}$

$$z = \underbrace{(z_0, \dots, z_n)}_{w} \underbrace{z_{n+1}}_{\sqrt{1-\|w\|^2}}$$

with  $\|w\| \leq 1$

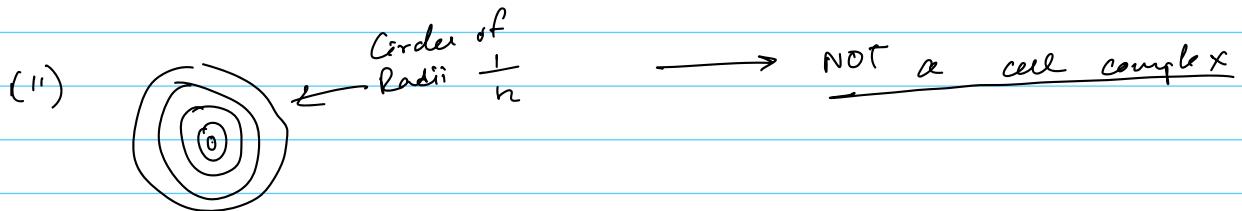
$$D_+^{2n} = \text{graph } (w \mapsto \sqrt{1-\|w\|^2})$$

$$\therefore \mathbb{C}\mathbb{P}^n = D_+^{2n} / (w \sim \lambda w \text{ if } w \in S^{2n-1})$$

$$= \mathbb{C}^{2n} \cup \left( S^{2n-1} / (w \sim \lambda w) \right)$$

$$= \mathbb{C}\mathbb{P}^{n-1} \cup \mathbb{C}^{2n}$$

$$= e^0 \cup e^2 \cup \dots \cup e^{2n}$$



## Properties of CW Complexes

- (1) They are normal ( $\therefore$  also Hausdorff)
- (2) Any finite cell complex is compact
- (3) A compact subspace of a cell cx is contained in a finite subcomplex
- (4) Closure finiteness  $\rightarrow$  The closure of each cell  $\ell$  meets only finitely many cells.
- (5) Locally contractible:  
 $\forall x \in X, \exists x \text{ open}, \exists V \subset U \text{ with } x \in V$   
s.t.  $V$  is contractible

(6)

Recall:

Top manifolds  $\rightarrow$  2<sup>nd</sup> Countable, Hausdorff, locally Euclidean  
Smooth manifolds  $\rightarrow$

Theorem: Every smooth manifold is homeomorphic to a cell complex.

Theorem: Every topological manifold is homotopy equivalent to a cell complex.

Theorem: Every top manifold of dimension  $\neq 4$  is homeomorphic to a cell complex  
(unknown in dim 4)

### Def: Characteristic Map

Each cell  $e_\alpha^n$  in a cell complex  $X$  has a characteristic map

$$\varphi_\alpha : D_\alpha^n \xrightarrow{\sim} X$$

which extends the attaching map  $\varphi_\alpha$  and is a homeomorphism from the interior of  $D_\alpha^n$  onto  $e_\alpha^n$

→  $\varphi_\alpha$  is the composition

$$D_\alpha^n \hookrightarrow X^{n-1} \coprod_\alpha D_\alpha^n \xrightarrow{\quad} X^n \hookrightarrow X$$

↑  
the quotient  
map that  
defines  $X^n$

### Example of characteristic map:

(1) Recall:  $S^n$  can be constructed by two cells:  $e^0$  and  $e^n$  ← just one point

where  $e^n$  is attached to  $e^0$  by

$$\varphi_\alpha : S^{n-1} \rightarrow e^0$$

Then, the characteristic map of  $e^n$  is

$$\varphi_\alpha : D_\alpha^n \rightarrow S^n \text{ which collapses } \partial D_\alpha^n \text{ to } e^0$$

Def: Subcomplex

A subcomplex of a cell complex  $X$  is a closed subspace  $A \subset X$  that is a union of cells of  $X$ .

$\rightarrow$  As  $A$  is closed, for each cell in  $A$ ,

the image of its characteristic map } contained in  $A$   
the image of its attaching map }

$\therefore A$  is a cell complex as well

Def : CW pair

A cell complex  $X$  and a subcomplex  $A$  forms a pair  $(X, A)$

Example of subcomplex

$\rightarrow$  Each skeleton,  $X^n$ , is a subcomplex.

$\rightarrow$  in  $\mathbb{R}\mathbb{P}^n$  and  $\mathbb{C}\mathbb{P}^n$ , the only subcomplexes are  $\mathbb{R}\mathbb{P}^k$  and  $\mathbb{C}\mathbb{P}^k$ ,  $\forall k \leq n$

Properties of subcomplexes

(1) Closure of a collection of cells is a subcomplex.

(2) Any union and intersection of subcomplexes is a subcomplex.

## Operations on Spaces

### Products

$X, Y \rightarrow \text{cell complexes}$

$X \times Y \rightarrow \text{cell complex with the cells } e_\alpha^m \times e_\beta^n$

cells of X      cells of Y

### Quotients

Given  $(X, A)$  a CW pair,  
the quotient space  $X/A$  also has a cell complex structure:

→ the cells of  $X/A$  are the cells of  $X-A$  and  
a new 0-cell which is the image of  
 $A$  in  $X/A$ .

→ for a cell  $e_\alpha^n$  of  $X-A$  attached by  
 $\varphi_\alpha: S^{n-1} \rightarrow X^{n-1}$ , the attaching map for the  
corresponding cell in  $X/A$  is the composition

$$S^{n-1} \rightarrow X^{n-1} \rightarrow X^{n-1}/A^{n-1}$$

Eg: ①  $D^n/S^{n-1} = S^n$

### Wedge Sum (for based spaces)

Given spaces  $X$  and  $Y$  with chosen points  $x_0 \in X$  and  $y_0 \in Y$ ,  
the wedge sum  $X \vee Y$  is the quotient of  $X \coprod Y$  by  
identifying  $x_0$  and  $y_0$  to a single point

→ Example:  $S^1 \vee S^1 = \infty$

$$X \vee Y = X \coprod Y / (x_0 \sim y_0)$$

→  $\bigvee_\alpha X_\alpha$  for an arbitrary collection of spaces  $X_\alpha$ :  
start with  $\coprod_\alpha X_\alpha$  and then identify  $x_\alpha \in X_\alpha$   
to one point.

→ If  $X_\alpha$  are cell complexes and the points  $x_\alpha$   
are 0-cells, then  $\bigvee_\alpha X_\alpha$  is a cell complex  
because we obtain it from the cell complex  $\coprod_\alpha X_\alpha$  and attach by

collapsing a subcomplex to a point.

→ For a cell complex  $X$ , the quotient  $X^n/X^{n-1}$  is a wedge sum of  $n$ -spheres  $\bigvee_{\alpha} S_{\alpha}^n$  with one sphere for each  $n$ -cell of  $X$

7) Smash Product  $X \wedge Y = (X \times Y) / ((x_0 \times Y) \cup (X \times y_0))$

Inside the product space  $X \times Y$ , there are copies of  $X$  and  $Y$ :  $X \times \{y_0\}$  and  $\{x_0\} \times Y$  for points  $y_0 \in Y$  and  $x_0 \in X$ .

These copies of  $X$  and  $Y$  intersect only at  $(x_0, y_0)$  so their union can be identified with the wedge sum  $X \vee Y$

$$\begin{aligned} \text{i.e. } (X \times \{y_0\}) \vee (\{x_0\} \times Y) &= X \vee Y \\ &= (X \amalg Y) / (x_0 \sim y_0) \end{aligned}$$

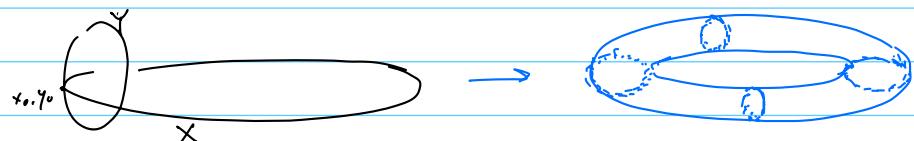
The smash product  $X \wedge Y$  is the quotient  $X \times Y / X \vee Y$

↳ i.e. we are collapsing away the separate factors  $X$  and  $Y$ .

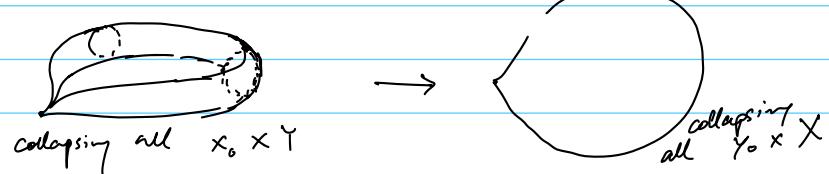
Eg:  $S^1 \wedge S^1 = S^2$   $\longrightarrow$   $S^1 = I / (0 \sim 1)$   
 $S^m \wedge S^n = S^{m+n}$

Why?

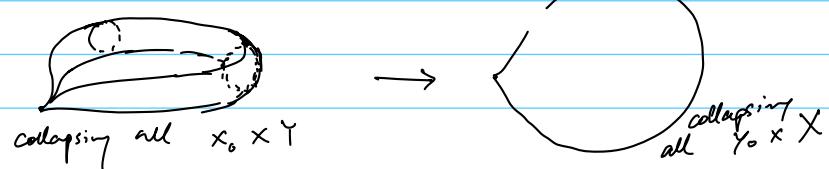
Firstly,  $S^1 \times S^1$  results in a torus  $T^2$



Secondly,  $S^1 \vee S^1 = \infty$



Then, quotienting:



II

### Suspension

for a space  $X$ , the suspension  $SX$  is the quotient of  $X \times I$  by collapsing  $X \times \{0\}$  to a point and  $X \times \{1\}$  to another.

#### Example

(i)  $X = S^n$

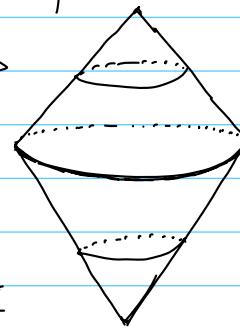
$SX = S^{n+1}$  with the two suspension points at North and South of  $S^{n+1}$

→ We can suspend maps too

$$f: X \rightarrow Y \rightsquigarrow Sf: SX \rightarrow SY$$

which is the quotient map of

$$f \times 1 : X \times I \rightarrow Y \times I$$



III

### Cone

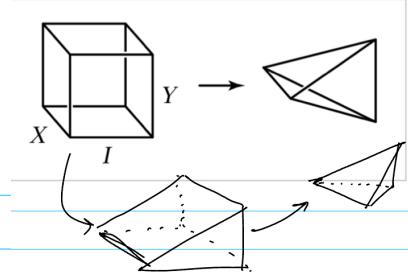
$$CX = (X \times I) / (X \times \{0\})$$



→ If  $X$  is a CW complex, then so are  $SX$  and  $CX$  as quotients of  $X \times I$  with its product cell structure with  $I$  given the standard cell structure of ~~two~~ two 0-cells joined by one 1-cell.

7

Join



Given  $X$  and  $Y$ , we can define the space of all line segments joining points in  $X$  to points in  $Y$ .

$$X * Y = (X \times Y \times I) / \left( \begin{array}{l} (x, y_1, 0) \sim (x, y_2, 0) \quad \forall x, x_1, x_2 \in X \\ (x_1, y_1, 1) \sim (x_2, y_1, 1) \quad \forall y, y_1, y_2 \in Y \end{array} \right)$$

$$\rightarrow pt * pt \longrightarrow \bullet \longrightarrow$$

$$pt * pt * pt \longrightarrow \triangle$$

$$pt * pt * \dots * pt = \Delta^n \rightarrow n\text{-simplex}$$

$\underbrace{\qquad\qquad\qquad}_{n+1 \text{ points}}$

④

Reduced Suspension:

$X \rightarrow \text{CW complex}$   
 $\{x_0\} \rightarrow \text{base point}$

$$SX = (X \times I) / (X \times \{0\}) \cup (X \times \{1\})$$

$$\Sigma X = SX / (\{x_0\} \times I)$$

⑤

## Criterion for Homotopy Equivalence

Recall:

Def: Homotopy Equivalence

A map  $f: X \rightarrow Y$  is a homotopy equivalence if  $\exists g: Y \rightarrow X$   
s.t.  $f \circ g \simeq id_Y$  and  $g \circ f \simeq id_X$

We say the spaces  $X$  and  $Y$  are homotopy equivalent  
and

$$X \simeq Y$$

→ can prove easily that this is an equivalence relation.



## Collapsing Subspaces

Theorem:

If  $(X, A)$  is a CW pair consisting of a CW complex  $X$  and a contractible subcomplex  $A$ , then the quotient map  $X \rightarrow X/A$  is a homotopy equivalence.

Example

(1) Graphs



→ they are homotopy equivalent

→ collapsing the middle edge of A and C produces B

(b) Let  $X$  be a graph with finitely many vertices and edges.

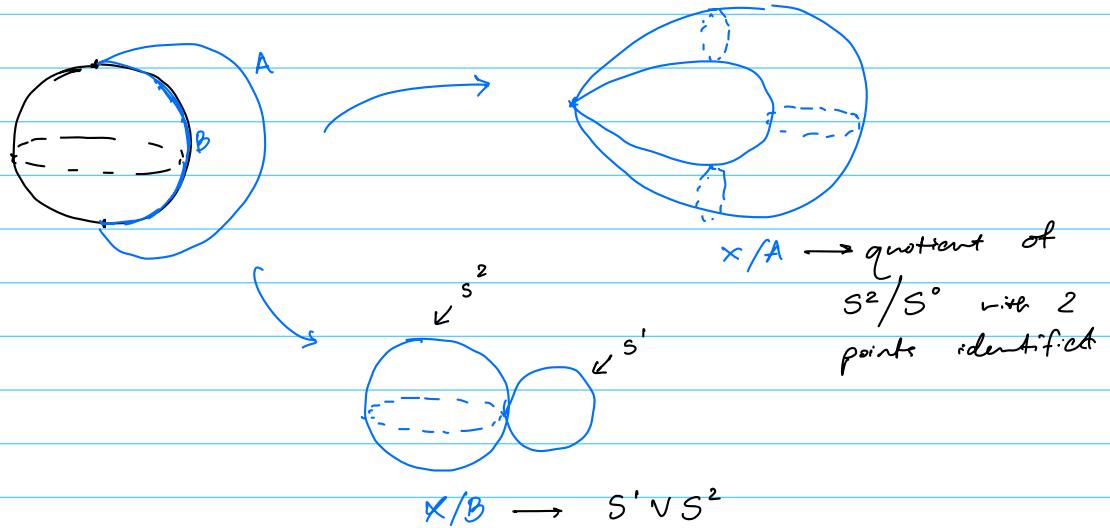
→ if the two endpoints of any edge are distinct, we can collapse it to a pt.



Leads to a homotopy equivalent graph with one less edge.

Can repeat until all edges are loops.

(2)  $X \rightarrow S^2$  but attach 2 ends of an arc  $A$  to  $N$  and  $S$  pole



### 7.1 Reduced Suspension

$$\Sigma X \cong SX$$



#### Attaching spaces

Start with space  $X_0$  and another space  $X_1$ , which we will attach to  $X_0$  by identifying points in a subspace  $A \subset X_1$ , with points of  $X_0$ .

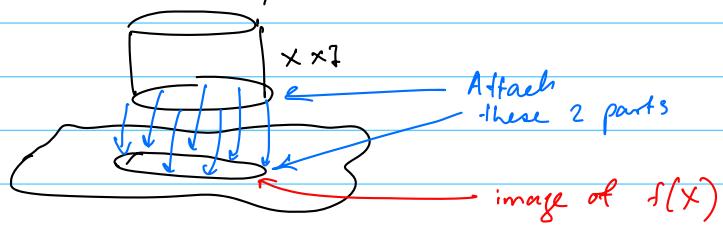
We do this using a map  $f: A \hookrightarrow X_0$  and then forming a quotient space of  $X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A)$

We denote

$$X_0 \sqcup_f X_1 = X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A) \text{ where } f: A \hookrightarrow X_0, A \subset X_1$$

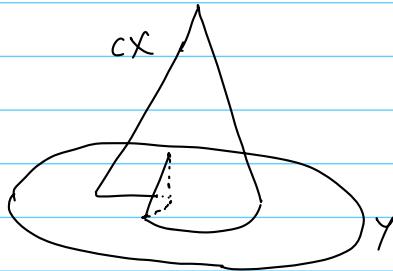
Example :

- (1) Mapping cylinder of a map  $f: X \rightarrow Y$  is  $M_f \rightarrow$  the space obtained from  $Y$  by attaching  $X \times I$  along  $X \times \{1\}$  via  $f$ .



- (2) Mapping Cone  $\rightarrow C_f = Y \sqcup_f CX$  where  $CX$  is the cone  $(X \times I) / (X \times \{0\})$

and we attach this to  $Y$  along  $X \times \{1\}$   
via  $(x, 1) \sim f(x)$



Example :  $X = S^{n-1}$

$C_f \rightarrow$  attaching to  $Y$  the  $n$ -cell  
via  $f: S^{n-1} \rightarrow Y$

Proposition

If  $(X_1, A)$  is a CW pair and the two attaching maps  
 $f, g: A \rightarrow X_0$  are homotopic, then  
 $X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$

## Homotopy Extension Property

Intuition:

Consider the map  $f_0 : X \rightarrow Y$ . Let  $A \subset X$  and consider the homotopy on  $A$   $f_t : A \rightarrow Y$  with  $f_0 = f|_A$ . We would like to extend this to a homotopy on  $X$  as a whole with  $f_t$ .

Def: Homotopy Extension

$A \subset X$

$(X, A)$  has the homotopy extension property (h.e.p)

if  $\forall Y, \forall f_0 : X \rightarrow Y, \forall$  homotopy  $g : A \times I \rightarrow Y,$   
 $g(a, 0) = f_0(a)$

we can extend  $g$  to a homotopy  $F : X \times I \rightarrow Y$

$$\text{i.e. } f_t(x, 0) = f_0(x)$$



$$f_0(x) = y$$

$(X, A)$  has the h.e.p if every pair of maps  $X \times \{0\} \rightarrow Y$  and

$A \times I \rightarrow Y$  that agree on  $A \times \{0\}$  can be extended to

$$f_t(a)$$

a map  $X \times I \rightarrow Y$

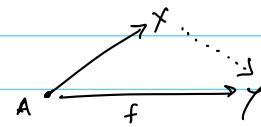
$$\hookrightarrow f_t : X \rightarrow Y$$

$$f_0 = f|_A$$

Lemma:

$A \subset X$  top space.

$\forall Y, \text{ any map } f : A \rightarrow Y \text{ extends to } X \rightarrow Y \text{ if and only if } A$  is a retract of  $X$



Proof:

$\Leftarrow$  Suppose  $A$  is a retract of  $X$  via  $r : X \rightarrow A$  s.t.  $r|_A = \text{id}_A$   
 Then  $(f \circ r) : X \rightarrow Y$  is our extension

$\Rightarrow$  Suppose,  $\forall Y$  and any map  $f : A \rightarrow Y$  extends to  $X \rightarrow Y$ .  
 i.e.  $f_t : X \rightarrow Y$  s.t.  $f|_A = f$

Then, let  $Y = A$  and  $f = \text{id}_A$  i.e.  $\text{id}_A : A \rightarrow A$  extends to  $f_t : X \rightarrow A$  s.t.  $f|_A = \text{id}_A \Rightarrow A$  is a retract of  $X$

Lemma :

A pair  $(X, A)$  has the h.e.p if and only if  $X \times \{0\} \cup A \times I$  is a retract of  $X \times I$ .

Proof : By hypothesis the identity map

$$\Rightarrow : X \times \{0\} \cup A \times I \hookrightarrow X \times \{0\} \cup A \times I \text{ extends to a map}$$

$$X \times I \hookrightarrow X \times \{0\} \cup A \times I$$

$\therefore X \times \{0\} \cup A \times I$  is a retract of  $X \times I$ .

$\Leftarrow$  if  $A$  is closed: consider any two maps  $X \times \{0\} \hookrightarrow Y$  and  $A \times I \hookrightarrow Y$  that agree on  $A \times \{0\}$ . They combine to give a map  $X \times \{0\} \cup A \times I \hookrightarrow Y$  which is continuous by continuity on the closed sets  $X \times \{0\}$  and  $A \times I$ .

Compose this map  $X \times \{0\} \cup A \times I \hookrightarrow Y$  with a retraction  $X \times I \hookrightarrow X \times \{0\} \cup A \times I$  (we have this via hypothesis)

We get an extension  $X \times I \hookrightarrow Y$

$\therefore (X, A)$  has the h.e.p.

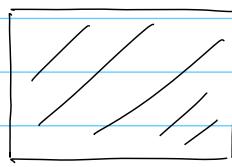
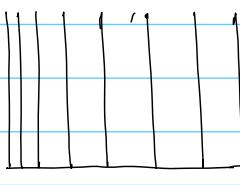
Properties

$$(1) \begin{array}{l} \text{H.e.p} \\ X - \text{normal iff} \end{array} \} \Rightarrow A \text{ is closed in } X$$

Non-example:  $(X, A)$  does not have h.e.p

(1)  $(I, A)$  where  $A = \{0, 1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\}$

There is no continuous retraction  $I \times I \hookrightarrow I \times \{0\} \cup A \times I$  because of the structure of  $(I, A)$  near 0.



Consider the ball  $B = B(x_0, r)$   
Then  $\exists \delta > 0$  s.t.  $B(x_0, \delta) \subset B$

$\gamma$  — path in  $B$  from  $x_0$  to  $x_1$

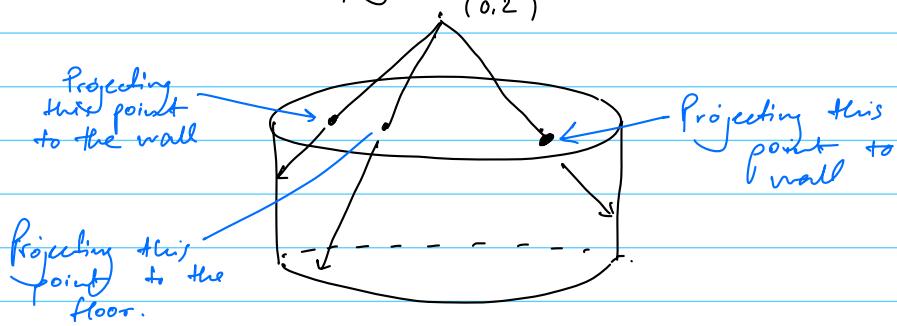
but  $x_0$  and  $x_1$  are in diff components  
 $\downarrow$  path at  $t=1$   $B(x_0, \delta)$  of  $C \cap B$

Proposition

If  $(X, A)$  is a CW pair, then  $X \times \{0\} \cup A \times I$  is a deformation retract of  $X \times I$ , hence  $(X, A)$  has the h.e.p.

Proof :

First, note that  $\exists$  a retraction  $r: D^n \times I \rightarrow D^n \times \{0\} \cup \partial D^n \times I$  for ex - radial projection from the point  $(0, 2) \in D^n \times \mathbb{R}$



Now, set  $r_t = tr + (1-t)\mathbb{1}$  is a deformation retraction of  $D^n \times I$  onto  $D^n \times \{0\} \cup \partial D^n \times I$ .

Now, with this, we have a deformation retraction of  $X^n \times I$  onto  $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$  since  $X^n \times I$  is obtained from  $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$  by attaching copies of  $D^n \times I$  along  $D^n \times \{0\} \cup \partial D^n \times I$ .

If we perform the def. ret. of  $X^n \times I$  onto  $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$  during the  $t$ -interval  $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$ , this infinite concatenation of homotopies is a def. ret. of  $X \times I$  onto  $X \times \{0\} \cup A \times I$ .

### Proposition

If the pair  $(X, A)$  satisfies h.e.p and  $A$  is contractible, then the quotient map  $q: X \rightarrow X/A$  is a homotopy equivalence.

### Proof :

Let  $f_t: X \rightarrow X$  be the homotopy extending a contraction of  $A$  with  $f_0 = \text{id}$ .

Now,  $f_t(A) \subset A \quad \forall t$ , so the composition

$$q \circ f_t: X \rightarrow X/A$$

sends  $A$  to a point and so factors as a composition

$$X \xrightarrow{q} X/A \longrightarrow X/A$$



Denote this by  $\bar{f}_t: X/A \rightarrow X/A$ )

$$\text{So, } q \bar{f}_t = \bar{f}_t q$$

$$X \xrightarrow{\bar{f}_t} X$$

$$\begin{array}{ccc} & & \\ q & \downarrow & \downarrow q \\ X/A & \xrightarrow{\bar{f}_t} & X/A \end{array}$$

When  $t=1$ ,  $f_1(A)$  equals to a point (since  $f_t$  is homotopy extension of the contraction of  $A$ ), so  $f_1$  induces a map  $g: X/A \rightarrow X$  with  $gq = f_1$

$$\begin{array}{ccc} & f_1 & \\ X & \xrightarrow{\quad} & X \\ q \downarrow & \nearrow g & \downarrow q \\ X/A & \xrightarrow{\bar{f}_1} & X/A \end{array}$$

$$\begin{aligned} \text{So, } qg &= \bar{f}_1 & \text{since } qg(\bar{x}) &= qg(x) \\ &&&= qf_1(x) \\ &&&= f_1(q(x)) \\ &&&= f_1(\bar{x}) \end{aligned}$$

The maps  $g$  and  $q$  are inverse homotopy equivalences as

$$gq = f_1 \simeq f_0 = 1 \text{ via } f_t \text{ and}$$

$$qg = f_1 \simeq \overline{f_0} = 1 \text{ via } \overline{f_t}.$$

Def:  $W \simeq Z \text{ rel } Y$

for  $(W, Y)$  and  $(Z, Y)$ , there are maps  $\varphi: W \rightarrow Z$  and  $\psi: Z \rightarrow W$  restricting to identity on  $Y$  s.t.  $\psi\varphi \simeq 1_W$  and  $\varphi\psi \simeq 1_Z$  via homotopies that restrict to the identity on  $Y$  at all times.

Proposition

If  $(X_1, A)$  is a CW pair and we have attaching maps  $f, g: A \hookrightarrow X_0$  that are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

Proof:

Let  $F: A \times I \rightarrow X_0$  is a homotopy from  $f$  to  $g$ , consider the space  $X_0 \sqcup_F (X_1 \times I)$ , which has both  $X_0 \sqcup_f X_1$  and  $X_0 \sqcup_g X_1$  as subspaces

We can deformation retract  $X_1 \times I$  onto  $X_1 \times \{0\} \cup A \times I$  which induces a def retraction of  $X_0 \sqcup_F (X_1 \times I)$  onto  $X_0 \sqcup_f X_1$

Similarly,  $X_0 \sqcup_f (X_1 \times I)$  def retracts onto  $X_0 \sqcup_g X_1$

Both of them are identity on  $X_0$  so we get the homotopy equivalence

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

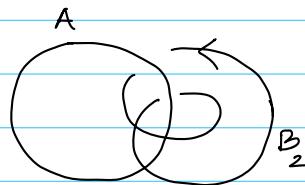
## Fundamental Group

### Intuition

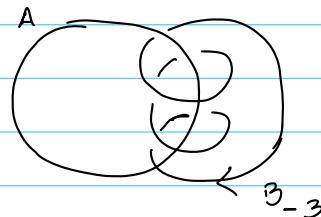
Two linked circles in  $\mathbb{R}^3$  :



Link B with A two times  
in the forward direction :



Link B with A three times  
in the backward direction :

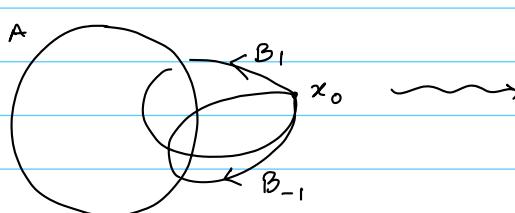
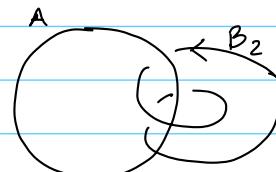
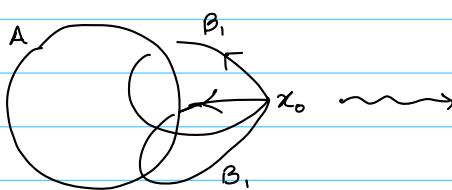


$B_2$  and  $B_{-3}$  are oriented circles/loops.

Two loops,  $B$  and  $B'$ , starting and ending at the same point  $x_0$  can be added to form a new loop that travels around  $B$  and  $B'$ .

$$\text{So, } B_1 + B_1 = B_2$$

$$B_1 + B_{-1} = B_0 \leftarrow \text{unlinked from A}$$



$$\text{More generally, } B_m + B_n = B_{m+n}$$

## Paths and Homotopy of paths

Def: Path in  $X$

A continuous map  $f: I \rightarrow X$  where  $I = [0, 1]$

Def: Homotopy

A family  $f_t: I \rightarrow X$  where  $t \in I$  s.t

(1)  $f_t(0) = x_0$  and  $f_t(1) = x_1 \forall t$

(2) The associated map  $F: I \times I \rightarrow X$   
is continuous

We say  $f_b \simeq f_1$ .

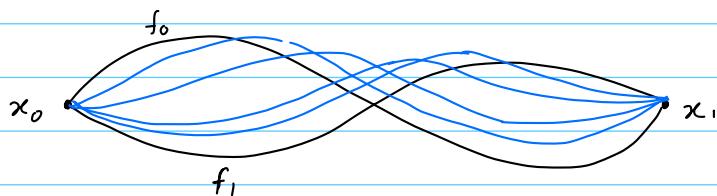
$\rightarrow f_0 \simeq f_1$  means homotopic rel.  $\partial I = \{0, 1\}$  as the endpoints are fixed.

Examples

(1) Linear homotopies in  $\mathbb{R}^n$ :

Any 2 paths  $f_0$  and  $f_1$  in  $\mathbb{R}^n$  with endpoints  $x_0$  and  $x_1$ ,  
are homotopic by  $f_t(x) = (1-t)f_0(x) + tf_1(x)$

Here,  $F(x, t) = f_t(x) = (1-t)f_0(x) + tf_1(x)$  is continuous  
since  $f_0$  and  $f_1$  are continuous, and sum & and scalar  
multiplication preserve continuity.



Non-example

$$f_0, f_1 : I \rightarrow S^1$$

$$\left. \begin{array}{l} f_0(t) = 1 \\ f_1(t) = e^{2\pi i t} \end{array} \right\} \text{They are not path homotopic}$$

### Proposition

The relation of homotopy on paths with fixed endpoints

in any space is an equivalence relation.

We denote the equivalence class of  $f$  by  $[f]$  and is called the homotopy class of  $f$ .

### Proof:

Reflexivity:  $f \simeq f$  by homotopy  $f_t = f$

Symmetry: If  $f_0 \simeq f_1$  via  $f_t$ , then  $f_1 \simeq f_0$  via  $f_{1-t}$ .

Transitivity: Suppose  $f_0 \simeq f_1$  via  $f_t$ . and if  $f_1 = g_0$  with  $g_0 \simeq g_1$  via  $g_t$ , then the homotopy

$$h_t = \begin{cases} f_{2t}, & t \in [0, \frac{1}{2}] \\ g_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}$$

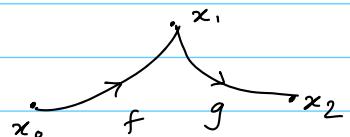
The associated function  $H(s, t) = h_t(s)$  is continuous.

A function on the union of 2 closed sets is continuous if it is continuous restricted to each of the 2 sets separately.

### Def: Product path

Given two paths  $f, g: I \rightarrow X$  s.t  $f(1) = g(0)$ , the product path  $f \cdot g$  first traverses  $f$  and then  $g$ :

$$f \cdot g(t) = \begin{cases} f(2t), & 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$



This product path preserves homotopy classes:

if  $f_0 \simeq f_1$  and  $g_0 \simeq g_1$  via  $f_t$  and  $g_t$  homotopies respectively

and if  $f_0(1) = g_0(0)$  so that  $f \cdot g_0$  is well-defined

then  $f_t \cdot g_t$  provides the homotopy

$$f \cdot g_0 \simeq f \cdot g_1$$

Def : Loop

Paths  $f: I \rightarrow X$  s.t  $f(0) = f(1) = x_0 \in X$

$x_0 \rightarrow$  basepoint

→ The set of all homotopy classes  $[f]$  of loops  $f: I \rightarrow X$  at the basepoint  $x_0 \in X$  is denoted  $\pi_1(X, x_0)$

Proposition :

$\pi_1(X, x_0)$  is a group w.r.t the product  
 $[f][g] = [f \cdot g]$

This group is called the fundamental group of  $X$  at basepoint  $x_0$ .

Proof :

Since the basepoint  $x_0 \in X$  is fixed, the product of any two paths,  $f$  and  $g$  in  $\pi_1(X, x_0)$  is defined.

Firstly, define reparametrisation of a path  $f$  to be a composition  $f\varphi$  where  $\varphi: I \rightarrow X$  is a continuous map r.t  $\varphi(0) = 0$  and  $\varphi(1) = 1$ .

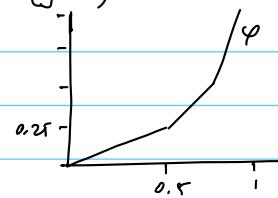
Reparametrisation preserves homotopy class of  $f$  since  $f\varphi \simeq f$  via homotopy  $f\varphi_t$  where  $\varphi_t(x) = (1-t)\varphi(x) + tx$  so  $\varphi_0(x) = \varphi(x)$  and  $\varphi_1(x) = x$

We often show that  $f$  is a reparametrisation of  $g$  to prove  $f \simeq g$ .

Given the paths  $f, g$  and  $h$  with  $f(1) = g(0)$  and  $g(1) = h(0)$ , then both  $(f \cdot g) \cdot h$  and  $f \cdot (g \cdot h)$  are defined.

Note  $(f \cdot g) \cdot h$  is a reparametrisation of  $f \cdot (g \cdot h)$  via  $f \cdot (g \cdot h) = (f \cdot g) \cdot h \varphi$  where  $\varphi$  is a continuous map s.t  $\varphi: [0, \frac{1}{2}] \rightarrow [0, \frac{1}{4}]$   $\varphi: [\frac{1}{2}, 1] \rightarrow [\frac{1}{4}, 1]$

$$\text{So, } (f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$$



Given a path  $f: I \hookrightarrow X$ , let  $c$  be the constant path at  $f(1)$  defined by  $c(s) = f(1)$ ,  $\forall s \in I$ . Then,  $f \cdot c$  is a reparametrisation of  $f$ :

$$f \cdot c(x) = \begin{cases} f(2x) & \text{for } x \in [0, \frac{1}{2}] \\ c(2x-1) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$\text{So, } f \cdot c = f \varphi \text{ where } \varphi: [0, \frac{1}{2}] \longrightarrow [0, 1]$$

$$\varphi: [\frac{1}{2}, 1] \longrightarrow \text{---}$$

$$\therefore f \cdot c \simeq f$$

Similarly  $c \cdot f \simeq f$  where  $c$  is constant path at  $f(0)$ .

Taking  $f$  to be a loop, the homotopy class of the constant path is a two-sided identity.

Now, let  $f$  be a path from  $x_0$  to  $x_1$ . Its inverse path is  $\bar{f}$  from  $x_1$  to  $x_0$  defined by  $\bar{f}(s) = f(1-s)$

Then,  $f \cdot \bar{f}$  is homotopic to a constant path via homotopy

$$h_t = f_t \cdot g_t$$

where  $f_t = f$  on  $[0, 1-t]$  and  $f_t = f(1-t)$  on  $[1-t, 1]$

$$\text{and } g_t = \bar{f}_t$$

Then,  $f_0 = f$  and  $f_1 = \text{constant path } c \text{ at } x_0$

So,  $h_t$  is a homotopy from  $f \cdot \bar{f}$  to  $c \cdot \bar{c}$

$$\text{as } h_0 = f_0 \cdot g_0 = \begin{cases} f & \text{for } x \in [0, \frac{1}{2}] \\ \bar{f} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

$$h_1 = f_1 \cdot g_1 = \begin{cases} c, & x \in [0, \frac{1}{2}] \\ \bar{c}, & x \in [\frac{1}{2}, 1] \end{cases}$$

$$\therefore f \cdot \bar{f} \simeq c \quad (\text{defining } c \cdot \bar{c} = c) \text{ where } c = x_0$$

Replacing  $f$  by  $\bar{f}$  gives  $\bar{f} \cdot f = c$

Take  $f$  to be the loop at  $x_0$ , then  $[\bar{f}]$  is a 2-sided inverse for  $[f]$  in  $\pi_1(X, x_0)$ .

Fundamental Group of  $X$  at  $x_0$ :  $\pi_1(X, x_0)$

$\pi_1(X, x_0) = \{ \text{loops home } x_0 \text{ to itself in } X \} / (\text{path homotopy})$

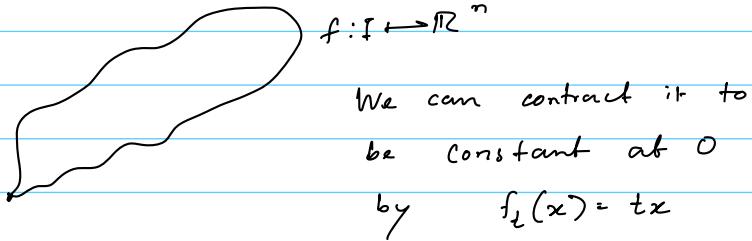
$$\rightarrow [f] \cdot [g] = [fg]$$

$$\rightarrow [f]^{-1} = [\bar{f}] \quad \text{where} \quad \bar{f}(t) = f(1-t)$$

$$\rightarrow [\text{constant}_{x_0}] = 1$$

### Examples

(i)  $\pi_1(\mathbb{R}^n, 0) = 1$



We say  $\pi_1(X) = 1$  if  $X$  is contractible

$$f_t(x) = r_t \circ f$$

homotopy from  $\text{id}_X$  to constant map

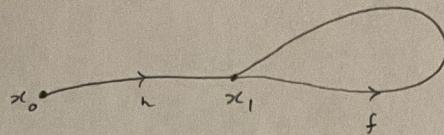
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### Change of basepoint

Let  $x_0$  and  $x_1$  lie in the same path-component of  $X$ .

Let  $h: I \rightarrow X$  be a path from  $x_0$  to  $x_1$ , with the inverse path  $\bar{h}(s) = h(1-s)$  from  $x_1$  to  $x_0$ .

Then, for each loop  $f$  based at  $x_1$ , define the loop  $h \cdot f \cdot \bar{h}$  based at  $x_0$ .



Alternatively, we can define a general  $n$ -fold product  $f_1, \dots, f_n$  in which the path  $f_i$  is traversed in  $[\frac{i-1}{n}, \frac{i}{n}]$ .

Then, define the change of basepoint map  $\beta_h: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  by  $\underline{\beta_h[f] = [h \cdot f \cdot \bar{h}]}$

Proposition: The map  $\beta_h: \pi_1(X, x_1) \rightarrow \pi_1(X, x_0)$  is an isomorphism.  
So,  $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Proof: Homomorphism as

$$\begin{aligned}\beta_h[f \cdot g] &= [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}] \\ &= \beta_h[f] \beta_h[g]\end{aligned}$$

This has the inverse  $\beta_{\bar{h}}$  as

$$\begin{aligned}\beta_h \beta_{\bar{h}}[f] &= \beta_h[\bar{h} \cdot f \cdot h] = [h \cdot \bar{h} \cdot f \cdot h \cdot \bar{h}] \\ &= [f]\end{aligned}$$

$$\text{Similarly, } \beta_{\bar{h}} \beta_h[f] = [f]$$

Def: Simply connected

A space is simply connected if it is path connected

and has trivial fundamental group.

i.e. the constant path

Proposition

A space  $X$  is simply connected iff there is  
a unique homotopy class of paths connecting  
any two points in  $X$ .

Proof :

$\Rightarrow$  : Need to show uniqueness.

Suppose

let  $f$  and  $g$  be 2 paths from  $x_0$  to  $x_1$ .

Then  $f \simeq f \cdot \bar{g} \cdot g \simeq g$  since the loops  $\bar{g} \cdot g$   
and  $f \cdot \bar{g}$  are each homotopic to constant  
loops, given  $\pi_1(X) = 0$

$\Leftarrow$  : If there is only one homotopy class of paths loops  
at  $x_0$ , then all loops at  $x_0$  are  
homotopic to the constant loop  
 $\therefore \pi_1(X, x_0) = \pi_1(X) = 0$

If  $X$  is path connected, then  $\pi_1(X, x_0)$  is independent  
of  $x_0$ . We write it as  $\pi_1(X)$ .

## Induced Homomorphism

### Def: Induced Homomorphism

Suppose,  $\varphi: X \rightarrow Y$  is a map taking basepoint  $x_0 \in X$  to the basepoint  $y_0 \in Y$

We say  $\varphi: (X, x_0) \mapsto (Y, y_0)$

Then,  $\varphi$  induces a homomorphism

$$\varphi_*: \pi_1(X, x_0) \mapsto \pi_1(Y, y_0)$$

defined by composing the loops  $f: I \rightarrow X$  based at  $x_0$  with  $\varphi$ :

$$\varphi_*([f]) = [\varphi f]$$

→ Well-defined:

Homotopy  $f_t$  of loops at  $x_0$  yields a homotopy  $\varphi f_t$  of loops based at  $y_0$ .

$$\therefore \varphi_*([f_0]) = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$$

→  $\varphi_*$  is a homomorphism:

$$\begin{aligned} \varphi_*(f \cdot g) &= \varphi(f \cdot g) && \xrightarrow{\text{both functions have values}} \\ &= \varphi f \cdot \varphi g && \varphi f(2s), \quad 0 \leq s \leq \frac{1}{2} \\ &= \varphi_*(f) \cdot \varphi_*(g) && \varphi g(2s-1), \quad \frac{1}{2} \leq s \leq 1 \end{aligned}$$

### Properties of induced homomorphisms

$$(1) \quad (X, x_0) \xrightarrow{\varphi} (Y, y_0) \xrightarrow{\varphi} (Z, z_0)$$

$$(\varphi\varphi)_* = \varphi_*\varphi_* : \pi_1(X, x_0) \longrightarrow \pi_1(Z, z_0)$$

Proof:

$$(\varphi\varphi)_f = \varphi(\varphi_f)$$

$$(2) \quad \mathbb{1}_* = \mathbb{1} \quad \text{which is saying } \mathbb{1}: X \rightarrow X \text{ induces } \mathbb{1}: \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$$

$$(3) \quad \text{If } \varphi \text{ is a homomorphism with inverse } \varphi^{-1}$$

then  $\varphi_*$  is an isomorphism with inverse  $(\varphi^{-1})_*$  since

$$\varphi_* (\varphi^{-1})_* = (\varphi \varphi^{-1})_* = \mathbb{1}_* = \mathbb{1} \quad \text{and similarly } \varphi^{-1}_* \varphi_* = \mathbb{1}$$

(4) Let  $\varphi, \psi: X \rightarrow Y$ .

If  $\varphi$  and  $\psi$  are homotopic, then  $\varphi_* = \psi_*$

Proof:

$$\varphi_* [f] = [\varphi f]$$

=  $[\psi f]$  (via homotopy of  $\varphi$  and  $\psi$ )

$$= \psi_* [f]$$

(5) Proposition:

If a space  $X$  retracts onto a subspace  $A$ , then the induced homomorphism  $i_*: \pi_1(A, x_0) \rightarrow \pi_1(X, x_0)$  induced by the inclusion  $i: A \hookrightarrow X$  is injective. If  $A$  is a deformation retract of  $X$ , then  $i_*$  is an isomorphism.

Proof:

Suppose,  $X$  retracts onto  $A \subset X$  via  $r: X \rightarrow A$

Then  $r_i = \text{id}_A$

$$\text{So, } (ri)_* = r_* i_* = \text{id}$$

Suppose  $i_*(f) = \text{id}$  for some  $f \in \pi_1(A, x_0)$

Therefore  $i_*$  is injective

Then,  $(r_* i_*)(f) = r_*(\text{id}) = \text{id}$ . But  $r_* i_* = \text{id}$   
so  $f = \text{id}$ .

Now, suppose  $X$  def. retracts onto  $A$  via  $r_t: X \rightarrow X$

$$\text{so, } r_0 = \text{id}_X, r_t|_A = \text{id}_A \text{ and } r_t(X) \subset A$$

then, for any loop  $f: I \rightarrow X$  based at  $x_0 \in A$ ,

the composition  $r_t f$  gives a homotopy of  $f$  to a loop in  $A$ , so  $i_*$  is also surjective.

$\hookrightarrow$  as  $r_t(X) \subseteq A$

$\hookrightarrow$  i.e. for any  $f: I \rightarrow X$ ,  
first def retract to  $f': I \rightarrow A$   
where  $f' = r_t f$ . Then  $i_*(f') = f' \in \pi_1(X, x_0)$   
and  $[f'] = [f]$  by  
the homotopy  $r_t$ .

Lemma 1.15

If a space  $X$  is the union of a collection of path connected open sets  $A_\alpha$ , each containing the basepoint  $x_0 \in X$  and if each intersection  $A_\alpha \cap A_\beta$  is path connected, then every loop in  $X$  at  $x_0$  is homotopic to a product of loops each of which is contained in a single  $A_\alpha$ .

Proof:

Consider a loop  $f: I \rightarrow X$  at  $x_0$ .

Partition  $I$  into  $0 = s_0 < s_1 < \dots < s_m = 1$  s.t. each

subinterval  $[s_{i-1}, s_i]$  is mapped by  $f$  to a single  $A_\alpha$

Since  $f$  is continuous, each  $s$  in  $I$  has an open nbhd  $V_s \subset I$  s.t.  $f$  maps  $V_s$  to  ~~$A_\alpha$~~  some  $A_\alpha$ . We can take  $V_s \subset I$  s.t.  $f$  maps  $\overline{V_s}$  (closure of  $V_s$ ) to a single  $A_\alpha$ .  
The endpoints of this finite set of intervals will define the partition  $0 = s_0 < s_1 < \dots < s_m = 1$ .

We denote  $A_i \dashv$  to be the set containing  $f([s_{i-1}, s_i])$  and we let  $f_i$  be the path obtained by restricting  $f|_{[s_{i-1}, s_i]}$ .

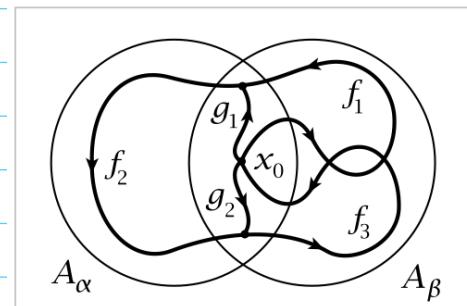
Now,  $f$  is the composition  $f_1 \cdot \dots \cdot f_m$  with  $f_i$  a path in  $A_i$ .

Since  $A_i \cap A_{i+1}$  is path connected, we can find a path  $g_i \in A_i \cap A_{i+1}$  from  $x_0$  to the point  $f(s_i) \in A_i \cap A_{i+1}$ .

Then, the loop

$$(f_1 \cdot \bar{g}_1) \cdot (\bar{g}_1 \cdot f_2 \cdot \bar{g}_2) \cdot \dots \cdot (g_{m-1} \cdot f_m)$$

is homotopic to  $f$  and is a composition of loops that each lie in a single  $A_i$ .



### Def : Basepoint Preserving Homotopy

Consider a homotopy  $\varphi_t$  taking  $A \subset X$  to a subspace  $B \subset Y$  for all  $t$ , then we speak of maps of pairs

$$\varphi_t : (X, A) \rightarrow (Y, B)$$

A basepoint-preserving homotopy  $\varphi_t : (X, x_0) \rightarrow (Y, y_0)$  is the case where  $\varphi_t(x_0) = y_0 \quad \forall t$ .

(6) If  $\varphi_t : (X, x_0) \rightarrow (Y, y_0)$  is a basepoint preserving homotopy, then  $\varphi_{0*} = \varphi_{1*}$

$$\begin{aligned} \text{Proof : } \varphi_{0*}[f] &= [\varphi_0 f] \\ &= [\varphi, f] \quad (\text{via homotopy } \varphi_t f) \\ &= \varphi_{1*}[f] \end{aligned}$$

### Def : Homotopy Equivalence for spaces with basepoints

We say  $(X, x_0) \simeq (Y, y_0)$  if there are maps  $\varphi : (X, x_0) \rightarrow (Y, y_0)$  and  $\psi : (Y, y_0) \rightarrow (X, x_0)$  with homotopies  $\varphi \psi \simeq \text{id}_{(Y, y_0)}$  and  $\psi \varphi \simeq \text{id}_{(X, x_0)}$ . through maps that fix the basepoint.

In this case, the induced maps on  $\pi_1$  satisfy

$$\varphi_* \psi_* = (\varphi \psi)_* = \text{id}_* = \text{id}$$

$$\psi_* \varphi_* = (\psi \varphi)_* = \text{id}_* = \text{id}$$

$\therefore \varphi_*$  and  $\psi_*$  are inverse isomorphisms

$$\therefore \pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

What if  $\varphi_t$  does not send  $x_0$  to a fixed  $y_0 \in Y$  for all  $t$ ? This means the basepoint in  $X$  is not always mapped to the same point by a homotopy.

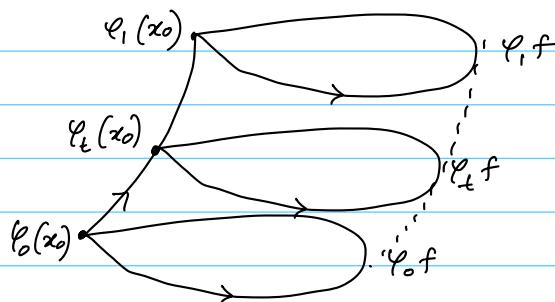
Lemma:

If  $\varphi_t : X \rightarrow Y$  is a homotopy and  $h$  is the path  $\varphi_t(x_0)$  formed by the images of a basepoint  $x_0 \in X$ , then the three maps in the diagram satisfy  $\varphi_{0*} = \beta_h \varphi_{1*}$

$$\begin{array}{ccc} & \varphi_{1*} & \rightarrow \pi_1(Y, \varphi_1(x_0)) \\ \pi_1(X, x_0) & \swarrow & \downarrow \beta_h \\ & \varphi_{0*} & \rightarrow \pi_1(Y, \varphi_0(x_0)) \end{array}$$

Proof:

Let  $h_t$  be the restriction of  $h$  to the interval  $[0, t]$  (with a reparametrization so that domain of  $h_t$  is  $[0, 1]$ ):  
So.  $h_t(s) = h(ts)$  where  $h : I \rightarrow Y$  with  $h(\tilde{t}) = \varphi_{\tilde{t}}(x_0)$



Then, if  $f$  is a loop in  $X$  at basepoint  $x_0$ , then the product  $h_t \cdot (\varphi_t f) \cdot \bar{h}$  gives a homotopy of loops at  $\varphi_0(x_0)$ .

Restricting this to  $t = 0$  and  $t = 1$ ,

$$\text{we see } \varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f]))$$

Theorem :

If  $\varphi: X \rightarrow Y$  is a homotopy equivalence, then the induced homomorphism  $\varphi_*: \pi_1(X, x_0) \rightarrow \pi_1(Y, \varphi(x_0))$  is an isomorphism for  $\forall x_0 \in X$ .

$$\therefore \pi_1(X, x_0) \xrightarrow{\sim} \pi_1(Y, \varphi(x_0))$$

Proof :

Let  $\varphi: X \rightarrow Y$  be a homotopy equivalence  $\Rightarrow$   
So, Let  $\psi: Y \rightarrow X$  be the homotopy inverse

$$\text{So, } \varphi \psi \simeq \text{id}$$

$$\psi \varphi \simeq \text{id}$$

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi \varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi \psi \varphi(x_0))$$

Given  $\psi \varphi \simeq \text{id}$ , then  $\psi_* \varphi_* = \text{id}$  for some  $h$  by the previous lemma.  $\Rightarrow \psi_* \varphi_*$  is an isomorphism

Since  $\psi_* \varphi_*$  is an isomorphism

$\varphi_*$  is injective.

Similarly, with  $\psi_* \varphi_*$ , we conclude  $\psi_*$  is injective.

$\therefore \varphi_*, \psi_*$  are injections and  $\psi_* \varphi_*$  is an isomorphism. so  $\varphi_*$  is a surjection too.

$$(4) \quad \pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

Proof :

A path  $I \rightarrow X \times Y$   
is a pair of paths  $(f: I \rightarrow X, g: I \rightarrow Y)$

## Fundamental Group of the Circle

Some preliminary tools:

- (1) Let  $w(s) = (\cos 2\pi s, \sin 2\pi s)$  for  $s \in I$  be a loop based at  $(1,0)$ .  
 Then,  $[w]^n = [w_n]$  where  $w_n(s) = (\cos 2\pi ns, \sin 2\pi ns)$  for  $n \in \mathbb{Z}$ .  
 by the definition of product path and the fact that product preserves homotopy.

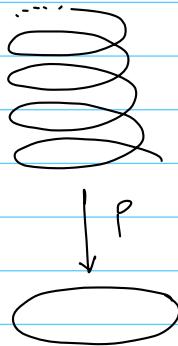
- (2) Compare paths in  $S^1$  with paths in  $\mathbb{R}$ :

→ Let  $p: \mathbb{R} \rightarrow S^1$  via  $p(s) = (\cos 2\pi s, \sin 2\pi s)$

Visualization: first, consider the helix  $s \mapsto (\cos 2\pi s, \sin 2\pi s, s)$

Then, project  $\mathbb{R}^3$  onto  $\mathbb{R}^2$  by  $(x,y,z) \mapsto (x,y)$

So, projecting the helix onto  $\mathbb{R}^2$  gives  $p$



→  $w_n(s) = p \tilde{w}_n(s)$  where  $\tilde{w}_n: I \rightarrow \mathbb{R}$  is the path  $\tilde{w}_n(s) = ns$

$\tilde{w}_n$  starts at 0 and ends at  $n$   
 $\tilde{w}_n$  is called the lift of  $w_n$ .

$p \tilde{w}_n(s)$  winds around the helix  $|n|$  times  $\rightarrow$  upwards if  $n > 0$  and downwards if  $n < 0$ .

- (3) Def: Covering Space

Given a space  $X$ , a covering space of  $X$  consists of a

space  $\tilde{X}$  and a map  $p: \tilde{X} \rightarrow X$  satisfying:

- (a) for each  $x \in X$ ,  $\exists$  open neighbourhood  $U \ni x$  in  $X$  st

$p^{-1}(U) = \coprod_{x \in U} V_x$  where each  $V_x$  is open and each

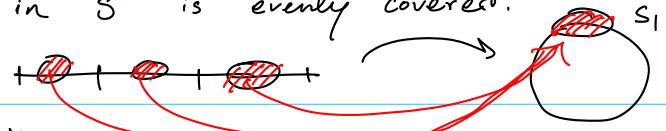
$V_x$  is mapped homeomorphically onto  $U$  by  $p$ .  
 $V_x$  is a union of disjoint open sets (each of

We say  $U$  is evenly covered.

$p|_{V_x}: V_x \rightarrow U$   
 is a homeomorphism

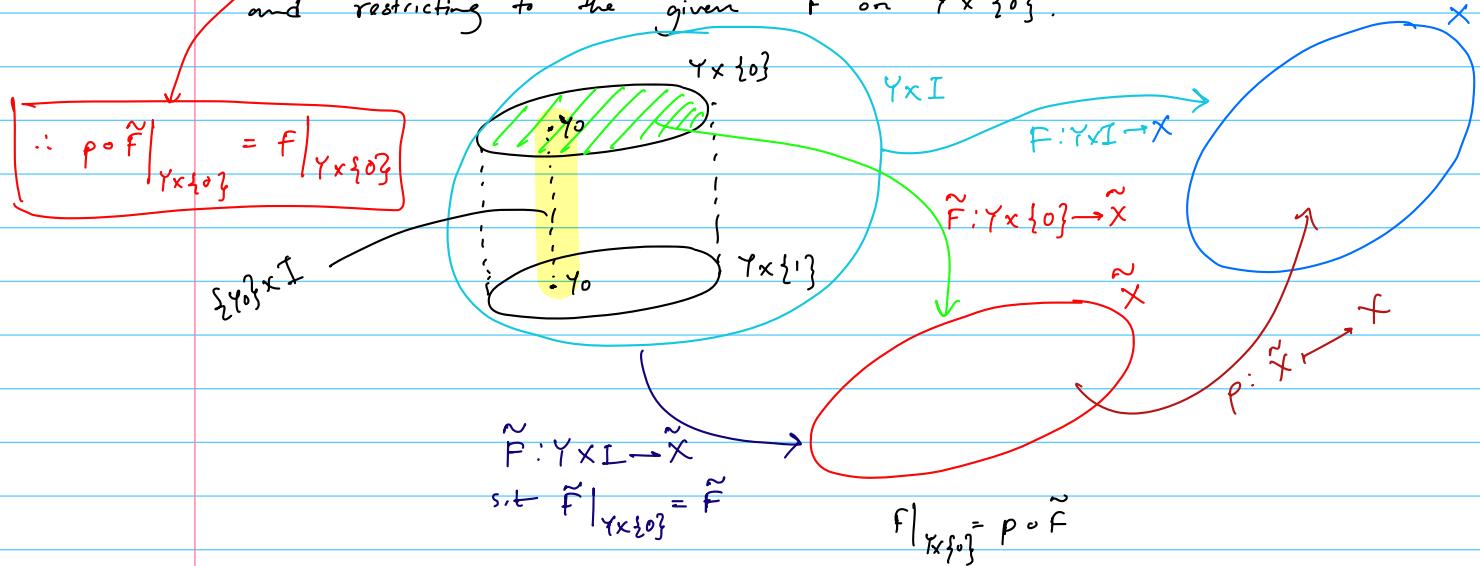
Example:

- (1)  $p: \mathbb{R} \rightarrow S^1$ , an open arc in  $S^1$  is evenly covered.  
Define it by  $p(\theta) = e^{2\pi i \theta}$ .



Lemma: Consider covering spaces  $p: \tilde{X} \rightarrow X$ .

Given a map  $f: Y \times I \rightarrow X$  and a map  $\tilde{F}: Y \times \{0\} \rightarrow \tilde{X}$  lifting  $F|_{Y \times \{0\}}$ , then there is a unique map  $\tilde{F}: Y \times I \rightarrow \tilde{X}$  lifting  $F$  and restricting to the given  $\tilde{F}$  on  $Y \times \{0\}$ .



Proof:

First, construct a lift  $\tilde{F}: N \times I \rightarrow \tilde{X}$  for some neighbourhood  $N$  of  $y_0 \in Y$ .

Given  $F$  is continuous,  $\forall (y_0, t) \in Y \times I$  has a product

neighbourhood  $N_t \times (a_t, b_t)$  s.t.  $F(N_t \times (a_t, b_t))$  is contained in an evenly covered neighbourhood of  $F(y_0, t)$ .

around  $F(y_0, t)$ ,  $\exists$  an evenly covered neighbourhood, since  $p: \tilde{X} \rightarrow X$  is a covering space.

By continuity, we can always shrink  $N_t \times (a_t, b_t)$  so that  $F(N_t \times (a_t, b_t))$  is inside this evenly covered nbd.

By compactness of  $\{y_0\} \times I$ , finitely many such  $N_t \times (a_t, b_t)$  products cover  $\{y_0\} \times I$ . Thus, we can choose one neighbourhood  $N$  of  $\{y_0\}$  and a partition  $0 = t_0 < t_1 < \dots < t_m = 1$  of  $I$  s.t. for each  $i$ ,  $F(N \times [t_i, t_{i+1}])$  is contained in an evenly covered neighbourhood  $U_i$ .

Assume, inductively,  $\tilde{F}$  has been constructed on  $N \times [0, t_i]$  starting with our given  $\tilde{F}$  on  $N \times \{0\}$ . Thus,  $F(N \times [t_i, t_{i+1}]) \subset U_i$ , so since  $U_i$  is evenly covered,  $\exists$  open set  $\tilde{U}_i \subset \tilde{X}$  projecting homeomorphically onto  $U_i$  by  $p$  and containing  $\tilde{F}(y_0, t_i)$  because

$\tilde{f}|_{N \times [0, t_i]}$  is a lift of  $f|_{N \times [0, t_i]}$   
so  $p(\tilde{f}(y_0, t_i)) = f(y_0, t_i)$   
we know it is a lift  
or we have already  
constructed the lift  
on  $N \times [0, t_i]$ .

We can extend  $\tilde{f}$  on  
 $N \times [t_i, t_{i+1}]$  by  
composing  $p^{-1}: U_i \rightarrow \tilde{U}_i$   
(since  $p$  is a homeomorphism)  
with  $f$ . For this to  
be continuous,  $\tilde{f}$  must  
agree with  $\tilde{f}$  on  
 $N \times [0, t_i]$ , in particular at  
 $(y_0, t_i)$ .

Replace  $N$  by a small enough nbd of  $y_0$ , we can get that  
 $\tilde{f}(N \times \{t_i\})$  is contained in  $\tilde{U}_i$  by replacing  $N \times \{t_i\}$  by  
its intersection with  $(\tilde{f}|_{N \times \{t_i\}})^{-1}(\tilde{U}_i)$ . Then define  $\tilde{f}$  on  
 $N \times [t_i, t_{i+1}]$  to be the composition of  $f$   
with  $p^{-1}: U_i \rightarrow \tilde{U}_i$ .

Continuing, we get  $\tilde{f}: N \times I \rightarrow \tilde{X}$ , a lift, for  
some neighbourhood  $N$  of  $y_0$ .

Next we show uniqueness of this lift. We prove for when  
 $Y$  is a point. Since  $Y$  is a point, we suppress  
it from our notation.

Let  $\tilde{f}$  and  $\tilde{f}'$  be 2 lifts of  $f: I \rightarrow X$   
s.t.  $\tilde{f}(0) = \tilde{f}'(0)$ .

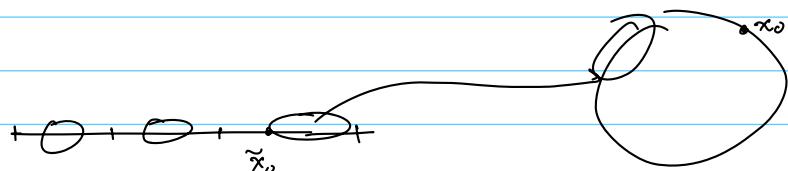
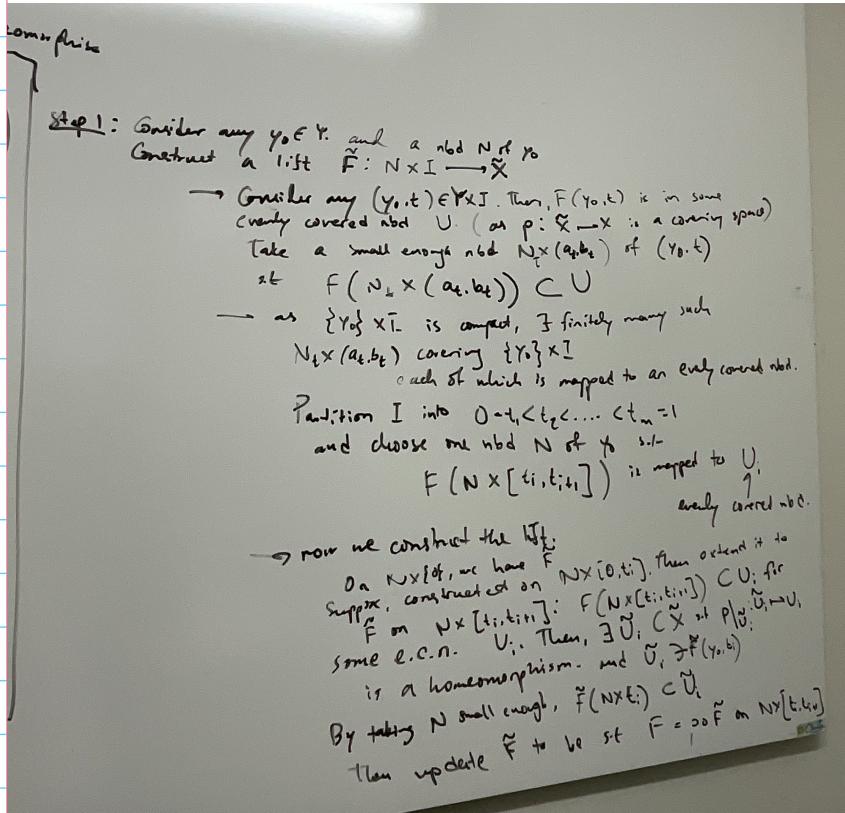
Again, we choose a partition  $\mathcal{D} = t_1 < t_2 < \dots < t_m = 1$   
of  $I$  s.t. for each  $i$ ,  $f([t_i, t_{i+1}])$  is contained in  
evenly covered nbd  $U_i$ .

Assume inductively that  $\tilde{f} = \tilde{f}'$  on  $[0, t_i]$ . As  $[t_i, t_{i+1}]$   
is connected, so is  $\tilde{f}([t_i, t_{i+1}]) \Rightarrow$  it must lie in single  
one of the disjoint open sets  $\tilde{U}_i$  projecting homeomorphically  
to  $U_i$ . By same logic,  $\tilde{f}'([t_i, t_{i+1}])$  lies in a  
single  $\tilde{U}_i$  and it must be the same one as  
 $\tilde{f}'(t_i) = \tilde{f}(t_i)$ . As  $p$  is injective on  $\tilde{U}_i$  and  
 $p \tilde{f} = p \tilde{f}'$ , we get  $\tilde{f} = \tilde{f}'$  on  $[t_i, t_{i+1}]$ . Continuing this way,  
 $\tilde{f} = \tilde{f}'$ .

Lastly, observe that since the lift  $\tilde{f}$  constructed on sets of the form  
 $N \times I$  is unique when restricted to each segment  $\{y\} \times I$ ,  
they must agree when two such sets  $N \times I$  overlap.

∴ we have a well-defined lift  $\tilde{F}$  on all of  $Y \times I$ .

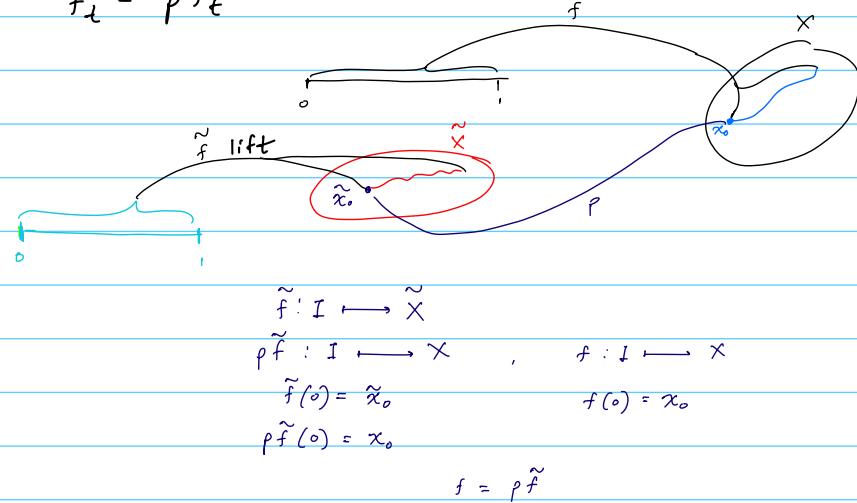
$\tilde{F}$  is continuous as it is continuous on each  $N \times I$ , and unique as it is unique on each segment  $\{y\} \times I$ .



### Path Lifting Property

!! Lemma : Consider covering spaces  $p: \tilde{X} \rightarrow X$ .

- (1) For each path  $f: I \rightarrow X$  s.t  $f(0) = x_0 \in X$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$ ,  $\exists$  a unique lift  $\tilde{f}: I \rightarrow \tilde{X}$  starting at  $\tilde{x}_0$ .  
Hence,  $f = p\tilde{f}$ .
- (2) For each homotopy  $f_t: I \rightarrow X$  of paths s.t  $f_t(0) = x_0$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$ ,  $\exists$  a unique lifted homotopy  $\tilde{f}_t: I \rightarrow \tilde{X}$  of paths starting at  $\tilde{x}_0$ .  
Hence,  $f_t = p\tilde{f}_t$



Proof :

(1) follows from prev. lemma when  $Y$  is a point

(2) Let  $Y = I$ .

Then for the homotopy  $f_t: I \rightarrow X$ , we have a map  $F: I \times I \rightarrow X$  with  $F(s, t) = f_t(s)$ .

We get a unique lift  $\tilde{F}: I \times \{0\} \rightarrow \tilde{X}$  using part (1).

Then, by prev. lemma, we get a unique lift

$$\tilde{F}: I \times I \rightarrow \tilde{X}$$

The restrictions  $\tilde{F}|_{\{0\} \times I}$  and  $\tilde{F}|_{\{1\} \times I}$  are paths

lifting constant paths, so they must also be constant by uniqueness of part (1).

So,  $\tilde{f}_t(s) = \tilde{F}(s, t)$  is a homotopy of paths

and  $\tilde{f}_t$  lifts  $f_t$  as  $F = p\tilde{F}$

We set  $\tilde{X}$  to be  $\mathbb{R}$   
here or  
 $p: \mathbb{R} \rightarrow S'$  is a  
covering space.

Theorem:  $\pi_1(S')$  is an infinite cyclic group generated by the homotopy class of the loop  $w(s) = (\cos 2\pi s, \sin 2\pi s)$  based at  $(1, 0)$ . So,  $\pi_1(S') \cong \mathbb{Z}$  as a group.

Proof: Let  $f: I \rightarrow S'$  be a loop at the basepoint  $x_0 = (1, 0)$  which is one element of the group  $\pi_1(S', x_0)$ .

Then,  $\tilde{f}$  starting at 0 and must end at some integer  $n$  since  $p\tilde{f}(1) = f(1) = x_0$  and  $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$   
recall  $p(s) = e^{2\pi i s}$ .

Another path in  $\mathbb{R}$  from 0 to  $n$  is  $\tilde{w}_n$  and  $\tilde{f} \simeq \tilde{w}_n$  via the

linear homotopy  $(1-t)\tilde{f} + t\tilde{w}_n$ . Compose the homotopy with  $p$  gives the homotopy  $\tilde{f} \simeq \tilde{w}_n$  so  $[f] = [w_n]$ .  
 $\therefore$  for any loop  $f$ ,  $f = [w_n]$ . Is  $n$  fixed here? Yes.

Next, we show that  $n$  is uniquely determined by  $[f]$ : Suppose  $f \simeq w_m$  and  $f \simeq w_n$ . Let  $f_t$  be a homotopy from  $w_m = f_0$  to  $w_n = f_1$ .

Then,  $f_t$  lifts to a homotopy  $\tilde{f}_t$  of paths starting at 0  
by previous lemma (2)

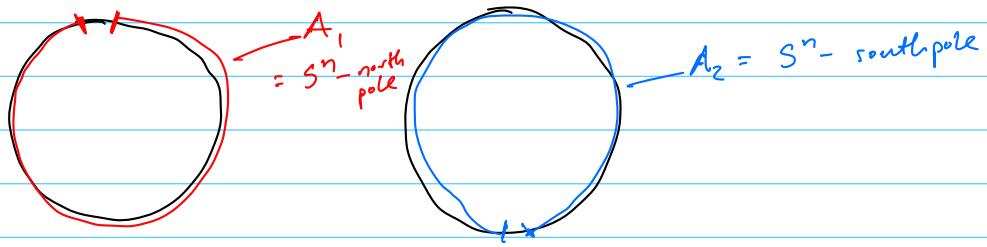
The uniqueness of  $\tilde{f}$  (by prev. lemma) implies that  $\tilde{f}_0 = \tilde{w}_m$  and  $\tilde{f}_1 = \tilde{w}_n$ . Since  $\tilde{f}_t$  is a homotopy of paths, the endpoint  $\tilde{f}_t(1)$  is independent of  $t$ . for  $t=0$ , the endpoint is  $m$  and for  $t=1$ , it is  $n$ . So,  $m=n$ .

The fact that this group is generated by  $w(s)$  is obvious from noting that  $[w]^n = [w_n]$

Proposition:  $\pi_1(S^n) = 0$  for  $n \geq 2$ .

Proof:

Write  $S^n$  as  $S^n = A_1 \cup A_2$  where  $A_1, A_2$  are open and each homeomorphic to  $\mathbb{R}^n$  (recall stereographic projection) and  $A_1 \cap A_2$  is homeomorphic to  $S^{n-1} \times \mathbb{R}$



Choose a basepoint  $x_0 \in A_1 \cap A_2$

Let  $n \geq 2$ . Then  $A_1 \cap A_2$  is path connected. Then by lemma 1.15 (Hatcher), every loop in  $S^n$  based at  $x_0$  is homotopic to a product of loops in  $A_1$  or  $A_2$ .

Since  $\pi_1(A_1) = 0 = \pi_1(A_2)$  (as  $A_1 \cong \mathbb{R}^n \cong A_2$ ), this product is nullhomotopic.

Theorem: Fundamental Theorem of Algebra

Every non-constant polynomial with coefficients in  $\mathbb{C}$  has a root in  $\mathbb{C}$ .

Proof:

Consider an arbitrary polynomial  $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n$ . Suppose,  $p(z)$  has no roots in  $\mathbb{C}$  (for contradiction)

Since  $p(z)$  has no roots in  $\mathbb{C}$ , then  $\forall r \in \mathbb{R}$ ,

$$f_r(s) = \frac{p(re^{2\pi is})/p(r)}{|p(re^{2\pi is})/p(r)|} \quad \text{is a loop in } S^1 \subset \mathbb{C} \text{ based at 1.}$$

$$\hookrightarrow f_r(0) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1$$

$$f_r(1) = \frac{p(r \cos(2\pi) + ri \sin(2\pi))/p(r)}{(\dots)} = 1$$

Then, as  $r$  varies,  $f_r$  is a homotopy of loops with basepoint 1.

for  $r=0$ ,  $f_0$  is the trivial loop constant at 1.

$$\therefore [f_r] = 0 \quad \forall r \text{ in } \pi_1(S^1)$$

$$\therefore p(z) \xleftarrow{\partial \in \pi_1(S^1)} \longrightarrow \textcircled{1}$$

Now, consider a large  $r$  s.t.  $r > |a_1| + \dots + |a_n|$  and  $r > 1$

Then, for  $|z|=r$ ,  $p(z)$  has no solution in  $|z|=r$ :

$$|z^n| > (|a_1| + \dots + |a_n|) |z^{n-1}| > |a_1 z^{n-1}| + \dots + |a_n| \geq |a_1 z^{n-1} + \dots + a_n|$$

$$\Rightarrow |z|^n > |a_1 z^{n-1} + \dots + a_n|$$

$$\Rightarrow p_t(z) := z^n + (a_1 z^{n-1} + \dots + a_n) \cdot \forall t \in I \text{ has no}$$

root on the circle  $|z|=r \longrightarrow$  this is a deformation of our polynomial to  $z^n$

Then, redefine  $f_r(s) := \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|}$

Let  $t$  go from 1 to 0, we find a homotopy from the loop  $f_r$  to  $w_n(s) = e^{2\pi i n s}$ .

$$\text{But } [\omega_n] = [\omega]^n \therefore p(z) \longleftrightarrow n \in \pi_1(S') \longrightarrow (2)$$

$$\Rightarrow [\omega_n] = [f_r] = 0 \quad \text{using (1) and (2)}$$

$\therefore n=0$ .  $\rightarrow$  contradiction as we assumed the degree was  $n$ .

Theorem: Brower Fixed Point Theorem in 2 dimensions

Every continuous map  $h: D^2 \rightarrow D^2$  has a fixed point i.e  
 $x \in D^2$  s.t  $h(x) = x$ .

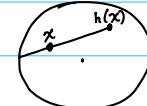
Proof:

Suppose,  $\forall x \in D^2, h(x) \neq x$ .

Define  $r: D^2 \rightarrow S^1$  (where  $\partial D^2 = S^1$ )

to be the point where the line from  $h(x)$  through  $x$  meets  $S^1$ :

$$r(x) = \frac{x - h(x)}{\|x - h(x)\|}$$



Clearly,  $r$  is continuous. Also,  $r(x) = x \quad \forall x \in S^1$ .

Thus,  $r$  is a retraction of  $D^2$  onto  $S^1$ .

However, no such retraction exists.

Let  $f_0 \in \pi_1(S')$

In  $D^2$ ,  $f_0 \cong$  constant loop by linear homotopy

$$f_t(x) = (1-t)f_0(x) + tx_0 \quad \nwarrow \text{basepoint of } f_0$$

Since  $r = \text{id}$  on  $S^1$ ,  $r \circ f_t$  is a homotopy in  $S^1$  from  $r \circ f_0 = f_0$  to the constant loop at  $x_0$ , since  $r$  is a retraction of  $D^2$  onto  $S^1$ .

But this contradicts the fact that  $\pi_1(S')$  is non-zero.

Theorem: Borsuk-Ulam Theorem in 2 dimensions

for every continuous map  $f: S^2 \mapsto \mathbb{R}^2$ ,  $\exists$  a pair of antipodal points  $x$  and  $-x$  in  $S^2$  s.t.  $f(x) = f(-x)$ .

OR

Weather Theorem

At any moment, there exists a pair of antipodal points on Earth s.t. they have

Proof:

Suppose not for  $f: S^2 \mapsto \mathbb{R}^2$

Define  $g: S^2 \mapsto S^1$  by  $g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$ . Notice:  $-g(x) = g(-x)$ .

Let the loop  $\eta$  in  $S^2 \subseteq \mathbb{R}^3$  be  $\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)$

and let  $h: I \mapsto S^1$  be the composed loop  $h = g \circ \eta$

Now,  $g(-x) = -g(x) \Rightarrow h(s + \frac{1}{2}) = -h(s) \quad \forall s \in [0, \frac{1}{2}]$ .

circle  
the  
equator  
of  
S  
once

Now, the loop  $h$  can be lifted to  $\tilde{h}: I \mapsto \mathbb{R}$ .

Since  $h(s + \frac{1}{2}) = -h(s)$ ,  $\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$  for some odd integer.

Now,  $q$  is independent of  $s$ :  $q$  depends on  $s \in [0, \frac{1}{2}]$  continuously but can take on odd integer values  $\Rightarrow$  it must be constant.

$$\text{Also, } \tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$$

$\therefore h$  represents  $q$  times the generator of  $\pi_1(S^1)$

Since  $q$  is odd,  $h$  is not nullhomotopic

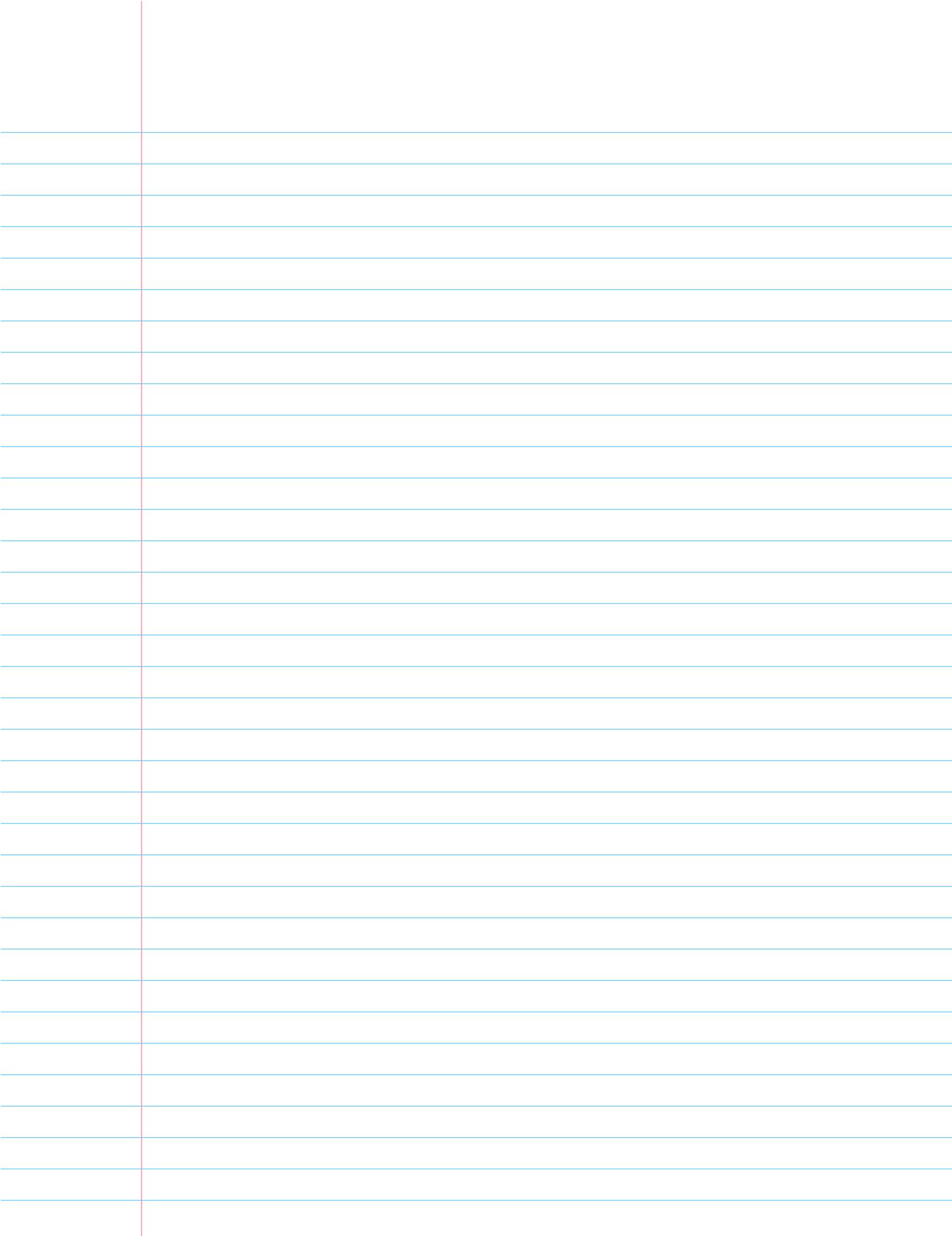
But  $h \circ g \circ \eta: I \mapsto S^2 \mapsto S^1$  and  $\eta$  is nullhomotopic in  $S^2$



Shrink it along the surface.

$\therefore g \circ \eta$  is nullhomotopic

$\therefore h$  is nullhomotopic  $\rightarrow$  contradiction.



## Van Kampen Theorem

### Free Product of Groups

First, we fix some notation:

(1)  $G = \langle X | R \rangle$  is a group.

$X \rightarrow$  set of generators

$R \rightarrow$  set of relations

Example 1:  $G = \langle a, b \mid a^5 b^{-1} ab^3 = 1, b^7 a^9 = 1 \rangle$

$= \langle a, b \rangle / \text{normal subgroup generated by } a^5 b^{-1} ab^3, b^7 a^9$

Example 2:  $\mathbb{Z} = \langle g \rangle$

$\mathbb{Z}/n = \langle g | g^n \rangle$

(2) Product of groups:

Given a collection of groups  $G_\alpha, \alpha \in A$ , the product is  $\prod_{\alpha \in A} G_\alpha$  which can be regarded as functions  $\alpha \mapsto g_\alpha \in G_\alpha$ .

↪ Suppose  $(g_1, g_2, g_3, \dots) \in \prod_{\alpha \in A} G_\alpha$

Then, this corresponds to a function  $f$

s.t.  $f(\alpha) = g_\alpha \in G_\alpha$ . So,  $f(1) = g_1, f(2) = g_2, \dots$

↪  $(g_1, g_2, \dots) \cdot (h_1, h_2, \dots) = f \cdot h$

as  $f(\alpha) \cdot h(\alpha) = g_\alpha \cdot h_\alpha, f(i) \cdot h(i) = g_i \cdot h_i$

¶ Problem with direct sum  $\bigoplus_\alpha G_\alpha$  or  $\prod_\alpha G_\alpha$ :

Elements of different subgroups  $G_\alpha$  commute with each other.

E.g.:  $G_1 = \mathbb{Z}_2$

$G_2 = \mathbb{Z}_3$

$$G_1 \times G_2 = \{(a, b) : a \in G_1, b \in G_2\}$$

thus, consider subgroups  $\{0\} \times \mathbb{Z}_3$  and  $\mathbb{Z}_2 \times \{0\}$

$$\text{Let } x_1 = (1, 0) \in \mathbb{Z}_2 \times \{0\}$$

$$x_2 = (0, 1) \in \{0\} \times \mathbb{Z}_3$$

$$x_1 \cdot x_2 = (1, 0) \cdot (0, 1) = (1, 1) = (0, 1) \cdot (1, 0) = x_2 \cdot x_1$$

As such, we will work with free products.

(3) Free Product:

$\ast G_\alpha$  consists of elements of the form  $g_1 g_2 \dots g_m$  for finite  $m \geq 0$  set:

(1) each  $g_i \in G_{\alpha_i}$

(2)  $g_i \neq 1_{G_i}$

(3)  $g_i$  and  $g_{i+1}$  belong to different groups (i.e.  $\alpha_i \neq \alpha_{i+1}$ )

→ words " $g_1 g_2 \dots g_m$ " satisfying these conditions are called reduced

→ unreduced words can be simplified to reduced ones by writing adjacent letters in the same  $G_\alpha$  as a single letter and by cancelling trivial letters.

→ empty word = identity of  $\ast G_\alpha$ .

→ Group operation:  $(g_1 \dots g_m)(h_1 \dots h_n) = g_1 \dots g_m h_1 \dots h_n$  and this should be simplified to reduced form i.e. if  $g_m h_1 \in G_\alpha$  then write  $(g_m h_1)$  as one letter and if it is identity, we cancel it.

Ex:  $(g_1 \dots g_m)(g_m^{-1} \dots g_1^{-1}) = \text{identity/empty word}$ .

Associative:

Let  $W$  be the set of reduced words  $g_1 \dots g_m$  including empty word.

for each  $g \in G_\alpha$ , we associate the function  $L_g: W \rightarrow W$

by multiplication on the left:  $L_g(g_1 \dots g_m) = g g_1 \dots g_m$  (to simplify)

Property of this association  $g \mapsto L_g$  is that  $L_{gg'} = L_g L_{g'}$

for  $g, g' \in G_\alpha$  i.e.  $g(g'(g_1 \dots g_m)) = (gg')(g_1 \dots g_m) \rightarrow$  this associativity follows from associativity in  $G_\alpha$ .

Now  $L_{gg'} = L_g L_{g'} \Rightarrow L_g$  is invertible with the inverse  $L_g^{-1}$ .

The association  $g \mapsto L_g$  is, thus, a homomorphism from  $G_\alpha$  to the group  $P(W)$  of all permutations of  $W$ . More generally, we can define:

$L: W \rightarrow P(W)$  by  $L(g_1 \dots g_m) = L_{g_1} \dots L_{g_m}$  for

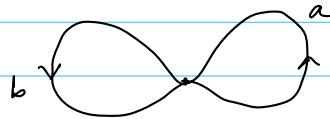
each reduced word  $g_1 \dots g_m \in W$ .

$L$  is injective as the permutation  $L(g_1 \dots g_m)$  sends the empty word to  $g_1 \dots g_m$ .

Now, the product operation in  $W$  corresponds under  $L$  to composition in  $P(W)$  as  $L_{gg'} = L_g L_{g'}$ . Since composition in  $P(W)$  is associative, the product in  $W$  is associative.

Eg:

(1)  $\mathbb{Z} * \mathbb{Z}$ :



Consider circles A and B, at the basepoint  $x_0$ .

Suppose  $\pi_1(A)$  is generated by a

$\pi_1(B)$  is generated by b

Then  $a^5 b^2 a^{-3} b$  is a loop in the A VB described above  
 $\underbrace{a^5 b^2 a^{-3} b}_{\text{go around A 5 times, around B 2 times, inverse around A 3 times, around B once}}$

This is a word in  $\mathbb{Z} * \mathbb{Z}$

Multiplication:  $(b^4 a^5 b^2)(a^5 b^{-1} a) = b^4 a^5 b^2 a^3 b^{-1} a$

This is an example of a free group.

the free product of any no. of copies of  $\mathbb{Z}$  (can be infinite)

→ one generator for each  $\mathbb{Z}$

→ the generators are called a basis for the free group

→ no. of basis elements = rank of the free group.

(2)  $\mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow$  not a free group

Here,  $a^2 = b^2 = \text{identity}$

$\mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow$  alternating words like a, b, ab, ba, aba, bab, ... and empty word.

Consider  $\varphi: \mathbb{Z}_2 * \mathbb{Z}_2 \rightarrow \mathbb{Z}_2$  which outputs the length of the word mod 2. Then  $\varphi$  is surjective, and its kernel is the set of words of even length. → These words of even length form an infinite cyclic subgroup generated by ab or  $(ba) = (ab)^{-1} \in \mathbb{Z}_2 * \mathbb{Z}_2$ .

called the infinite dihedral group.

(\*) Now, for a free product  $\ast_{\alpha} G_\alpha$ , each group  $G_\alpha$  can be identified with a subgroup of  $\ast_{\alpha} G_\alpha$  consisting of the empty word and the non-identity one letter words  $g \in G_\alpha$ .  
 $\rightarrow \because$  the empty word is the common identity element of all the subgroups  $G_\alpha$  (which are otherwise disjoint).

(\*\*) A consequence of associativity is that any product  $g_1 \cdots g_m$  of elements  $g_i \in G_\alpha$  has a unique reduced form.

Proposition :

for the free product  $\ast_{\alpha} G_\alpha$ , any collection of homomorphisms  $\varphi_\alpha : G_\alpha \rightarrow H$  extends uniquely to a homomorphism  $\varphi : \ast_{\alpha} G_\alpha \rightarrow H$   
 $\text{s.t. } \varphi(g_1 \cdots g_n) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$

Example: for a free product  $G * H$ , the inclusions  $G \hookrightarrow G * H$  and  $H \hookrightarrow G * H$  induce a surjective homomorphism  $G * H \rightarrow G * H$ .

### Amalgamated Free Product

$G_1, G_2, H \rightarrow \text{groups}$

$f_1 : H \rightarrow G_1$   
 $f_2 : H \rightarrow G_2$

} homomorphism

Amalgamated free product :  $G_1 *_H G_2 = G_1 * G_2 / f_1(h) = f_2(h) \forall h \in H$

Ex:  $\underbrace{\mathbb{Z}}_{f_1} \star \underbrace{\mathbb{Z}}_{f_2} = \langle g_1, g_2 \mid g_1^m = g_2^m \rangle$

$$f_1(h) = g_1^m$$

$$f_2(h) = g_2^m$$

## Van Kampen's Theorem

Suppose, the space  $X$  can be decomposed as the union of a collection of path-connected, open subsets  $A_\alpha$ , each of which contains the basepoint  $x_0 \in X$ .

Consider the inclusion  $A_\alpha \hookrightarrow X$  which induces the homomorphisms

$$j_\alpha : \pi_1(A_\alpha) \rightarrow \pi_1(X).$$

This can be extended to the homomorphism

$$\Phi : *_{\alpha} \pi_1(A_\alpha) \rightarrow \pi_1(X)$$

$$\text{st } \Phi(f_1, f_2, \dots, f_n) = j_{\alpha_1}(f_1) \cdots j_{\alpha_n}(f_n)$$

Consider the inclusion  $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$  inducing  $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \rightarrow \pi_1(A_\alpha)$ .

Then,  $j_\alpha i_{\alpha\beta}(w) = j_\beta i_{\alpha\beta}(w)$  for any loop in  $A_\alpha \cap A_\beta$ .

and both of them are induced by the inclusion

$$A_\alpha \cap A_\beta \hookrightarrow X.$$

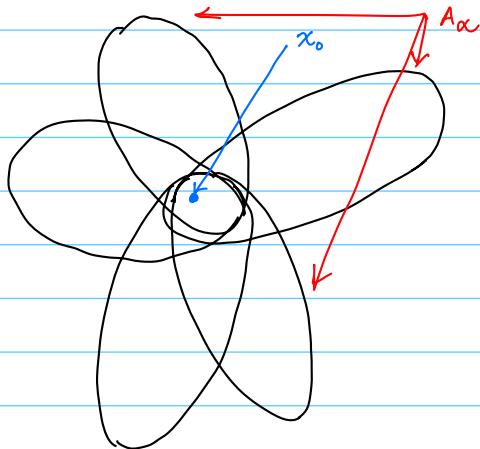
$\therefore$  kernel of  $\Phi$  contains all elements of the form

$$i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1} \text{ for } w \in \pi_1(A_\alpha \cap A_\beta).$$

$$\hookrightarrow \Phi(i_{\alpha\beta}(w) i_{\beta\alpha}(w)^{-1})$$

$$= \Phi(\text{empty word}) \quad \downarrow \text{since we are going to define it to be 1}$$

$$= \text{constant loop} \quad (\text{since } \Phi$$



$$A_\alpha \hookrightarrow X$$

$$\downarrow$$

$$j_\alpha : \pi_1(A_\alpha) \rightarrow \pi_1(X)$$

$$A_\alpha \cap A_\beta \hookrightarrow A_\alpha$$

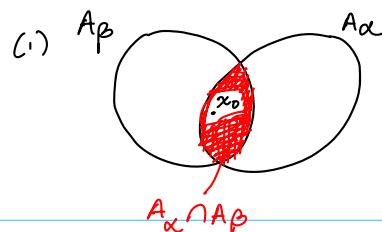
$$\downarrow$$

$$i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \rightarrow \pi_1(A_\alpha)$$

$$\Phi : *_{\alpha} \pi_1(A_\alpha) \rightarrow \pi_1(X)$$

by

$$\Phi(f_1, f_2, \dots, f_n) = j_{\alpha_1}(f_1) \cdots j_{\alpha_n}(f_n)$$



(2)

### Theorem: Seifert-van Kampen Theorem

- (1) If  $X$  is the union of path connected open sets  $A_\alpha$  each containing the basepoint  $x_0 \in X$  and if each intersection  $A_\alpha \cap A_\beta$  is path connected, then the homomorphism  $\Phi: *_\alpha \pi_1(A_\alpha) \rightarrow \pi_1(X)$  is surjective.

{ a loop in  $X$  can be thought of as composition of loops in each  $A_\alpha$  }

- (2) In addition, if each intersection  $A_\alpha \cap A_\beta \cap A_\gamma$  is path connected, then the kernel of  $\Phi$  is the normal subgroup  $N$  generated by elements of the form  $i_{\alpha\beta}(w)i_{\beta\alpha}(w)^{-1}$  for  $w \in \pi_1(A_\alpha \cap A_\beta)$  and, hence,  $\Phi$  induces an isomorphism  $\pi_1(X) \cong *_\alpha \pi_1(A_\alpha)/N$ .

### Proof:

- (1) is true by the following :

#### Lemma:

If a space  $X$  is the union of a collection of path connected open sets  $A_\alpha$ , each containing the basepoint  $x_0 \in X$  and if each intersection  $A_\alpha \cap A_\beta$  is path connected, then every loop in  $X$  at  $x_0$  is homotopic to a product of loops each of which is contained in a single  $A_\alpha$ .

- (2). We need to prove that  $\ker(\Phi)$  is  $N$ .

#### Def: Factorization of a loop

Factorization of  $[f] \in \pi_1(X)$  is a formal product  $[f_1] \cdots [f_k]$  s.t. (1) each  $f_i \in A_\alpha$  for some  $\alpha$  at basepoint  $x_0$  and  $[f_i] \in \pi_1(A_\alpha)$  (2) the loop  $f$  is homotopic to  $f_1 \cdots f_k$  in  $X$ .

The factorization of  $[f]$  is a word in  $*_\alpha \pi_1(A_\alpha)$ , possibly unreduced that is mapped to  $[f]$  by  $\Phi$ .

Surjectivity of  $\Phi$  is equivalent to saying that every  $[f] \in \pi_1(X)$  has a factorization.

#### Def: Equivalent factorizations

Two factorizations are equivalent if they are related by sequences of the following two moves or their inverses:

(move 1): combine adjacent terms  $[f_i][f_{i+1}]$  into  $[f_i \cdot f_{i+1}]$  if  $[f_i], [f_{i+1}]$  both belong to the same  $\pi_1(A_\alpha)$

(move 2): regard  $[f_i] \in \pi_1(A_\alpha)$  as lying in  $\pi_1(A_\beta)$  instead if  $f_i$  is a loop in  $A_\alpha \cap A_\beta$

move 1 does not change the element in  $\ast_A x$  w.r.t the definition of factorization

move 2 does not change the image of this element in the quotient group  $Q := \ast_A \pi_1(Ax)/N$

We want to prove that any two factorizations of  $f$  are equivalent. Then, we will have proven that  $Q \hookrightarrow \pi_1(X)$  is injective  $\Rightarrow \text{ker } \phi = 0 \Rightarrow Q \cong \pi_1(X)$ .

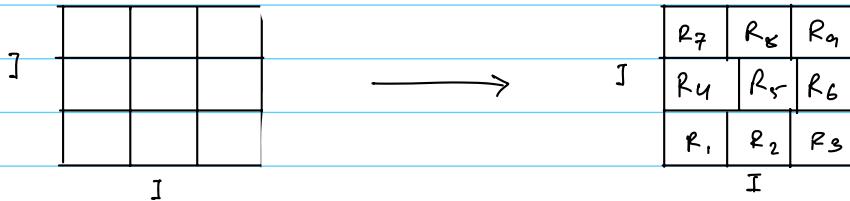
Let  $[f_1] \dots [f_k]$  and  $[f'_1] \dots [f'_k]$  be two factorizations of  $[f]$ .

Then, the composed paths  $f_1 \dots f_k$  and  $f'_1 \dots f'_k$  are homotopic via  $F: I \times I \rightarrow X$ .

Now,  $I$  partitions  $0 = s_0 < s_1 < \dots < s_m = 1$  and  $0 = t_1 < t_2 < \dots < t_n = 1$  s.t each rectangle  $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$  is mapped by  $f$  into a single  $A_x$  called  $A_{ij} \rightarrow$  we get these partitions by covering  $I \times I$  by finitely many rectangles  $[a, b] \times [c, d]$  each mapping to a single  $A_x$  and then partitioning  $I \times I$  by the union of all vertical and horizontal lines containing edges of these rectangles.

→ The  $s$ -partition subdivides these partitions to give the products  $f_1 \dots f_k$  and  $f'_1 \dots f'_k$ .

Now,  $f$  maps a nbhd of  $R_{ij}$  to  $A_{ij}$ , so we may perturb the vertical sides of the rectangles  $R_{ij}$  so that each point in  $I \times I$  is in at most three  $R_{ij}$ 's:



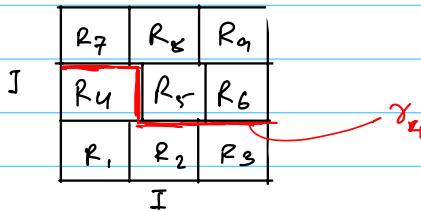
We are perturbing only the middle rows (not the first and last - we are assuming there are at least three).

Label the rectangles  $R_1, R_2, \dots, R_{mn}$ .

If  $\gamma$  is a path in  $I \times I$  from the left to the right edge, then the restriction  $F|_\gamma$  is a loop at the basepoint  $x_0$  since  $F$  maps both the left and right edges of  $I \times I$  to  $x_0$ .

Let  $\gamma_r$  be the path separating the first  $r$  rectangles

from the rest.



Then,  $\gamma_r$  is the bottom edge of  $I \times I$

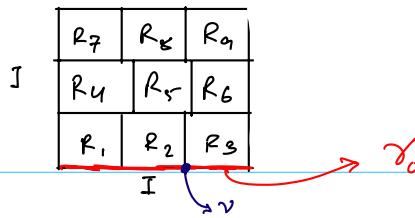
$\gamma_{mn}$  is the top edge.

We go from  $\gamma_r$  to  $\gamma_{r+1}$  by pushing across the rectangle  $R_{r+1}$ .

Now, consider the vertices of  $R_r$ . For each vertex  $v$  with  $F(v) \neq x_0$ , we choose a path  $g_v$  from  $x_0$  to  $F(v)$  that lie in the intersection of the two or three  $A_{ij}$ 's corresponding to the  $R_r$ 's containing  $v$ . (for a visualization, see the proof the surjectivity).

Then, we have a factorization of  $[F|_{\gamma_r}]$  by inserting the appropriate paths  $\bar{g}_v g_v$  into  $F|_{\gamma_r}$  at successive vertices (similar to the way we did it in the proof of surjectivity). This factorization depends on our choices : consider the path between two successive vertices which can lie in 2 different  $A_{ij}$ 's since the path may be in 2 different  $R_i$ 's. However, different choices of  $A_{ij}$ 's here gives equivalent factorizations (using move 2). Also, the factorization for successive paths  $\gamma_r$  and  $\gamma_{r+1}$  are equivalent since pushing  $\gamma_r$  across  $R_{r+1}$  to  $\gamma_{r+1}$  changes  $F|_{\gamma_r}$  to  $F|_{\gamma_{r+1}}$  by a homotopy within the  $A_{ij}$  corresponding to  $R_{r+1}$  and we can choose this  $A_{ij}$  for all the segments of  $\gamma_r$  and  $\gamma_{r+1}$  in  $R_{r+1}$ .

This shows that the factorisation associated with all  $\gamma_r$  are equivalent.



We can arrange so that the factorization associated to  $\gamma_0$  is equivalent to the factorization  $[f_1] \dots [f_k]$  by choosing the path  $g_v$  for each vertex  $v$  along the lower edge of  $I \times I$  to lie not just in the two  $A_{ij}$ 's corresponding to the  $R_i$ 's containing  $v$  but also in the  $A_\alpha$  for the  $f_i$  containing  $v$  in its domain.

→ in case  $v$  is the common endpoint of the domains of two ~~cont~~  $f_i$ 's,  $F(v) = x_0$ , so there is no need to choose a  $g_v$  here.

Similarly, assume that the factorization associated to the final  $\gamma_m$  is equivalent to  $[f'_1] \dots [f'_k]$ .

Since the factorization associated to all the  $\gamma_i$ 's are equivalent,  $[f_1] \dots [f_k]$  and  $[f'_1] \dots [f'_k]$  are equivalent.

Seifert-Van-Kampen; in amalgamated free product notation:

$$\pi_1(X) := \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$$

provided  $U, V$  are open + path connected,  $X = U \cup V$ ,  $U \cap V$  = path connected

More generally,

$$\pi_1(X) = \ast_{\alpha} \pi_1(U_\alpha) / \left( (i_{\alpha\beta})_* (w) = (i_{\beta\alpha})_* (w), \forall w \in \pi_1(U_\alpha \cap U_\beta) \right)_{\forall \alpha, \beta}$$

$$i_{\alpha\beta} : U_\alpha \cap U_\beta \hookrightarrow U_\alpha$$

$$i_{\beta\alpha} : U_\alpha \cap U_\beta \hookrightarrow U_\beta$$

→ Loops in  $U_\alpha \cap U_\beta$  must be interpreted  
the same way, regardless of whether  
we see them as loops in  $U_\alpha$  or  $U_\beta$ .

Applying Van Kampen's Theorem to compute fundamental groups:

(1) Wedge sum of  $X_\alpha$ :  $\pi_1(\bigvee_\alpha X_\alpha)$

Let the basepoints be  $x_\alpha \in X_\alpha$ .

for each  $x_\alpha \in X_\alpha$ , if  $x_\alpha$  is a deformation retract of an open neighbourhood  $U_\alpha \subset X_\alpha$ , then  $X_\alpha$  is a deformation

retract of its open neighbourhood  $A_\alpha = X_\alpha \cup_{\beta \neq \alpha} U_\beta$

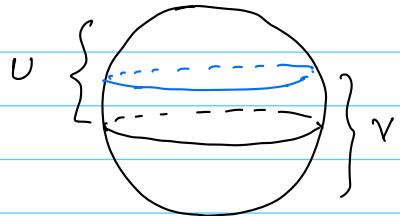
$\therefore X_\alpha \cong A_\alpha$ . Here,  $A_\alpha$  will be our open cover.

The intersection of two or more distinct  $X_\alpha$  is  $\bigvee_\alpha U_\alpha$ , which deformation retracts to a point where we wedge all  $X_\alpha$ . So, the intersection of  $A_\alpha$  is trivial as it is trivially path connected. This also means  $N$  is the trivial subgroup.

$$\therefore \pi_1\left(\bigvee_\alpha X_\alpha\right) \cong \ast_\alpha \pi_1(X_\alpha)$$

$$\rightarrow \pi_1\left(\bigvee_\alpha S_\alpha^1\right) \cong \ast_\alpha \pi_1(S_\alpha^1) = \ast_\alpha \mathbb{Z} \rightarrow \text{the free group}$$

(2)  $\pi_1(S^n)$ :



Note  $U \cong B^n$ ,  $V \cong B^n$ ,  $U \cap V = S^{n-1} \times I \cong S^{n-1}$

For  $n \geq 2$ :

$$\pi_1(S^n) = \pi_1(U) *_{\pi_1(S^{n-1})} \pi_1(V)$$

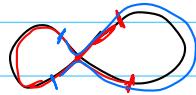
$$= \pi_1(B^n) *_{\pi_1(S^{n-1})} \pi_1(B^n)$$

$$= 1 *_{\pi_1(S^{n-1})} 1$$

$$= 1$$

(for  $n < 2$ ,  $S^{n-1}$  is not path connected :  $S^0 = \{-1, 1\}$ )

$$(3) \pi_1(\underbrace{\infty})$$



$s' \vee s'$

$$\therefore \pi_1(\infty) = \pi_1(\text{X}) * \pi_1(\text{S}) = \mathbb{Z} * \mathbb{Z} = F_2$$

## Applying Van-Kampen's theorem to Cell Complexes.

Intuition:

Consider a path connected space  $X$ .

Suppose, we attach a bunch of 2-cells  $e_\alpha^2$  to  $X$  via  $\varphi_\alpha : S' \mapsto X$  (since the boundary of  $e_\alpha^2$  is  $S'$ ). Let the basepoint of  $S'$  be  $s_0$ .

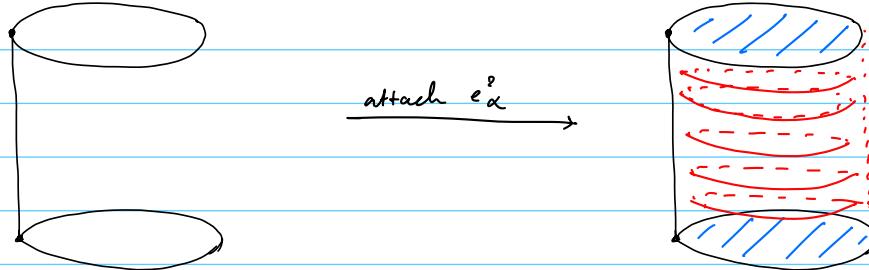
Then,  $\varphi_\alpha : S' \mapsto X$  is a loop at  $\varphi_\alpha(s_0)$ .

Call this loop  $\gamma_\alpha$  → although a loop would be  $f : I \rightarrow X$   
we use the shorthand  $\gamma_\alpha : S' \mapsto X$   
do refer to this loop at  $\varphi_\alpha(s_0)$ .

For each  $\alpha$ , we get a different loop at each  $\varphi_\alpha(s_0)$   
since the basepoints  $\{\varphi_\alpha(s_0) : \forall \alpha\}$  may not all be  
the same.

To fix this, we choose a basepoint  $x_0 \in X$  and a path  $\gamma_\alpha$  in  $X$   
from  $x_0 \in X$  to  $\varphi_\alpha(s_0)$  for each  $\alpha$ . Then,  $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$  is a loop  
at  $\varphi_\alpha(s_0)$ , for each  $\alpha$ .

These loops may not be nullhomotopic in  $X$  but they will  
be after the cell  $e_\alpha^2$  is attached.



!!! [ ∵ The normal subgroup  $N \subset \pi_1(X, x_0)$  generated by all the loops  
 $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$  for each  $\alpha$  lies in the kernel of the  
map  $\pi_1(X, x_0) \rightarrow \pi_1(Y, x_0)$  induced by the inclusion  $i : X \hookrightarrow Y$  ]

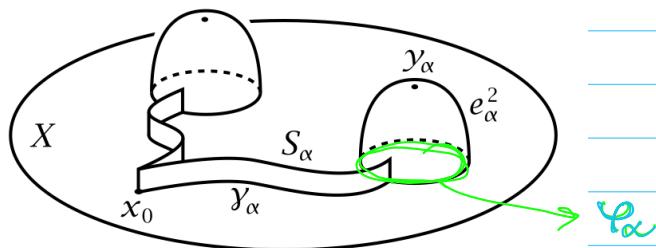
Proposition :

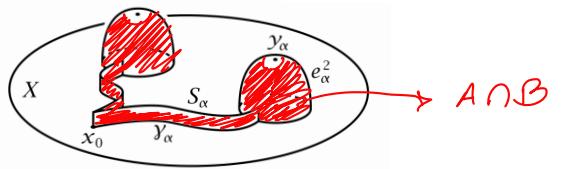
- (a) If  $Y$  is obtained from  $X$  by attaching 2 cells as described, then the inclusion  $i: X \hookrightarrow Y$  induces a surjection  $\pi_1(X, x_0) \rightarrow \pi_1(Y, x_0)$ . whose kernel is  $N$ .  
 $\therefore \pi_1(Y) \approx \pi_1(X)/N$
- (b) If  $Y$  is obtained from  $X$  by attaching  $n$ -cells for a fixed  $n > 2$ , then the inclusion  $i: X \hookrightarrow Y$  induces an isomorphism  $\pi_1(Y) \approx \pi_1(X)$ .
- (c) for a path connected cell complex  $X$ , the inclusion of the 2-skeleton  $i: X^2 \hookrightarrow X$  induces an isomorphism  $\pi_1(X^2, x_0) \rightarrow \pi_1(Y, x_0)$ .

Note : in (a),  $N$  is independent of the choice of our paths  $\gamma_\alpha$  since if  $\gamma_\alpha \gamma_\alpha \bar{\gamma}_\alpha$  is in  $N$ , and  $\gamma_\alpha \bar{\gamma}_\alpha \gamma_\alpha$  is another possible path, then  $(\gamma_\alpha \bar{\gamma}_\alpha) \gamma_\alpha \bar{\gamma}_\alpha \gamma_\alpha (\bar{\gamma}_\alpha \gamma_\alpha) = \gamma_\alpha \bar{\gamma}_\alpha \gamma_\alpha$  i.e they are conjugate to each other.

Proof :

- (a) Suppose,  $Y$  is obtained from  $X$  by attaching 2 cells.  
 Expand  $Y$  to a slightly larger space  $Z$  s.t  $Z$  def retracts onto  $Y$ . (so,  $\pi_1(Z) \approx \pi_1(Y)$ )  
 ↳ build  $Z$  by doing : attach rectangular strips  $S_\alpha = I \times I$  with the lower edge  $I \times \{0\}$  attached along  $\gamma_\alpha$  and the right edge  $\{1\} \times I$  attached along an arc starting from  $\gamma_\alpha(s)$  and going radially into  $e_\alpha^2$  and the left edges of every strip (ie for each  $\alpha$ ) are identified together.  
 → Since the top edge is not attached to anything, we can def retract  $Z$  onto  $Y$ .





Suppose, in each 2-cell  $e_\alpha^2$ , we choose a basepoint  $y_\alpha$  s.t  $y_\alpha$  is not in the arc along which  $S_\alpha$  is attached.

Let  $A = Z - \bigcup_\alpha \{y_\alpha\}$  → this def retracts onto  $X$

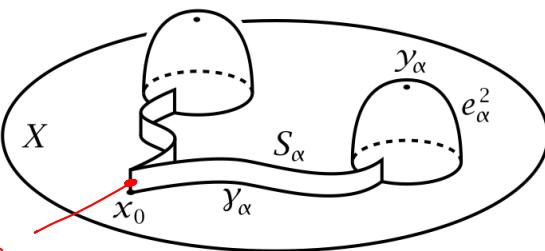
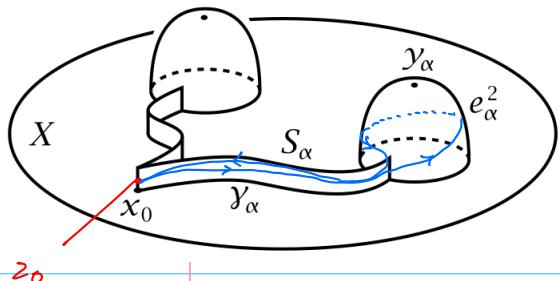
$$\therefore A \cong X$$

$$B = Z - X \rightarrow \text{contractible} \Rightarrow \pi_1(B) = 0$$

$\therefore \pi_1(Y) \approx \pi_1(Z) \approx \pi_1(A)/N \approx \pi_1(X)/N \rightarrow$  a normal subgroup generated by loops in  $A \cap B$

Now, cover  $Z$  by  $A \cup B$ . Since  $\pi_1(B) = 0$ ,  $\therefore \pi_1(Z) \approx \pi_1(A)/N$   
where  $N$  is the generated by the image of the map  
 $\pi_1(A \cap B) \rightarrow \pi_1(A)$ : since  $B$  is contractible

↳ specifically, let  $z_0 \in A \cap B$  near  $x_0$  on the segment where all  $S_\alpha$  intersect



Now, choose loops  $\delta_\alpha \in \pi_1(A \cap B, z_0)$  based at  $z_0$  representing elements in  $\pi_1(A, z_0)$  that correspond to  $[\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha] \in \pi_1(A, x_0)$  after a basepoint shift from  $x_0 \rightarrow z_0$  along the edge connecting all  $S_\alpha$ .

↳  $\therefore$  we make these loops  $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$  generate  $N$ :

Claim:  $\pi_1(A \cap B, z_0)$  is generated by loops  $\delta_\alpha$ .

→ Use Van Kampen's theorem again but to the cover of  $A \cap B$  by open sets

$$A_\alpha = A \cap B - \bigcup_{\beta \neq \alpha} e_\beta^2$$

Given  $A_\alpha$  deformation retracts onto a circle in  $e_\alpha^2 - \{y_\alpha\}$ ,

$\pi_1(A_\alpha, z_0) \approx \mathbb{Z}$  generated by  $\delta = \gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$   
but these

we already saw prior to the theorem:

↳ The normal subgroup  $N \subset \pi_1(X, x_0)$  generated by all the loops  $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$  for each  $\alpha$  lies in the kernel of the map  $\pi_1(X, x_0) \rightarrow \pi_1(Y, x_0)$  induced by the inclusion  $i: X \hookrightarrow Y$

(b) Same proof as before, but here  $A_\alpha$  def retracts onto a sphere  $S^{n-1}$  so,  $\pi_1(A_\alpha) = 0$  if  $n \geq 2$  (as  $\pi_1(S^n) = 0$  for  $n \geq 2$ )  
 $\therefore \pi_1(A \cap B) = 0 \Rightarrow$  the normal subgroup generated by loops in  $\pi_1(A \cap B)$  is trivial.

(c) follows from (b) by induction when  $X$  is finite dimensional, ie  $X = X^n$ .  
 (recall we go from  $X^2$  to  $X$  by attaching  $e_\alpha^n$  for  $n \geq 2$ )

Now, suppose  $X$  is not finite dimensional.

Let  $f: I \rightarrow X$  be a loop at the basepoint  $x_0 \in X^2$ . The image of  $f$  is compact which must lie in  $X^n$  for some  $n$ .

Then, by part (b),  $f$  is homotopic to a loop in  $X^2$ .  
 ↳ as  $X^2 \hookrightarrow X^n$  is surjective  $\Rightarrow f$  is homotopic to a loop in  $X^2$

Thus,  $\pi_1(X^2, x_0) \hookrightarrow \pi_1(X, x_0)$  is surjective

Claim:  $\pi_1(X^2, x_0) \hookrightarrow \pi_1(X, x_0)$  is surjective also injective

Suppose  $f$  is a loop in  $\pi_1(X^2, x_0)$  that is nullhomotopic in  $X$  via  $F: [0, 1] \times I \rightarrow X$ .

Then  $F$  has a compact image lying in some  $X^n$  and we can assume  $n \geq 2$ .

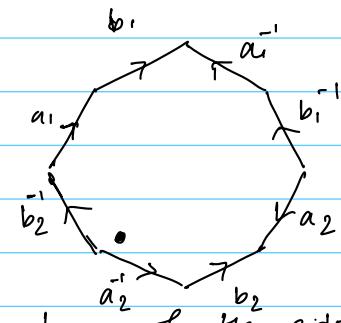
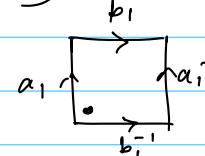
Since  $\pi_1(X^2, x_0) \hookrightarrow \pi_1(X^n, x_0)$  is injective by (b),  $f$  is nullhomotopic in  $X^2$ .

(4) orientable surface of genus  $g$ :

$$\Sigma_g : \underbrace{\text{---} \curvearrowleft \text{---} \curvearrowleft \text{---}}_{\text{genus } g}$$

$$\Sigma_1 :$$

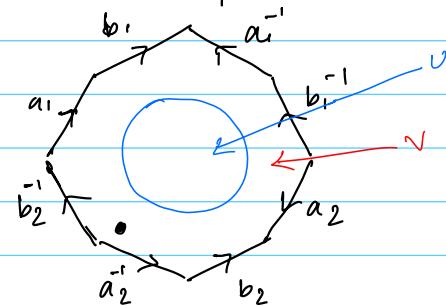
$$\Sigma_2 :$$



Generally,  $\Sigma_g$  can be constructed from a polygon of  $4g$  sides

labelled  $a_1, b_1, a_1^{-1}, b_1^{-1}, \dots, a_g, b_g, a_g^{-1}, b_g^{-1}$

and gluing  $a_i$  to  $a_i^{-1}$ ,  $b_i$  to  $b_i^{-1}$ . Glue them so that all loops meet at the same basepoint



$$U \cap V = S' \Rightarrow \pi_1(S') \cong \mathbb{Z}$$

$$V = B^2 \Rightarrow \pi_1(B^2) \cong 1 \text{ as } B^2 \text{ is contractible}$$

$$\pi_1(U) = \pi_1(V S')$$

$$\therefore \pi_1(\Sigma_g) = \pi_1\left(\bigvee_{2g} S'\right) * 1$$

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid [a_1, b_1] [a_2, b_2] \dots [a_g, b_g] = 1 \rangle$$

$$a_1, b_1, a_1^{-1}, b_1^{-1}$$

Another way to see this is as from Hatcher:

As a first application we compute the fundamental group of the orientable surface  $M_g$  of genus  $g$ . This has a cell structure with one 0-cell,  $2g$  1-cells, and one 2-cell, as we saw in Chapter 0. The 1-skeleton is a wedge sum of  $2g$  circles, with fundamental group free on  $2g$  generators. The 2-cell is attached along the loop given by the product of the commutators of these generators, say  $[a_1, b_1] \cdots [a_g, b_g]$ . Therefore

$$\pi_1(M_g) \approx \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$$

where  $\langle g_\alpha \mid r_\beta \rangle$  denotes the group with generators  $g_\alpha$  and relators  $r_\beta$ , in other words, the free group on the generators  $g_\alpha$  modulo the normal subgroup generated by the words  $r_\beta$  in these generators.

Corollary :

The surface  $M_g$  is not homotopy equivalent or homeomorphic to  $M_n$  if  $g \neq n$ .

(5) Non-orientable surface of genus  $g$ :

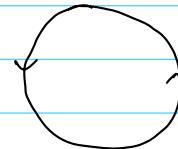
we create them by → (1) take the wedge sum of  $g$  circles

$$\pi_1(\vee^g S^1) = \underbrace{\mathbb{Z} * \mathbb{Z} * \dots * \mathbb{Z}}_{g \text{ times}}$$

let the generator of each of these groups be  $a_i$ . So, the generators are  $a_1, a_2, \dots, a_g$ .

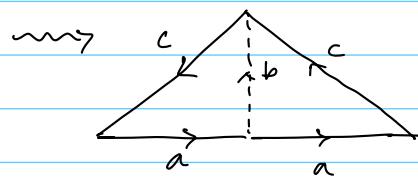
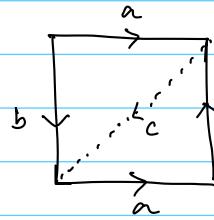
(2) attach a 2-cell to this wedge sum along the path  $a_1^2 a_2^2 \dots a_g^2$

$N_1: RR^2 \longrightarrow$



generated by  $a$ ,

$N_2: \text{Klein bottle} \longrightarrow$

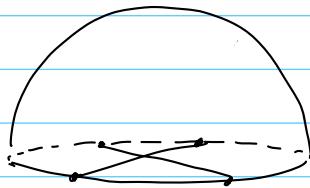


cut the square along  $c$   
then reassemble

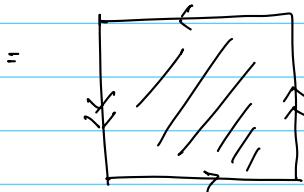
By our proposition,

$$\pi_1(N_g) \approx \langle a_1, \dots, a_g \mid a_1^2 \dots a_g^2 \rangle$$

(s)  $\mathbb{RP}^2$



$$\mathbb{RP}^2 = D^2 \cup \text{circle with a cross}$$



$$= D^2 \xrightarrow{\text{trivial}} \text{circle} \approx S^1/n$$

$$V \cap V = S^1$$

Then,  $\pi_1(\mathbb{RP}^2) =$

(6) Non-orientable surface of genus  $g$ :

$$\pi_1(N_g) = \pi_1\left(\text{Diagram of a non-orientable surface}\right) = \langle a_1, \dots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 = 1 \rangle$$

Then,  $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$

Corollary:

(a)  $\Sigma_g \not\cong \Sigma_h$  for  $g \neq h$

(b)  $N_g \not\cong N_h$  for  $g \neq h$

(c)  $\Sigma_g \not\cong N_g$

Proof: We want to say that they have different fundamental groups.

First, we do Abelianisation: Given  $G$  a group,

$$\text{Ab } G := G / \langle [g, h] \rangle$$

generated by  
all  $[g, h]$

Classification for finitely generated abelian groups:

$$\mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}/p_1^{m_1} \oplus \mathbb{Z}/p_2^{m_2} \oplus \cdots \oplus \mathbb{Z}/p_k^{m_k}$$

$p_i \rightarrow$  prime, not necessarily distinct.

Now, (a)  $\text{Ab } \pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid 1 \rangle = \mathbb{Z}^{2g}$

Notice  $\mathbb{Z}^{2g} \neq \mathbb{Z}^{2h}$  for  $g \neq h$

(b)  $\text{Ab } \pi_1(N_g) : \langle a_1, \dots, a_g \mid 2a_1 + 2a_2 + \cdots + 2a_g = 0 \rangle$

$a_i \rightarrow$  commute

Let  $b = a_1 + \cdots + a_g$

$$\text{Ab } \pi_1(N_g) = \langle a_1, \dots, a_{g-1}, b \mid 2b = 0 \rangle$$

$$= \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2$$

More generally.

Let  $X$  be path connected,

Attach an  $n$ -cell  $\rightsquigarrow Y$

$$\text{Then, } \pi_1(Y) = \pi_1(X) * \pi_1(S^{n-1})^1 = \pi_1(X) \text{ if } n > 2$$

= quotient of  $\pi_1(X)$  if  $n = 2$

$$\pi_1(RP^n) = \pi_1\left(\underbrace{e^0 \cup e^1 \cup e^2 \cup e^3 \cup \dots \cup e^n}_{RP^2}\right)$$

Suppose  $n \geq 2$

$$= \pi_1(RP^2)$$

$$= \mathbb{Z}/2$$

$$\pi_1(CP^n) = \pi_1\left(\underbrace{e^0 \cup e^2 \cup \dots \cup e^{2n}}_{S^2}\right)$$

$$= \pi_1(S^2)$$

$$= 1$$

### Corollary

For every group  $G$ ,  $\exists$  a 2-dimensional cell complex  $X_G$  with  $\pi_1(X_G) \approx G$ .

Proof:

Let  $G = \langle g_\alpha | r_\beta \rangle$  be a representation. This exists as every group is a quotient of a free group

→ start with a free group  $F$  generated by the set  $\{g_\alpha\}$  without any relations other than group axioms. So,  $F$  is just words that can not be reduced.

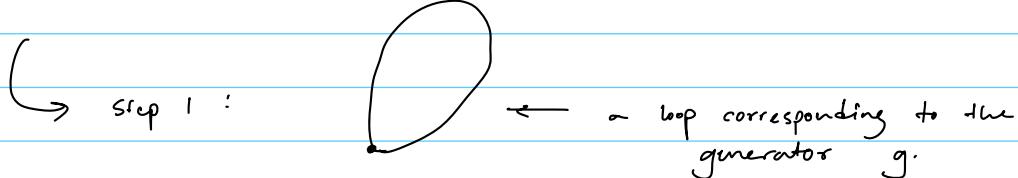
Then,  $G = F/N$  where  $N = \text{normal closure of } \{r_\beta\}$   
So,  $r_\beta = \text{ker}(\Phi)$  where  $\Phi : F \rightarrow G$ .

Construct  $X_G$  from  $\bigvee S_\alpha^1$  by attaching 2-cells  $e_\beta^2$  by the loops specified by the word  $r_\beta$ .

→ each circle corresponds to a generator  $g_\alpha$

Example :

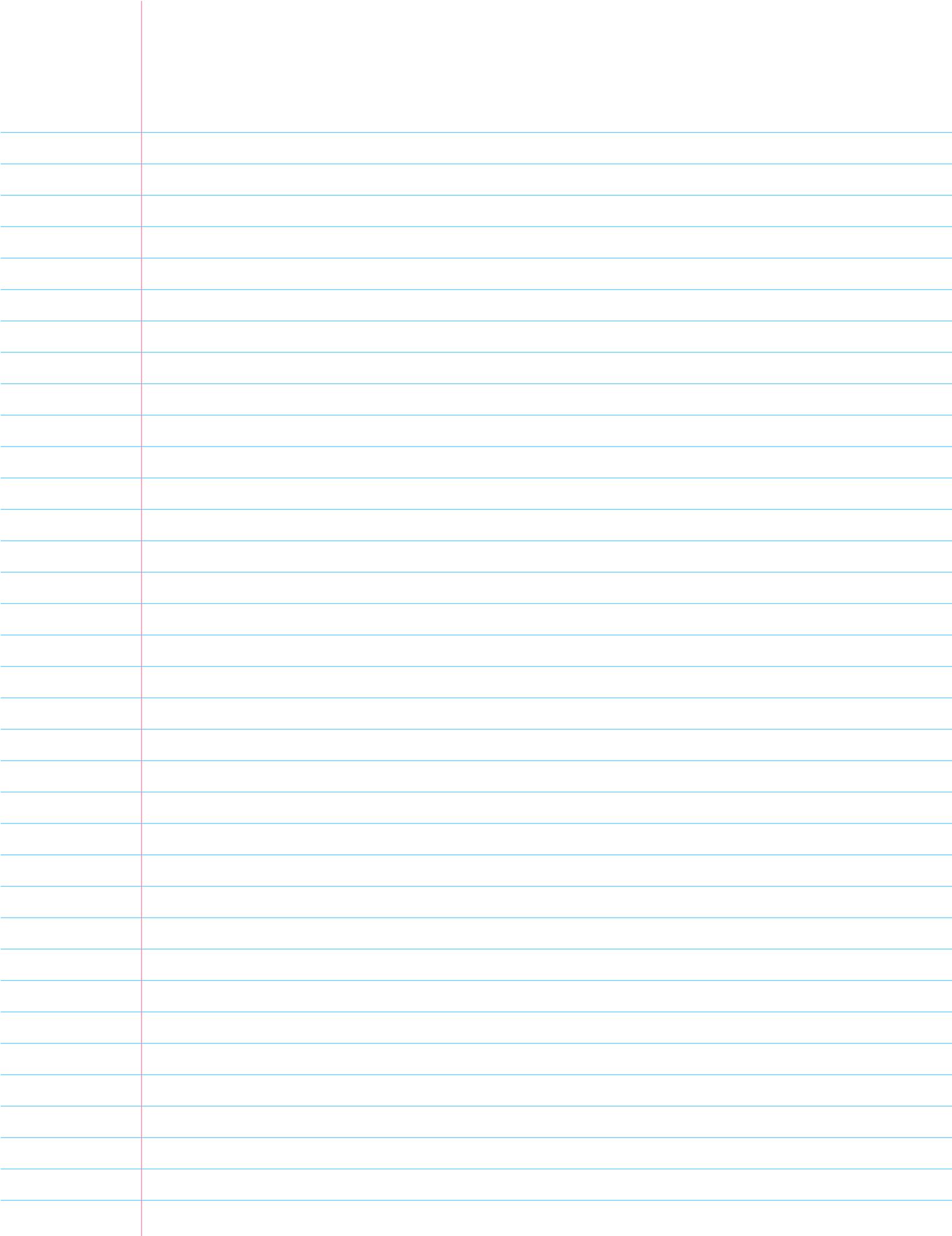
(1)  $\mathbb{Z}_2 = \langle g | g^2 = 1 \rangle$



Step 2 : attach a disk so that as we trace the boundary of the disk, we trace this loop twice.

∴ antipodal points are glued.

∴ we get  $\mathbb{RP}^2$  and  $\mathbb{Z}_2 \approx \mathbb{RP}^2$



## Covering Spaces

Goal: Classify covering spaces of  $X$  in terms of  $\pi_1(X)$ .

Algebraic aspects of the fundamental group  $\leftrightarrow$  geometric language of covering spaces

Review:

Def: Covering Spaces

$p: \tilde{X} \xrightarrow{\sim} X$  is covering means  $\forall x \in X, \exists$  nbhd  $U$  of  $x$  s.t.  $U$  is evenly covered i.e.  $p^{-1}(U) = \bigsqcup_{\tilde{x}} V_{\tilde{x}}$

s.t.  $p|_{V_{\tilde{x}}} : V_{\tilde{x}} \xrightarrow{\sim} U$  is a homeomorphism.

$U \rightarrow$  called "evenly covered",  $V_{\tilde{x}} \rightarrow$  called sheets of  $\tilde{X}$  over  $U$ .

$\rightarrow$  If  $U$  is connected,  $V_{\tilde{x}}$  are the connected components of  $p^{-1}(U)$ .

$\rightarrow$  When  $U$  is not connected, the decomposition of  $p^{-1}(U)$  may not be unique.

$\rightarrow p$  need not be surjective as  $p^{-1}(U)$  can be empty.

Example:

$$(1) p: \mathbb{R} \rightarrow S^1 \text{ by } p(s) = e^{2\pi i s} \in S^1$$

Some results we have already proven...

Provided covering spaces  $p: \tilde{X} \xrightarrow{\sim} X$

called the homotopy lifting property

(1) Given a map  $f: Y \times I \rightarrow X$  and a map  $\tilde{f}: Y \times \{0\} \xrightarrow{\sim} \tilde{X}$  that lifts  $f|_{Y \times \{0\}}$  (i.e.  $p \circ \tilde{f} = f|_{Y \times \{0\}}$  on  $Y \times \{0\}$ ), there exists a unique map  $\tilde{F}: Y \times I \rightarrow \tilde{X}$  lifting  $F: Y \times I \rightarrow X$  (i.e.  $p \circ \tilde{F} = F$  on  $Y \times I$ ) s.t. it agrees on  $Y \times \{0\}$

called the path lifting property

(2) For each path  $f: I \rightarrow X$  s.t.  $f(0) = x_0 \in X$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$ ,  $\exists$  a unique lift  $\tilde{f}: I \rightarrow \tilde{X}$  s.t.  $\tilde{f}(0) = \tilde{x}_0$  and so,  $f = p \circ \tilde{f}$

path homotopy lifting property

(3) For each homotopy  $f_t: I \rightarrow X$  of paths s.t.  $f_t(0) = x_0$  and each  $\tilde{x}_0 \in p^{-1}(x_0)$ ,  $\exists$  a unique lifted homotopy  $\tilde{f}_t: I \rightarrow \tilde{X}$  of paths s.t.  $\tilde{f}_t(0) = \tilde{x}_0$ .  $\therefore f_t = p \circ \tilde{f}_t$

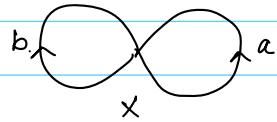
## More examples of covering spaces

(1) Consider  $S \subset \mathbb{R}^3$  consisting of points  $(s \cos(2\pi t), s \sin(2\pi t), t)$  for  $(s, t) \in (0, \infty) \times \mathbb{R}$

Then  $\rho: S \rightarrow \mathbb{R}^2 - \{0\}$  via the map  $(x, y, z) \mapsto (x, y)$

(2)  $\rho: S^1 \rightarrow S^1$  via  $\rho(z) = z^n$ ,  $z \in \mathbb{C}$  with  $|z|=1$   
 $n \in \mathbb{Z}_{>0}$ .

(3) Covering spaces of  $S^1 \times S^1 =: X$



Consider  $X$  to be a graph with one vertex and edges  $a$  and  $b \rightarrow a$  and  $b$  have some orientation.

Now, consider  $\tilde{X}$  to be another graph with four ends of edges at each vertex, similar to  $X$ , and, again, each edge is labelled either  $a$  or  $b$  and oriented in a way that there is

- 1  $a$ -edge going away
- 1  $a$ -edge going into
- 1  $b$ -edge going away
- 1  $b$ -edge going into

*can be  $> 1$  vertices*

Call  $\tilde{X}$  a 2-oriented graph.

Examples :

(1)	 $\langle a, b^2, bab^{-1} \rangle$	(2)	 $\langle a^2, b^2, ab \rangle$
(3)	 $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$	(4)	 $\langle a, b^2, ba^2b^{-1}, baba^{-1}b^{-1} \rangle$
(5)	 $\langle a^3, b^3, ab^{-1}, b^{-1}a \rangle$	(6)	 $\langle a^3, b^3, ab, ba \rangle$
(7)	 $\langle a^4, b^4, ab, ba, a^2b^2 \rangle$	(8)	 $\langle a^2, b^2, (ab)^2, (ba)^2, ab^2a \rangle$

Then, we can construct  $p: \tilde{X} \rightarrow X$  s.t  $p$  sends all vertices of  $\tilde{X}$  to the one vertex in  $X$  and sending each edge of  $\tilde{X}$  to an edge in  $X$  with the same label s.t  $p$  is a homeomorphism from the interior of the edge and preserves orientation.

## Lifting Properties

II Proposition: Homotopy Lifting Property (HLP) / Covering Homotopy Property

Given a covering space  $p: \tilde{X} \rightarrow X$ , a homotopy  $f_t: Y \rightarrow X$  and a map  $\tilde{f}_0: Y \rightarrow \tilde{X}$  lifting  $f_0$ , then there exists a unique homotopy  $\tilde{f}_t: Y \rightarrow \tilde{X}$  of  $\tilde{f}_0$  that lifts  $f_t$ .

Proof:

We already proved:

Given a map  $F: Y \times I \rightarrow X$  and a map  $\tilde{F}: Y \times \{0\} \rightarrow \tilde{X}$  that lifts  $F|_{Y \times \{0\}}$  (i.e.,  $p \circ \tilde{F} = F|_{Y \times \{0\}}$  on  $Y \times \{0\}$ ), there exists a unique map  $\tilde{F}: Y \times I \rightarrow \tilde{X}$  lifting  $F: Y \times I \rightarrow X$  (i.e.,  $p \circ \tilde{F} = F$  on  $Y \times I$ ) s.t. it agrees on  $Y \times \{0\}$ .

II By the path lifting property's uniqueness, every lift of the constant path is constant.

Proposition: (a) The map  $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \rightarrow \pi_1(X, x_0)$  induced by  $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$

1.31 is injective

(b)  $\text{Im}(p_*(\pi_1(\tilde{X}, \tilde{x}_0)))$  in  $\pi_1(X, x_0)$  consists of a homotopy class of loops in  $X$  based at  $x_0$  whose lifts to  $\tilde{X}$  starting at  $\tilde{x}_0$  are loops.

$$\text{Im}(p_*) = \{ [f] \in \pi_1(X, x_0) \mid f \text{ lifts to a loop at } \tilde{x}_0 \}$$

Proof:

(a)  $\text{ker}(p_*)$  consists of loops s.t. each belongs to the homotopy class of a loop  $\tilde{f}_0: I \rightarrow \tilde{X}$  with a homotopy  $f_t: I \rightarrow X$  of  $f_0 = p \circ \tilde{f}_0$  to the trivial loop  $f_1$ . Then, we can find a homotopy  $\tilde{f}_t: I \rightarrow \tilde{X}$  s.t.  $\tilde{f}_0$  starting from  $\tilde{x}_0$  and ending with the constant loop (since the lift of the constant loop is constant)  $\rightarrow$  we do this using path homotopy lifting property  $\therefore [\tilde{f}_0] = 0$  in  $\pi_1(\tilde{X}, \tilde{x}_0) \Rightarrow p_*$  is injective

(b)  $\supseteq$ : this is obvious as if  $f$  lifts to a loop  $\tilde{f}$  in  $\tilde{x}_0$   
 then,  $p_*(\tilde{f}) = p \circ \tilde{f} = f$  by definition

$\subseteq$ : Suppose.  $[f] \in \pi_1(X, x_0)$  represents an element of  
 the image of  $p_*$ .

Then  $[f] \cong$  a loop in the image of  $p_*$ .

Let  $[f] \cong \underbrace{p_*(\tilde{g})}_{\text{a loop that lifts to a loop in } \tilde{X}}$

Then by <sup>path</sup> homotopy lifting,  $[f]$  itself can be lifted  
 to a loop in  $X$ .

### Proposition:

The no. of sheets of a covering space  $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$   
 with  $X$  and  $\tilde{X}$  path-connected equals the  
 $\frac{\text{index}}{\text{L}} \text{ of } p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \text{ in } \pi_1(X, x_0)$   
 $\hookrightarrow$  no. of distinct cosets of the subgroup

### Proof:

Let  $g$  be a loop based at  $x_0$  in  $X$

Let  $\tilde{g}$  be the lift of  $g$  in  $\tilde{X}$  starting at  $\tilde{x}_0$ .

A product  $h \cdot g$  with  $[h] \in H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$  has the lift  
 $\tilde{h} \cdot \tilde{g}$  ending at the same point as  $\tilde{g}$  since  $\tilde{h}$  is a loop.

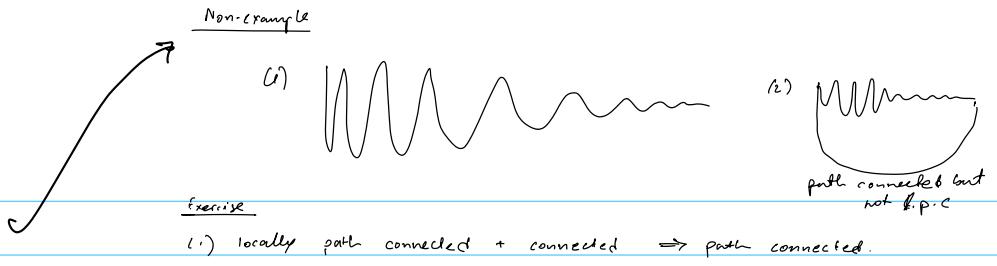
$\therefore$  We can define  $\Phi$  from cosets  $H[g]$  to  $p^{-1}(x_0)$  by  
 sending  $H[g]$  to  $\tilde{g}(1)$ .

$\rightarrow \Phi$  is surjective:  $\tilde{X}$  is path connected, so  $\tilde{x}_0$  can  
 be joined to any point in  $p^{-1}(x_0)$   
 by a path  $\tilde{g}$  projecting to a loop  
 $g$  at  $x_0$

$\rightarrow \Phi$  is injective:  $\Phi(H[g_1]) = \Phi(H[g_2])$

$\Rightarrow g_1 \cdot \overline{g_2}$  lifts to a loop in  $\tilde{X}$   
 based at  $\tilde{x}_0$ , so

$$[g_1][g_2]^{-1} \in H \Rightarrow H[g_1] = H[g_2]$$



Def: Locally path connected

A space  $Y$  is called locally path connected if  $\forall y \in Y$ ,  $\forall$  nbhd  $V$  of  $y$ ,  $\exists V \subset U$  open.  $y \in V$ ,  $V$  is path connected.

Proposition: (Lifting Criterion)

Suppose we have a covering space  $p: (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$  and a map  $f: (Y, y_0) \rightarrow (X, x_0)$  with  $Y$  path connected and locally path-connected. Then, a lift  $\tilde{f}: (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$  of  $f$  exists if and only if  $f_* (\pi_1(Y, y_0)) \subset p_* (\pi_1(\tilde{X}, \tilde{x}_0))$ .

Proof:

If  $\exists$  a lift  $\tilde{f}: (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ , of  $f$ , then

$$p \circ \tilde{f} = f$$

$$f_* = p_* \circ \tilde{f}_*$$

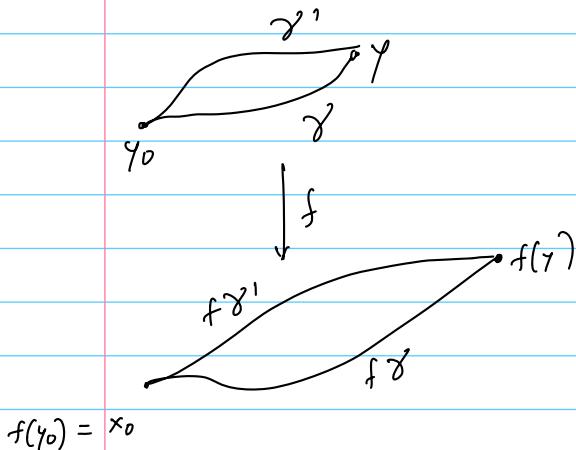
$$\text{So, } f_* \in \text{Im } p_*$$

Conversely, let  $y \in Y$  and let  $\gamma$  be a path in  $Y$  from  $y_0$  to  $y$ .

The path  $f\gamma$  in  $X$  from  $x_0$  has the unique lift (by path lifting property)  $\tilde{f}\gamma$  starting from  $\tilde{x}_0$ .

$$\text{Let } \tilde{f}(y) = \tilde{f}\gamma(1)$$

This is  $\rightarrow$  well-defined:



Let  $\gamma'$  be a different path from  $y_0$  to  $y$ . Then  $(f\gamma') \cdot \overline{(f\gamma)}$  is a loop at  $x_0$  as shown in the diagram on the left.

Call the loop  $h_0 := (f\gamma') \cdot \overline{(f\gamma)} : I \rightarrow X$

Hence  $[h_0] \in f_* (\pi_1(Y, y_0)) \subset p_* (\pi_1(\tilde{X}, \tilde{x}_0))$

by the def as  
 $h_0$  is defined by  
a loop in  $X$

by hypothesis

$\therefore$  There exists a homotopy  $h_t$  from  $h_0$  to a loop  $h_1$  in  $X$ . By path lifting property,  $h_1$  lifts to  $\tilde{h}_1$  in  $\tilde{X}$  based at  $\tilde{x}_0$ . Then, by path homotopy lifting,  $h_t$  lifts to  $\tilde{h}_t$ .

Now,  $\tilde{h}_i$  is a loop based at  $\tilde{x}_0$ , so  $\tilde{h}_0$  is also a loop based at  $\tilde{x}_0$ .  $\rightarrow$  in fact  $\tilde{h}_t$  are all loops at  $\tilde{x}_0$ .

By uniqueness of lifted paths, the first half of  $\tilde{h}_0$  is  $\tilde{f}\tilde{\gamma}'$  and the second half is  $(\tilde{f}\tilde{\gamma})$

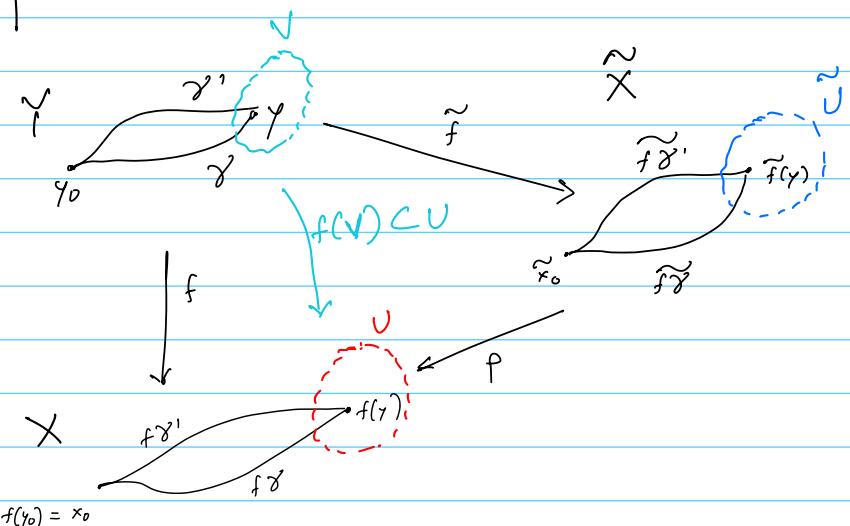
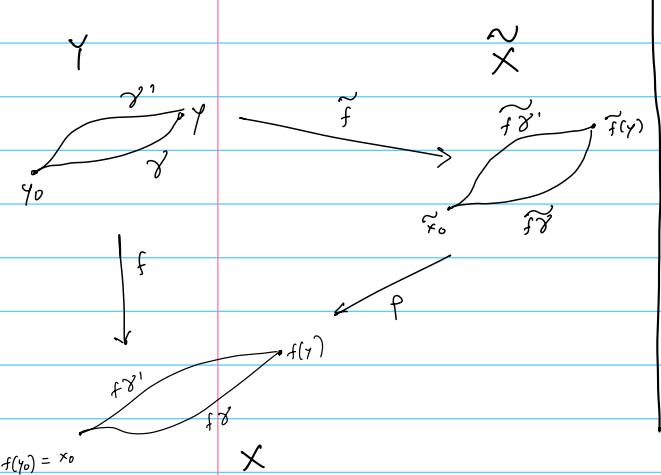
and the common midpoint is  ~~$\tilde{f}(\tilde{y})$~~

$$\tilde{f}\tilde{\gamma}'(1) = \tilde{f}\tilde{\gamma}(1)$$

$\therefore \tilde{f}(\tilde{y})$  is well-defined

$\rightarrow$  This map is continuous:

Let  $U \subset X$  be an open neighbourhood of  $f(y)$  having a lift  $\tilde{U} \subset \tilde{X}$  containing  $\tilde{f}(y)$  s.t  $p: \tilde{U} \rightarrow U$  is a homeomorphism. Now, choose a path connected open nbhd  $V$  of  $y$  with  $f(V) \subset U$ .



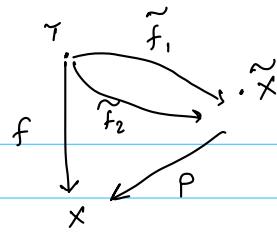
For paths from  $y_0$  to points  $y' \in V$ , take a path  $\gamma'$  from  $y_0$  to  $y$  (as above) and then a path  $\eta$  from  $y$  to  $y' \in V$  inside  $V$ .

Then, the paths  $(f\tilde{\gamma}).(\tilde{f}\eta)$  in  $X$  have lifts  $(\tilde{f}\tilde{\gamma}).(\tilde{f}\eta)$  where

$$\tilde{f}\tilde{\eta} = p^{-1}f\eta \text{ and } p^{-1}: V \xrightarrow{\sim} \tilde{V}$$

$$\therefore \tilde{f}(V) \subset \tilde{U} \text{ and } \tilde{f}|_V = p^{-1}f$$

$\therefore \tilde{f}$  is continuous at  $y$



Proposition: Unique Lifting Criterion

Given a covering space  $p: \tilde{X} \rightarrow X$  and a map  $f: Y \rightarrow X$ , if two lifts  $\tilde{f}_1, \tilde{f}_2: Y \rightarrow \tilde{X}$  of  $f$  agree at one point of  $Y$  and  $Y$  is connected, then  $\tilde{f}_1$  and  $\tilde{f}_2$  agree on all of  $Y$ .

Proof: For a point  $y \in Y$ , let  $U$  be an openly covered nbhd of  $f(y)$  in  $X$ . Then  $p^{-1}(U) = \coprod_{\alpha} \tilde{U}_{\alpha}$  s.t.  $\tilde{U}_{\alpha} \cong U$

Let  $\tilde{U}_1$  and  $\tilde{U}_2$  be the sheets containing  $\tilde{f}_1(y)$  and  $\tilde{f}_2(y)$  respectively.

By continuity of  $\tilde{f}_1$  and  $\tilde{f}_2$ ,  $\exists$  a neighbourhood  $N \ni y$  s.t.  $\tilde{f}_1$  maps  $N$  to  $\tilde{U}_1$  and  $\tilde{f}_2$  maps  $N$  to  $\tilde{U}_2$ .

Now, if  $\tilde{f}_1(y) \neq \tilde{f}_2(y)$ , then  $\tilde{U}_1 \neq \tilde{U}_2 \Rightarrow \tilde{U}_1$  and  $\tilde{U}_2$  are disjoint.

Take the union of all such  $N$  for every point  $y \in Y$  where  $\tilde{f}_1 \neq \tilde{f}_2$ , then the complement (where they agree) is closed.

If  $\tilde{f}_1(y) = \tilde{f}_2(y)$ , then  $\tilde{U}_1 = \tilde{U}_2 \Rightarrow \tilde{f}_1 = \tilde{f}_2$  on  $N$

as each  $\tilde{U}_{\alpha}$  is disjoint

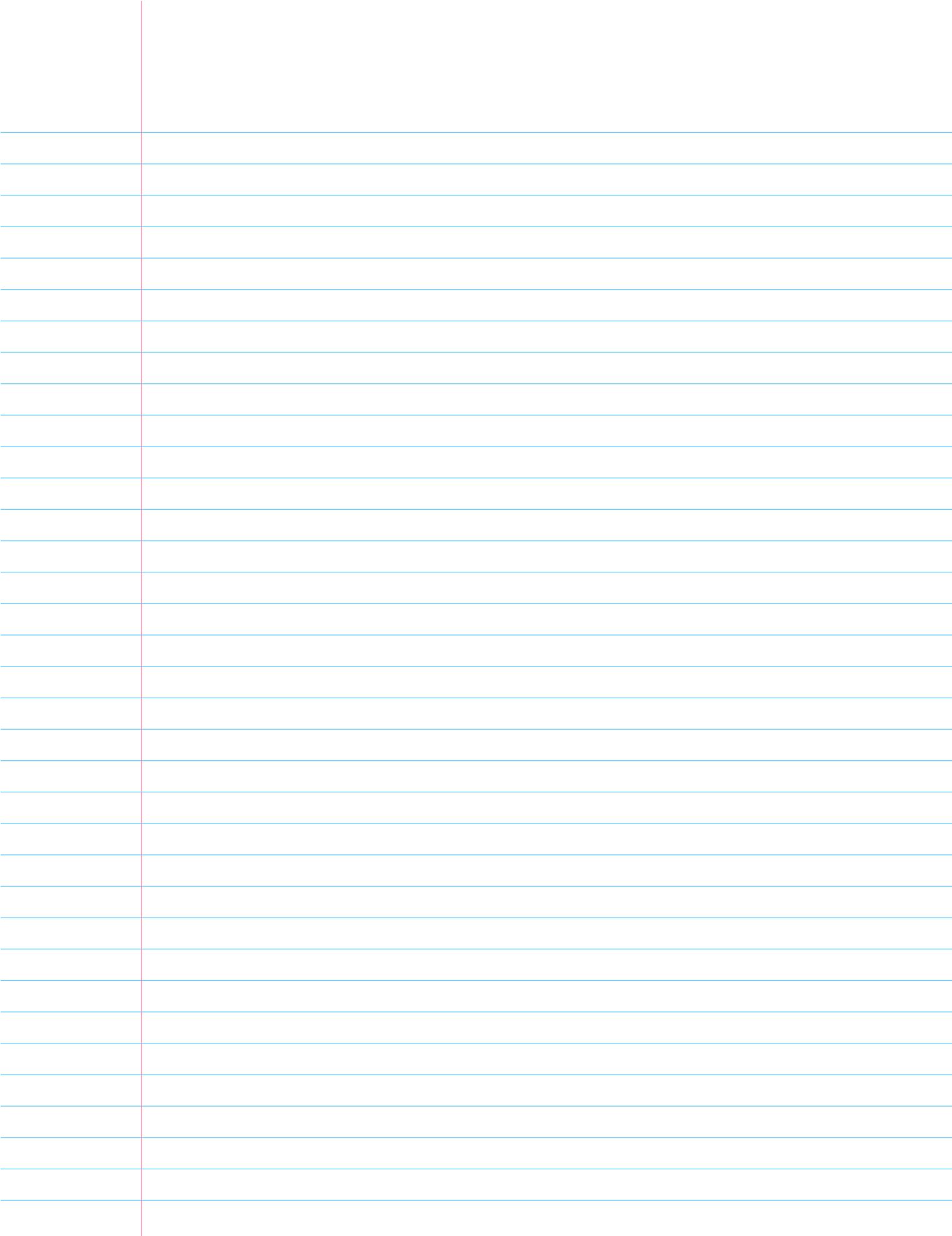
because  $p\tilde{f}_1 = p\tilde{f}_2$

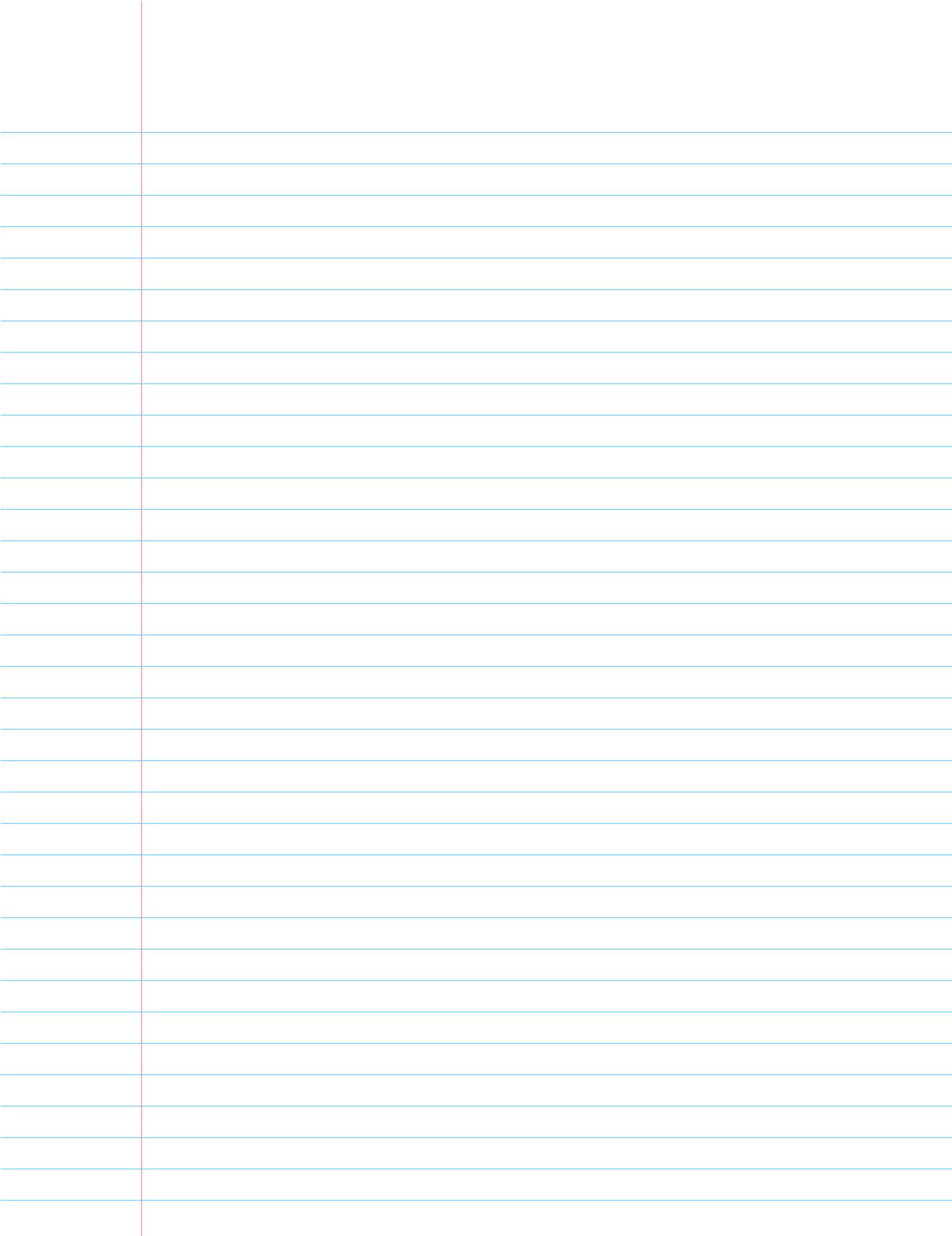
and  $p$  is injective on  $\tilde{U}_1 = \tilde{U}_2$

as both are equal to  $f$

Here  $N$  is open. Take the union of all such  $N$ , we get an open set.  $\therefore$  The set of points where  $\tilde{f}_1 = \tilde{f}_2$  is both open and closed in  $Y$

Since  $Y$  is connected, this set is all of  $Y$ .





### The lifting correspondence

$$\Phi : \pi_1(X, x_0) \longrightarrow \text{path } p^{-1}(x_0)$$
$$[f] \longmapsto \tilde{f}(\cdot) \quad \text{where } \tilde{f} \text{ is a lift of } f$$

Suppose,  $X, \tilde{X} \rightarrow$  path connected  
we had.  $\Phi$  is onto

$$\ker(\Phi) = \{[f] \mid f \text{ lift } f \text{ a loop}$$
$$= \text{Im}(p_*)$$

$$\therefore p^{-1}(x_0) = \pi_1(X, x_0) / \text{Im}(p_*)$$

Recall: If  $\pi_1(\tilde{X}) = 1$ , then  $\Phi$  is bijection  
when does such a cover exist?

Def: Universal covering space of  $X$  is a simply  
connected covering space.