UNIVERSITY OF OSLO

General relativistic model of a spinning cosmic string

Bjørn Jensen and Harald H. Soleng

Department of Physics, University of Oslo,
Box 1048 Blindern, N-0316 Oslo 3, Norway

Report 91-30
ISSN-0332-5571

Received: 1991-11-21

DEPARTMENT OF PHYSICS

REPORT SERIES

General relativistic model of a spinning cosmic string

Bjørn Jensen and Harald H. Soleng

Department of Physics, University of Oslo,
Box 1048 Blindern, N-0316 Oslo 3, Norway

Report 91-30
ISSN-0332-5571

Received: 1991-11-21

OUP--91-30

General relativistic model of a spinning cosmic string

Bjørn Jensen and Harald H. Soleng

Institute of Physics, University of Oslo,
P.O. Box 1048 Blindern, N-0316 Oslo 3, Norway

Abstract

We investigate the infinite, straight, rotating cosmic string within the framework of Einstein's General Theory of Relativity. A class of exact interior solutions is derived for which the source satisfies the weak and the dominant energy conditions. The interior metric is matched smoothly to the exterior vacuum.

A subclass of these solutions has closed time-like curves both in the interior and the exterior geometry.

PACS nos.: 04.20.-q 04.20.Jb 98.80.Dr

1 Introduction

The geometry outside a straight, static cosmic string with vanishing radius [1-3] represents a four-dimensional extension of the point particle solution of (2+1)-dimensional Einstein-gravity [4]. In three spacetime dimensions the Riemannian curvature depends algebraically on the Einstein curvature. Hence, it can be expressed entirely in terms of the local distribution of matter and energy. This makes vacuum solutions of three dimensional Einstein theory differ from Minkowski space-time only in their global topological properties [5].

Just as a static three-dimensional point particle solution becomes a four-dimensional cosmic string model, the three-dimensional "Kerr" metric [4,5], may be interpreted as a vacuum solution outside a cosmic string carrying angular momentum [6]. This geometry has attracted considerable interest, because of its interesting global topology. In addition to the conical topology of the usual string solution, there is a helical structure of time which gives rise to the possibility of closed time-like curves (CTC) near the source, as well as a gravitational time-delay [7]. When considering quantum theory on such a "spinning cone" [8], the causality violating region gives rise to unitarity problems [8], and it is appearantly responsible for making the Dirac Hamiltonian lose its self-adjoint character [9,10]. Semiclassical gravitational effects on a spinning cone have been considered by Matsas [11], who showed that the vacuum expectation value of the angular momentum of a massless conformally coupled scalar field is nonzero in this background geometry.

Recently a number of authors have pointed out that spinning point particles (in 2+1 dimensions), or strings (in 3+1 dimensions), behave as gravitational analogs to anyons — gravitational anyons [12-16]. Such particles, which display fractional statistics, were first shown to exist within the framework of quantum mechanics in two space dimensions [17].

Because of the patologies associated with the idealised infinitely thin string model, one should construct a more realistic model where the singularity at the tip of the cone is avoided. In the realistic model one might hopefully avoid CTC's. Thus it represents a space-time in which neither the problem of unitarity nor the lack of self-adjointness of the Dirac Hamiltonian appear. Moreover, such a space-time manifold is the proper setting

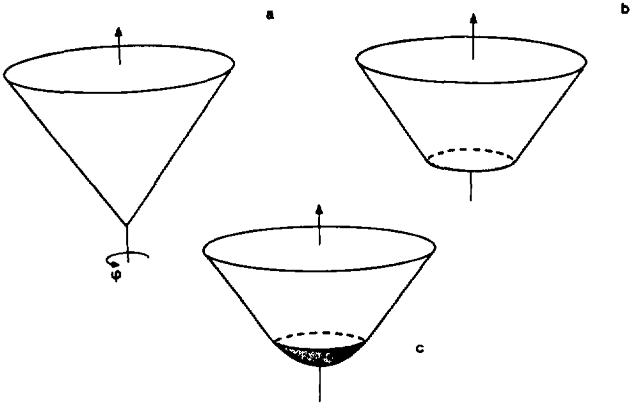

Figure 1: The figures show two-dimensional projections of the string geometry. (a) The source is a point singularity on the tip of a cone. (b) In the "flower-pot" model, the source is a ring at a finite radius. (c) In the "ballpoint pen" model the Einstein-curvature is distributed over a smooth cap surface.

for the study of semiclassical gravitational effects outside a spinning cosmic string. The singular tip of the cone could be replaced by an extended interior region with a nonvanishing Einstein curvature [18-20]. Hence, the tip is replaced by a smooth cap, producing a "ballpoint pen" model [21]. Alternatively the source could be concentrated on a ring of finite radius. Then the geometry is that of a "flower-pot" [21], see fig. 1).

It has been shown that when regarded as a line singularity, the space-time geometry for a spinning string exhibits torsion at the location of the source [22]. A "ballpoint pen" source for this geometry has been found within the Einstein-Cartan theory [23]. Here the source is a homogeneous cylinder with spin-polarisation along the axis of symmetry.

The belief that physically realistic sources will not produce CTC's, is often encountered. In [4], refering to three dimensional space-time, it is stated that "One can show that such closed time-like contours are not possible in a space with n moving spinless particles, (...)". Waelbroeck [24] has shown that a pair of cosmic strings with relative angular momentum does not have CTC's. A more general conviction is found in reference [25]: "We believe that this is a general result: closed time-like curves are absent for reasonable geometries." In reference [26] entitled *"Physical cosmic strings do not generate closed time-like curves"* it is proved, in the case of point particles, that "there are no CTC if the space-times have physically acceptable global structure, which they do for physically acceptable sources." A four-dimensional rotating dust cylinder, can produce closed time-like curves [27,28], however. This solution must be dismissed, it might be argued, since the exterior is not asymptotically flat, and hence unphysical. On the other hand it would be surprising if the causal structure of a realistic spinning cosmic string model differs fundamentally from that of a spinning dust cylinder. After all, the existence of CTC's is more of a topological issue than a geometrical one.

Here we consider both types of extended string models. We find that a rotating cosmic string cannot be given by a rotating "flower-pot". For the "ballpoint pen", we find exact solutions of Einstein's field equations. In the latter model we show that closed time-like curves can exist even if both the weak and the dominant energy conditions hold in the interior of the string. The loop-hole in the proof of non-existence of CTC's in reference [26] is the restriction to a source of n point particles, combined with the lack of gravitational attraction between such particles (strings) in three (four) spacetime dimensions.

2 Geometry of the spinning string

The line-element in both the exterior and interior regions will be assumed to be on the form

$$ds^2 = -(dt + Md\phi)^2 + dr^2 + A^2 d\phi^2 + dz^2 \qquad (1)$$

where M and A are functions of r only. Note that Lorentz invariance along the z-axis, implies that any solution of the four-dimensional field equations may also be interpreted as a solution of the corresponding (2+1)-dimensional equations.

In the exterior region, the metric is a flat vacuum solution [4]. Here

$$M = m \quad \text{and} \quad A = B(r + r_0) \quad , \tag{2}$$

where m, B and r_0 are constants. Locally this is the Minkowski metric in disguise ($t \equiv T - m\phi$), but the global topology is different. Firstly $m \neq 0$ induces a helical structure of time, and secondly $B < 1$ produces a conical topology. The constant, m, is determined by the angular momentum per length, J, by

$$m = 4GJ \ . \tag{3}$$

B is a measure of angle deficit of the cone, which is determined by the mass per length, μ, by

$$B \equiv 1 - 4G\mu \ . \tag{4}$$

Note that if $r < m$, the ϕ-coordinate becomes timelike, and because it is periodic, closed timelike curves do exist here. r_0 is a constant determining the origin of the exterior radial coordinate, so that the radial coordinates coincide in the interior and exterior coordinate systems.

For later convenience we define an orthonormal tetrad frame by

$$\left.\begin{aligned} \omega^0 &= dt + M d\phi \ , \\ \omega^1 &= A d\phi \ , \\ \omega^2 &= dz \ , \\ \omega^3 &= dr \ . \end{aligned}\right\} \tag{5}$$

We will match the interior region to the exterior vacuum, and thereby determine the properties of the string surface by means of Israel's formalism *of singular surfaces* [31]. The *energy-momentum surface density* of this surface is given by Lanczos energy-momentum tensor:

$$8\pi G S^i{}_j = [K^i{}_j] - \delta^i{}_j [K] \ . \tag{6}$$

where $K^i{}_j$ are components of the external curvature tensor, x^i are the coordinates in the hyperspace (t, ϕ, z), and $[\quad]$ signifies the discontinuity at the junction radius.

Since we are using a radial coordinate where $g_{rr} = 1$, the exterior curvature is given simply by

$$K_{ij} = -\frac{1}{2} g_{ij,r} \ . \tag{7}$$

Transforming K_{ij} to the tetrad basis (5), we get[1]

$$8\pi G S^0{}_0 = 8\pi G S^2{}_2 = [\frac{A'}{A}] \tag{8}$$

and

$$8\pi G S^0{}_1 = -8\pi G S^1{}_0 = -[\frac{M'}{2A}] \ . \tag{9}$$

3 The "flower-pot" model

The space-time region inside a uniformly rotating infinitely thin hollow cylinder is flat [29,30]. Therefore the interior space-time metric is given by

$$M = a \quad \text{and} \quad A = r^2 \tag{10}$$

A nonzero a is necessary to allow a continuous matching to the exterior vacuum solution.

The projection of the metric into the junction surface must be a continuous function over the junction. This gives

$$a = m \quad \text{and} \quad r_s = B(r_s + r_0) \tag{11}$$

where the subscript s denotes the value of the radial coordinate r at the junction surface.

Note that since the helical structure of time extends to the center of the interior solution, there is still a singularity at the axis. Consequently, a "flower-pot" model cannot provide a viable model of a non-singular extended string.

[1] Throughout the paper a prime denotes a derivative with respect to r

This conclusion is confirmed by calculating the Lanczos tensor of the string surface. Inserting the metric functions of the exterior and interior solutions into Eqs. (8) and (9), we get

$$8\pi G S^0{}_0 = 8\pi G S^2{}_2 = \frac{1}{r_s + r_0} - \frac{1}{r_s} = -\frac{4G\mu}{r_s} \tag{12}$$

and

$$8\pi G S^0{}_1 = -8\pi G S^1{}_0 = 0 \ . \tag{13}$$

This shows that the "flower-pot" model is a possible source of the angle deficit, because the mass per length of the singular surface

$$\mu_s = -\int_0^{2\pi} S^0{}_0 r_s d\phi = \mu \tag{14}$$

is exactly the one measured in the exterior vacuum, but the singular surface is not a source of the time-helical topology. Therefore we turn to a "ballpoint pen" model.

4 The "ballpoint pen" model

With the definition

$$\Omega \equiv \frac{M'}{2A} \ , \tag{15}$$

Einstein's field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ take the form

$$8\pi G\rho = 3\Omega^2 - \frac{A''}{A} \tag{16}$$

$$8\pi G p_\phi = \Omega^2 \tag{17}$$

$$8\pi G p_r = -\Omega^2 + \frac{A''}{A} \tag{18}$$

$$8\pi G p_z = \Omega^2 \tag{19}$$

$$8\pi G q_\phi = \Omega' \tag{20}$$

where ρ, p and q are the energy-density, the pressure and the heat flow relative to the reference frame (5), respectively. Due to Lorentz invariance along the z-axis, the same field equations, except (19), are valid in (2+1)-dimensions. Note that it is the rotation which allows us to construct a finite (2+1)-dimensional model with pressure. In the hydrostatic case such models do not exist [32]. It is this connection between rotation and pressure which rules out a rotating "flower-pot" as a source for the time-helical topology.

Petti [33] has shown that discrete rotating masses in General Relativity imply the presence of translational holonomy which is transformed into torsion by the limiting process which transforms discrete masses into continuous matter fields. According to the Einstein-Cartan theory, it is the *canonical* energy-momentum tensor which gives the correct local description of the energy and momentum of matter, whereas the *combined* energy-momentum tensor is the source of the metric field [34]. In simple classical spin fluid models one finds that the effective gravitational mass and pressure densities are the canonical ones minus the spin density squared. By Petti's correspondence one should expect that the exterior gravitational field of a limited rotating mass distribution should be determined by the *combined* density and pressures $\tilde{\rho} = \rho - \Omega^2$ and $\tilde{p} = p - \Omega^2$. These quantities imply the proper equation of state for a cosmic string

$$\tilde{\rho} = -\tilde{p}_z \quad \text{and} \quad \tilde{p}_\phi = \tilde{p}_r = 0 \ . \tag{21}$$

Note, that if we calculate the Tolman mass using these quantities we get zero, which is consistent with a flat exterior geometry.

To be able to integrate the field equations, we assume that the energy density is of the simple form

$$8\pi G\rho = \Lambda + 3\Omega^2 \ . \tag{22}$$

where Λ is a positive constant. Then it follows from Einstein's field equations that the weak energy condition is satisfied.

With the condition that the metric is Minkowski on the axis, the solution for A is

$$A = \frac{1}{\sqrt{\Lambda}} \sin(\sqrt{\Lambda} r) \ . \tag{23}$$

Without further specifications of the physical properties of the string, Ω remains undetermined. One alternative is to select a simple form of q_ϕ, and integrate the field equations. Choosing q_ϕ = constant, and demanding that the dominant energy condition

$$-T^0_{\ 0} - |T^0_{\ 1}| \geq 0 \tag{24}$$

is satsified, the energy flux is restricted by $|8\pi G q_\phi| = |\Omega'| \leq \lambda$. Thus

$$\Omega' = -\alpha\lambda \ , \tag{25}$$

where $|\alpha| \leq 1$.

To match this solution with an external vacuum we must demand that the radial pressure vanishes at the junction radius. Hence, if we let r_s stand for the value of r at the radius of the string, we have to set $\Omega(r_s) = 0$ or

$$\Omega = \alpha\lambda(r_s - r) \ . \tag{26}$$

With the definition (15), and the condition that $M(0) = 0$, we find

$$M(r) = 2\alpha\left[(r - r_s)\cos(\sqrt{\lambda}r) - \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}r) + r_s\right] \tag{27}$$

The condition for existence of closed time-like curves is $g_{\varphi\varphi} < 0$ for some values of r. Then the periodic coordinate φ becomes time-like. Hence, we must demand

$$A^2 - M^2 < 0 \ . \tag{28}$$

By choosing the parameters α and r_s one may construct models with and without CTC's, which satisfy the weak and the dominant energy conditions. In figure (2) we present a graph of $g_{\phi\phi}$ as function of r for a model with CTC's.

The total effective mass per unit length μ, and angular momentum per unit length J of the source, is given by the integrals

$$\mu = \frac{1}{8\pi G}\int_0^{2\pi}\int_0^{r_s} \omega^1\omega^2 \tag{29}$$

and

$$J = \int_0^{2\pi}\int_0^{r_s} \sigma\omega^1\omega^2 \tag{30}$$

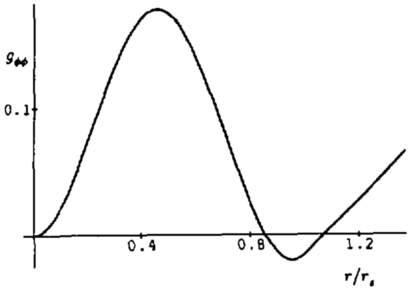

Figure 2: The figure shows $g_{\phi\phi}$ in units of λ^{-1} as a function of r/r_s with the parameter values $\alpha = 1$, and $\sqrt{\lambda}r_s = 3/2$. There are CTC's in the region where $g_{\phi\phi}$ is negative.

where $\sigma = \Omega/4\pi G$ is the angular momentum density, and $\omega^1\omega^2 = A dr d\phi$ is the area element of the string section. This gives

$$\mu = \frac{1}{4G}[1 - \cos(\sqrt{\lambda}r_s)] \tag{31}$$

and

$$J = \frac{1}{4G}M(r_s) \ . \tag{32}$$

This solution is to be matched to an exterior vacuum solution. The metric projected into the junction surface has to be a continuous function of r. This yields

$$M(r_s) = m \quad \text{and} \quad A(r_s) = B(r_s + r_0) \ . \tag{33}$$

Also the first derivatives of the metric must be continuous. Hence, the solution must also satisfy the conditions

$$[A'] = 0 \quad \text{and} \quad [M'] = 0 \tag{34}$$

The conditions (33) and (34) are met provided

$$r_0 = [\frac{(1-(1-4G\mu)^2)^{1/2}}{(1-4G\mu)\arccos(1-4G\mu)} - 1]r_s \tag{35}$$

We have constructed a physical source for all the three constants of the exterior vacuum solution. $B = 1 - 4G\mu$ is a measure of the total mass per length of the string, and the spin parameter $m = 4GJ$ measures the total angular momentum per unit length of the string.

In the limit $m \to 0$ and $\Omega \to 0$, the model reduces to the Gott-Linet-Hiscock model [18-20].

5 Conclusion

The fact that the exterior metric is flat, is consistent with the fact that the Tolman mass of the string is zero. Therefore, a string as discussed above, has no gravitational mass, and all the gravitational effects are of topological origin. Hence, a freely falling test-particle feels no gravitational attraction outside the string, but because of the conical topology, two parallel straight lines passing on different sides of the string will converge and cross each other.

As opposed to the Einstein-Cartan spinning string model there is no homogeneous interior solution in Einstein's theory. Both in Einstein's theory, and in the Einstein-Cartan theory, there is no fundamental law which forbids causality violating regions in the space-time of a spinning cosmic string. This, however, does not mean that such objects really exists in nature. On the contrary, for realistic strings the angular momentum density is too small for the phenomenon to exist [23].

But even if the angular momentum density is large enough to produce CTC's there are other mechanisms present which could prevent their formation. There are for instance indications that rotating objects will radiate away sufficient angular momentum to prevent CTC's to come into existence [35]. It may also be that quantum distortions of the classical space-time prevent the creation of CTC's [36]. Furthermore, as pointed out in [25], it is reasonable to believe that the history of formation of a system should be given by a Cauchy evolution of space-like surfaces. This puts severe restrictions on the possibility of occurence of CTC's as shown by Tipler [37]; "(...) closed time-like lines cannot evolve from regular initial data in a singularity-free asymptotically flat spacetime" (which satisfies the weak energy condition and the generic condition). On the other hand, however appealing the idea that the history of any system should be constructable by a Cauchy evolution might be, there do not exist compelling reasons based on physical arguments that the Universe really admits a Cauchy surface [38]. One should also note that Tipler's theorem does not hold for the spinning string, because the generic condition states that any non space-like geodesics must feel tidal forces in at least one point. This condition cannot be satisfied in the space-time exterior to a finite matter distribution in three dimensional space-time, because of the complete absence of tidal forces in vacuum. An infinite straight string is of course an idealisation. It is tempting to assume that space-time close to a very long string does not differ much from the space-time outside an infinte one. But it is risky to claim that properties of an infinite cylinder also holds true for a finite one [39]. This is particularly evident here, since the space-time geometry exterior to a *finite* string is curved, and therefore satisfies the generic condition.

References

[1] L. Marder, Proc. Roy. Soc. London, **252**, 45 (1959).

[2] D. D. Sokolov and A. A. Starobinskii, Sov. Phys. Dokl. **22**, 312 (1977).

[3] A. Vilenkin, Phys. Rev. D **23**, 852 (1981).

[4] S. Deser, R. Jackiw, G. 't Hooft, Ann. Phys. **152**, 220 (1984).

[5] S. Giddings, J. Abbott, and K. Kuchar, Gen. Rel. Grav. **16**, 751 (1984).

[6] P. O. Mazur, Phys. Rev. Lett. **57**, 929 (1986).

[7] D. Harari and A. P. Polychronakos, Phys. Rev. D **38**, 3320 (1988).

[8] P. de Sousa Gerbert and R. Jackiw, Commun. Math. Phys. **124**, 229 (1989).

[9] P. Gerbert, Phys. Rev. D **40**, 1346 (1989).

[10] R. Jackiw, *Five lectures on planar gravity*, SILARG VII, Cocoyoc, Mexico, December (1990).

[11] G. E. A. Matsas, Phys. Rev. D **42**, 2927 (1990).

[12] S. Deser, Phys. Rev. Lett. **64**, 611 (1990).

[13] S. Deser and J. G. McCarthy, Nucl. Phys. **B344**, 747 (1990)

[14] M. Reuter, Phys. Rev. D **44**, 1132 (1991).

[15] M. E. Oritz, Nucl. Phys. **B363**, 185 (1991).

[16] Y. M. Cho, D. H. Park and C. G. Han, Phys. Rev. D **43**, 1421 (1991).

[17] J. M. Leinaas and J. Myrheim, Nuovo Cimento **B37**, 1 (1977).

[18] J. R. Gott III, Astrophys. J. **288**, 422 (1985).

[19] B. Linet, Gen. Rel. Grav. **17**, 1109 (1985).

[20] W. A. Hiscock, Phys. Rev. D **31**, 3288 (1985).

[21] B. Allen and A. C. Ottewill, Phys. Rev. D **42**, 2669 (1990).

[22] P. de Sousa Gerbert, Nucl. Phys. **B346**, 440 (1990).

[23] H. H. Soleng, *A Spinning String*, Gen. Rel. Grav. (in press).

[24] H. Waelbroeck, Gen. Rel. Grav. **23**, 219 (1991).

[25] R. Jackiw, *Update on planar gravity. (≈ Physics of infinite cosmic strings)*, 6th Marcel Grossmann Meeting, Kyoto, Japan, June (1991).

[26] S. Deser, R. Jackiw, and G. 't Hooft, *Physical cosmic strings do not generate closed time-like curves.* Center for Theoretical Physics, MIT preprint CTP 2011 (1991).

[27] F. J. Tipler, Phys. Rev. D **9**, 2203 (1974).

[28] W. Bonnor, J. Phys. A: Math. Gen. **13**, 2121 (1980).

[29] H. Davies and T. A. Caplan, Proc. Camb. Phil. Soc. **69**, 325 (1971).

[30] E. Frehland, Commun. Math. Phys. **26**, 307 (1972).

[31] W. Israel, Il Nuovo Cimento **44B**, 1 (1966).

[32] *N. J. Cornish and N. E. Frankel, Phys. Rev. D* **43**, *2555 (1991).*

[33] R. J. Petti, Gen. Rel. Grav. **18**, 441 (1986).

[34] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).

[35] N. J. Charlton, J. Phys. A: Math. Gen. **11**, 2207 (1978).

[36] S.-W. Kim and K. S. Thorne, Phys. Rev. D **43**, 3929 (1991).

[37] F. J. Tipler, Phys. Rev. Lett. **37**, 879 (1976).

[38] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time* Ch. 6, (Cambridge University Press, Cambridge, 1973).

[39] K. S. Thorne, Comments Astrophys. and Space Phys. **2**, 191 (1970)

FYSISK INSTITUTT FORSKNINGS-GRUPPER

Biofysikk

Elektronikk

Elementærpartikkel-fysikk

Faste stoffers fysikk

Kjernefysikk

Plasma-, molekylar- og kosmisk fysikk

Strukturfysikk

Teoretisk fysikk

DEPARTMENT OF PHYSICS RESEARCH SECTIONS

Biophysics

Electronics

Experimental Elementary Particle physics

Condensed Matter physics

Nuclear physics

Plasma-, Molecular and Cosmic physics

Structural physics

Theoretical physics

ISSN - 0332 - 5571