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A summary of The Capacity of Hopfield Associative Memory
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1 Motivation

The purpose of the paper is to determine the capacity of Hopfield Neural network structure for information storage. Given an m-dimensional set of vectors, called fundamental memory. The authors seek to evaluate how large m can be. By using, an intial *probe* vector and a Hopfield connection matrix, they explore how much memory can be recovered.

2 Introduction

The authors liken Hopfield networks to the human brain, stating that the processing nodes are like human neurons connected with synaptic conduits. In Hopfield networks these *neurons* are represented by bistable elements, which take a value of +1 or -1, and by extension the state of the system can be represented as a binary n-tuple. A Neuron i would transmit information to a neuron j, via an interconnection weight T_{ij}. The current state of the system is represented with a *state vector*. The system is updated constantly at each neuron using the following rule:
Rule 1:

$$x_i^{'} = sgn(\sum_{j=1}^{n} T_{ij}x_j) = \{+1if \sum T_{ij}x_j \geq 0, -1if \sum T_{ij}x_j < 0\}$$

It is also worth of note to highlight the important features of these associative networks:

- A robust distributed information processing that results naturally from collective system dynamics
- The simplicity of the neurons or individual processing nodes evidenced by the simple bistable values the nodes can have and simplicity of the state change rules.
- The magnificent parallelism in information processing from the flow of information in the system, that is, several operations happening at the same time.

3 States and Memory

There are two methods of state change explored in this paper - a synchronous operation, in which each neuron is updated simultaneously using rule 1 above and an asynchronous operation in which the components of the current state vector are updated one at a time. One of the goals of the network is that the final state is as close to the stored memory as possible. There are two essentials components of the association i.e the network; a memory encoding rule (how to store memory) and the capacity of the network to recall the stored memories with error correction. To encode memory, a connection weight matrix is built.

$$\text{Let } \mathbf{x} = \{x^1, x^2....x^m\}$$

Each x is n-dimensional and referred to as the **fundamental memory**. For each x^α we form:

$$T_\alpha = x^\alpha (x^\alpha)^T - I_n$$

The Hopfield connection matrix is the sum of T_α over "m", which is also referred to as the **sum of outer products** because each T_α is constructed from the outer product of x^α with itself.

4 Stability

In exploring stability, the authors assume a **forced choice** model, where some of the memories are known and others have to be recovered. It is assumed that $(1-\rho)n$ elements are known and ρn elements are wrong. The largest possible ρn is called the **radius of attraction**. There two ways to achieve stability or convergence in the asynchronous operation discussed in this paper:

- Every state change or transition is a step in the right direction or the radius of attraction is directly attracted to the fundamental memory.
- A random step is in the right direction, and after enough steps the probe(the initial state vector) is close enough to the fundamental memory with a high probability(close to 1, but not 1), then all following changes are in the right direction.

5 Capacity

Capacity here refers to the rate of growth rather than an exact number. This is because an information theory approach is being used. The goal is to find the upper bound of recoverable memories. We choose $m = m(n)$ memories, where n is the length of the memory and each memory can take a value of either +1 or -1. We assume that we have a fixed ρ where $0 \leq \rho < \frac{1}{2}$. The largest rate of growth, i.e. our the capacity of the network is $m(n), n \to \infty$

The authors present a simple derivation for the rate of growth, note that there are several cases of the capacity derivation, all of which will be shown in the conclusion. We assume that the first memory $x^{(1)}$ has only values of +1. We model $n(m-1)$ components of the remaining $(m-1)$ memories as an independently and identically distributed random variables. The goal of this simple derivation is to find the probability that x is a fixed point. This is proved to be approximately equal to

$$\beta = exp\{-nQ(\sqrt{\frac{n}{m}}\}$$

If we require that the probability is a fixed number, close to 1, we will be able to get an approximately value for m, our capacity such that:

$$m \ \frac{n}{2logn}$$

6 Conclusion and Unsolved problems

There are four expressions that can be used to describe the asymptotic capacity of a Hopfield network.

- For a Hopfield memory of length n when a random, independent probability is shown with probability 1/2 fundamental memories to store and when probing with an n-tuple probe at a maximum distance of ρn away from a fundamental memory $(0 \leq \rho < 1/2)$ the capacity is defined by:

$$\frac{(1-2\rho)^2}{2}\frac{n}{logn}$$

- If no fundamental memory can be exceptional, the capacity is defined by:

$$\frac{(1-2\rho)^2}{4}\frac{n}{logn}$$

- If $0 \leq \rho < \frac{1}{2}$, ρ is given, some errors are permitted and we have a small fraction of fundamental memories. The capacity is defined by:

$$\frac{n}{2logn}$$

- If no fundamental memory can be exceptional and no wrong moves are permitted, the capacity is defined by:

$$\frac{n}{4logn}$$

In the above four equations, we can see that each pair is similar, this is because if no fundamental memory is exceptional the capacity doubles. We must also note that the capacities are the same in both the synchronous and asynchronous models. The authors also note that there are fixed points, which are not fundamental memories. This is not fully understood and they do not know precisely how to calculate the exact number of fixed points.