This theorem applies to a single threshold element acting on a weighted set of inputs.

"A satisfactory assignment of weights from the associator units is defined as an assignment resulting in a response +1 for signals of Class I and −1 for signals of Class II"

When there is a misclassification or what the author would describe as an unsatisfactory assignment, the weights are re-adjusted by the method known as "error correction".

"The theorem asserts ~~that~~ no matter what assignments of weights we begin with the process of re-adjusting the weights by the method known as "error-correction" will terminate after a finite number of corrections in a satisfactory assignment, provided such satisfactory assignment exists"

Definition of symbols.

ω_i — activity of associators

s_i — stimulus

y — satisfactory assignment.

$\{\omega_{i_n}\}$ - training sequence

$\{v_n\}$ — weights.

θ — threshold

$(\omega_i, v) > \theta$ correct output / classification

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$(w_i, v) \leq \theta$ wrong output/classification.

The theorem basically implies that if a linear seperator exists in the training sequence you will not have to continuously perform "error correction", that is you will not have to continuously update the weights

Proof of theorem

Let $w_1, \ldots w_N$ be a set of vectors in a Euclidean space of fixed finite dimension.

Our Remark.

A training sequence is always finite.

$$(w_i, y) > \theta > 0 \quad i = 1, \ldots, N \quad — ①$$

v_0 is arbitrary

Our remark.

Initial weights are picked based on the scientists discretion.

$$v_n = \begin{cases} v_{n-1} & \text{if } (w_{i_n}, v_{n-1}) > \theta \\ v_{n-1} + w_{i_n} & \text{if } (w_{i_n}, v_{n-1}) \leq \theta \end{cases} \quad — ②$$

The theorem asserts that v_n is convergent.

Terms where $v_n = v_{n-1}$ are omitted. This represents points that have been correctly classified, hence are in no need of error correction.

$$v_n = v_{n-1} + w_{i_n} \text{ and } (w_{i_n}, v_{n-1}) \leq \text{ for each } n \quad \text{— ③}$$

n - number of corrections made up to nth step.

The theorem asserts that ① implies ④ below

$$\|v_n\|^2 > Cn^2$$

for a suitable choice of the positive constant C, and n sufficiently large.

if $v_n = v_0 + w_{i_1} + \dots w_{i_n}$

then $(v_n, y) > (v_0, y) + n\theta$

Cauchy Schwarz inequality states that for two vectors x and y

$$\|x\|^2 \|y\|^2 \geq (x, y)^2$$

Thus

$$\|v_n\|^2 \|y\|^2 \geq (v_n, y)^2 > [(v_0, y) + n\theta]^2$$

$$\|v_n\|^2 \geq \frac{(v_n, y)^2}{\|y\|^2} > \frac{[(v_0, y) + n\theta]^2}{\|y\|^2}$$

$$\frac{[(v_0, y) + n\theta]^2}{\|y\|^2} = \frac{1}{\|y\|^2}\left[[(v_0, y) + n\theta][(v_0, y) + n\theta] \right]$$

$$= \frac{1}{\|y\|^2}\left[(v_0,y)^2 + 2(v_0,y)n\theta + n^2\theta^2\right]$$

$$= \frac{\theta^2}{\|y\|^2}\left[\frac{(v_0,y)^2}{\theta} + \frac{2(v_0,y)n}{\theta} + n^2\right]$$

since $a^2 + 2ab + b^2 = (a+b^2)$

let $a = n$

$b = \frac{(v_0,y)}{\theta}$

$$= \frac{\theta^2}{\|y\|^2}\left[n + \frac{(v_0,y)}{\theta}\right]^2$$

If $(v_0,y) \geq 0$ we may choose $C = \frac{\theta^2}{\|y\|^2}$

since $\frac{v_0,y}{\theta}$ ~~would~~ can be discarded

if n is sufficiently large (one of our initial assumptions).

The proof asserts that (3) ~~shows to~~ implies

$$\|v_n\|^2 \leq \|v_0\|^2 + (2\theta + M)n \quad (5)$$

where

$$M = \max_{i=1,\dots,N} \|w_i\|^2$$

from ③ we have,

$$\|V_n\|^2 = (V_{n-1} + w_n)^2$$

$$\|V_n\|^2 = \|V_{n-1}\|^2 + 2(V_{n-1}, w_n) + \|w_n\|^2$$

$$\|V_n\|^2 - \|V_{n-1}\|^2 = 2(V_{n-1}, w_n) + \|w_n\|^2$$

since $(V_{n-1}, w_n) \leq \theta$ & $\|w_n\|^2 \leq M$

$$2(V_{n-1}, w_n) + \|w_n\|^2 \leq 2\theta + M$$

For each k, the following is satisfied

$$\|V_k\|^2 - \|V_{k-1}\|^2 = 2(V_{k-1}, w_{i_k}) + \|w_{i_k}\|^2 \leq 2\theta + M \quad — ⑥$$

If we sum ⑥ above for $k = 1, 2, \dots, n$
since $V_n = V_0 + w_{i_1} + \dots + w_{i_n}$

$$V_n = \sum_{i=1}^{n} V_0 + w_{i_n}$$

$$\|V_n\|^2 \leq \sum \|V_{n-1}\|^2 + 2\theta + M$$

$$\|V_n\|^2 \leq \|V_0\|^2 + \sum_{i=1}^{n} (2\theta + M)$$

$$\|V_n\|^2 \leq \|V_0\|^2 + (2\theta + M)n . \quad — ⑤$$

⑤ above shows that the sequence V_n is convergent and since we have shown ⑤ to be true, we can assert that the proof is correct.

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