Problem 1

Given, $J(n) = \frac{1}{2}|e(n)|^2 = \frac{1}{2}\left[y_d(n) - y(n)\right]^2$

$y(n) = \sum_{K=1}^{N} w_k(n)\, \phi\{x(n), c_K, \sigma_K\}$ and $e(n) = \left[y_d(n) - y(n)\right]$

(i) The weight update equation is given by -

$$w(n+1) = w(n) - \mu_w \frac{\partial}{\partial w} J(n)\Big|_{w=w(n)}$$

Substituting the value of $J(n)$ and $y(n)$ in above equation,

$$w(n+1) = w(n) - \mu_w \frac{\partial}{\partial w}\left[\frac{1}{2}\left[y_d(n) - \sum_{K=1}^{N} w_k(n)\, \phi\{x(n), c_K, \sigma_K\}\right]^2\right]$$

$$= w(n) - \mu_w \left[\frac{1}{\cancel{2}} \times \cancel{2}\left[y_d(n) - \left(w_1(n)\,\phi\{x(n), c_1, \sigma_1\} + w_2(n)\,\phi\{x(n), c_2, \sigma_2\} + \ldots + w_n(n)\,\phi\{x(n), c_N, \sigma_N\}\right)\right]\right]$$

$$\cdot\left[-\phi\{x(n), c_1, \sigma_1\} \ldots\ldots \phi\{x(n), c_N, \sigma_N\}\right]$$

$$= w(n) - \mu_w\left[y_d(n) - y(n)\right]\cdot\left[-\Psi(n)\right]$$

where, $\Psi(n) = \left[\phi\{x(n), c_1, \sigma_1\} \ldots \phi\{x(n), c_N, \sigma_N\}\right]$

and $y(n) = w_1(n)\,\phi\{x(n), c_1, \sigma_1\} + \ldots + w_n(n)\,\phi\{x(n), c_N, \sigma_N\}$

$$w(n+1) = w(n) - \mu_w [y_d(n) - y(n)] \cdot [-\Psi(n)]$$

$$w(n+1) = w(n) - \mu_w \, e(n) \cdot [-\Psi(n)] \qquad \left[\begin{array}{l} \text{Since,} \\ y_d(n) - y(n) = e(n) \end{array}\right]$$

$$w(n+1) = w(n) + \mu_w \, e(n) \, \Psi(n)$$

Hence, $\boxed{w(n+1) = w(n) + \mu_w \, e(n) \, \Psi(n)}$

(ii) Given,

$$J(n) = \frac{1}{2}\left[y_d(n) - \sum_{k=1}^{N} w_k(n) \, \phi\{x(n), c_k(n), \sigma_k(n)\}\right]^2$$

For gaussian kernel,

$$J(n) = \frac{1}{2}\left[y_d(n) - \sum_{k=1}^{N} w_k(n) \exp\left(-\frac{\|x(n) - c_k(n)\|^2}{2\sigma_k^2(n)}\right)\right]^2$$

The centre vector update equation is given by -

$$c_k(n+1) = c_k(n) - \mu_c \frac{\partial}{\partial c_k} J(n) \Big|_{c_k = c_k(n)} \quad -①$$

Computing $\frac{\partial}{\partial c_k} J(n) \Big|_{c_k = c_k(n)}$,

$$\frac{\partial}{\partial c_k} J(n) \Big|_{c_k = c_k(n)} = \frac{\partial}{\partial c_k} \frac{1}{2} [e(n)]^2$$

$$= \frac{1}{\cancel{2}} \times \cancel{2} \times e(n) \frac{\partial}{\partial c_k} e(n)$$

$$= e(n) \frac{\partial}{\partial c_K} e(n)$$

$$= e(n) \frac{\partial}{\partial c_K} \left[y_d(n) - \sum_{K=1}^{N} w_k(n) \exp\left(\frac{-\|x(n) - c_k(n)\|^2}{2\sigma_k^2(n)} \right) \right]$$

$$= e(n) \times \left[-w_k(n) \exp\left(\frac{-\|x(n) - c_k(n)\|^2}{2\sigma_k^2(n)} \right) \times \frac{(-2)\{x(n) - c_k(n)\}(-1)}{2\sigma_k^2(n)} \right]$$

$$= e(n) \times \left[-w_k(n)\ \phi\{x(n), c_K, \sigma_K\} \left[\frac{x(n) - c_k(n)}{\sigma_k^2(n)} \right] \right]$$

Hence, $\frac{\partial}{\partial c_K} J(n) \Big|_{c_K = c_K(n)} = \frac{-e(n)\, w_k(n)\, \phi\{x(n), c_K, \sigma_K\} [x(n) - c_K(n)]}{\sigma_k^2(n)}$

Substituting this value back in equation (1), we get

$$c_K(n+1) = c_K(n) + \mu_c \frac{e(n)\, w_k(n)\, \phi\{x(n), c_k(n), \sigma_K\} [x(n) - c_K(n)]}{\sigma_k^2(n)}$$

Hence,

$$\boxed{c_K(n+1) = c_k(n) + \mu_c \frac{e(n)\, w_k(n)}{\sigma_k^2(n)} \phi\{x(n), c_k(n), \sigma_K\} [x(n) - c_k(n)]}$$

(iii) Spread paramcter update equation is given by -

$$\sigma_K(n+1) = \sigma_K(n) - \mu_\sigma \frac{\partial}{\partial \sigma_K} J(n) \Big|_{\sigma_K = \sigma_K(n)} \quad \text{—— (2)}$$

Computing $\frac{\partial}{\partial \sigma_K} J(n) \Big|_{\sigma_K = \sigma_K(n)}$

$$\frac{\partial}{\partial \sigma_K} J(n)\Big|_{\sigma_K = \sigma_K(n)} = \frac{\partial}{\partial \sigma_K} \frac{1}{2}\left[e(n)\right]^2 = \frac{1}{2} \times 2\, e(n) \frac{\partial}{\partial \sigma_K} e(n)$$

$$= e(n) \times \frac{\partial}{\partial \sigma_K}\left[y_d(n) - \sum_{K=1}^{N} w_K(n) \exp\left(-\frac{\|x(n) - c_K(n)\|^2}{2\sigma_K^2(n)}\right)\right]$$

$$= e(n) \times -w_K(n) \exp\left(\frac{-\|x(n) - c_K(n)\|^2}{2\sigma_K^2(n)}\right) \times \left(\frac{-\|x(n) - c_K(n)\|^2}{2}\right) \times \left(\frac{-2}{\sigma_K^3(n)}\right)$$

$$= \frac{e(n) \times -w_K(n)\ \phi\{x(n), c_K(n), \sigma_K\}}{\sigma_K^3(n)} \left[\|x(n) - c_K(n)\|^2\right]$$

Substituting this value back in equation ②, we get,

$$\sigma_K(n+1) = \sigma_K(n) + u_\sigma \frac{e(n) \times w_K(n)}{\sigma_K^3(n)} \phi\{x(n), c_K(n), \sigma_K\} \left[\|x(n) - c_K(n)\|^2\right]$$

Hence,

$$\boxed{\sigma_K(n+1) = \sigma_K(n) + u_\sigma \frac{e(n)\, w_K(n)}{\sigma_K^3(n)} \phi\{x(n), c_K(n), \sigma_K\} \|x(n) - c_K(n)\|^2}$$