ECE/CS/ME 539 – Fall 2024 — Activity 8

Setting Up and Solving the PCA Optimization Problem

In this exercise, you will perform Principal Component Analysis (PCA) by setting up and solving the optimization problem that defines the first principal component. You will work with a simple 2D dataset and derive the principal component by solving an optimization problem similar to the one shown below:

$$\text{maximize}_{\phi_{11},\ldots,\phi_{p1}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{j=1}^{p} \phi_{j1} x_{ij} \right)^2 \right\} \text{ subject to } \sum_{j=1}^{p} \phi_{j1}^2 = 1.$$

Dataset Consider the following 2D dataset with four data points:

Point	x_1	x_2
1	1	1
2	2	2
3	3	3

Step 1: Center the Data First, calculate the mean of each feature and center the data by subtracting the mean from each data point.

$$\text{Mean of } x_1 : \mu_1 = 2$$

$$\text{Mean of } x_2 : \mu_2 = 2$$

The centered data is:

Point	$x_1 - \mu_1$	$x_2 - \mu_2$
1	-1	-1
2	0	0
3	1	1

Step 2: Set Up the PCA Optimization Problem To find the first principal component, set up the following optimization problem:

$$\text{maximize}_{\phi_{11},\phi_{21}} \left\{ \frac{1}{3} \sum_{i=1}^{3} \left(\phi_{11}(x_{i1} - \mu_1) + \phi_{21}(x_{i2} - \mu_2) \right)^2 \right\}$$

subject to the constraint:

$$\phi_{11}^2 + \phi_{21}^2 = 1.$$

Step 3: Solve the Optimization Problem Using Lagrange multipliers, solve the optimization problem. Set up the Lagrangian function:

$$\mathcal{L}(\phi_{11}, \phi_{21}, \lambda) = \frac{1}{3}\sum_{i=1}^{3} \left(\phi_{11}(x_{i1} - \mu_1) + \phi_{21}(x_{i2} - \mu_2)\right)^2 - \lambda\left(\phi_{11}^2 + \phi_{21}^2 - 1\right)$$

Take the partial derivatives with respect to ϕ_{11}, ϕ_{21}, and λ and set them to zero to find the optimal values of ϕ_{11} and ϕ_{21}.

Step 4: Compute the Principal Component Use the values of ϕ_{11} and ϕ_{21} to compute the first principal component. Project the centered data onto this principal component.

Step 5: Interpret the Results

1. What are the values of ϕ_{11} and ϕ_{21}?
2. How much of the variance in the data is captured by this principal component?
3. Project the original centered data onto the first principal component and provide the 1D representation of the data.

Step 6: Connect to Eigenvalue Decomposition Can you find the connection between the steps above and eigenvalue decomposition?