ECE/CS/ME 539 – Fall 2024 — Activity Solution 2

Problem 1

(a) **Proof that the Transpose of the Transpose of Any Matrix is the Matrix Itself**

Given any matrix $\mathbf{A}$, we want to prove that:

$$(\mathbf{A}^\top)^\top = \mathbf{A}.$$

Proof:

The transpose of a matrix $\mathbf{A}$, denoted $\mathbf{A}^\top$, is defined such that if $\mathbf{A} = [a_{ij}]$, then:

$$\mathbf{A}^\top = [a_{ji}].$$

Now, the transpose of $\mathbf{A}^\top$, denoted $(\mathbf{A}^\top)^\top$, is obtained by swapping the rows and columns of $\mathbf{A}^\top$:

$$(\mathbf{A}^\top)^\top = [(a_{ji})^\top] = [a_{ij}] = \mathbf{A}.$$

Hence, we have shown that:

$$(\mathbf{A}^\top)^\top = \mathbf{A}.$$

(b) **Proof that the Transpose of the Sum of Two Matrices is Equal to the Sum of Their Transposes**

Given two matrices $\mathbf{A}$ and $\mathbf{B}$, we want to prove:

$$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top.$$

Proof:

Let $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$. Then, the sum of $\mathbf{A}$ and $\mathbf{B}$ is:

$$\mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}].$$

Taking the transpose of the sum:

$$(\mathbf{A} + \mathbf{B})^\top = [a_{ij} + b_{ij}]^\top = [a_{ji} + b_{ji}].$$

But this is exactly:

$$[a_{ji}] + [b_{ji}] = \mathbf{A}^\top + \mathbf{B}^\top.$$

Thus, we have:

$$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top.$$

(c) **Determine if $\mathbf{A} + \mathbf{A}^\top$ is Always Symmetric for Any Square Matrix $\mathbf{A}$**

To check if $\mathbf{A} + \mathbf{A}^\top$ is symmetric, we need to show that:

$$(\mathbf{A} + \mathbf{A}^\top)^\top = \mathbf{A} + \mathbf{A}^\top.$$

Proof:

Using the result from part (b):

$$(\mathbf{A} + \mathbf{A}^\top)^\top = \mathbf{A}^\top + (\mathbf{A}^\top)^\top.$$

From part (a), we know that:

$$(\mathbf{A}^\top)^\top = \mathbf{A}.$$

So:

$$(\mathbf{A} + \mathbf{A}^\top)^\top = \mathbf{A}^\top + \mathbf{A} = \mathbf{A} + \mathbf{A}^\top.$$

Since the transpose of $\mathbf{A} + \mathbf{A}^\top$ is equal to itself, the matrix $\mathbf{A} + \mathbf{A}^\top$ is indeed symmetric.

(d) **Show that $\mathbf{D}\mathbf{D}^\top$ is Always Symmetric for Any Square Matrix $\mathbf{D}$**

We want to prove that:

$$(\mathbf{D}\mathbf{D}^\top)^\top = \mathbf{D}\mathbf{D}^\top.$$

Proof:

Using the property of transposes for matrix products, we have:

$$(\mathbf{D}\mathbf{D}^\top)^\top = (\mathbf{D}^\top)^\top \mathbf{D}^\top.$$

From part (a), $(\mathbf{D}^\top)^\top = \mathbf{D}$, so:

$$(\mathbf{D}\mathbf{D}^\top)^\top = \mathbf{D}\mathbf{D}^\top.$$

Since the transpose of $\mathbf{D}\mathbf{D}^\top$ equals itself, $\mathbf{D}\mathbf{D}^\top$ is symmetric.

(e) **Prove that if $\mathbf{A} + \mathbf{B} = \mathbf{C}$ and $\mathbf{A}$ and $\mathbf{B}$ are Symmetric, then $\mathbf{C}$ Must Also be Symmetric**

Given that:

$$\mathbf{A} + \mathbf{B} = \mathbf{C}, \quad \text{and} \quad \mathbf{A} = \mathbf{A}^\top, \mathbf{B} = \mathbf{B}^\top,$$

we want to prove that $\mathbf{C}$ is symmetric.

Proof:

Take the transpose of $\mathbf{C}$:

$$\mathbf{C}^\top = (\mathbf{A} + \mathbf{B})^\top.$$

Using the result from part (b):

$$\mathbf{C}^\top = \mathbf{A}^\top + \mathbf{B}^\top.$$

Since $\mathbf{A} = \mathbf{A}^\top$ and $\mathbf{B} = \mathbf{B}^\top$, we have:

$$\mathbf{C}^\top = \mathbf{A} + \mathbf{B} = \mathbf{C}.$$

Thus, $\mathbf{C}$ is symmetric.

Problem 2

2a)

$$w_1x_1 + w_2x_2 + b = 0, \quad w_1^2 + w_2^2 = 1 \implies (5,0) \implies 5w_1 + b = 0, \quad (0,4) \implies 4w_2 + b = 0$$

$$\implies w_1 = 0.8w_2 \implies (0.8w_2)^2 + w_2^2 = 1 \implies w_2 = \frac{1}{\sqrt{1.64}} = 0.7809 \implies w_1 = 0.6247, \quad b = -3.1235$$

2b) For any point x, let the closest point to it in H be x'. Similar to $\mathbf{w}$, the vector $x' - x$ is perpendicular to the hyperplane. Thus, $x' - x$ and $\mathbf{w}$ must be parallel, and their cosine similarity must be:

$$d_{\cos}(x - x', \mathbf{w}) = \frac{\mathbf{w}^T(x - x')}{||\mathbf{w}|| \cdot ||x - x'||} = 1.$$

On the other hand, we know that x' lies in the hyperplane, so it must satisfy the hyperplane equation:

$$\mathbf{w}^T x' + b = 0 \implies \mathbf{w}^T x' = -b.$$

Replacing the latter equation into the former, we get the desired equation:

$$r = |x' - x| = \frac{\mathbf{w}^T(x - x')}{||\mathbf{w}||} = \frac{\mathbf{w}^T x - \mathbf{w}^T x'}{||\mathbf{w}||} = \frac{\mathbf{w}^T x + b}{||\mathbf{w}||}.$$

2c) Assuming that $||\mathbf{w}|| = 1$, we get:

$$r = \frac{\mathbf{w}^T x + b}{||\mathbf{w}||} = \frac{\mathbf{w}^T[0,0] + b}{||\mathbf{w}||} = \frac{b}{||\mathbf{w}||} = b.$$

Thus, the distance of the hyperplane to the origin is given by $r = |b|$.

2d) Given the formula in (2b), and substituting the coordinates of point $C = (4.5, 3)$, we obtain:

$$r = 4.5w_1 + 3w_2 + b = 2.03035$$

Note that if we substitute the coordinate of the origin $(0,0)$ into $g(x)$, we have:

$$g(0,0) = b = -3.1235.$$

Thus, the distance r of any point on the side of the origin O with respect to the hyperplane H shall have a negative value. Hence, the positive sign of the estimated r corresponding to point C implies that C is on the opposite side of the origin with respect to the hyperplane (line) H.