ECE/CS/ME 539 – Fall 2024 — Activity 3

1.

Outcomes of tossing a coin with a Head (H) side and a Tail (T) side leads to a Bernoulli distribution: Let p be the probability of showing a Head (outcome $x = 1$).

$$P_X(x) = \begin{cases} p & \text{if } x = 1, \\ 1 - p & \text{if } x = 0. \end{cases}$$

If one repeatedly tosses the same coin N times, and denote the outcomes to be $x_1, x_2, \ldots, x_N$, where $x_i = 1$ if outcome = Head, and $x_i = 0$ if outcome = Tail. Thus, each x_i is a random variable whose value (0 or 1) is a function of the outcome (Head or Tail).

(a) Compute the expectation of x_i. Hint: It should be a function of p.

Solution:

The expectation of a Bernoulli random variable x_i with probability p of being 1 (Head) is given by:

$$E[x_i] = 1 \cdot p + 0 \cdot (1 - p) = p.$$

(b) The probability p may be estimated empirically as a function of $\{x_i\}$. This estimate can be expressed as:

$$\hat{p}(x_1, \ldots, x_N) = \frac{1}{N} \sum_{i=1}^{N} x_i$$

Show that this is an unbiased estimate. Namely,

$$E[\hat{p}(x_1, \ldots, x_N)] = p$$

Solution:

The estimator $\hat{p}$ is the sample mean of the x_i's. Since each x_i is an independent Bernoulli random variable with expectation p, we have:

$$E[\hat{p}] = E\left[\frac{1}{N} \sum_{i=1}^{N} x_i\right] = \frac{1}{N} \sum_{i=1}^{N} E[x_i] = \frac{1}{N} \sum_{i=1}^{N} p = \frac{N \cdot p}{N} = p.$$

Thus, $\hat{p}$ is an unbiased estimator of p.

(c) Define an event A as all the experiments where at least one of the 20 trials yields a Head. Find an expression of $p(A)$ in terms of p.

Solution:

The event A is the complement of the event that all 20 trials yield Tails. The probability that a single trial yields a Tail is $1-p$. The probability that all 20 trials yield Tails is $(1-p)^{20}$. Therefore, the probability of event A is:

$$p(A) = 1 - (1-p)^{20}.$$

2.

Consider a Random Variable X with a Uniform PDF:

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

Find the mean $\mu_X = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$ and variance $\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 \cdot f(x)\, dx$

Solution:

The mean μ_X of a uniform distribution over $[a, b]$ is given by:

$$\mu_X = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{b+a}{2}.$$

The variance σ_X^2 is calculated as follows:

$$\sigma_X^2 = \int_a^b (x-\mu_X)^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{(x-\mu_X)^3}{3}\right]_a^b.$$

Substituting $\mu_X = \frac{b+a}{2}$, the variance simplifies to:

$$\sigma_X^2 = \frac{(b-a)^2}{12}.$$

3.

Consider a Regular Deck of 52 Playing Cards. A regular deck contains 4 different suits of cards (spades, clubs, hearts, and diamonds), and 13 different cards within each suit (2 through 10, jack, queen, king, and ace).

(a) If you draw a card and it's a red suit (hearts or diamonds), what is the probability that it's a king?

Solution:

There are 26 red cards (hearts and diamonds), and there are 2 red kings (one in each suit). The probability of drawing a king given that the card is red is:

$$P(\text{King} \mid \text{Red}) = \frac{\text{Number of red kings}}{\text{Number of red cards}} = \frac{2}{26} = \frac{1}{13}.$$

(b) If you draw a card and it's a black suit (spades or clubs), what is the probability that it's an ace?

Solution:

There are 26 black cards (spades and clubs), and there are 2 black aces (one in each suit). The probability of drawing an ace given that the card is black is:

$$P(\text{Ace} \mid \text{Black}) = \frac{\text{Number of black aces}}{\text{Number of black cards}} = \frac{2}{26} = \frac{1}{13}.$$

(c) Are the events "card is red" and "card is a king" statistically independent? Justify your answer.

Solution:

Two events A and B are independent if $P(A \cap B) = P(A) \cdot P(B)$. Here, let A be the event "card is red" and B be the event "card is a king".

$P(A) = \frac{26}{52} = \frac{1}{2}$ $P(B) = \frac{4}{52} = \frac{1}{13}$ $P(A \cap B) = \frac{2}{52} = \frac{1}{26}$

Now, check independence:

$$P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}.$$

Since $P(A \cap B) = P(A) \cdot P(B)$, the events "card is red" and "card is a king" are statistically independent.