Prepatory Lab Work, Lab 2

Part I

$$\frac{\Theta_l}{E_a} = \frac{nK}{s[(R_a + sL_a)(J_{eq}s + B_{eq}) + K_b K]}$$

Where $J_{eq} = J_m + n^2 J_l$ and $B_{eq} = B_m + n^2 B_l$

From Appendix A in the lab manual, solving for Ea,

$$L_a \frac{dI_a}{dt} + R_a I_a + E_b = E_a ,$$

We know $E_b = K_b \dot{\Theta}_m$,

$$L_a \frac{dI_a}{dt} + R_a I_a + K_b \dot{\Theta}_m = E_a$$

↳ convert into Laplace domain

$$E_a(s) = I_a(s)[R_a + sL_a] + K_b s \Theta_m(s)$$

From the lab manual, we know:

$$T_m = K_1 \Phi I_a$$

but since the transfer function is that of a peramanent magnet,

$$T_m = K I_a \rightarrow I_a = \frac{T_m}{K}$$

We can sub this into the previous equation ⟶

$$E_a(s) = [R_a + sL_a]\frac{T_m(s)}{K} + K_b s \Theta_m(s)$$

From Appendix A, we also know that:

$$T_m(t) = J_m \frac{d^2\Theta_m}{dt^2} + B_m \frac{d\Theta_m}{dt} + T_1$$

↳ Put in Laplace domain

$$T_m(s) = J_m s^2 \Theta_m + B_m s \Theta_m(s) + T_1$$

If we sub this into the previous equation,

$$E_a(s) = \frac{[R_a + sL_a]}{K}[J_m s^2 \Theta_m \cancel{(s)+} \; B_m s \Theta_m(s)] + K_b \Theta_m(s)$$

$$\frac{E_a(s)}{\Theta_m(s)} = \frac{[R_a + sL_a]}{K}[J_m s^2 + B_m s] + K_b$$

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{1}{\frac{[R_a + sL_a]}{K}[J_m s^2 + B_m s] + K_b}$$

$$\frac{\Theta_m(s)}{E_a(s)} = \frac{K}{[R_a + sL_a][J_m s^2 + B_m s] + K_b K}$$

From Appendix A we also know:

$$\Theta_l = n\Theta_m$$

Therfor we can sub in Θ_l →

$$\frac{\Theta_L}{E_a} = \frac{nK}{s[(R_a + sL_a)(J_m s + B_m s) + K_b K]},$$

2.

$K_a = 0.45 \rightarrow$ motor #1

The third-order transfer function model of the motor:

$$G_p(s) = \frac{\Theta(s)}{V_a(s)} : K_a = \frac{0.45nK}{s[(R_a + sL_a)(J_{eq}s + B_{eq}) + K_b K}$$

3.

By ignoring Armature Inductance, $L_a = 0$:

$$G_p(s) = \frac{\Theta(s)}{V_a(s)} = \frac{0.45nK}{s[(R_a)(J_{eq}s + B_{eq}) + K_b K]}$$

$$= \frac{0.45nK}{s[R_a J_{eq} s + R_a B_{eq} + K_b K]}$$

$$= \frac{0.45nK}{(R_a B_{eq} + K_b K)s\left[\frac{R_a J_{eq} s +}{R_a B_{eq} + K_b K} + 1\right]}$$

$$= \frac{\frac{0.45nK}{R_a B_{eq} + K_b K}}{s\left[\frac{R_a J_{eq} s}{R_a B_{eq} + K_b K} + 1\right]}$$

4.

Assuming a ZOH of the function $G_p(s)$:

$$G(z) = (1-z^{-1})Z\left(\frac{G(s)}{s}\right)$$

$$= (1-z^{-1})Z\left(\frac{0.45nK/R_aB_{eq}+K_bK}{s^2\left[\frac{R_aJ_{eq}s}{R_aB_{eq}+K_bK}+1\right]}\right)$$

↳ Sub in variable values

$$= (1-z^{-1})Z\left[\frac{\frac{0.45(1/17.2)(6.09\times10^{-3})}{3(1.23\times10^{-7})+(6.09\times10^{-3})(6.09\times10^{-3})}}{s^2\left[\frac{3(0.25\times10^{-6})+(1/17.2)^2(2\times10^{-3})}{3(1.23\times10^{-7})+(6.09\times10^{-3})(6.09\times10^{-3})}\right]s+1}\right]$$

$$= (1-z^{-1})Z\left[\frac{4.25}{s^2(0.025s+1)}\right]$$

$$\frac{4.25}{s^2(0.025s+1)} = \frac{A}{s^2} + \frac{B}{s+40} + \frac{C}{s}$$

↳ using partial fraction expansion,

$A = 4.25$
$B = 0.10625$
$C = -0.10625$

$$\rightarrow G(z) = (1-z^{-1})Z\left[\frac{4.25}{s^2} + \frac{0.10625}{s+40} - \frac{0.10625}{s}\right]$$

↳

$$G(z) = \frac{z-1}{z}\left[\frac{4.25z}{(z-1)^2} + \frac{0.10625z}{z - e^{-40t}} - \frac{0.10625z}{z-1}\right]$$

$$= \frac{4.225z - 3.226}{z^2 - 1.765z + 0.765}$$

$1/150 =$ 0.0067

Part B

1.

The relationship between the PID parameters is that they are the same parameters rewritten.

Equation 2.1

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s$$

Equation 2.4

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

In equation 2.1, the gain is already multiplied by the values of K_p, and it is filtering it out in equation 2.4. For the integrator value, in equation 2.1 and equation 2.4, the denominator both contain the parameter s, the value of $K_i = K_p / T_i$. T_i acts as the integral action time or reset rate. The final value $K_d = K_p \cdot T_d$, they both contain s in the numerator and act as the derivative action time or derivative rate.

→

2.

From the frequency response,

$K_u = 10^{GM/20} \rightarrow K_u = 10^{37.4/20} \rightarrow K_u = 74$

The frequency of oscillation P_u,

$P_u = \frac{2\pi}{W_{pc}}$ -> from the frequency response $W_{pc} = 105$

$= \frac{2\pi}{105} \rightarrow P_u \approx 0.06$

-> From the ultimate sensity method,

$K_p = 0.6K_u \rightarrow 0.6(74) \rightarrow K_p = 44.4$

$T_i = P_u/2 \rightarrow (0.06)/2 \rightarrow T_i = 0.03$

$T_d = P_u/8 \quad (0.06)/8 \rightarrow T_d = 0.0075$

Part C

1.

$$\phi_{CL}(s) = \frac{G_p(s)\,G_c}{1 + G_p(s)\,G_c}$$

$$G_p(s)\cdot G_c(s) = \frac{K_m}{s(T_m s+1)} \cdot K_p + \frac{K_i}{s} + \frac{K_d s}{1+\beta s}$$

$$= \frac{K_m K_p}{s(T_m s+1)} + \frac{K_m K_i}{s^2(T_m s+1)} + \frac{K_m K_d s}{s(T_m s+1)(1+\beta s)}$$

$$= \frac{K_m K_p}{T_m s^2 + s} + \frac{K_m K_i}{T_m s^3 + s^2} + \frac{K_m K_d s}{T_m s^2 + s + T_m s \beta s + \beta s^2}$$

$$\phi_{CL}(s) = \frac{\dfrac{K_m K_p}{T_m s^2 + s} + \dfrac{K_m K_i}{T_m s^3 + s^2} + \dfrac{K_m K_d s}{T_m s^2 + s + T_m \beta s^2 + \beta s^2}}{1 + \dfrac{K_m K_p}{T_m s^2 + s} + \dfrac{K_m K_i}{T_m s^3 + s^2} + \dfrac{K_m K_d s}{T_m s^2 + s + T_m \beta s^2 + \beta s^2}}$$

$$\phi_{CL} = \frac{K_m K_p + K_m K_i + K_m K_d s}{[(T_m s^2 + s + T_m s^3 + s^2 + T_m s^2 + s + T_m \beta s^2 + \beta s^2) + (K_m K_p + K_m K_i + K_m K_d s)}$$

↳ By expanding and simplifying the terms →

$$\phi_{CL} = s^4 + \left(\frac{1}{T_m} + \frac{1}{\beta}\right)s^3 + \left(\frac{K_m K_b}{T_m} + \frac{K_m K_d}{T_m \beta} + \frac{1}{T_m \beta}\right)s^2 + \left(\frac{K_m K_p}{T_m \beta} + \frac{K_m K_i}{T_m}\right)s + \left(\frac{K_m K_i}{T_m \beta}\right)$$

2.

a) $\zeta = \dfrac{-\ln(PO/100)}{\sqrt{\pi^2 + (\ln(PO/100))^2}}$

$= \dfrac{-\ln(15/100)}{\sqrt{\pi^2 + (\ln(15/100))^2}}$

$= 0.5169$

≈ 0.517

b) $T_s(1\%) = \dfrac{4.6}{\zeta W_n}$

$0.25(0.517) W_n = 4.6$

$W_n = 35.59$

Therefore, the pair of dominant closed loop poles which satisfy the design requirement are:

$S_d = -\zeta W_n \pm j W_n \sqrt{1-\zeta^2}$

$= -18.396 \pm j30.465$

3.

The 2 additional closed loops are:

$S = -a = -4\zeta W_n = -4(0.517)(35.59) = -73.6$

$S = -b = -6\zeta W_n = -6(0.517)(35.59) = -110.4$

By equating equation 2.7 and 2.8,

$$s^4 + \left(\frac{1}{T_m} + \frac{1}{\beta}\right)s^3 + \left(\frac{K_m K_p}{T_m} + \frac{K_m K_d}{T_m \beta} + \frac{1}{T_m \beta}\right)s^2 + \left(\frac{K_m K_p}{T_m \beta} + \frac{K_m K_i}{T_m}\right)s + \left(\frac{K_m K_i}{T_m \beta}\right) = (S+a)(S+b)(S^2 + 2\zeta W_n S + W_n^2)$$

$\longrightarrow$

We know the values of $K_m, T_m, a, b, \zeta, \omega_n$:

$$K_m = \frac{0.45 n K}{R_a B_{eq} + K_b K} = \frac{0.45(1/17.2)(6.09\times10^{-3})}{3(1.23\times10^{-7}) + (6.09\times10^{-3})(6.09\times10^{-3})}$$
$$= 4.25$$

$$T_m = \frac{R_a J_{eq}}{R_a B_{eq} + K_b K} = \frac{3(0.25\times10^{-6} + (1/17.2)^2(2\times10^{-5}))}{3(1.23\times10^{-7}) + (6.09\times10^{-3})(6.09\times10^{-3})}$$
$$= 0.025$$

$a = 73.6$
$b = 110.4$
$\zeta = 0.517$
$\omega_n = 35.59$

$$s^4 + \left(\frac{1}{0.025} + \frac{1}{\beta}\right)s^3 + \left(\frac{4.25K_p}{0.025} + \frac{4.25K_d}{0.025\beta} + \frac{1}{0.025\beta}\right)s^2 + \left(\frac{4.25K_p}{0.025\beta} + \frac{4.25K_i}{0.025}\right)s$$
$$+ \left(\frac{4.25K_i}{0.025\beta}\right) = (s+73.6)(s+110.4)(s^2 + 2(0.517)(35.59)s + (35.59)^2)$$

↳ expanding the right side

$$= (s+73.6)(s+110.4)(s^2 + 36.8s + 1266.65)$$
$$= (s^2 + 184 + 8125.44)(s^2 + 36.8s + 1266.65)$$
$$= s^4 + 220.8s^3 + 16\,163.3s^2 + 532\,080s + 1.02921\times10^7$$

↳ Since the s^3 has 1 missing variable, solve for β

$$\left(\frac{1}{0.025} + \frac{1}{\beta}\right)\cancel{s^3} = 220.8\cancel{s^3}$$
$$40 + \frac{1}{\beta} = 220.8$$

$\beta = 0.00553097$
≈ 0.0055

→

→ Since s^0 has 1 missing variable solve for K_i

$$\left(\frac{4.25K_i}{0.025 \times \beta}\right) = 1.02921 \times 10^7 \quad \rightarrow K_i = 332.98$$

$$\left(\frac{4.25(K_i)}{0.025(0.0055)}\right) = 1.02921 \times 10^7$$

→ Since s^1 has 1 missing variable solve for K_p

$$\left(\frac{4.25K_p}{0.025\beta} + \frac{4.25K_i}{0.025}\right)\cancel{s} = 532\,080\cancel{s}$$

$$\left(\frac{4.25(K_p)}{(0.025)(0.0055)} + \frac{4.25(332.98)}{0.025}\right) = 532\,080$$

$$K_p = 15.383$$

→ Since s^2 has 1 missing variable solve for K_d

$$\left(\frac{4.25K_p}{0.025} + \frac{4.25K_d}{0.025\beta} + \frac{1}{0.025\beta}\right)\cancel{s^2} = 16\,163.3\cancel{s^2}$$

$$\left(\frac{4.25(15.383)}{0.025} + \frac{4.25(K_d)}{0.025(0.0055)} + \frac{1}{0.025(0.0055)}\right) = 16\,163$$

$$K_d = 0.20302$$

Part D

→ By using the forward and backward rectangle rule;

$$G_c(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

$$U(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right) E(s)$$

apply rectangle rule

$$= K_p s E(s) + + \frac{K_p}{T_i} E(s) + K_p T_d s^2 E(s)$$

$$\hookrightarrow \frac{U(t) - U(t-1)}{T} = \left(K \frac{e(t) - e(t-1)}{T}\right) + \frac{K_p}{T_i} + \boxed{T_d\left(\frac{\dot{e}(t) - \dot{e}(t-1)}{T}\right)}$$

↳ for the derivative term, b/c of the s^2, apply rectangle rule again

$$T_d\left(\frac{\frac{e(t) - e(t-1)}{T} - \frac{e(t-1) - e(t-2)}{T}}{T}\right)$$

↳ Simplify the equation

$$\frac{u(t) - u(t-1)}{T} = \left[K\left(\frac{e(t) - (t-1)}{T}\right)\right] - \frac{K_p}{T_i} + T_d\left[K\left[\frac{e(t) - e(t-1) - e(t-1) - e(t-2)}{T^2}\right]\right]$$