Big Maths

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1 Introduction

This book is designed as revision/introduction to advanced methods used in undergraduate science degrees, primarily maths and physics. To make the most of this book, the reader should consult this text in chronological order because as we progress to the harder topics, formulas that have been addressed previously will be known as trivial.

This manual has been written with the intention to make it as comprehensive and simple as possible so that a larger audience can gain from it and not just the mathematically gifted. Hopefully by the end of this volume, the reader will have a greater understanding of many topics within science.

This guide has been designed to make you think, rather than blindly copying out the same formula for every problem. This is with the intent for the reader to really understand the mathematics.

It is recommended that you have taken A-level Further Maths before you read the text so that you get the most out of it.

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3 The Limit

The following formula below is fundamental to understanding the essence of calculus.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We can use this limit in order to derive the derivatives of a range of functions

- Example: Prove the power rule

$$f'(x) = nx^{n-1}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

using the binomial:

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \qquad n, m \in \mathbb{Q}^+ \quad n \geq m$$

$$f'(x) = \lim_{h \to 0} \frac{x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + ... - x^n}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + ...}{h}$$

$$f'(x) = \lim_{h \to 0} [nx^{n-1} + \frac{n(n-1))}{2}x^{n-2}h + ...]$$

As h tends to zero, this yields the following result:

$$f'(x) = [nx^{n-1} + \frac{n(n-1))}{2}x^{n-2}(0) + ...]$$

So now we are left with the following:

$$f'(x) = nx^{n-1}$$

A similar approach may be taken in order to prove other rules such as the chain rule:

$$\frac{\mathrm{dy}}{\mathrm{d}x} = \frac{\mathrm{dy}}{\mathrm{d}u}\frac{\mathrm{du}}{\mathrm{d}x}$$

The concept is exactly the same as how we derived the power rule. But, in order to get used to university notation, let's write the limit of the derivative in a more formal matter, that is:

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

As you can see, exactly the same just a small substitution. Now, since the chain rule is formed by the product of two limits, we must therefore evaluate two limits in order to get to our required result. We can prove the chain rule by doing the following:

$$\frac{\mathrm{dy}}{\mathrm{d}u} = \lim_{\Delta u \to 0} \frac{y(u + \Delta u) - y(u)}{\Delta u}, \frac{\mathrm{du}}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

I hope you can see where this change in variables comes from. Now the derivative on the left of the above equation can be written as

$$\frac{dy[u(x)]}{dx} = \frac{y[u(x + \Delta x)] - y[u(x)]}{\Delta x}$$

Next, if we multiply the fraction by

$$\frac{u(x + \Delta x) - u(x)}{u(x + \Delta x) - u(x)}$$

which is perfectly valid as we are effectively multiplying our fraction by 1. It is clear therefore that the derivative is equal to

$$\lim_{\Delta u \to 0} \frac{y(u + \Delta u) - y(u)}{\Delta u} \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x}$$

Notice that we are back to where we started therefore we have now proved that

$$\frac{\mathrm{dy}}{\mathrm{d}x} = \frac{\mathrm{dy}}{\mathrm{d}u} \frac{\mathrm{du}}{\mathrm{d}x}$$

4 L'Hôpital's Rule

- Theorem: L'Hôpital's Rule states that if a limit of a quotient is indeterminate, then under certain conditions the derivative of the two functions that make the quotient can be found independently to find the limit of the function.

Let our quotient where we intend to find the limit be described as followed

$$\lim_{x\to 1} \frac{e^x - 1}{x^2 - 1}$$

As you can see, if we were to "plug in" the x value of 1 into the quotient, our answer would be

$$\frac{0}{0}$$

But how do we interpret this? The short answer is that we can't. So we have to use L'Hôpital's Rule in order to evaluate the limit.

$$\lim_{x\to 1} = \frac{\frac{\mathrm{d}}{\mathrm{d}x}e^x - 1}{\frac{d}{dx}x^2 - 1} = \lim_{x\to 1} \frac{e^x}{2x} = \frac{1}{2}$$

That wasn't so bad now was it?

There are multiple times when a limit is indeterminate and L'Hôpital's Rule must be used to go further. Below is the list of all the times that a quotient is indeterminate:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 * \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$$

All other quotients are ok!

Now time for a harder example:

- Calculate the limit of the following quotient

$$\lim_{n\to\infty}[(1+\frac{1}{n})^n]$$

Now, in order to evaluate this limit, we must take the natural logarithm of both sides. This yields:

$$\lim_{n\to\infty} n\ln(1+\frac{1}{n})$$

We must undergo some algebraic manipulation in order to turn the function into a quotient, we can do this by writing n as

$$\frac{1}{\frac{1}{n}}$$

which is the same as n.

Therefore we now get the following result:

$$\ln A = \lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})}{\frac{1}{n}}$$

Now applying L'Hôpital's Rule yields:

$$\ln A = \lim_{n\to\infty}\frac{1}{n+1} = 1$$

then by taking the exponential on both sides leaves us with the solution.

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

5 Challenge Problems

- Calculate the following limit

$$\lim_{x \to 0} [x^{sin(x)}]$$

- Calculate

$$\lim_{n \to 0} \binom{7n}{6n} - n^n$$

- Prove the Product Rule using the formal limit of the derivative

$$f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

- Show that the following integral

$$\int_0^1 \ln(x)dx = -1$$

- By forming an appropriate limit, show that

$$0^0 = 1$$

- Without applying L'Hôpital's Rule, evaulate the following limit, Hint: Use the Squeeze Theorem.

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

6 Calculus

In school, most of you learnt how to take the derivative of a one dimensional function. At the end of this section, hopefully you will be able to take the derivative of 2+ dimensional functions. We can accomplish this by introducing Partial Derivatives - These allow us to find the gradient of functions along the x y z plane. However, when we are computing these slopes, it is very important that we only find the gradient along one axis at a time, this is because the function depends on the variables. Let's start off with a very simple example so that we can start to understand the concept:

- Find all first order partial derivatives of the following function:

$$f(x, y, z) = \frac{7x}{8} + \arctan(y) + 7ysin(z) - xcos(x)$$

Let's first find the partial derivative of$f(x, y, z)$ with respect to x

$$\frac{\partial}{\partial x} = \frac{7}{8} - cos(x) + xsin(x)$$

As you can see, $/arctan(y) + 7y/sin(z)$ has disappeared, this is because when we find a partial derivative with respect to a certain variable, all other variables are treated as constants.

Now let's apply the same technique to the partial derivatives of y and z

$$\frac{\partial}{\partial y} = \frac{1}{1 + y^2} + 7sin(z)$$

$$\frac{\partial}{\partial z} = 7ycos(z)$$

Now that we understand the basic concepts of partial differentiation, let's move onto a harder example:

- The function $f(x, y, z)$ is described as follows

$$f(x, y, z) = 7ln(z) + xe^z - y\csc(x)$$

Find $\frac{\partial^2}{\partial x \partial y}$

First find the first order partial derivative with respect to x

$$\frac{\partial}{\partial x} = e^z + y \csc(x) \cot(x)$$

Now we find the partial derivative with respect to y using our first order partial derivative with respect to x

$$\frac{\partial^2}{\partial x \partial y} = \csc(x) \cot(x)$$

Would the answer be the same if we found the first partial with respect to y then the partial with respect to x? I'll let you figure that out.

With this you can follow the same rules to find implicit functions.

Now, Let's try to find the stationary points of partial derivatives and their nature:

As always let's start with a basic example.

Compute the stationary points and their nature of the following equation:

$$z = 2x^3 - 16y^3 + 4xy^2 - 8x + 16y + 12$$

First let's find the first partial derivative with respect to x and the first partial derivative with respect to y. This yields

$$\frac{\partial z}{\partial x} = 6x^2 + 4y^2 - 8$$

$$\frac{\partial z}{\partial y} = -48y^2 + 8xy + 16$$

Taking the second derivative with respect to x and y respectably yields:

$$\frac{\partial^2 z}{\partial x^2} = 12x$$

$$\frac{\partial^2 z}{\partial y^2} = -96y + 8x$$

And finally

$$\frac{\partial z}{\partial x \partial y} = 8y$$

Now, in order to find the stationary points of the function, we have to set our first derivative partials equal to 0. Then we can solve using simultaneous equations. This is up to the reader to evaluate so that you can truly understand the mathematics rather than blindly reading through the example. Once you have done that. Use the following formula to find the nature of the stationary points

$$D(\alpha, \beta) = f_{xx}(\alpha, \beta) f_{yy}(\alpha, \beta) - f_{xy}(\alpha, \beta)$$