Part 1 Theoretical Analysis

Suppose that $A = (a_{ij})$ and $B = (b_{ij})$ are square $n \times n$ matrices, then in the product $C = A \cdot B$, we define the entry c_{ij}, for $i, j = 1, 2, \cdots, n$, by

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \tag{1}$$

We must compute n^2 matrix entries, and each is the sum of n values. The following procedure takes $n \times n$ matrices A and B and multiplies them, returning their $n \times n$ product C. We assume that each matrix has an attribute *rows*, giving the number of rows in the matrix.

Square Matrix Multiply

Pseudo code

SQUARE-MATRIX-MULTIPLY (A, B)

1 $n = A.rows$

2 let C be a new $n \times n$ matrix

3 for $i = 1$ to n

4   for $j = 1$ to n

5         $c_{ij} = 0$

6   for $k = 1$ to n

7             $c_{ij} = c_{ij} + a_{ik} \cdot b_{kj}$

8 return C

Proof

The SQUARE-MATRIX-MULTIPLY procedure works as follows. The for loop of lines 3–7 computes the entries of each row i, and within a given row i, the for loop of lines 4–7 computes each of the entries c_{ij} , for each column j . Line 5 initializes c_{ij} to 0 as we start computing the sum given in equation(1), and each iteration of the for loop of lines 6–7 adds in one more term of equation(1).

Because each of the triply-nested for loops runs exactly n iterations, and each execution of line 7 takes constant time, the SQUARE-MATRIX-MULTIPLY procedure takes $\Theta(n^3)$time.

Strassen algorithm

Pseudo code

Suppose that we partition each of A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Create 10 matrix which is defined by

$$\begin{aligned}
S_1 &= B_{12} - B_{22} \\
S_2 &= A_{11} + A_{12} \\
S_3 &= A_{21} + A_{22} \\
S_4 &= B_{21} - B_{11} \\
S_5 &= A_{11} + A_{22} \\
S_6 &= B_{11} + B_{22} \\
S_7 &= A_{12} - A_{22} \\
S_8 &= B_{21} + B_{22} \\
S_9 &= A_{11} - A_{21} \\
S_{10} &= B_{11} + B_{12}
\end{aligned}$$

Method Strassen(A,B)
$n = A.rows$

$P_1 = Strassen(A_{11}, B_{12} - B_{22})$
$P_2 = Strassen(A_{11} + A_{12}, B_{22})$
$P_3 = Strassen(A - 21 + A_{22}, B_{11})$
$P_4 = Strassen(A_{22}, B_{21} - B_{11})$
$P_5 = Strassen(A_{11} + A_{22}, B_{11} + B_{22})$
$P_6 = Strassen(A_{12} - A_{22}, B_{21} + B_{22})$
$P_7 = Strassen(A_{11} - A_{21}, B_{11} + B_{12})$

let C be a new $m \times n$ matrix
if $n == 1$
  $c_{11} = a_{11} * b_{11}$
else partition A, B and C as above
  $C_{11} = P_5 + P_4 - P_2 + P_6$
  $C_{12} = P_1 + P_2$
  $C_{21} = P_3 + P_4$
  $C_{22} = P_1 + P_5 - P_3 - P_7$
  $C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$

return C

Proof

Step 1

At fisrst, we assume that n is an exact power of 2 in each of the $n \times n$ matrices.

We make this assumption because in each divide step, we will divide $n \times n$ matrices into four $n/2 \times n/2$ matrices, and by assuming that n is an exact power of 2, we are guaranteed that as long as $n \geq 2$, the dimension $n/2$ is an integer.

Suppose that we partition each of A, B, and C into four $n/2 \times n/2$ matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{2}$$

so that we can rewrite the equation $C = A \cdot B$ as

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{3}$$

The equation(3) corresponds to the four equations

$$C_{11} = A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \tag{4}$$

$$C_{12} = A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \tag{5}$$

$$C_{21} = A_{21} \cdot B_{11} + A_{12} \cdot B_{21} \tag{6}$$

$$C_{22} = A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \tag{7}$$

Each of these four equations specifies two multiplications of $n/2 \times n/2$ matrices and the addition of their $n/2 \times n/2$ produts.

This step take $\Theta(1)$ time by index calculation.

Step 2

Create 10 matrices $S_1, S_2, \cdots, S_{10}$, each of which is $n/2 \times n/2$ and is the sum or difference of two matrices created in step 1.

And the ten matrices are shown as follows:

$$\begin{aligned} S_1 &= B_{12} - B_{22} \\ S_2 &= A_{11} + A_{12} \\ S_3 &= A_{21} + A_{22} \\ S_4 &= B_{21} - B_{11} \\ S_5 &= A_{11} + A_{22} \\ S_6 &= B_{11} + B_{22} \\ S_7 &= A_{12} - A_{22} \\ S_8 &= B_{21} + B_{22} \\ S_9 &= A_{11} - A_{21} \\ S_{10} &= B_{11} + B_{12} \end{aligned}$$

Since we need to add or subtract $n/2 \times n/2$ matrices 10 times, this step does indeed take $n/2 \times n/2$ time.

Step 3

Using the submatrices created in step 1 and the 10 matrices created in step 2, recursively compute seven matrix products $P_1, P_2, \cdots, P_7$. Each matrix P_i is $n/2 \times n/2$.

And the seven matrices are shown as follows:

$$\begin{gathered} P_1 = A_{11} \cdot S_1 = A_{11} \cdot B_{12} - A_{11} \cdot B_{22} \\ P_2 = S_2 \cdot B_{22} = A_{11} \cdot B_{22} + A_{12} \cdot B_{22} \\ P_3 = S_3 \cdot B_{11} = A_{21} \cdot B_{11} + A_{22} \cdot B_{11} \\ P_4 = A_{22} \cdot S_4 = A_{22} \cdot B_{21} - A_{22} \cdot B_{11} \\ P_5 = S_5 \cdot S_6 = A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22} \\ P_6 = S_7 \cdot S_8 = A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22} \\ P_7 = S_9 \cdot S_{10} = A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12} \end{gathered}$$

Step 4

Compute the desired submatrices $C_{11}, C_{12}, C_{21}, C_{22}$ of the result matrix C by adding and subtracting various combinations of the P_i matrices. We can compute all four submatrices in $\Theta(n^2)$ time.

The calculation process is shown as follows

$$\begin{aligned} C_{11} &= A_{11} \cdot B_{11} + A_{12} \cdot B_{21} \\ &= (A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}) + (A_{22} \cdot B_{21} - A_{22} \cdot B_{11}) \\ &\quad - (A_{11} \cdot B_{22} + A_{12} \cdot B_{22}) + (A_{12} \cdot B_{21} + A_{12} \cdot B_{22} - A_{22} \cdot B_{21} - A_{22} \cdot B_{22}) \\ &= P_5 + P_4 - P_2 + P_6 \end{aligned}$$

$$\begin{aligned} C_{22} &= A_{21} \cdot B_{12} + A_{22} \cdot B_{22} \\ &= (A_{11} \cdot B_{11} + A_{11} \cdot B_{22} + A_{22} \cdot B_{11} + A_{22} \cdot B_{22}) + (A_{11} \cdot B_{12} - A_{11} \cdot B_{22}) \\ &\quad - (A_{21} \cdot B_{11} + A_{22} \cdot B_{11}) - (A_{11} \cdot B_{11} + A_{11} \cdot B_{12} - A_{21} \cdot B_{11} - A_{21} \cdot B_{12}) \\ &= P_5 + P_1 - P_3 - P_7 \end{aligned}$$

$$\begin{aligned} C_{12} &= A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ &= (A_{11} \cdot B_{12} - A_{11} \cdot B_{22}) + (A_{11} \cdot B_{22} + A_{12} \cdot B_{22}) \\ &= P_1 + P_2 \end{aligned}$$

$$\begin{aligned} C_{21} &= A_{21} \cdot B_{11} + A_{12} \cdot B_{21} \\ &= (A_{21} \cdot B_{11} + A_{22} \cdot B_{11}) + (A_{22} \cdot B_{21} - A_{22} \cdot B_{11}) \\ &= P_3 + P_4 \end{aligned}$$

Altogether, we add or subtract $n/2 \times n/2$ matrices eight times in step 4, and so this step indeed take $\Theta(n^2)$ time.

Conclusion

Now we already have enough information to set up a recurrence for the running time of Strassen's method. Let us

assume that once the matrix size n gets down to 1, we perform a simple scalar multiplication.

When $n > 1$, steps 1, 2, and 4 take a total of $\Theta(n^2)$ time, and step 3 requires us to perform seven multiplications of $n/2 \times n/2$ matrices. Hence, we obtain the following recurrence for the running time $T(n)$ of Strassen's algorithm:

$$T(n) = \begin{cases} \Theta(1) & n = 1 \\ 7T(n/2) + \Theta(n^2) & n > 1 \end{cases} \tag{8}$$

Then by the master method

Since $a = 7, b = 2$, $f(n) = \Theta(n^2)$, and $n^{log_b^a} = n^{log_2 7}$, Rewriting $log_2 7$ as $lg7$ and recalling that $2.80 < lg7 < 2.81$, we see that $f(n) = O(n^{lg7-\varepsilon})$ for $\varepsilon = 0.8$.

Thus, case 1 applies, and we have the solution $T(n) = \Theta(n^{lg7})$.