Proposition 3.9. *Let $R(t)$ denote the orientation of the rotating frame as seen from the fixed frame. Denote by* w *the angular velocity of the rotating frame. Then*

$$\dot{R}R^{-1} = [\omega_s], \tag{3.37}$$
$$R^{-1}\dot{R} = [\omega_b], \tag{3.38}$$

where $\omega_s \in \mathbb{R}^3$ is the fixed-frame vector representation of w *and $[\omega_s] \in so(3)$ is its 3×3 matrix representation, and where $\omega_b \in \mathbb{R}^3$ is the body-frame vector representation of* w *and $[\omega_b] \in so(3)$ is its 3×3 matrix representation.*

It is important to note that ω_b is *not* the angular velocity relative to a moving frame. Rather, ω_b is the angular velocity relative to the *stationary* frame {b} that is instantaneously coincident with a frame attached to the moving body.

It is also important to note that the fixed-frame angular velocity ω_s *does not depend on the choice of body frame*. Similarly, the body-frame angular velocity ω_b *does not depend on the choice of fixed frame*. While Equations (3.37) and (3.38) may appear to depend on both frames (since R and $\dot{R}$ individually depend on both {s} and {b}), the product $\dot{R}R^{-1}$ is independent of {b} and the product $R^{-1}\dot{R}$ is independent of {s}.

Finally, an angular velocity expressed in an arbitrary frame {d} can be represented in another frame {c} if we know the rotation that takes {c} to {d}, using our now-familiar subscript cancellation rule:

$$\omega_c = R_{cd}\omega_d.$$

3.2.3 Exponential Coordinate Representation of Rotation

We now introduce a three-parameter representation for rotations, the **exponential coordinates for rotation**. The exponential coordinates parametrize a rotation matrix in terms of a rotation axis (represented by a unit vector $\hat{\omega}$) and an angle of rotation θ about that axis; the vector $\hat{\omega}\theta \in \mathbb{R}^3$ then serves as the three-parameter exponential coordinate representation of the rotation. Writing $\hat{\omega}$ and θ individually is the **axis-angle** representation of a rotation.

The exponential coordinate representation $\hat{\omega}\theta$ for a rotation matrix R can be interpreted equivalently as:

- the axis $\hat{\omega}$ and rotation angle θ such that, if a frame initially coincident with {s} were rotated by θ about $\hat{\omega}$, its final orientation relative to {s} would be expressed by R; or

- the angular velocity $\hat{\omega}\theta$ expressed in {s} such that, if a frame initially coincident with {s} followed $\hat{\omega}\theta$ for one unit of time (i.e., $\hat{\omega}\theta$ is integrated over this time interval), its final orientation would be expressed by R; or
- the angular velocity $\hat{\omega}$ expressed in {s} such that, if a frame initially coincident with {s} followed $\hat{\omega}$ for θ units of time (i.e., $\hat{\omega}$ is integrated over this time interval) its final orientation would be expressed by R.

The latter two views suggest that we consider exponential coordinates in the setting of linear differential equations. Below we briefly review some key results from linear differential equations theory.

3.2.3.1 Essential Results from Linear Differential Equations Theory

Let us begin with the simple scalar linear differential equation

$$\dot{x}(t) = ax(t), \tag{3.39}$$

where $x(t) \in \mathbb{R}$, $a \in \mathbb{R}$ is constant, and the initial condition $x(0) = x_0$ is given. Equation (3.39) has solution

$$x(t) = e^{at}x_0.$$

It is also useful to remember the series expansion of the exponential function:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots .$$

Now consider the vector linear differential equation

$$\dot{x}(t) = Ax(t), \tag{3.40}$$

where $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is constant, and the initial condition $x(0) = x_0$ is given. From the above scalar result one can conjecture a solution of the form

$$x(t) = e^{At}x_0 \tag{3.41}$$

where the **matrix exponential** e^{At} now needs to be defined in a meaningful way. Again mimicking the scalar case, we define the matrix exponential to be

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots \tag{3.42}$$

The first question to be addressed is under what conditions this series converges, so that the matrix exponential is well defined. It can be shown that if A is constant and finite then this series is always guaranteed to converge to a finite limit;

the proof can be found in most texts on ordinary linear differential equations and is not covered here.

The second question is whether Equation (3.41), using Equation (3.42), is indeed a solution to Equation (3.40). Taking the time derivative of $x(t) = e^{At}x_0$,

$$\begin{aligned}\dot{x}(t) &= \left(\frac{d}{dt}e^{At}\right)x_0 \\ &= \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots\right)x_0 \\ &= \left(A + A^2t + \frac{A^3t^2}{2!} + \cdots\right)x_0 \\ &= Ae^{At}x_0 \\ &= Ax(t),\end{aligned} \tag{3.43}$$

which proves that $x(t) = e^{At}x_0$ is indeed a solution. That this is a unique solution follows from the basic existence and uniqueness result for linear ordinary differential equations, which we invoke here without proof.

While $AB \neq BA$ for arbitrary square matrices A and B, it is always true that

$$Ae^{At} = e^{At}A \tag{3.44}$$

for any square A and scalar t. You can verify this directly using the series expansion for the matrix exponential. Therefore, in line four of Equation (3.43), A could also have been factored to the right, i.e.,

$$\dot{x}(t) = e^{At}Ax_0.$$

While the matrix exponential e^{At} is defined as an infinite series, closed-form expressions are often available. For example, if A can be expressed as $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$ then

$$\begin{aligned}e^{At} &= I + At + \frac{(At)^2}{2!} + \cdots \\ &= I + (PDP^{-1})t + (PDP^{-1})(PDP^{-1})\frac{t^2}{2!} + \cdots \\ &= P(I + Dt + \frac{(Dt)^2}{2!} + \cdots)P^{-1} \\ &= Pe^{Dt}P^{-1}.\end{aligned} \tag{3.45}$$

If moreover D is diagonal, i.e., $D = \mathrm{diag}\{d_1, d_2, \ldots, d_n\}$, then its matrix exponential is particularly simple to evaluate:

$$e^{Dt} = \begin{bmatrix} e^{d_1 t} & 0 & \cdots & 0 \\ 0 & e^{d_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n t} \end{bmatrix}. \tag{3.46}$$

We summarize the results above in the following proposition.

Proposition 3.10. *The linear differential equation $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0$, where $A \in \mathbb{R}^{n\times n}$ is constant and $x(t) \in \mathbb{R}^n$, has solution*

$$x(t) = e^{At}x_0 \tag{3.47}$$

where

$$e^{At} = I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots. \tag{3.48}$$

The matrix exponential e^{At} further satisifies the following properties:

(a) $d(e^{At})/dt = Ae^{At} = e^{At}A$.

(b) *If $A = PDP^{-1}$ for some $D \in \mathbb{R}^{n\times n}$ and invertible $P \in \mathbb{R}^{n\times n}$ then $e^{At} = Pe^{Dt}P^{-1}$.*

(c) *If $AB = BA$ then $e^A e^B = e^{A+B}$.*

(d) $(e^A)^{-1} = e^{-A}$.

The third property can be established by expanding the exponentials and comparing terms. The fourth property follows by setting $B = -A$ in the third property.

3.2.3.2 Exponential Coordinates of Rotations

The exponential coordinates of a rotation can be viewed equivalently as (1) a unit axis of rotation $\hat{\omega}$ ($\hat{\omega} \in \mathbb{R}^3$, $\|\hat{\omega}\| = 1$) together with a rotation angle about the axis $\theta \in \mathbb{R}$, or (2) as the 3-vector obtained by multiplying the two together, $\hat{\omega}\theta \in \mathbb{R}^3$. When we represent the motion of a robot joint in the next chapter, the first view has the advantage of separating the description of the joint axis from the motion θ about the axis.

Referring to Figure 3.11, suppose that a three-dimensional vector $p(0)$ is rotated by θ about $\hat{\omega}$ to $p(\theta)$; here we assume that all quantities are expressed

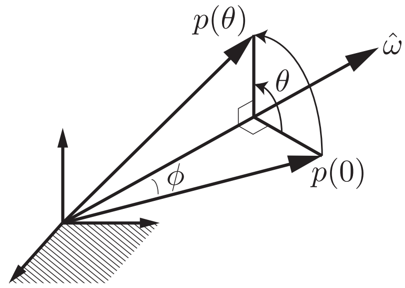

Figure 3.11: The vector $p(0)$ is rotated by an angle θ about the axis $\hat{\omega}$, to $p(\theta)$.

in fixed-frame coordinates. This rotation can be achieved by imagining that $p(0)$ rotates at a constant rate of 1 rad/s (since $\hat{\omega}$ has unit magnitude) from time $t = 0$ to $t = \theta$. Let $p(t)$ denote the path traced by the tip of the vector. The velocity of $p(t)$, denoted $\dot{p}$, is then given by

$$\dot{p} = \hat{\omega} \times p. \tag{3.49}$$

To see why this is true, let ϕ be the constant angle between $p(t)$ and $\hat{\omega}$. Observe that p traces a circle of radius $\|p\| \sin\phi$ about the $\hat{\omega}$-axis. Then $\dot{p}$ is tangent to the path with magnitude $\|p\| \sin\phi$, which is equivalent to Equation (3.49).

The differential equation (3.49) can be expressed as (see Equation (3.30))

$$\dot{p} = [\hat{\omega}]p \tag{3.50}$$

with initial condition $p(0)$. This is a linear differential equation of the form $\dot{x} = Ax$, which we studied earlier; its solution is given by

$$p(t) = e^{[\hat{\omega}]t} p(0).$$

Since t and θ are interchangeable, the equation above can also be written

$$p(\theta) = e^{[\hat{\omega}]\theta} p(0).$$

Let us now expand the matrix exponential $e^{[\hat{\omega}]\theta}$ in series form. A straightforward calculation shows that $[\hat{\omega}]^3 = -[\hat{\omega}]$, and therefore we can replace $[\hat{\omega}]^3$ by $-[\hat{\omega}]$, $[\hat{\omega}]^4$ by $-[\hat{\omega}]^2$, $[\hat{\omega}]^5$ by $-[\hat{\omega}]^3 = [\hat{\omega}]$, and so on, obtaining

$$\begin{aligned} e^{[\hat{\omega}]\theta} &= I + [\hat{\omega}]\theta + [\hat{\omega}]^2 \frac{\theta^2}{2!} + [\hat{\omega}]^3 \frac{\theta^3}{3!} + \cdots \\ &= I + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)[\hat{\omega}] + \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)[\hat{\omega}]^2. \end{aligned}$$

Now recall the series expansions for $\sin\theta$ and $\cos\theta$:

$$\begin{aligned}\sin\theta &= \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \\ \cos\theta &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\end{aligned}$$

The exponential $e^{[\hat{\omega}]\theta}$ therefore simplifies to the following:

Proposition 3.11. *Given a vector $\hat{\omega}\theta \in \mathbb{R}^3$, such that θ is any scalar and $\hat{\omega} \in \mathbb{R}^3$ is a unit vector, the matrix exponential of $[\hat{\omega}]\theta = [\hat{\omega}\theta] \in so(3)$ is*

$$\mathrm{Rot}(\hat{\omega},\theta) = e^{[\hat{\omega}]\theta} = I + \sin\theta\,[\hat{\omega}] + (1-\cos\theta)[\hat{\omega}]^2 \in SO(3). \tag{3.51}$$

Equation (3.51) is also known as **Rodrigues' formula** for rotations.

We have shown how to use the matrix exponential to construct a rotation matrix from a rotation axis $\hat{\omega}$ and an angle θ. Further, the quantity $e^{[\hat{\omega}]\theta}p$ has the effect of rotating $p \in \mathbb{R}^3$ about the fixed-frame axis $\hat{\omega}$ by an angle θ. Similarly, considering that a rotation matrix R consists of three column vectors, the rotation matrix $R' = e^{[\hat{\omega}]\theta}R = \mathrm{Rot}(\hat{\omega},\theta)R$ is the orientation achieved by rotating R by θ about the axis $\hat{\omega}$ in the fixed frame. Reversing the order of matrix multiplication, $R'' = Re^{[\hat{\omega}]\theta} = R\,\mathrm{Rot}(\hat{\omega},\theta)$ is the orientation achieved by rotating R by θ about $\hat{\omega}$ in the body frame.

Example 3.12. The frame {b} in Figure 3.12 is obtained by rotation from an initial orientation aligned with the fixed frame {s} about a unit axis $\hat{\omega}_1 = (0, 0.866, 0.5)$ by an angle $\theta_1 = 30° = 0.524$ rad. The rotation matrix representation of {b} can be calculated as

$$\begin{aligned}R &= e^{[\hat{\omega}_1]\theta_1} \\ &= I + \sin\theta_1[\hat{\omega}_1] + (1-\cos\theta_1)[\hat{\omega}_1]^2 \\ &= I + 0.5\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix} + 0.134\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} 0.866 & -0.250 & 0.433 \\ 0.250 & 0.967 & 0.058 \\ -0.433 & 0.058 & 0.899 \end{bmatrix}.\end{aligned}$$

The orientation of the frame {b} can be represented by R or by the unit axis $\hat{\omega}_1 = (0, 0.866, 0.5)$ and the angle $\theta_1 = 0.524$ rad, i.e., the exponential coordinates $\hat{\omega}_1\theta_1 = (0, 0.453, 0.262)$.

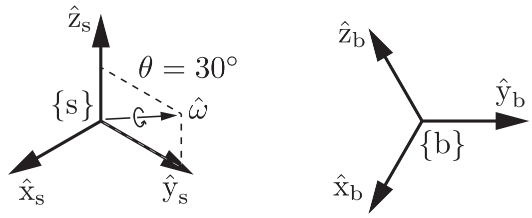

Figure 3.12: The frame {b} is obtained by a rotation from {s} by $\theta_1 = 30°$ about $\hat{\omega}_1 = (0, 0.866, 0.5)$.

If {b} is then rotated by θ_2 about a fixed-frame axis $\hat{\omega}_2 \neq \hat{\omega}_1$, i.e.,

$$R' = e^{[\hat{\omega}_2]\theta_2} R,$$

then the frame ends up at a different location than that reached were {b} to be rotated by θ_2 about an axis expressed as $\hat{\omega}_2$ in the body frame, i.e.,

$$R'' = Re^{[\hat{\omega}_2]\theta_2} \neq R' = e^{[\hat{\omega}_2]\theta_2} R.$$

Our next task is to show that for any rotation matrix $R \in SO(3)$, one can always find a unit vector $\hat{\omega}$ and scalar θ such that $R = e^{[\hat{\omega}]\theta}$.

3.2.3.3 Matrix Logarithm of Rotations

If $\hat{\omega}\theta \in \mathbb{R}^3$ represents the exponential coordinates of a rotation matrix R, then the skew-symmetric matrix $[\hat{\omega}\theta] = [\hat{\omega}]\theta$ is the **matrix logarithm** of the rotation R.[4] The matrix logarithm is the inverse of the matrix exponential. Just as the matrix exponential "integrates" the matrix representation of an angular velocity $[\hat{\omega}]\theta \in so(3)$ for one second to give an orientation $R \in SO(3)$, the matrix logarithm "differentiates" an $R \in SO(3)$ to find the matrix representation of a constant angular velocity $[\hat{\omega}]\theta \in so(3)$ which, if integrated for one second, rotates a frame from I to R. In other words,

$$\begin{aligned} \exp: &\quad [\hat{\omega}]\theta \in so(3) &\rightarrow&\quad R \in SO(3), \\ \log: &\quad R \in SO(3) &\rightarrow&\quad [\hat{\omega}]\theta \in so(3). \end{aligned}$$

[4]We use the term "*the* matrix logarithm" to refer both to a specific matrix which is a logarithm of R as well as to the algorithm that calculates this specific matrix. Also, while a matrix R can have more than one matrix logarithm (just as $\sin^{-1}(0)$ has solutions $0, \pi, 2\pi$, etc.), we commonly refer to "the" matrix logarithm, i.e., the unique solution returned by the matrix logarithm algorithm.

To derive the matrix logarithm, let us expand each entry for $e^{[\hat{\omega}]\theta}$ in Equation (3.51),

$$\begin{bmatrix} c_\theta + \hat{\omega}_1^2(1-c_\theta) & \hat{\omega}_1\hat{\omega}_2(1-c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1-c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1-c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1-c_\theta) & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1-c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1-c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1-c_\theta) \end{bmatrix}, \tag{3.52}$$

where $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$, and we use again the shorthand notation $s_\theta = \sin\theta$ and $c_\theta = \cos\theta$. Setting the above matrix equal to the given $R \in SO(3)$ and subtracting the transpose from both sides leads to the following:

$$\begin{aligned} r_{32} - r_{23} &= 2\hat{\omega}_1 \sin\theta, \\ r_{13} - r_{31} &= 2\hat{\omega}_2 \sin\theta, \\ r_{21} - r_{12} &= 2\hat{\omega}_3 \sin\theta. \end{aligned}$$

Therefore, as long as $\sin\theta \neq 0$ (or, equivalently, θ is not an integer multiple of π), we can write

$$\begin{aligned} \hat{\omega}_1 &= \frac{1}{2\sin\theta}(r_{32} - r_{23}), \\ \hat{\omega}_2 &= \frac{1}{2\sin\theta}(r_{13} - r_{31}), \\ \hat{\omega}_3 &= \frac{1}{2\sin\theta}(r_{21} - r_{12}). \end{aligned}$$

The above equations can also be expressed in skew-symmetric matrix form as

$$[\hat{\omega}] = \begin{bmatrix} 0 & -\hat{\omega}_3 & \hat{\omega}_2 \\ \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\ -\hat{\omega}_2 & \hat{\omega}_1 & 0 \end{bmatrix} = \frac{1}{2\sin\theta}\left(R - R^{\mathrm{T}}\right). \tag{3.53}$$

Recall that $\hat{\omega}$ represents the axis of rotation for the given R. Because of the $\sin\theta$ term in the denominator, $[\hat{\omega}]$ is not well defined if θ is an integer multiple of π.[5] We address this situation next, but for now let us assume that $\sin\theta \neq 0$ and find an expression for θ. Setting R equal to (3.52) and taking the trace of both sides (recall that the trace of a matrix is the sum of its diagonal entries),

$$\operatorname{tr} R = r_{11} + r_{22} + r_{33} = 1 + 2\cos\theta. \tag{3.54}$$

The above follows since $\hat{\omega}_1^2 + \hat{\omega}_2^2 + \hat{\omega}_3^2 = 1$. For any θ satisfying $1 + 2\cos\theta = \operatorname{tr} R$ such that θ is not an integer multiple of π, R can be expressed as the exponential $e^{[\hat{\omega}]\theta}$ with $[\hat{\omega}]$ as given in Equation (3.53).

[5]Singularities such as this are unavoidable for any three-parameter representation of rotation. Euler angles and roll–pitch–yaw angles suffer from similar singularities.

Let us now return to the case $\theta = k\pi$, where k is some integer. When k is an even integer, regardless of $\hat{\omega}$ we have rotated back to $R = I$ so the vector $\hat{\omega}$ is undefined. When k is an odd integer (corresponding to $\theta = \pm\pi, \pm 3\pi, \ldots$, which in turn implies $\operatorname{tr} R = -1$), the exponential formula (3.51) simplifies to

$$R = e^{[\hat{\omega}]\pi} = I + 2[\hat{\omega}]^2. \tag{3.55}$$

The three diagonal terms of Equation (3.55) can be manipulated to give

$$\hat{\omega}_i = \pm\sqrt{\frac{r_{ii}+1}{2}}, \qquad i = 1, 2, 3. \tag{3.56}$$

The off-diagonal terms lead to the following three equations:

$$\begin{aligned} 2\hat{\omega}_1\hat{\omega}_2 &= r_{12}, \\ 2\hat{\omega}_2\hat{\omega}_3 &= r_{23}, \\ 2\hat{\omega}_1\hat{\omega}_3 &= r_{13}, \end{aligned} \tag{3.57}$$

From Equation (3.55) we also know that R must be symmetric: $r_{12} = r_{21}$, $r_{23} = r_{32}$, $r_{13} = r_{31}$. Equations (3.56) and (3.57) may both be necessary to obtain a solution for $\hat{\omega}$. Once such a solution has been found then $R = e^{[\hat{\omega}]\theta}$, where $\theta = \pm\pi, \pm 3\pi, \ldots$

From the above it can be seen that solutions for θ exist at 2π intervals. If we restrict θ to the interval $[0, \pi]$ then the following algorithm can be used to compute the matrix logarithm of the rotation matrix $R \in SO(3)$.

Algorithm: Given $R \in SO(3)$, find a $\theta \in [0, \pi]$ and a unit rotation axis $\hat{\omega} \in \mathbb{R}^3, \|\hat{\omega}\| = 1$, such that $e^{[\hat{\omega}]\theta} = R$. The vector $\hat{\omega}\theta \in \mathbb{R}^3$ comprises the exponential coordinates for R and the skew-symmetric matrix $[\hat{\omega}]\theta \in so(3)$ is the matrix logarithm of R.

(a) If $R = I$ then $\theta = 0$ and $\hat{\omega}$ is undefined.

(b) If $\operatorname{tr} R = -1$ then $\theta = \pi$. Set $\hat{\omega}$ equal to any of the following three vectors that is a feasible solution:

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix} \tag{3.58}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix} \tag{3.59}$$

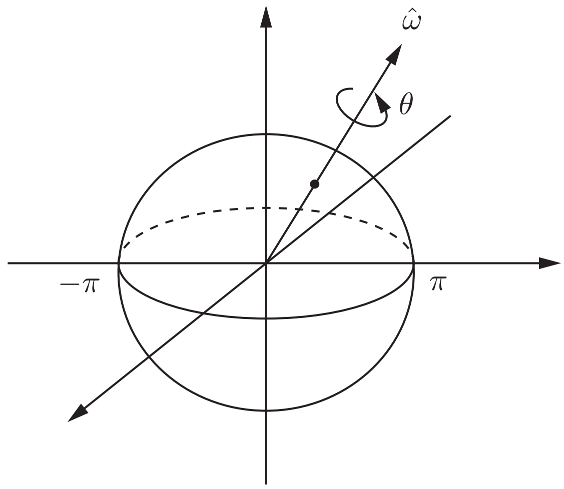

Figure 3.13: $SO(3)$ as a solid ball of radius π. The exponential coordinates $r = \hat{\omega}\theta$ may lie anywhere within the solid ball.

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}. \tag{3.60}$$

(Note that if $\hat{\omega}$ is a solution, then so is $-\hat{\omega}$.)

(c) Otherwise $\theta = \cos^{-1}\left(\frac{1}{2}(\operatorname{tr} R - 1)\right) \in [0, \pi)$ and

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^{\mathrm{T}}). \tag{3.61}$$

Since every $R \in SO(3)$ satisfies one of the three cases in the algorithm, for every R there exists a matrix logarithm $[\hat{\omega}]\theta$ and therefore a set of exponential coordinates $\hat{\omega}\theta$.

Because the matrix logarithm calculates exponential coordinates $\hat{\omega}\theta$ satisfying $||\hat{\omega}\theta|| \leq \pi$, we can picture the rotation group $SO(3)$ as a solid ball of radius π (see Figure 3.13): given a point $r \in \mathbb{R}^3$ in this solid ball, let $\hat{\omega} = r/\|r\|$ be the unit axis in the direction from the origin to the point r and let $\theta = \|r\|$ be the distance from the origin to r, so that $r = \hat{\omega}\theta$. The rotation matrix corresponding to r can then be regarded as a rotation about the axis $\hat{\omega}$ by an angle θ. For any $R \in SO(3)$ such that $\operatorname{tr} R \neq -1$, there exists a unique r in the interior of the solid ball such that $e^{[r]} = R$. In the event that $\operatorname{tr} R = -1$, $\log R$ is given by two antipodal points on the surface of this solid ball. That is, if there exists some r such that $R = e^{[r]}$ with $\|r\| = \pi$ then $R = e^{[-r]}$ also holds; both r and $-r$ correspond to the same rotation R.

3.3 Rigid-Body Motions and Twists

In this section we derive representations for rigid-body configurations and velocities that extend, but otherwise are analogous to, those in Section 3.2 for rotations and angular velocities. In particular, the homogeneous transformation matrix T is analogous to the rotation matrix R; a screw axis $\mathcal{S}$ is analogous to a rotation axis $\hat{\omega}$; a twist $\mathcal{V}$ can be expressed as $\mathcal{S}\dot{\theta}$ and is analogous to an angular velocity $\omega = \hat{\omega}\dot{\theta}$; and exponential coordinates $\mathcal{S}\theta \in \mathbb{R}^6$ for rigid-body motions are analogous to exponential coordinates $\hat{\omega}\theta \in \mathbb{R}^3$ for rotations.

3.3.1 Homogeneous Transformation Matrices

We now consider representations for the combined orientation and position of a rigid body. A natural choice would be to use a rotation matrix $R \in SO(3)$ to represent the orientation of the body frame {b} in the fixed frame {s} and a vector $p \in \mathbb{R}^3$ to represent the origin of {b} in {s}. Rather than identifying R and p separately, we package them into a single matrix as follows.

Definition 3.13. The **special Euclidean group** $SE(3)$, also known as the group of **rigid-body motions** or **homogeneous transformation matrices** in $\mathbb{R}^3$, is the set of all 4×4 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.62}$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$ is a column vector.

An element $T \in SE(3)$ will sometimes be denoted (R, p). In this section we will establish some basic properties of $SE(3)$ and explain why we package R and p into this matrix form.

Many robotic mechanisms we have encountered thus far are planar. With planar rigid-body motions in mind, we make the following definition: