期末复习

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Reference:
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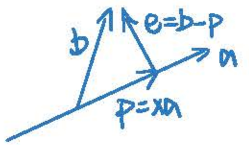
1. Basic Linear Algebra.

1.1 Angles between vectors. (projection)

① Projection (MIT)

vector project to vector $P = \frac{aa^T}{a^Ta}$



$a^T(b-ax)=0$ (求 x)

$a^Tb - a^Tax = 0$

$x = \frac{a^Tb}{a^Ta}$

$\Rightarrow p = \underline{\frac{aa^T}{a^Ta}} b$ projection matrix

vector project to plane $P = A(A^TA)^{-1}A^T$

② Find the matrix that projects every point in the plane onto the line $x-2y=0$

direction vector of $x-2y=0$ is $[2,1]^T$

b is a point or vector $\vec{OB}$, $|\vec{OB}||v|\cos\theta = \vec{OB}\cdot\vec{v}$

$|\vec{OB}|\cos\theta \cdot \frac{v}{\|v\|} = \vec{v}^T\cdot\vec{OB}\cdot\frac{\vec{v}}{\|v\|^2} = \frac{\vec{v}\vec{v}^T}{\|v\|^2}\vec{OB}$

③ Find a set of vectors of the plane that perpendicular to the plane $x+y-z=3$ in $\mathbb{R}^3$.

intersect points, $(3,0,0)$, $(0,3,0)$, $(0,0,-3)$

$\Rightarrow l_1 = [-1,1,0]^T$ $l_2 = [1,0,1]^T$ $v = [1,1,-1]^T$

v is one of the set, the other could be arbitrary as long as it perpendicular to v

1.2 Linearly independence ⇐ base

A set of vectors (Vector space) $\triangleq V$, let it be the column space of A. $Ax=0$'s null space only have 0. s.t. $c_1v_1 + c_2v_2 + \cdots c_nv_n$ is 0 only if $c_1, c_2 \cdots c_n$ all equal 0.

1.3 Range. column space ...

1.4 Rank ...

1.5 Conservation of dim ...

1.6 Null space.

① find the null space

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{array}{c} \begin{bmatrix} 1 & \boxed{2} & 0 & \boxed{-2} \\ 0 & \boxed{0} & 1 & \boxed{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \uparrow & & \uparrow & \\ p_1 & & p_2 & \end{matrix} \end{array}$$

$$\text{Null space} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1.9* eigenvalues and eigenvetor ↓

1.7* diagonalize, Jodan form.

① diagonalize $A = \begin{bmatrix} 6 & 2 & 4 \\ 2 & 3 & 2 \\ 4 & 2 & 6 \end{bmatrix}$

$$\lambda I - A = \begin{bmatrix} \lambda-6 & -2 & -4 \\ -2 & \lambda-3 & -2 \\ -4 & -2 & \lambda-6 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{\begin{matrix} \lambda-6 & -10 \\ -2 & \lambda-7 \end{matrix}} & \begin{matrix} -4 \\ -2 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & \boxed{\lambda-2} \end{bmatrix}$$

$|\lambda I - A| = 0 \Rightarrow \lambda_1 = \lambda_2 = 2 \quad \lambda_3 = 11$

substitute to $\lambda I - A$

(1) $\lambda = 2$ $\begin{bmatrix} -4 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$\text{Null space} = \begin{bmatrix} -1/2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

(2) $\lambda = 11 \Rightarrow v_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

$T = [v_1 \; v_2 \; v_3] \quad \Lambda = T^{-1} A T = P A P^{-1}$

$$P = T^{-1} = \frac{1}{9} \begin{bmatrix} -1 & -4 & 2 \\ 4 & -2 & 1 \\ -1 & 5 & 2 \end{bmatrix}$$

② Jordan Form. of $A = \begin{bmatrix} 3 & -6 & 0 & 2 \\ 4 & -5 & -2 & 6 \\ 0 & 0 & 3 & -2 \\ 0 & 0 & 2 & -1 \end{bmatrix}$

$|\lambda I - A| \rightarrow \quad \lambda_1 = \lambda_2 = 1 \quad \lambda_3 = \lambda_4 = -1$

Find Jordan Block:

(1) $\lambda = 1 \quad B_1 = I - A$

$\text{rank}(B_1) = 3 \quad \text{rank}(B_1^2) = 2 \quad \text{rank}(B_1^n) = 2 \; (n > 2)$

num of block (rk=1) $[n - rk(B_1)] - [rk(B_1) - rk(B_1^2)] = 0$

num of block (rk=2) $[rk(B_1) - rk(B_1^2)] - [rk(B_1^2) - rk(B_1^3)] = 1$

(2) $\lambda = -1 \Rightarrow$ num of block (rk=2) = 1

$$\Rightarrow \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

1.8* inverse matrix

① dim = 2 $= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

② Adjugate matrix

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{A} \xrightarrow[(-1)^{i+j}]{det} \begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{bmatrix} \xrightarrow{C^T} \underbrace{\begin{bmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 3 & -3 \end{bmatrix}}_{adj(A)}$$

$$A^{-1} = \frac{1}{|A|} adj(A)$$

2. State space models.

2.1 Data Based modeling (LSE)

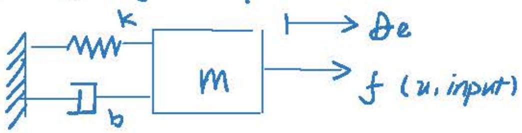
① continuous-time state model

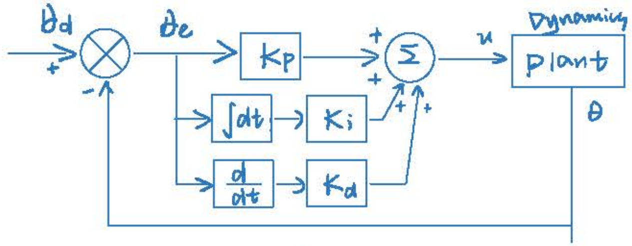
$$\begin{cases} \dot{x} = f(x,u) & \text{vector field} \\ y = h(x,u) \end{cases}$$

discrete-time state space model

$$\begin{cases} x(k+1) = f(x(k), u(k)) \\ y(k) = h(x(k), u(k)) \end{cases}$$

③ PID spring damper.





Dynamics: $f = m\ddot{\theta}_e + b\dot{\theta}_e + k\theta_e$, $u = f = g(x)$

(f is what to control)

PID: $u = K_p \theta_e + K_d \dot{\theta}_e + K_i \int \theta_e dt$

$0 = m\ddot{\theta}_e + (b - K_d)\dot{\theta}_e + (k - K_p)\theta_e - K_i \int \theta_e dt$

$$x = \begin{bmatrix} \int \theta_e dt \\ \theta_e \\ \dot{\theta}_e \end{bmatrix} \Rightarrow \dot{x} = f(x,u) = f(x, g(x))$$

⑬ Continuous to Discrete.

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases}$$

$$\begin{cases} x(k+1) = x(k) + f(x, u)\Delta t \\ y(k) = h(x, u) \end{cases}$$

If Δt is too large $\Rightarrow$ unstable

If $\dot{x} = Ax + Bu$

$$x(k+1) = (I + A\Delta t)\, x(k) + u(k)$$

(HW 3, 4) (Proj 2 4)

⑭ Nonlinear to Linear.

linearize around $(\hat{x}, \hat{u})$

$$\begin{cases} \Delta x = x - \hat{x} \\ \Delta u = u - \hat{u} \\ \Delta y = y - h(\hat{x}, \hat{u}) \end{cases} \qquad f(\hat{x}, \hat{u}) - \hat{x}$$

$$\Delta x(k+1) \approx \hat{A}\Delta x(k) + \hat{B}\Delta u(k) + C$$

Jacobian:

$$\frac{\partial f}{\partial z} = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \frac{\partial f_1}{\partial z_3} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_2}{\partial z_3} \end{bmatrix}$$ (care about the order)

$$df = \frac{\partial f}{\partial z} dz = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \frac{\partial f_1}{\partial z_2} & \frac{\partial f_1}{\partial z_3} \\ \frac{\partial f_2}{\partial z_1} & \frac{\partial f_2}{\partial z_2} & \frac{\partial f_2}{\partial z_3} \end{bmatrix} \begin{bmatrix} dz_1 \\ dz_2 \\ dz_3 \end{bmatrix}$$

e.g. $f(z) = \begin{bmatrix} 2z_1 + e^{z_2} \\ \log(z_3) + \frac{1}{z_2} \end{bmatrix} \quad \hat{z} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\frac{\partial f}{\partial z} = \begin{bmatrix} 2 & e^{z_2} & 0 \\ 0 & -\frac{1}{z_2^2} & \frac{1}{z_3} \end{bmatrix}\Bigg|_{z=\hat{z}} = \begin{bmatrix} 2 & e^2 & 0 \\ 0 & -\frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

$$f(z) = f(\hat{z}) + \left(\frac{\partial f}{\partial z}(z)\Big|_{z=\hat{z}}\right)\Delta z + \text{H.O.T}$$

(given Δz, find $f(z)$)

$$f(x,u) \approx f(\hat{x}, \hat{u}) + \underbrace{\left(\frac{\partial f(x,u)}{\partial x}\Big|_{x=\hat{x}, u=\hat{u}}\right)}_{\hat{A}}(x - \hat{x}) + \underbrace{\left(\frac{\partial f(x,u)}{\partial u}\Big|_{x=\hat{x}, u=\hat{u}}\right)}_{\hat{B}}(u - \hat{u})$$

Could also be written as:

$$x = f(x, u) = f(\hat{x}, \hat{u}) + \left[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial u}\right]\begin{bmatrix}\Delta x \\ \Delta u\end{bmatrix}$$

$$\Delta x_{k+1} \triangleq x_{k+1} - \hat{x}$$
$$= f(x_k, u_k) - \hat{x}$$
$$= \hat{A}\Delta x_k + \hat{B}\Delta u_k + \cancel{f(\hat{x},\hat{u}) - \hat{x}}$$

is zero at equilibrium

$$\Delta y \triangleq \underbrace{\frac{\partial h}{\partial x}}_{\hat{C}}\Delta x_k + \underbrace{\frac{\partial h}{\partial u}}_{\hat{D}}\Delta u_k$$

(Proj 2 3)

(HW3 4)

(Lecture Note 2 23 example)

2.4 Least square problems.

① Linear LSE Problem

Solution: $\theta = (X^T X)^{-1} X^T b$

② Curve fitting (HW3, 5)

$y = be^{ax}$ (LN3 15) $y = \alpha_0 + \alpha_1 x + \alpha_2 x^2 \ldots$ (LN3 16)

2.1.1 estimate parameters of ARX model

2.2.3 ARX model

Auto Regression:

$$y(k) + \alpha_1 y(k-1) + \cdots + \alpha_p y(k-p)$$
$$= \beta_0 u(k) + \beta_1 u(k-1) + \cdots + \beta_q u(k-q) + \underbrace{v(k)}_{\text{noise}}$$

Model parameter: $\theta = [\alpha_1, \ldots, \alpha_p, \beta_0, \beta_1, \ldots, \beta_q]^T$

given θ

One-step predictor:

$$\hat{y}(k|\theta) = -\alpha_1 y(k-1) - \cdots - \alpha_p y(k-p)$$
$$+ \beta_0 u(k) + \beta_1 u(k-1) + \cdots + \beta_q u(k-q)$$

$$y\left\{\begin{bmatrix} y(k_0) \\ \vdots \\ y(m)\end{bmatrix}\right. = \underbrace{\begin{bmatrix}\phi^T(k_0) \\ \vdots \\ \phi^T(m)\end{bmatrix}}_{H}\theta + \begin{bmatrix} v(k_0) \\ \vdots \\ v(m)\end{bmatrix}$$

$\boxed{k_0 \text{ start from } \max(p, q) + 1}$

① find best estimate a, b. $G(z) = \frac{z^2 + b}{z^3 + az}$

Given dataset $\{u_i, y_i\}$ $i = 1, 2, \cdots, 20$

$$Y(z) = G(z)U(z)$$
$$(1 + az^{-2})Y(z) = (z^{-1} + bz^{-3})U(z)$$
$$y(k) + a\,y(k-2) = u(k-1) + b\,u(k-3)$$

$$k_0 = \max(p, q) + 1 = 3 + 1 = 4$$

$$y(4) - u(3) = \begin{bmatrix} -y(2) & u(1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} y(4) - u(3) \\ y(5) - u(4) \\ \vdots \\ y(20) - u(19) \end{bmatrix} = \begin{bmatrix} -y(2) & u(1) \\ -y(3) & u(2) \\ \vdots & \vdots \\ -y(18) & u(17) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$y = H\theta$$

$$\hat{\theta}_{LS} = (H^T H)^{-1} H^T y$$

(Quiz 1) 将参数全部放右侧

2.2 Transformation among models

2.2.1 Transfer function

$$H(z) = C(zI - A)^{-1}B + D$$

SISO system:

$$H(z) = \frac{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}{z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0}$$

Write the above transfer function in controllable canonical form. (multiple sln)

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 - a_0 b_n & b_1 - a_1 b_n & \cdots & b_{n-1} - a_{n-1} b_n \end{bmatrix}$$

$$D = [b_n]$$

one solution

A more common form is:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} \end{bmatrix}$$

$$D = [0]$$

another solution

① Example in (LN3 28)

② (HW4 5) (a) (b)

3. State space System properties

3.1 state space solution

$$x(k+1) = A(k)\,x(k) + B(k)\,u(k)$$
$$y(k) = C(k)\,x(k) + D(k)\,u(k)$$

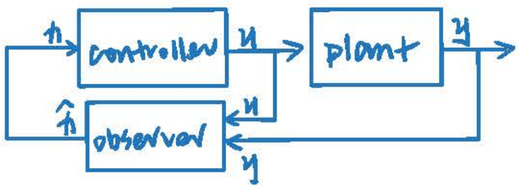
state trajectory: $x(0) = \hat{x}$

$$x(k) = A^k \hat{x} + \sum_{j=0}^{k-1} A^{k-j-1} B u(j)$$

e.g. $x(2) = A(A\hat{x} + Bu(0)) + Bu(1)$
$$= A^2\hat{x} + [ABu(0) + Bu(1)]$$

$$y(k) = \underbrace{CA^k\hat{x}}_{\text{zero input res}} + \underbrace{\sum_{j=0}^{k-1} CA^{k-1-j} Bu(j) + Du(k)}_{\text{zero state res}}$$

$\downarrow$
output trajectory



3.2 Internal stability & BIBO (External)

	definition	test
Internal (Asym)	$u_k \equiv 0$, $\|x_k\| \to 0 \;\; \forall x_0$	$\Rightarrow$ $\|\mathrm{eig}(A)\| < 1$
		$A^k \xrightarrow[\text{Jordan form}]{\text{Diagonalize}} A$
		$\Downarrow \quad \not\Uparrow$
External (BIBO)	Bounded $\|u_k\|$ $\Rightarrow$ Bounded $\|y_k\|$	$\Rightarrow$ poles of $H(z) < 1$
		$H(z) = C(zI - A)^{-1}B + D$

poles 是 eig 的一部分
有的可能分子分母消去了

(HW4 1) (HW4 2)

3.3 K-step reachability

Given (A, B), final state x_f, initial x_0

$$x(k) - A^k x_0 = [B \;\; AB \;\cdots\; A^{k-1}B] \begin{bmatrix} u(k-1) \\ u(k-2) \\ \vdots \\ u(0) \end{bmatrix}$$

that is to solve $x_f - A^k x_0 = M_k u_k$.

(Quiz 2) (if $x_f - A^k x_0 \in \mathrm{Col}(M_k)$)

3.4 controllability (def and test)

Controllability:

definition	test
Any $x_0 \to$ Any x_K in finite time $\Rightarrow$	$rank(M_c) = n$

$$M_c = [B \ AB \ A^2B \cdots A^{k-1}B]$$ (HW4 3)

3.5 Observability (def and test)

Observability:

definition	test
$\{u_k, y_k\} \to x_0$ determine $\Rightarrow$	$rank(M_o) = n$

$$M_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{k-1} \end{bmatrix} \qquad \begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k-1) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix} x_0$$

(Assume $u(k)=0$)

There is also K-step observable; like (HW4 4(b))

observable (A, C) $\Leftrightarrow$ controllable (A^T, C^T)

3.6 similarity transform

$A \Leftrightarrow PAP^{-1}$ $\quad \hat{M}_c = P^{-1}M_c$ (LN4 28)

4. Eigenvalue Assignment controller design

4.1 closed-loop dynamics under state feedback

state feedback controller = mapping

$$u_k = \mu(x_k)$$

closed loop system: $\begin{cases} x_{k+1} = f(x_k, \mu(x_k)) \triangleq f_{cl}(x_k) \\ y_k = h(x_k, \mu(x_k)) \triangleq h_{cl}(x_k) \end{cases}$

4.2 controllable canonical form ...

4.3 eigenvalue assignment in controllable canonical form

Linear controller: $u_k = -\underbrace{K}_{\text{feedback gain}} x_k$

$\Rightarrow$ for a linear closed loop system

$$x_{k+1} = (A - BK) x_k$$

Use "eigenvalue assignment" to design gain

$\Leftrightarrow$ Find K s.t. $eig(A-BK) = eig_{desired}$

Steps:

(1) If system (A, B) is controllable

(2) Find $\bar{K} = [K_1\ K_2]$ s.t. $eig(A - BK) = eig_{desired}$

(3) Check new $\bar{A}_{cl}$

(1) Example (LN5 13)

$$\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \lambda^*_{1,2} = \pm 0.5$$

step 1: $\Delta_{desired}(\lambda) = (\lambda - 0.5)(\lambda + 0.5) = \lambda^2 - 0.25$

$\Rightarrow \alpha_0^* = -0.25 \quad \alpha_1^* = 0$

step 2: $\alpha_0 = 1 \quad \alpha_1 = 2 \quad$ find $\bar{K} = [K_1\ K_2]$

$A - BK \Rightarrow$ new $\alpha_i^* \Rightarrow -\alpha_i - K_{i+1} = -\alpha_i^*$

$\Rightarrow K_{i+1} = \alpha_i^* - \alpha_i$

$K_1 = -0.25 - 1 = -1.25 \qquad K_2 = 0 - 2 = -2$

$K = [-1.25\ \ -2]$

step 3: $\bar{A}_{cl} = \bar{A} - \bar{B}\bar{K} = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} \Rightarrow \Delta_{cl} = \lambda^2 - 0.25$

$\Rightarrow \lambda_{cl} = \pm 0.5$

4.4 eigenvalue assignment for system not in canonical, how to assign eig

(1) General SISO system (A, B)

Under similarity transform:

$$\begin{cases} x(k) = P\bar{x}(k) \\ \bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) \\ \bar{A} = P^{-1}AP \qquad \bar{B} = P^{-1}B \\ \bar{M}_c = P^{-1}M_c \\ eig(A) = eig(\bar{A}) \\ \Delta_A(\lambda) = \Delta_{\bar{A}}(\lambda) \end{cases}$$

How to find P

$P^{-1} = \bar{M}_c M_c^{-1} \Rightarrow P = M_c \bar{M}_c^{-1}$

Step 5: (LN5 15)

(1) similarity transform, find P

$\Delta_A(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda + \alpha_0$

$$\Delta_{\bar{A}}(\lambda) = \Delta_A(\lambda) \longrightarrow \bar{A} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \cdots & 1 \\ -\alpha_0 & -\alpha_1 & \cdots & -\alpha_{n-1} \end{pmatrix} \quad \bar{B} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\bar{M}_c \leftarrow (\bar{A}, \bar{B})$ $\quad M_c \leftarrow (A, B)$

$P = M_c \bar{M}_c^{-1} \longrightarrow$ found P

(2) Find $\bar{K}$ to assign $eig_{desired}$

(3) compute $K = \bar{K}P^{-1}$, found K

(4) compute $P^{-1}(A-BK)P$ to check

$\bar{A} - \bar{B}\bar{K} = P^{-1}AP - P^{-1}BKP = P^{-1}(A-BK)P$

(Project 2 b)

If B has multiple input, use 1 col (LNJ 17)

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how to assign eig (LNJ 19)

Suppose we wand $T_s \le 5s$ $PO \le 35\%$

$$\begin{cases} \frac{4}{\zeta\omega_n} \le 5 \\ 100 e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \le 35 \end{cases} \Rightarrow \begin{cases} \zeta\omega_n \ge 0.8 \\ \zeta \ge 0.32 \end{cases}$$

choose $\zeta = 0.5$ $\quad \zeta\omega_n = 1 \Rightarrow \omega_n = 2$.

$p_{1,2} = -1 \pm j\sqrt{3} \quad \Rightarrow T = 0.03$, DT: $z_{1,2} = e^{(-1 \pm \sqrt{3}j) 0.03}$

(LNJ 20)

5. Observer design

5.1 Observer structure

state vector is not available

because (A, B) is actually plugin (?)

$\Rightarrow$ generate estimate according to known system dynamics (directly use sensor data to compute would enlarge noise)

5.2 Error dynamics

$$\hat{x}(k+1) = \underbrace{A\hat{x}(k) + Bu(k)}_{\text{estimate at } k} + \underbrace{L[y(k) - C\hat{x}(k) - Du(k)]}_{\text{"suprise" correction term}}$$

L actually similar to the design of kalman gain

state estimate error: $e(k) = x(k) - \hat{x}(k)$

error dynamics: $e(k+1) = (A-LC)e(k)$

$$\begin{aligned} e(k+1) &= x(k+1) - \hat{x}(k+1) \\ &= Ax(k) + Bu(k) - \hat{x}(k+1) = \dots \\ &= A(x(k) - \hat{x}(k)) - LC(x(k) - \hat{x}(k)) \\ &= (A-LC)e(k) \end{aligned}$$

(LNJ 23)

5.3 Design observer gain matrix

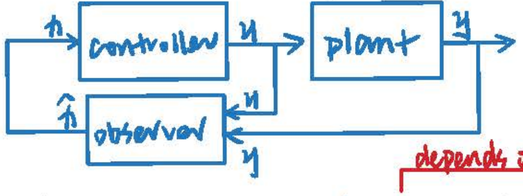
According to Duality Theorem

(A, C) observable $\Leftrightarrow$ (A^T, C^T) controllable

$\Rightarrow$ Design $eig(A^T - C^T\tilde{K}) = eig_{desired}$

$L = \tilde{K}^T$ (Proj2 8)

5.4 Output feedback design.



depends on

Design K, L to let $(A-BK)$, $(A-LC)$ meet the desired eigenvalues

$eig(A-CL)$ need to be faster than $eig(A-BK)$

Closed loop dynamics: (LN5 28)

$\begin{bmatrix} x(k) \\ e(k) \end{bmatrix} \Rightarrow$ joint state vector $\begin{bmatrix} x(k+1) \\ e(k+1) \end{bmatrix} = \begin{bmatrix} A-BK & BK \\ 0 & A-CL \end{bmatrix} \begin{bmatrix} x(k) \\ e(k) \end{bmatrix}$

$eig\left(\begin{bmatrix} A-BK & BK \\ 0 & A-CL \end{bmatrix}\right) = eig(A-BK)\, eig(A-CL)$

$\Rightarrow$ prove that K, L can be designed seperately

6. Probability

6.1 probability and conditional probability

① Probability Space $(\Omega, \mathcal{F}, P)$

event is a subset of sampling space Ω

event space $\mathcal{F}$ is all the subset of Ω

$P: \mathcal{F} \to [0,1]$ mapping $\mathcal{F}$ to $[0,1]$

(HW5 1)

② Conditional prob (LN7(1) 11)

(HW5 2) (HW5 3)

6.2 random variables and random vectors

6.3 jointly distributed random vectors and conditional expectation

① Expectation properties

$E(AX+BY) = AE(X) + BE(Y)$

$$E(X) = \begin{bmatrix} E(X_1) \\ \vdots \\ E(X_n) \end{bmatrix} \qquad E(X) \triangleq \int_{R^n} x f(x)\,dx$$
$$E(X) \triangleq \sum_x x \cdot \text{Prob}(X=x)$$

② jointly distributed

$$(X, Y) \sim f_{XY}(x, y)$$
$$p(x, y) = \int_A f_{XY}(x,y)\,dx\,dy$$

marginal density: (LN 7(1) 19)

$X \sim f_X(x)$ $\qquad$ $Y \sim f_Y(y)$

$$f_X(x) = \int_{R^m} f_{XY}(x,y)\,dy \qquad f_Y(y) = \int_{R^n} f_{XY}(x,y)\,dx$$

③ conditional density (LN 7(1) 20)

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow p_{X|Y}(x=i \mid Y=j) = \frac{p_{XY}(x=i, Y=j)}{\underbrace{\sum_i p_{XY}(x=i, Y=j)}_{\text{Prob}(Y=j)}}$$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} \qquad f_X(x) = \int_{R^m} f_{X|Y}(x|y) f_Y(y)\,dy$$

(If X is independent of Y, $X \perp Y$, $f_{XY}(x,y) = f_X(x) f_Y(y)$)

④ conditional expectation

$$E(X|Y=y) \triangleq \int_{R^n} x f_{X|Y}(x,y)\,dx$$
$$E(X|Y=y) \triangleq \sum_i i \cdot \text{Prob}(x=i \mid Y=y)$$

$$E(X) = \sum_y E(X|Y=y) \cdot p_Y(Y=y)$$
$$E(g(X,Y)) = \sum_y E(g(X,Y)|Y=y) \cdot p_Y(Y=y)$$

(LN 7(1) 21-25)

6.4 covariance matrix

$$\text{Cov}(X, Y) = E\left((X - E(X))(Y - E(Y))^T\right)$$

meaning: correlation, if $\text{cov}(X, Y) = 0$, uncorrelated

$$\text{cov}(X, Y) = \begin{bmatrix} \text{cov}(X_1, Y_1) & \text{cov}(X_1, Y_2) & \cdots & \text{cov}(X_1, Y_m) \\ \text{cov}(X_2, Y_1) & \text{cov}(X_2, Y_2) & \cdots & \text{cov}(X_2, Y_m) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_n, Y_1) & \text{cov}(X_n, Y_2) & \cdots & \text{cov}(X_n, Y_m) \end{bmatrix}$$

properties:

$$\text{Cov}(X+a, Y+b) = \text{Cov}(X, Y)$$
$$\text{Cov}(X, Y) = \text{Cov}(Y, X)^T$$

$Cov(X_1+X_2, Y) = Cov(X_1, Y) + Cov(X_2, Y)$

$Cov(AX, BY) = A\,Cov(X, Y)\,B^T$

$Cov(X, Y) = 0$ if $X \perp Y$

$Cov(X) \triangleq Cov(X, X)$ is positive semidefinite

$Cov(AX+BY) = A\Sigma_X A^T + A\Sigma_{XY}B^T + B\Sigma_{XY}^T A^T + B\Sigma_Y B^T$

(LN 7 (1) 32)

7. Kalman filter

7.1 Minimum Mean squared estimation

An estimator is a function that maps each measurement $Y=y$ to an estimate $\hat{x}$

We use MSE (mean-squared error) to judge the performance of an estimator $E(\|\phi(Y)-X\|^2)$

$$y \longrightarrow \underset{\phi(y)}{\boxed{\text{estimator}}} \longrightarrow \hat{x}$$

Example (LN 7 (2) 5) (LN 7 (2) 17-18)

$\Rightarrow$ MMSE is an estimator that minimize MSE

$E(\|\phi(Y)-X\|^2 \mid Y=y)$ minimize when

$\hat{x}_{MMSE} = \phi_{MMSE}(y) = E(X \mid Y=y)$

7.2 Gaussian Random Vectors (condition)

Calculate $E(X \mid Y=y)$ for jointly Gaussian random vector $\longrightarrow$ simple form ~~($\int x f_{X|Y}(x|y)dx$)~~

$X \sim N(\mu, \sigma)$

1D Gau pdf: $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

nD Gau pdf:

$$f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}}(\det(\Sigma))^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right\}$$

σ^2 and Σ are covariance or variance, not standard deviation σ (Qui 8.4 2)

... is also Gaussian (LN 7 (2) 11-12)

$X \sim N(\mu, \Sigma) \longrightarrow Z = AX+b \sim N(A\mu+b, A\Sigma A^T)$

Conditional distribution of X given $Y=y$

$X \mid Y=y \sim N(\mu_{X|Y=y}, \Sigma_{X|Y=y})$

where

$$\begin{cases} \mu_{x|y=y} = \mu_x + \Sigma_{xy}\Sigma_y^{-1}\underbrace{(y-\mu_y)}_{\text{surprise term}} \\ \Sigma_{x|y=y} = \Sigma_x - \Sigma_{xy}\Sigma_y^{-1}\Sigma_{yx} \end{cases}$$

(Quiz 4 3) (LN7 (2) 15-16)

7.3 Kalman Filter Derivations

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k + w_k \\ y_k = C_k x_k + D_k u_k + v_k \end{cases}$$

$w_k \sim N(0, Q_k)$ process noise (tuned)

$v_k \sim N(0, R_k)$ measurement noise

Prediction

Use prior knowledge to predict k state: $\underbrace{\hat{x}_{k|k-1}}_{\text{prior knowledge of k state}} = A_{k-1}\underbrace{\hat{x}_{k-1}}_{\text{posterior of k-1 state}} + B_{k-1}u_{k-1}$

the covariance of the prior prediction result: $\underbrace{P_{k|k-1}}_{\text{prior covariance of } \hat{x}_{k|k-1}} = A_{k-1}\underbrace{P_{k-1}}_{\text{posterior covariance of } \hat{x}_{k-1}}A_{k-1}^T + Q_{k-1}$

Measurement Update

Compute Kalman gain:

$$K_k = P_{k|k-1}C_k^T(C_kP_{k|k-1}C_k^T + R_k)^{-1}$$

$$= \frac{P_{k|k-1}C_k^T}{C_kP_{k|k-1}C_k^T + R_k} \quad (R_k \text{ is known})$$

$P_{k|k-1}C_k^T \longrightarrow cov(x_{k+1}, y_{k+1} | Y_k)$

$\underbrace{C_kP_{k|k-1}C_k^T}_{\text{covariance of prediction}} \longrightarrow cov(y_{k+1}|Y_k)$ vs. $\underbrace{R_k}_{\text{covariance of measure}}$

Update estimate and error convariance:

$$\underbrace{\hat{x}_k}_{\text{posterior}} = \underbrace{\hat{x}_{k|k-1}}_{\text{prior}} + K_k\underbrace{(y_k - C_k\hat{x}_{k|k-1} - D_ku_k)}_{\text{surprise}}$$

$$\Downarrow$$

$$P_k = (I - K_kC_k)P_{k|k-1}$$

A 1-D example: (LN7 (2) 34)

7.4 EKF (LN7 (B) 7)

EKF Prediction step: $F_k \triangleq \frac{\partial f}{\partial x}\Big|_{\hat{x}_k, u_k}$

$$\hat{x}_{k|k-1} = f(\hat{x}_{k-1}, u_{k-1})$$

$$P_{k|k-1} = F_{k-1} P_{k-1} F_{k-1}^T + Q_{k-1}$$

EKF Measurement Update: $H_k = \frac{\partial h}{\partial x}\Big|_{\hat{x}_{k|k-1}, u_k}$

$$K_k = P_{k|k-1} H_k^T (H_k P_{k|k-1} H_k^T + R_k)^{-1}$$

$$\hat{x}_k = \hat{x}_{k|k-1} + K_k (y_k - h(\hat{x}_{k|k-1}, u_k))$$

$$P_k = (I - K_k H_k) P_{k|k-1}$$

7.5 Kalman filter problems

7.5.1 tracking

example of tracking (LN7 (B) 11)

example of joint state and paramter estimation

(LN7 (B) 14)

(Project 3) K_k is between 0 and C^{-1}

8. Dynamic Programming and LQR

8.1 general discrete-time optimal control problem

8.2 Dynamical programming

$$x \rightarrow \boxed{\mu_k^*(x_k)} \rightarrow u$$

Encode objective into "Cost Function + constraint"

$$\underbrace{J_N(z, u)}_{\text{cost function}} = \sum_{k=0}^{N-1} \{ l(x_k, u_k), g(x_N) \}$$

s.t. ©
$$\begin{cases} x_{k+1} = f(x_k, u_k) \rightarrow \text{dynamics} \\ x_k \in X \rightarrow \text{robot not fall down, state ©...} \\ u_k \in U(x_k) \rightarrow \text{state dependent constraint} \end{cases}$$

Shortest path problem

DP: $V_j(z) \triangleq \min_{\substack{u_0 \cdots u_{j-1} \\ \text{s.t. ©}}} J_j(z, u)$

Start from 0 horizon: $V_0(z) = g(z)$

From j to $j+1$: $V_{j+1}(z) = \min_{u \in U(z)} \{ l(z, u), \underbrace{V_j(f(z, u))}_{\text{future state}} \}$

(for all z, value iteration)

$\downarrow$

curse of dim for $z \in \mathbb{R}^n$ (n too large)

$$\mu_{j+1}(z) = \arg\min_{u \in U(z)} \{ l(z, u), V_j(f(z, u)) \}$$

8.4 Linear Quadratic Regulator

Exception / special case of former DP:

$$J_N(z,u) = \Big\{\underbrace{\sum_{k=0}^{N-1}\left(x_k^T Q x_k + u_k^T R u_k\right)}_{\text{running cost}}\Big\} + \underbrace{x_N^T Q_f x_N}_{\text{terminal cost}}$$

s.t. ©
$$\begin{cases} x_{k+1} = A x_k + B u_k & \text{(dynamics)} \\ x_k \in \mathbb{R}^n \quad u_k \in \mathbb{R}^m & \text{(no constraint)} \end{cases}$$

$V_j(z) = z^T P_j z \quad \mu_j(z) = -K_j z$ (optimal !)

start from $P_0 = Q_f \qquad P_{j+1} = \underbrace{\rho(P_j)}_{\text{Riccati mapping (map } j \text{ to } j+1)}$

N-horizon: $x_0 = z \xrightarrow[\mu_N^*(x_0)]{u_0^*} x_1^* \xrightarrow[\mu_{N-1}^*(x_1^*)]{u_1^*} x_2^* \longrightarrow \cdots$

control law def: when you have N-1 steps left you are at state x_1^*, you optimal control should be u_1^*; If add a noise to $x_1^* \longrightarrow x_1' = x_1^* + \delta$, the optimal control becomes $\hat{u}_1^* = \mu_{N-1}^*(x_1')$

① example of shortest path (1):

g			

Problem

Running cost: 1

terminal cost: $\begin{cases} 0 & g \\ \infty & \text{otherwise} \end{cases}$

0	∞	∞	∞
∞	∞	∞	∞
∞	∞	∞	∞
0	∞	∞	∞

→

0	1	∞	∞
1	∞	∞	∞
1	∞	∞	∞
0	1	∞	∞

→

0	1	2	∞
1	2	∞	∞
1	2	∞	∞
0	1	2	∞

↓

0	1	2	3
1	2	3	∞
1	2	3	∞
0	1	2	3

←

0	1	2	3
1	2	3	4
1	2	3	4
0	1	2	3

←

0	1	2	3
1	2	3	4
1	2	3	4
0	1	2	3

② (HW7 1,2)