# Math 1014
# Midterm Exam Spring 2022

## Part I: Multiple Choice Questions

1. Find the area under the graph of the function $f(x) = 3x\sqrt{4-x^2}$ over the interval $[0, 2]$.

**Solution** $\int_0^2 3x\sqrt{4-x^2}dx = \Big[-(4-x^2)^{3/2}\Big]_0^2 = 4^{3/2} = 8$

Or let $u = 4 - x^2$, $du = -2xdx$, then $\int_0^2 3x\sqrt{4-x^2}dx = \int_4^0 -\frac{3}{2}u^{1/2}du = \Big[-u^{3/2}\Big]_4^0 = 8.$

Or use the trigonometric substitution, $x = 2\sin u$, $dx = 2\cos u du$:

$$\int_0^2 3x\sqrt{4-x^2}dx = \int_0^{\frac{\pi}{2}} 3(2\sin u)(2\cos u)2\cos u du$$

$$= \int_0^{\frac{\pi}{2}} 24\sin u\cos^2 u du = \Big[-8\cos^3 u\Big]_0^{\frac{\pi}{2}} = 8$$

(Basic substitution: M2/Math1012/Math1013 level integral, Sample Final, Q3.)

2. Evaluate the integral $\int_0^2 4\cos(\pi x)\cos(2\pi x)\cos(3\pi x)dx$.

**Solution** $\int_0^2 4\cos(\pi x)\cos(2\pi x)\cos(3\pi x)dx = \int_0^2 2[\cos(\pi x) + \cos(3\pi x)]\cos(3\pi x)dx$

$$= \int_0^2 [\cos(2\pi x) + \cos(4\pi x) + \cos(6x) + 1]dx$$

$$= \Big[\frac{1}{2\pi}\sin(2\pi x) + \frac{1}{4\pi}\sin 4\pi x + \frac{1}{6\pi}\sin(6x) + x\Big]_0^2 = 2$$

(Product to sum formula: M2 to Math1014 level integral; Homework 2, Question 9.)

3. Evaluate the integral $\int_0^\pi (\pi - x)\sin x\cos^2 x dx$.

**Solution** $\int_0^\pi (\pi - x)\sin x\cos^2 x dx = \int_0^\pi -\frac{1}{3}(\pi - x)d\cos^3 x$

$$= -\frac{1}{3}(\pi - x)\cos^3 x\Big|_0^\pi - \int_0^\pi \cos^3 x dx = \frac{\pi}{3}$$

since $\int_0^\pi \cos^3 x dx = 0$ either by the cancellation of the +ve and -ve area between the graph of $\cos^3 x$ and the $x$-axis, or by using the reduction formula

$$\int_0^\pi \cos^3 x dx = \frac{\cos^2 x\sin x}{3}\Big|_0^\pi + \frac{2}{3}\int_0^\pi \cos x dx = 0$$

(Integration by parts, and/or reduction formula.)

4. Evaluate the integral $\int_2^\infty \frac{5}{x^2\sqrt{x^2+5}}dx$.

**Solution** Let $x = \sqrt{5}\tan u$, $du = \sqrt{5}\sec^2 u du$, and then

$$\int_2^\infty \frac{5}{x^2\sqrt{x^2+5}}dx = \int_{\tan^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \frac{5\sqrt{5}\sec^2 u}{5\tan^2 u\sqrt{5}\sec u}du$$

$$= \int_{\tan^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \frac{\cos u}{\sin^2 u}du = \Big[-(\sin u)^{-1}\Big]_{\tan^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} = -1 + \frac{3}{2} = \frac{1}{2}$$

(Trigonometric substitution: Homework 2, Question 13, 14, 15.)

5. For which constant $k$ can the improper integral $\displaystyle\int_0^\infty \left(\frac{3x^2+x-1}{2x^3+1} - \frac{kx+2}{2x^2+5}\right) dx$ be convergent.

   **Solution** $\displaystyle\int_0^\infty \frac{(6-2k)x^4 + \text{lower order term}}{(2x^3+1)(2x^2+5)} dx$ will diverge if $6-2k \neq 0$; hence $k=3$.

   (Homework 4, Question 7.)

6. Which of the following improper integral is convergent?

   (i) $\displaystyle\int_1^\infty \frac{\sqrt{x}}{1+x} dx$ **Solution** (i) $\displaystyle\geq \int_1^\infty \frac{1}{2x} dx = \infty$

   (ii) $\displaystyle\int_1^\infty \frac{\ln x^2}{4+x^2} dx$ convergent ((ii) $\displaystyle\leq \int_1^\infty \frac{4\ln\sqrt{x}}{4+x^2} dx < \int_1^\infty \frac{4\sqrt{x}}{x^2} dx < \infty$)

   (iii) $\displaystyle\int_1^\infty \frac{2^x}{x+2^x} dx$ (iii) $\displaystyle\geq \int_1^\infty \frac{2^x}{2\cdot 2^x} dx = \infty$

   (iv) $\displaystyle\int_1^\infty \frac{1}{1+x\ln x} dx$ (iv) $\displaystyle\geq \int_1^\infty \frac{1}{10x\ln x} dx = \infty$

   (v) $\displaystyle\int_1^\infty \frac{1}{x\ln\sqrt{x}} dx$ (iv) $\displaystyle\geq \int_2^\infty \frac{1}{\frac{1}{2}x\ln x} dx = \infty$

   Divergence of (i), (iii), (iv), (v) are all given by basic examples of divergent improper integral, if (ii) is considered tricky.

   (Homework 4, Questions 9, 10, 11, 12.)

7. The region under the graph of $y = 2xe^{-x^3/6}$ over the interval $[0,\infty)$ is rotated about the $x$-axis to generate a solid of revolution. Find the volume of the solid..

   **Solution** The volume is

$$V = \int_0^\infty \pi(2xe^{-x^3/6})^2 dx = \int_0^\infty 4\pi x^2 e^{-x^3/3} dx = 4\pi\Big[-e^{-x^3/3}\Big]_0^\infty = 4\pi$$

   (Volume of a solid of revolution: Sample Final Question 18, same setup, easier integral.)

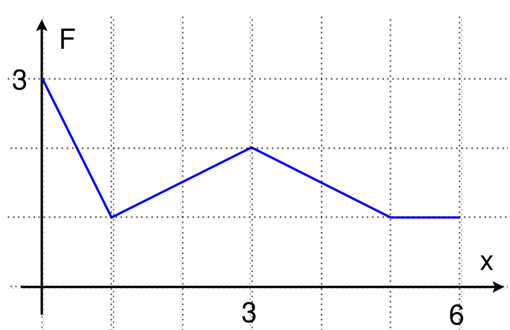
8. The base of a solid sitting on the $xy$-plane is the region bounded enclosed by the graphs of $y = 9\sin x$ and $y = \sin x$, where $0 \leq x \leq \pi$. Suppose that the cross sections of the solid perpendicular to the $x$-axis are semi-discs. Find the volume of the solid.

   **Solution** Since a cross section as $x$ is a semi-disc with diameter $9\sin x - \sin x = 8\sin x$, by the method of slicing, the volume is

$$V = \int_0^\pi \frac{1}{2}\pi(4\sin x)^2 dx = 8\pi\int_0^\pi \sin^2 x dx = 4\pi\int_0^\pi (1-\cos 2x) dx = 4\pi^2$$

   (Volume by slicing: Homework 5, Question 5 and 6, Sample Midterm Question 7.)

9. The graph of a force function (in newtons) is given as below. How much work (in Joules) is done by the force in moving an object from $x=0$ to $x=5$ (in meters)?



**Solution** The work is given by

$$W = \int_0^5 F(x)dx = \text{area under the graph over } [0,5] = 8$$

(Definition of work, and counting of the number of "squares".)

10. The length of the graph of a positive continuous function $y = f(x)$ over the interval $[a, b]$ is 2 units. Suppose the area of the surface of revolution obtained by rotating the graph of $f$ about the $x$-axis is $2\pi$ square units. Find the area of the surface of revolution obtained by rotating the graph of $y = f(x) + 1$ about the $x$-axis.

**Solution** The area of the surface of revolution is

$$A = \int_a^b 2\pi(f(x)+1)\sqrt{1+[(f(x)+1)']^2}dx$$

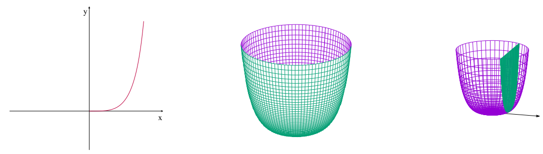
$$= \underbrace{\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}dx}_{surface\ area} + 2\pi \underbrace{\int_a^b \sqrt{1+[f'(x)]^2}dx}_{arc\ length}$$

$$= 2\pi + 2\pi \cdot 2 = 6\pi$$

(Formuals for the Area of a surface of revolution and arc length.)

## Part II

1. ([25 points]) A bowl is in the shape of a surface of revolution obtained by rotating the graph of the function $y = 6\tan^2 x^2$ about the $y$-axis, where $0 \le x \le \frac{\sqrt{\pi}}{2}$. ($x$, $y$ are in meters.)



(a) Find the volume of the bowl. [14 pts]

**Solution** By the cylindrical shell method, the volume of the bowl is

$$V = \int_0^{\frac{\sqrt{\pi}}{2}} 2\pi x \cdot (6 - 6\tan^2(x^2))dx = 2\pi \int_0^{\frac{\sqrt{\pi}}{2}} x \cdot (12 - 6\sec^2(x^2))dx$$

$$= 2\pi \Big[6x^2 - 3\tan(x^2)\Big]_0^{\frac{\sqrt{\pi}}{2}} = 2\pi \Big[\frac{3}{2}\pi - 3\Big] = 3\pi[\pi - 2]$$

Or you could let $u = x^2$ for the evaluation of the volume integral

$$V = \int_0^{\frac{\sqrt{\pi}}{2}} 2\pi x \cdot (6 - 6\tan^2(x^2))dx = \pi \int_0^{\frac{\pi}{4}} (6 - 6\tan^2 u)du$$

Now, if you insist to use the slicing method, then

$$V = \int_0^6 \pi x^2 dy = \int_0^6 \pi \tan^{-1}\sqrt{\frac{y}{6}}\, dy$$

Let $u = \frac{\sqrt{y}}{\sqrt{6}}$, $\sqrt{6}du = \frac{1}{2\sqrt{y}}dy$, $dy = 12u du$,

$$V = 6\pi \int_0^1 2u \tan^{-1} u du = 6\pi \left[ u^2 \tan^{-1} u \Big|_0^1 - \int_0^1 \frac{u^2}{1+u^2} du \right]$$

$$= 6\pi \left[ \frac{\pi}{4} - \int_0^1 (1 - \frac{1}{1+u^2})du \right] = 6\pi \left[ \frac{\pi}{4} - \Big[u - \tan^{-1} u\Big]_0^1 \right] = 3\pi(\pi - 2)$$

(Sample Midterm Exam, Q12; same setup; Homework 5, Q7.)

(b) Consider the cross sections of the solid region contained by the bowl which are perpendicular to the $x$-axis. Find the average value of their areas. [4 pts]

**Solution** Let $A(x)$ be the cross section area function. Then the average value of the cross section area function is given by

$$\frac{1}{\sqrt{\pi}} \int_{-\frac{\sqrt{\pi}}{2}}^{\frac{\sqrt{\pi}}{2}} A(x)dx = \frac{\text{volume of the bowl}}{\sqrt{\pi}} = \frac{1}{\sqrt{\pi}} \cdot 3\pi(\pi - 2) = 3\sqrt{\pi}(\pi - 2)$$

(Sample Midterm Exam, Q6.)

(c) Suppose the bowl is full of water. Express the work required to pumped all water in the bowl to an outlet at the top of the bowl by a definite integral. **Do not need to evaluate the integral**. (You may denote the density of water by $\rho$, and the gravity acceleration by $g$, both in SI units.) [7 pts]

**Solution** $\tan^2 x^2 = \dfrac{y}{6}$, $\tan x^2 = \sqrt{\frac{y}{6}}$, $x^2 = \tan^{-1}\sqrt{\frac{y}{6}}$.

The work required is

$$W = \int_0^6 \pi\rho g\Big(\tan^{-1}\sqrt{\tfrac{y}{6}}\Big)(6 - y)\, dy$$

(Sample Midterm Exam, Q13: similar setup; Sample Final Exam, Q8: same setup, different function. See also Homework 6, Q10.)

## Part III

1. ([25 points]) Let $I_n = \displaystyle\int_0^2 \frac{1}{(x^2+4)^n}dx$, where $n = 1, 2, 3, \ldots$ is a positive integer.

   (a) Using integration by parts, or otherwise, find $A(n)$, $B(n)$, which are expressions depending on $n$, such that

   $$I_{n+1} = A(n)I_n + B(n) \ .$$

   (Hint: Start with $I_n$.) [12 pts]

   **Solution**

   $$I_n = \int_0^2 \frac{1}{(x^2+4)^n}dx = x \cdot \frac{1}{(x^2+4)^n}\bigg|_0^2 - \int_0^2 x d(x^2+4)^{-n}$$
   $$= x \cdot \frac{1}{(x^2+4)^n}\bigg|_0^2 + \int_0^2 \frac{2nx^2}{(x^2+4)^{n+1}}dx$$
   $$I_n = \frac{2}{8^n} + 2n\int_0^2 \frac{(x^2+4)-4}{(x^2+4)^{n+1}}dx = \frac{2}{8^n} + 2nI_n - 8nI_{n+1}$$
   $$I_{n+1} = \frac{2n-1}{8n}I_n + \frac{2}{n8^{n+1}}$$

   Otherwise, let $x = 2\tan u$, $dx = 2\sec^2 udu$, and use the reduction formula for $\cos^{2n} x$ to find

   $$I_{n+1} = \int_0^{\frac{\pi}{4}} \frac{1}{4^{n+1}\sec^{2n+2} x} 2\sec^2 x dx = \frac{2}{4^{n+1}}\int_0^{\frac{\pi}{4}} \cos^{2n} x dx$$
   $$= \frac{2}{4^{n+1}}\left[\frac{1}{2n}\cos^{2n-1} x \sin x\bigg|_0^{\frac{\pi}{4}} + \frac{2n-1}{2n}\int_0^{\frac{\pi}{4}} \cos^{2n-2} x dx\right]$$
   $$= \frac{2}{n \cdot 8^{n+1}} + \frac{2n-1}{n4^{n+1}}\int_0^{\frac{\pi}{4}} \cos^{2n-2} x dx$$

   Since

   $$I_n = \frac{2}{4^n}\int_0^{\frac{\pi}{4}} \cos^{2n-2} x dx, \quad \int_0^{\frac{\pi}{4}} \cos^{2n-2} x dx = \frac{4^n}{2}I_n$$
   $$I_{n+1} = \frac{2}{n \cdot 8^{n+1}} + \frac{2n-1}{8n}I_n$$

   (Apply integration by parts, or otherwise copy the reduction formula.)

   (b) Using (a), or otherwise, evaluate the integral [7 pts]

   $$\int_0^2 \left[\frac{8}{(x^2+4)^5} - \frac{7}{4(x^2+4)^4}\right]dx \ .$$

   **Solution**

   $$I_5 = \frac{7}{8 \cdot 4}I_4 + \frac{2}{4 \cdot 8^5}$$
   $$8I_5 - \frac{7}{4}I_4 = \frac{2}{4 \cdot 8^4} = \frac{1}{2^{13}}$$

   Or, let $x = 2\tan u$, $dx = 2\sec^2 udu$, and then apply the reduction formula:

   $$\int_0^2 \left[\frac{8}{(x^2+4)^5} - \frac{7}{4(x^2+4)^4}\right]dx = \frac{2}{4^5}\left[\int_0^{\frac{\pi}{4}} 8\cos^8 udu - \int_0^{\frac{\pi}{4}} 7\cos^6 udu\right]$$
   $$= \frac{2}{4^5}\left[\cos^7 u \sin u\bigg|_0^{\frac{\pi}{4}} + 7\int_0^{\frac{\pi}{4}} \cos^6 udu - 7\int_0^{\frac{\pi}{4}} \cos^6 udu\right]$$
   $$= \frac{2}{4^5}\cos^7 u \sin u\bigg|_0^{\frac{\pi}{4}} = \frac{1}{4^5 \cdot 2^3} = \frac{1}{2^{13}}$$

(c) If Simpson's rule on four subintervals is used to approximate

$$\pi = \int_0^2 \frac{8}{x^2+4}dx \ ,$$

a rational approximate value of $\pi$ can be found as

$$\pi \approx \frac{1}{3}\left[1 + \frac{64}{a} + \frac{8}{b} + \frac{64}{c} + \frac{1}{d}\right]$$

where $a, b, c, d$ are positive integers. Find $a, b, c, d$.

[6 pts]

**Solution**

$$S_4 = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)]$$

$$S_4 = \frac{\frac{2}{4}}{3}\left[\frac{8}{4} + \frac{4\cdot 8}{\frac{1}{2^2}+4} + \frac{2\cdot 8}{1^2+4} + \frac{4\cdot 8}{\frac{3^2}{2^2}+4} + \frac{8}{2^2+4}\right]$$

$$= \frac{1}{3}\left[1 + \frac{2\cdot 8}{\frac{1}{2^2}+4} + \frac{8}{1^2+4} + \frac{2\cdot 8}{\frac{3^2}{2^2}+4} + \frac{4}{2^2+4}\right]$$

$$= \frac{1}{3}\left[1 + \frac{64}{17} + \frac{8}{5} + \frac{64}{25} + \frac{1}{2}\right] \qquad (\approx 3.1415686)$$

$a = 17, \quad b = 5, \quad c = 25, \quad d = 2$, or $a = 25, \quad b = 5, \quad c = 17, \quad d = 2$

(Straightforward usage of the Simpson' Rule formula.)