9.1-9.2 Sequences

Definition 1 (Sequence) A **sequence** $\{a_n\}$ is an ordered list of numbers of the form

$$\{a_1, a_2, a_3, \ldots, a_n\}$$

A sequence may be generated by a

- **recurrence relation** of the form $a_{n+1} = f(a_n)$, where a_1 is given, or
- **explicit formula** of the form $a_n = f(n)$,

for $n = 1, 2, 3, \ldots$.

Definition 2 (Limit of a Sequence) If the terms of a sequence $\{a_n\}$ approach a unique number L as n increases, then we say $\lim_{n\to\infty} a_n = L$ exists, and the sequence **converges** to L. If the terms of the sequence do not approach a single number as n increases, the sequence has no limit, and the sequence **diverges**.

Definition 3 (Terminology for Sequences) These are the common terminologies for sequences:

1. **Non-decreasing Sequence/Increasing Sequence**
   A sequence $\{a_n\}$ in which each term is greater than or equal to its predecessor $(a_{n+1} \geq a_n)$ is said to be non-decreasing/increasing.
2. **Non-increasing Sequence/Decreasing Sequence**
   A sequence $\{a_n\}$ in which each term is less than or equal to its predecessor $(a_{n+1} \leq a_n)$ is said to be non-increasing/decreasing.
3. **Monotonic Sequence**
   A sequence that is either non-increasing or non-decreasing is said to be monotonic.
4. **Bounded Sequence**
   A sequence whose terms are all less than or equal to some finite number in magnitude ($|a_n| \leq M$, for some real number M) is said to be bounded.

Theorem 1 (Properties of Limits of Sequences) *Assume that the sequences $\{a_n\}$ and $\{b_n\}$ have limits A and B, respectively. Then,*

(a) $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

(b) $\lim_{n\to\infty} ca_n = cA$, *where c is a real number*

(c) $\lim_{n\to\infty} a_n b_n = AB$

(d) $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$, *provided $B \neq 0$*

Theorem 2 (Geometric Sequences) *Let r be a real number. Then,*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } |r| < 1 \\ 1 & \text{if } r = 1 \\ \text{does not exist} & \text{if } r \leq 1 \text{ or } r > 1 \end{cases}$$

If $r > 0$, then $\{r_n\}$ converges or diverges monotonically. If $r < 0$, then $\{r_n\}$ converges or diverges by oscillation.

Theorem 3 (Methods to Find the Limit of Sequence) *You may try to use the following 3 theorems to find the limit of a sequence.*

1. ***Limits of Sequences from Limits of Functions***
   Suppose f is a function such that $f(n) = a_n$ for all positive integers n. If $\lim_{x\to\infty} f(x) = L$, then the limit of the sequence $\{a_n\}$ is also L.
2. ***Squeeze/Sandwich Theorem for Sequences***
   Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N. If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.
3. ***Monotone Convergence Theorem***
   A bounded monotonic sequence converges.

Theorem 4 (Growth Rates of Sequences) *The following sequences are ordered according to increasing growth rates as $n \to \infty$; that is, if $\{a_n\}$ appears before $\{b_n\}$ in the list, then*

- $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$, *and*
- $\lim_{n\to\infty} \dfrac{b_n}{a_n} = \infty$

The sequence is:

$$\{ln^q n\} \ll \{n^p\} \ll \{n^p ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}$$

where it applies for all positive real numbers p, q, r, s, and $b > 1$.

Definition 4 (Limit of a Sequence) The sequence $\{a_n\}$ converges to L provided the terms of an can be made arbitrarily close to L by taking n sufficiently large. More precisely, $\{a_n\}$ has the unique limit L if given any tolerance $\epsilon > 0$, it is possible to find a positive integer N (depending only on ϵ) such that

$$|a_n - L| < \epsilon \text{ whenever } n > N$$

If the limit of a sequence is L, we say the sequence **converges** to L, written

$$\lim_{n\to\infty} a_n = L$$

A sequence that does not converge is said to **diverge**.

9.3 Infinite Series

Definition 5 (Infinite Series) Given a set of numbers $\{a_1, a_2, a_3, \ldots\}$, the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

is called an **infinite series**. Its **sequence of partial sums** $\{S_n\}$ has the terms

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \text{ for } n = 1, 2, 3, \ldots$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series converges to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = L.$$

If the sequence of partial sums diverges, the infinite series also diverges.

Theorem 5 (Geometric Series) *Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

If $|r| \geq 1$, then the series diverges.

Definition 6 (Telescoping Series) A series of the form

$$\sum_{k=1}^{\infty} (a_n - a_{n+1})$$

is called a **telescoping series** and its partial sums can be expressed as

$$S_n = (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_n - a_{n+1}) = a_1 - a_{n+1}$$

therefore we have

$$\sum_{k=1}^{\infty} (a_n - a_{n+1}) = \lim_{n\to\infty} (a_1 - a_{n+1})$$

Examples or Exercises

Example 1 (Basic Skills)
Determine if the limit of the sequence $\left\{(-1)^n \frac{n}{n+1}\right\}$ exists. If it exists, find its limit.

$n = 2k-1$

$$= \lim_{k\to\infty} (-1)^{2k-1} \frac{2k-1}{2k}$$

$$= \lim_{k\to\infty} -\frac{2k-1}{2k}$$

$$= \lim_{k\to\infty} -\frac{2-\frac{1}{k}}{2}$$

$$= -1$$

$n = 2k$

$$= \lim_{k\to\infty} (-1)^{2k} \frac{2k}{2k+1}$$

$$= \lim_{k\to\infty} \frac{2k}{2k+1}$$

$$= \lim_{k\to\infty} \frac{2}{2+\frac{1}{k}}$$

$$= 1$$

divergent

Example 2 (Limits of Sequences from Limits of Functions)
Determine if the limit of the sequence

$$\left\{\frac{\ln(1/n)}{n}\right\}$$

exists. If it exists, find its limit.

$$f(x) = \frac{\ln\frac{1}{x}}{x} \qquad f(n) = \frac{\ln\frac{1}{n}}{n}$$

$$\lim_{x\to\infty} \frac{\ln\frac{1}{x} \to \infty}{x \to \infty} \qquad \frac{\infty}{\infty}$$

L'hospital's rule $$= \lim_{x\to\infty} \frac{x \cdot \left(-\frac{1}{x^2}\right)}{1}$$

$$= \lim_{x\to\infty} -\frac{1}{x}$$

$$= 0$$

$$\lim_{n\to\infty} \frac{\ln\frac{1}{n}}{n} = 0$$

Example 3 (Squeeze/Sandwich Theorem)
Evaluate the limit of the sequence by using Squeeze theorem: $\left\{\dfrac{2\tan^{-1} n}{n^3+4}\right\}$.

$$\arctan x \le \frac{\pi}{2}$$

$$0 \le \frac{2\tan^{-1} n}{n^3+4} \le \frac{2\cdot\frac{\pi}{2}}{n^3+4}$$

$$0 \to 0, \qquad \frac{2\cdot\frac{\pi}{2}}{n^3+4} \to 0$$

By squeezing thm

$$\lim_{n\to\infty} \frac{2\tan^{-1} n}{n^3+4} = 0$$

Example 4 (Monotone Convergence Theorem)
Define $a_0 = 1$ and $a_{n+1} = \sqrt{1+a_n}$ for $n = 0, 1, 2, \ldots$. Find the value for $L = \lim_{n\to\infty} a_n$ provided that the limit does exist.

a_n ① bounded

② monotonic.

$a_0 = 1 \quad a_1 = \sqrt{1+1} = \sqrt{2} \quad a_2 = \sqrt{1+\sqrt{2}} \ \ldots$

Claim: $a_n \le 2$. $a_{n+1} \ge a_n$

Pf of Claim (Induction)

① When $n=0$ $a_1 = \sqrt{2} \le 2$. $\sqrt{2} \ge 1$

② Assume when $n=k$. claim true

. $a_{k+1} \ge a_k$.

$\sqrt{1+a_{k+1}} \ge \sqrt{1+a_k}$

$a_{k+2} \ge a_{k+1}$

. $a_k \le 2$

$\sqrt{1+a_k} \le \sqrt{3}$

$a_{k+1} \le \sqrt{3} \le 2$

When $n=k+1$. claim true ✓

$\{a_n\}$ bounded. monotonic

By Bounded Convergence Thm

$\{a_n\}$ converges.

Let $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$

$L = \sqrt{1+L} \Rightarrow L = \dfrac{1+\sqrt{5}}{2}$ or $\cancel{\dfrac{1-\sqrt{5}}{2}}$

Example 5 (Geometric Series)
Evaluate

$$\sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k 5^{6-k}$$

$$\sum_{k=0}^{\infty} 4^{-k} 5^{6-k}$$

$$= 5^6 \sum 4^{-k} \cdot 5^{-k}$$

$$= 5^6 \underbrace{\sum \frac{1}{20^k}}$$

$$= 5^6 \frac{1}{1-\frac{1}{20}} = 5^6 \cdot \frac{20}{19} = \frac{312500}{19}$$

□

Example 6 (Telescoping Technique)
Evaluate

$$\sum_{k=0}^{\infty} \frac{1}{(3k+1)(3k+4)}$$

$$\frac{1}{3k+1} - \frac{1}{3k+4} = \frac{3}{(3k+1)(3k+4)}$$

$$= \sum_{k=0}^{\infty} \frac{1}{3}\left(\frac{1}{3k+1} - \frac{1}{3k+4}\right)$$

$$= \lim_{n\to\infty} \frac{1}{3} \sum_{k=0}^{n} \left(\frac{1}{3k+1} - \frac{1}{3k+4}\right)$$

$$= \lim_{n\to\infty} \frac{1}{3}\left[\left(1 - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{4}} - \cancel{\tfrac{1}{7}}\right) + \left(\cancel{\tfrac{1}{7}} - \tfrac{1}{10}\right) + \cdots + \left(\cancel{\tfrac{1}{3n+\ldots}}\right.\right.$$

$= \lim_{n\to\infty} \frac{1}{3}\left(1 - \frac{1}{3n+4}\right) = \frac{1}{3}$

Exercise 1 (Limits of Sequences from Limits of Functions)
Determine if the limit of the sequence

$$\left\{\frac{3^{n+1}+3}{3^n}\right\}$$

exists. If it exists, find its limit.

Exercise 2 (Limits of Sequences from Limits of Functions)
Determine if the limit of the sequence

$$\left\{\left(\frac{1}{n}\right)^{\frac{1}{n}}\right\}$$

exists. If it exists, find its limit.

Exercise 3 (Squeeze/Sandwich Theorem)
Evaluate the limit of the sequence

(a) $\left\{\frac{\cos n}{n}\right\}$

(b) $\left\{\frac{\sin 6n}{5n}\right\}$

Exercise 4 (Monotone Convergence Theorem)
Define $a_0 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for $n = 0, 1, 2, \ldots$. Find the value for $L = \lim_{n\to\infty} a_n$ provided that the limit does exist.

Exercise 5 (Geometric Series)
Evaluate

$$\sum_{k=0}^{\infty} \left(\frac{e}{\pi}\right)^k$$

Exercise 6 (Telescoping Technique)
Evaluate

$$\sum_{k=0}^{\infty} \frac{1}{16k^2 + 8k - 3}$$

Suggested Practice Problems in Textbook
Ex 9.1 13, 19, 33, 35, 37, 45, 46, 63, 67, 76, 77, 82
Ex 9.2 13, 19, 25, 33, 41, 56, 57, 60, 61, 67, 90, 99
Ex 9.3 25, 33, 37, 51, 57, 61, 67, 68, 73, 77, 87, 91, 98