MATH 1014 Tutorial Notes

Week 4

Basic Integration Formulas

Theorem 1 (Basic Integration Formulas)**.** *The following are the basic integration formulas you need to know.*

$$\int k\,dx = kx + C \qquad \int x^p\,dx = \frac{x^{p+1}}{p+1} + C \textit{ where } p \neq -1$$

$$\int \cos ax\,dx = \frac{\sin ax}{a} + C \qquad \int \sin ax\,dx = -\frac{\cos ax}{a} + C$$

$$\int \sec^2 ax\,dx = \frac{\tan ax}{a} + C \qquad \int \csc^2 ax\,dx = -\frac{\cot ax}{a} + C$$

$$\int \sec ax \tan ax\,dx = \frac{\sec ax}{a} + C \qquad \int \csc ax \cot ax\,dx = -\frac{\csc ax}{a} + C$$

$$\int \sinh ax\,dx = \frac{\cosh ax}{a} + C \qquad \int \cosh ax\,dx = \frac{\sinh ax}{a} + C$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C \qquad \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Integration by Parts

Theorem 2 (Integration by Parts)**.** *Suppose that u and v are differentiable functions of x. Then*

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$$

Or more precisely,

$$\int udv = uv - \int vdu$$

Theorem 3 (Integration by Parts for Definite Integrals)**.** *Let u and v be differentiable of x. Then*

$$\int_a^b u(x)dv(x) = [u(x)v(x)]_a^b - \int_a^b v(x)du(x)$$

Procedures of Integration by Parts

1. Try to "put" one part of the integrand into the differential "dx".
2. Use integration by part to interchange the integrand and the function inside the differential.
3. See whether the integration is easier, you may need to repeat integration by parts several times to get the answer.
4. If you find that the integration is not simpler than before, then you may try to "put" another part of the integrand in step 1 and repeat the procedures again.

Other Techniques of Integration by Parts

1. Repeat Itself
   If you find that the resulting integration after integration by parts is the same as the original integration, then you may setup a formula to solve for the integration.
2. Reduction Formula
   Sometimes we will express an integration in terms of n as I_n and express it in terms of I_{n-1} or I_{n-2}, then use these reduction formulas repeatly to express I_n in terms of I_0 or I_1.

Examples

Example 1 (Integration by Parts)**.** Evaluate the following integrals.

(a) **(Famous Formula)** $\int \ln x \, dx$

(b) **(Integration of Inverse Trigonometric Function)** $\int \sin^{-1} x \, dx$

(c) **(Repeat Itself)** $\int e^x \cos x \, dx$

Example 2 (Reduction Formula)**.** Let $I_n = \int x^n e^{ax} \, dx$ for $a \neq 0$.

(a) Use integration by parts to derive the reduction formulas

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

(b) Use the reduction formulas in (a) to evaluate $\displaystyle\int x^2 e^{3x}\,dx$.

Example 3 (Harder Example of Integration by Parts)**.** Show that if f is a function satisfying $f(1) = 3$, $f'(1) = 2$ and f'' is continuous on the interval $[0, 1]$ with $|f''(x)| \leq 4$ for $0 \leq x \leq 1$, then

$$\left| \int_0^1 f(x)dx \right| \leq \frac{8}{3}.$$