1. Ratio, Root & Comparison Test

$\sum a_k$, $\sum b_k$ are infinite series with positive terms

(1) Ratio test

$$r = \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

① $0 \le r < 1$, $\sum a_k$ converges

② $r > 1$, $\sum a_k$ diverges

③ $r = 1$, ?

(2) Root test

$$\rho = \lim_{k\to\infty} \sqrt{a_k}$$

① $0 \le \rho < 1$, $\sum a_k$ converges

② $\rho > 1$, $\sum a_k$ diverges

③ $\rho = 1$, ?

(3) Comparison test

$0 \le a_k \le b_k$

① $\sum b_k$ converges $\Rightarrow$ $\sum a_k$ converges

② $\sum a_k$ diverges $\Rightarrow$ $\sum b_k$ diverges

(4) Limit comparison test

$$L = \lim_{k\to\infty} \frac{a_k}{b_k}$$

① If $0 < L < \infty$, $\sum a_k$. $\sum b_k$ both converge (diverge).

② $L = 0$, $\sum b_k$ converges $\Rightarrow$ $\sum a_k$ converges

③ $L = \infty$. $\sum b_k$ diverges $\Rightarrow$ $\sum a_k$ diverges.

2. Alternating Series and Alternating Series Test

(1) Alternating Series Test

$\sum (-1)^{k+1} a_k$ converges if

① $0 < a_{k+1} \le a_k$ ($a_k \downarrow$) for k large enough

② $\lim_{k\to\infty} a_k = 0$

(2) Remainder

$R_n = |S - S_n|$ $\quad S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$. $\quad S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$

$\Rightarrow R_n \le a_{n+1}$

(3) Two kind of convergence

Assume $\sum a_k$ converges

① $\sum a_k$ converges absolutely if $\sum |a_k|$ converges

② $\sum a_k$ converges conditionally otherwise.

(4) Absolute convergence test

① $\sum |a_k|$ converges $\Rightarrow$ $\sum a_k$ converges

② $\sum a_k$ diverges $\Rightarrow$ $\sum |a_k|$ diverges

Remark : Absolute convergence $\Rightarrow$ convergence

Example 1. Prove that the series

$$\sum_{n=3}^{\infty} \frac{1}{(\ln \ln n)^{\ln n}} \quad \text{vs} \quad \sum \frac{1}{n^{\lambda}} \qquad \sum \frac{1}{n^2}$$

is convergent.

Solution: ① $a^b = e^{\ln a^b} = e^{b \ln a}$

$$(\ln \ln n)^{\ln n} = e^{\ln n \cdot \ln(\ln \ln n)} = n^{\ln \ln \ln n}$$

② compare $\ln \ln \ln n$ with const $= 2$

Let $n > e^{e^{e^2}}$

$$\ln \ln \ln n > 2$$

$$n^{\ln \ln \ln n} > n^2 \Rightarrow \frac{1}{n^{\ln \ln \ln n}} < \frac{1}{n^2}$$

③ $\sum \frac{1}{n^{\ln \ln \ln n}} < \sum \frac{1}{n^2}$ convergent

Example 2. Study the convergence of the series

$$\sum_{n=1}^{\infty} a^n \left(1 + \frac{1}{n}\right)^n$$

where a is a given positive number.

$$x_n = a^n \left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty} \sqrt[n]{x_n} = a$$

① $a < 1$ convergent

② $a > 1$ divergent

③ $a = 1$ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$ $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$ divergent

$\square$

Example 3. Let a be a positive number. Prove that the series

$$\sum_{n=1}^{\infty} \frac{a^n}{\sqrt{n!}}$$

is convergent.

$$x_n = \frac{a^n}{\sqrt{n!}}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{n+1}\sqrt{n!}}{a^n\sqrt{(n+1)!}} = \lim_{n\to\infty} \frac{a}{\sqrt{n+1}} = 0 < 1$$

By ratio test. convergent

$\square$

an ↓

Example 4. Suppose that $a_1 > a_2 > \cdots$, and $\lim_{n\to\infty} a_n = 0$. Define

$$S_n = \sum_{j=n}^{\infty}(-1)^{j-n}a_j = a_n - a_{n+1} + a_{n+2} - \cdots .$$

Prove that the series $\sum_{n=1}^{\infty} S_n^2$, $\sum_{n=1}^{\infty} a_nS_n$, and $\sum_{n=1}^{\infty} a_n^2$ converge or diverge together.

$$\sum_{n=1}^{\infty} S_n^2 < \sum_{n=1}^{\infty} a_nS_n < \sum_{n=1}^{\infty} a_n^2$$

Prove

① $\sum a_n^2$ converges $\Rightarrow$ $\sum S_n^2$ converges

by alternating series test. $\sum a_n^2$ converges

Since $\sum a_n^2 > \sum S_n^2$. $\sum S_n^2$ converges

② Prove $\sum S_n^2$ converges $\Rightarrow$ $\sum a_n^2$ converges

$S_n = a_n - S_{n+1} \Rightarrow a_n = S_n + S_{n+1}$

Also $(x+y)^2 \le 2(x^2+y^2)$

$\sum a_n^2 = \sum (S_n + S_{n+1})^2 \le \sum 2(S_n^2 + S_{n+1}^2) = 2\sum S_n^2 + 2\sum S_{n+1}^2$

$< 4 \sum S_n^2$

$\sum S_n^2$ converges $\Rightarrow$ $\sum a_n^2$ converges

Example 5. Suppose $(a_n)_{n\geq 1}$ is a decreasing sequence of positive numbers and for each natural number n, define $b_n = 1 - a_{n+1}/a_n$. Then the sequence $(a_n)_{n\geq 1}$ converges to zero if and only if the series $\sum_{n=1}^{\infty} b_n$ diverges.

$M > N$

$$0 \leq \frac{a_N - a_{M+1}}{a_N} \overset{①}{=} \frac{1}{a_N} \sum_{n=N}^{M} (a_n - a_{n+1}) \leq \sum_{n=N}^{M} b_n \leq \frac{1}{a_M} \cdot \sum_{n=N}^{M} (a_n - a_{n+1}) \overset{②}{=} \frac{a_N - a_{M+1}}{a_M}$$

"$\Rightarrow$" Assume $\sum b_n$ converges, $a_n \to 0$.

$M \to \infty$

$$\frac{a_N - a_{M+1}}{a_N} \to 1$$

$$\sum_{n=N}^{\infty} b_n \geq 1 \qquad \times$$

$a_n \to 0$, $\sum b_n$ diverges.

"$\Leftarrow$" If $a_n \to \alpha > 0$.

Let $N = 1$, $M \to \infty$

$$\sum_{n=1}^{\infty} b_n \leq \frac{a_1 - \alpha}{\alpha}$$

$\Rightarrow \sum b_n$ converges