Improper Integrals

I: Infinite Intevals

(a) If $\int_a^t f(x)\,dx$ exists for $\forall t \geq a$

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

(b) ... $\int_t^b f(x)\,dx$ exists for $\forall t \leq b$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to\infty} \int_t^b f(x)\,dx$$

$\int_a^{\infty} f(x)\,dx$. $\int_{-\infty}^{b} f(x)\,dx$ convergent if limit exists

divergent if limit does not exist.

II: Discontinuity

(a) f continuous on $[a,b)$, discontinuous at b.

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

(b) f continuous on $(a,b]$, discontinuous at a

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

(c) discontinuity at c, $a < c < b$.

both $\int_a^c f(x)\,dx$. $\int_c^b f(x)\,dx$ convergent

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Comparison Theorem

f, g Continuous functions. $f(x) \geq g(x) \geq 0 \quad x \in [a, +\infty)$

(a) If $\int_a^\infty f(x)\,dx$ is convergent $\Rightarrow$ $\int_a^\infty g(x)\,dx$ is convergent

(b) If $\int_a^\infty g(x)\,dx$ is divergent $\Rightarrow$ $\int_a^\infty f(x)\,dx$ is divergent

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} p > 1 & \text{convergent} \\ p \leq 1 & \text{divergent} \end{cases}$$