§1. Sequence

Def 1 (Sequence)

A Sequence $\{a_n\} = \{a_1, a_2, \cdots a_n\}$

generated by

- $a_{n+1} = f(a_n)$
- $a_n = f(n)$
- a_n can be randomly selected

Def (Limit of a Sequence)

$\{a_n\}$ limit exists, if $\exists L$

$$\lim_{n \to \infty} a_n = L$$

$\Rightarrow \{a_n\}$ converges to L

Otherwise, $\{a_n\}$ diverges

Def (Terminology)

① Non-decreasing sequence.

$\{a_n\}$ $\quad a_{n+1} \geq a_n$

② Non-increasing sequence

$a_{n+1} \leq a_n$

③ Monotonic sequence

① & ②

④ Bounded sequence

$|a_n| \leq M$.

Thm (Properties of Limits of Sequences)

$\{a_n\}$. $\{b_n\}$ $\quad \lim_{n\to\infty} a_n = A$. $\lim_{n\to\infty} b_n = B$

① $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

② $\lim_{n\to\infty} c \cdot a_n = c \cdot A \qquad c \in \mathbb{R}$

③ $\lim_{n\to\infty} a_n b_n = A \cdot B$

④ If $B \neq 0$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$$

Thm (Geometric sequence)

$\alpha \in \mathbb{R}$

$$\lim_{n\to\infty} \alpha^n \begin{cases} 0 & \text{if } |\alpha| < 1 \\ 1 & \text{if } \alpha = 1 \\ \text{does not exist} & \text{if } |\alpha| > 1 \end{cases}$$

If $\alpha > 0$. $\{\alpha^n\}$ converges / diverges monotonically

$\alpha < 0$ $\{\alpha^n\}$ converges / diverges by oscillation $+ - + -$

Thm (Method to find the limit of sequence)

① Approximate the limit of sequences from limit of function

$\{a_n\}$. If $\exists f(x)$ s.t. $f(n) = a_n$ for $n \in \mathbb{N}^+$.

$$\lim_{x\to\infty} f(x) = L \Rightarrow \lim_{n\to\infty} a_n = L$$

② Squeeze theorem

$\{a_n\}$ $\{b_n\}$ $\{c_n\}$. $\qquad a_n \le b_n \le c_n$

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

Then $\lim_{n\to\infty} b_n = L$

③ Monotonic Convergence Thm

$\{a_n\}$ bounded & monotonic $\Rightarrow$ $\{a_n\}$ converges

Thm (Growth Rates of Sequence)

$\{a_n\}, \{b_n\}$

growth rates of b_n > growth rates of a_n

$\lim_{n\to\infty} \frac{b_n}{a_n} = \infty$ or $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$

Denote as $b_n \gg a_n$

e.g. $p, q, r, s \in \mathbb{R}^+$

$\{(\ln n)^q\} \ll \{n^p\} \ll \{n^p \ln^r n\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll$

§2. Infinite Series

Def $\{a_1, a_2, a_3, \dots\}$

$\sum_{k=1}^{\infty} a_k$ ← infinite series

$S_1 = a_1$

$S_2 = a_1 + a_2$

$\vdots$

$S_n = a_1 + \dots + a_n = \sum_{k=1}^{n} a_k$

$\{S_n\}$ sequence of partial sums

$\lim_{n\to\infty} S_n = L \Leftrightarrow \lim_{n\to\infty} \sum_{k=1}^{n} a_n = L \Leftrightarrow \sum_{k=1}^{\infty} a_n = L$

If the sequence of $\{S_n\}$ diverges, then $\sum_{k=1}^{\infty} a_n$ diverges

Thm (Geometric Series)

$a \neq 0$. $r \in \mathbb{R}$. ① If $|r| < 1$

$$\sum_{k=0}^{\infty} a \cdot r^k = \frac{a}{1-r} \quad \text{convergent}$$

② If $|r| \geq 1$. divergent

Def (Telescoping Series)

$$\sum_{k=1}^{\infty} (a_n - a_{n+1})$$

$$S_n = (a_1 - a_2) + (a_2 - a_3) + \cdots + (a_n - a_{n+1}) = a_1 - a_{n+1}$$

$$\lim_{n\to\infty} S_n = \sum_{k=1}^{\infty} (a_n - a_{n+1}) = \lim_{n\to\infty} (a_1 - a_{n+1})$$