MATH 1014 Tutorial Notes

Week 2

1 Average Value of a Function

Definition 1. *The average value of f on the interval $[a, b]$ is defined as*

$$f_{ave} = \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x.$$

2 Mean Value Theorem for Integral

Theorem 1. *If f is continuous on $[a, b]$, then there exists a number c in $[a, b]$ such that*

$$f(c) = f_{ave} = \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x$$

that is,

$$\int_a^b f(x) \mathrm{d}x = f(c)(b-a).$$

3 Arc Length

Theorem 2 (Arc Length for $y = f(x)$)**.** *Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f1(a))$ to $(b, f(b))$ is .*

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx$$

Theorem 3 (Arc Length for $x = g(y)$)**.** *Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is*

$$L = \int_c^d \sqrt{1 + g'(y)^2} dy$$

4 Surface Area

Theorem 4 (Area of a Surface of Revolution)**.** *Let f be difforentiable and positive on the interval $[a, b]$. The area of the surface generated when the graph of f on the interval $[a, b]$ is revolved about the x-axis is*

$$S = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}dx$$

Theorem 5 (Area of a Surface of Revolution)**.** *Let $x = g(y)$ be differentiable and positive on the interval $[c, d]$. The area of the surface generated when the graph on the interval $[c, d]$ is revolved about the y-axis is*

$$S = \int_c^d 2\pi g(y)\sqrt{1 + g'(y)^2}dy$$

5 Physical Applications: Mass, Density and Work

Theorem 6 (Mass-Density Relation of a One-Dimensional Object)**.** *Suppose a thin bar or wire is represented by a line segment on the interval $a \leq x \leq b$ with a density function $\rho(x)$ (with units of mass per length). The mass of the object is*

$$m = \int_a^b \rho(x)dx$$

Theorem 7 (Work Done by a Variable Force)**.** *The work done by a variable force $F(x)$ in moving an object along a line from $x = a$ to $x = b$ in the direction of the force is*

$$W = \int_a^b F(x)dx$$

6 Steps for Solving Lifting Problems

1. Draw a y-axis in the vertical direction (parallel to gravity) and choose a convenient origin. Assume the interval $[a, b]$ corresponds to the vertical extent of the fluid.

2. For $a \leq y \leq b$, find the cross-sectional area $A(y)$ of the horizontal slices and the distance $D(y)$ the slices must be lifted.

3. The work required to lift the water is

$$W = \int_a^b \rho g A(y)D(y)dy$$

Example 1 (Length of Curves). *Find the arc length of the function* $y = f(x) = \frac{2}{3}(x-1)^{\frac{3}{2}}$ *over* $[1, 2]$.

Example 2 (Surface area of a torus)**.** *When the circle* $x^2 + (y - a)^2 = r^2$ *on the interval* $[-r, r]$ *is revolved about the* x*-axis, the result is the surface of a torus, where* $0 < r < a$*. Show that the surface area of the torus is* $S = 4\pi^2 ar$*.*

Example 3 (Mass of Two Bars)**.** *Two bars of length L have densities $\rho_A(x) = 4e^{-x}$ and $\rho_B(x) = 6e^{-2x}$ respectively. For what values of L is the bar A heavier than bar B ?*

Example 4 (A Nonlinear Spring)**.** *Consider a spring whose restoring force is given by* $F(x) = 16x - 0.1x^3$ *for* $7 \leq x \leq 7$*, where it is compressed or stretched* x *units from the equilibrium position. How much work is done in stretching the spring from its equilibrium position* $x = 0$ *to* $x = 2$ *?*