1. Ratio, Ruot & Comparison Test

Zak, Ibr are infinite series with positive terms

(1) Ratio test
$$r = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}$$

D 0 ≤ r < 1, [ax converges

1 r>1, Tak diverges

3 r=1 ,?

(2) Root test

O 0 = p < 1. Dan wnverges

(3) Comparison test

o≤an ≤ bn

D I be converges ⇒ Iak converges

② I Ok diverges => I bk diverges

14) Limit comparison test

$$L = \lim_{k \to \infty} \frac{a_k}{b_k}$$

0 If 0 < L < 00, Ear. Ibn both converge (diverge).

② L=0 . Ibk converges → Iak converges

3 $L=\infty$. I be diverges $\Rightarrow \Sigma$ an diverges.

2. Alternating Series and Alternating Series Test

11) Alternating Series Test

Σ(-1)k+1ak converges if

@ 0 < art = ar (ar) for k large enough

(2) lim ak =0

(2) Remainder

Rn = 15 - Sn $S = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ $S_n = \sum_{k=1}^{n} (-1)^{k+1} a_k$

⇒ Rn = antl

(3) Two kind of convergence

Assume Ian converges

O Ian converges absolutely if Ilanl converges

2 Iah converges conditionally otherwise.

14) Absolute convergence test

D Ilan Converges => Ian converges

② Iak diverges ⇒ Ilan diverges

Remark: Absolute convergence > sunvergence

Example 1. Prove that the series

$$\sum_{n=3}^{\infty} \frac{1}{(\ln \ln n)^{\ln n}} \quad \text{vs} \quad \overline{2} \quad \frac{1}{n^{\lambda}} \qquad \overline{2} \quad \frac{1}{n^{2}}$$

is convergent.

Solution:
$$0 a^b = e^{\ln a^b} = e^{\ln n}$$

 $(\ln \ln n)^{\ln n} = e^{\ln n} \ln (\ln \ln n) = n \ln \ln n$

② compare
$$\frac{\ln \ln \ln n}{\ln \ln n}$$
 with $\omega nst = 2$

Let $n > e^{e^2}$
 $\frac{\ln \ln \ln n}{n} > 2$
 $\frac{\ln \ln \ln n}{n} > n^2$
 $\frac{1}{n \ln \ln n} < \frac{1}{n^2}$
 $\frac{1}{n \ln \ln n} < \frac{1}{n^2}$
 $\frac{1}{n \ln \ln n} < \frac{1}{n^2}$ convergent

Example 2. Study the convergence of the series

$$\sum_{n=1}^{\infty} a^n \left(1 + \frac{1}{n} \right)^n$$

where a is a given positive number.

$$An = Q^n \left(1 + \frac{1}{n}\right)^n$$

Example 3. Let a be a positive number. Prove that the series

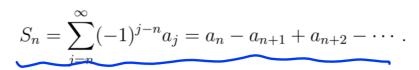
$$\sum_{n=1}^{\infty} \frac{a^n}{\sqrt{n!}}$$

is convergent.

$$\chi_{n} = \frac{\alpha^{n}}{\sqrt{n!}}$$

$$\lim_{n \to \infty} \frac{\chi_{n+1}}{\chi_{n}} = \lim_{n \to \infty} \frac{\alpha^{n+1} \sqrt{n!}}{\alpha^{n} \sqrt{(n+1)!}} = \lim_{n \to \infty} \frac{\alpha}{\sqrt{n+1}} = 0 < 1$$
By ratio test. Convergent

Example 4. Suppose that $\underline{a_1 > a_2 > \cdots}$ and $\underline{\lim}_{n \to \infty} a_n = 0$. Define



Prove that the series $\sum_{n=1}^{\infty} S_n^2$, $\sum_{n=1}^{\infty} a_n S_n$, and $\sum_{n=1}^{\infty} a_n^2$ converge or diverge together.

$$\sum_{n=1}^{\infty} S_n^2 < \sum_{n=1}^{\infty} a_n S_n < \sum_{n=1}^{\infty} a_n^2$$

(1) I an converges => I sur converges

by alternating senies test. $\overline{2ah}$ converges $\overline{2ah}$, $\overline{2sh}$.

2) Prove Isn' converges => 2 an' converges

$$S_h = \alpha_h - S_{h+1}$$
 $\Rightarrow \alpha_h = S_h + S_{h+1}$

$$S_{n} = \alpha_{n} - S_{n+1} \implies \alpha_{n} = S_{n} + S_{n+1}$$

$$(Also (x+y)^{2} \le 2(x^{2}+y^{2})$$

$$(S_{n}^{2} + S_{n+1})^{2} \le 2(S_{n}^{2} + S_{n+1}) = 22S_{n}^{2} + 22S_{n}^{2} + 22S_{n}^{2}$$

$$(42S_{n}^{2})$$

$$(2S_{n}^{2}) = 2S_{n}^{2} = 2S_{n}^{2}$$

Example 5. Suppose $(a_n)_{n\geq 1}$ is a decreasing sequence of positive numbers and for each natural number n, define $b_n = 1 - a_{n+1}/a_n$. Then the sequence $(a_n)_{n\geq 1}$ converges to zero if and by if the series $\sum_{n=1}^{\infty} b_n$ diverges.

$$0 \le \frac{QN - QN + 1}{QN} = \frac{1}{QN} \sum_{n=N}^{M} (an - Qn + 1) \le \sum_{n=N}^{M} bn \le \frac{1}{QN} \sum_{n=N}^{M} (an - Qn + 1) = \frac{QN - Qn + 1}{QN}$$

$$\frac{\alpha N - \alpha MH}{\alpha N} \rightarrow 1$$

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$$t$$
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Let $N=1$, $M \rightarrow \infty$

$$\sum_{n=1}^{\infty}b_n\leq\frac{a_1-d}{d}$$