MATH 1014 Tutorial Notes 1

6.1 Area Between Curves

Definition 1 (Area of a Region Between Two Curves) Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[a, b]$. The area of the region bounded by the graphs of f and g on $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]dx$$

Definition 2 (Area of a Region Between Two Curves with Respect to y) Suppose that f and g are continuous functions with $f(x) \geq g(x)$ on the interval $[c, d]$. The area of the region bounded by the graphs $x = f(y)$ and $x = g(y)$ on $[c, d]$ is

$$A = \int_c^d [f(y) - g(y)]dy$$

6.2 Volume by Slicing/Solid of Revolution

Theorem 1 (General Slicing Method) Suppose a solid object extends from $x = a$ to $x = b$ and the cross section of the solid perpendicular to the x-axis has an area given by a function $A(x)$ that is integrable on $[a, b]$. The volume of the solid is

$$V = \int_a^b A(x)dx$$

Theorem 2 (Disk Method) There are two types of disk methods to find the volume of solid of revolution:

(a) Disk Method

(i) Solid of revolution about the x-Axis Let f be continuous function with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi y^2 dx = \int_a^b \pi f(x)^2 dx$$

(ii) Solid of revolution about the y-Axis Let p be continuous function with $p(y) \geq 0$ on the interval $[c, d]$. If the region R bounded by $x = p(y)$, the y-axis, and the lines $y = c$ and $y = d$ is revolved about the y-axis, the volume of the resulting solid of revolution is

$$V = \int_c^d \pi x^2 dy = \int_c^d \pi p(y)^2 dy$$

(b) Generalized Disk Method (Washer Method)

(i) Solid of revolution about the x-Axis Let f and g be continuous functions with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b \pi \left[f(x)^2 - g(x)^2\right] dx$$

(ii) Solid of revolution about the y-Axis Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y), x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is

$$V = \int_c^d \pi \left[p(y)^2 - q(y)^2\right] dy$$

Theorem 3 (Shell Method) There are two types of shell methods to find the volume of solid of revolution:

(a) Shell Method

(i) Solid of revolution about the y-Axis Let f be continuous function with $f(x) \geq 0$ on the interval $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the y-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b 2\pi xy dx = \int_a^b 2\pi x f(x) dx$$

(ii) Solid of revolution about the x-Axis Let p be continuous function with $p(y) \geq 0$ on the interval $[c, d]$. If the region R bounded by $x = p(y)$, the y-axis, and the lines $y = c$ and $y = d$ is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_c^d 2\pi xy dy = \int_c^d 2\pi y p(y) dy$$

(b) Generalized Shell Method

(i) Solid of revolution about the y-Axis Let f and g be continuous functions with $f(x) \geq g(x)$ on $[a, b]$. Let R be the region bounded by $y = f(x), y = g(x)$, and the lines $x = a$ and $x = b$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is

$$V = \int_a^b 2\pi x[f(x) - g(x)]dx$$

(ii) Solid of revolution about the x-Axis Let p and q be continuous functions with $p(y) \geq q(y)$ on $[c, d]$. Let R be the region bounded by $x = p(y), x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the x-axis, the volume of the resulting solid of revolution is

$$V = \int_c^d 2\pi y[p(y) - q(y)]dy$$

Example 1 (Computation of Area)

Find the area enclosed by $y = x - 8$ and $x = \frac{(y-2)^2}{3}$.

Solve $\begin{cases} y = 8 - x \\ x = \frac{(y-2)^2}{3} \end{cases}$

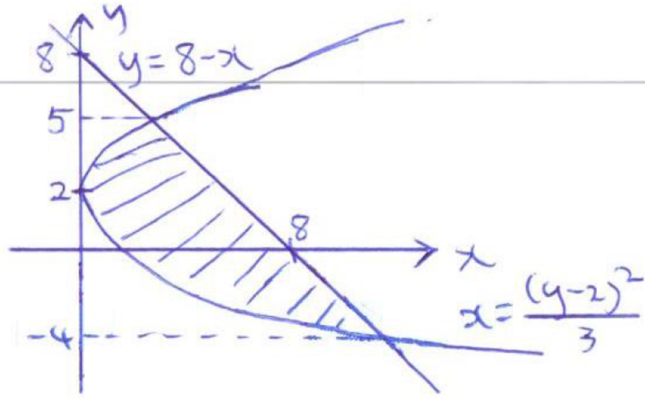
$3(8-y) = (y-2)^2$

$24 - 3y = y^2 - 4y + 4$

$y^2 - y - 20 = 0$

$y = -4$ or 5

$\therefore$ The y-coor of the intersection points are -4 and 5.



$\therefore$ Area

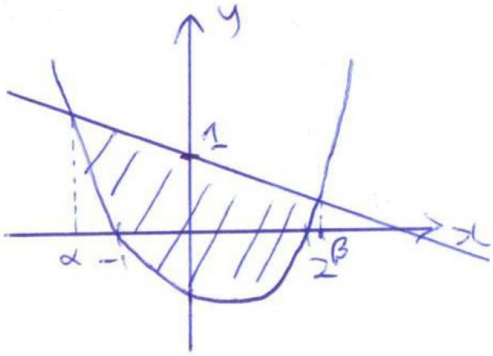
$= \int_{-4}^{5} (8-y) - \frac{(y-2)^2}{3}\, dy$

$= \frac{1}{3} \int_{-4}^{5} 20 + y - y^2\, dy$

$= \frac{1}{3} \left(20y + \frac{y^2}{2} - \frac{y^3}{3} \right) \Big|_{-4}^{5}$

$= \frac{81}{2}$

Example 2 (Computation of Area)
Graph the curves $y = (x+1)(x-2)$ and $y = ax + 1$. For what value of a is the area of the region between the two curves minimum?



Solve $\begin{cases} y = (x+1)(x-2) \\ y = ax+1 \end{cases}$

$x^2 - x - 2 = ax + 1$

$x^2 - (a+1)x - 3 = 0$

Let α, β be the roots of this equation where $\alpha < \beta$.

Then $\begin{cases} \alpha + \beta = a+1 \\ \alpha\beta = -3 \end{cases}$

Area

$$= \int_\alpha^\beta (ax+1) - (x+1)(x-2)\, dx$$

$$= \int_\alpha^\beta 3 + (a+1)x - x^2\, dx$$

$$= 3x + \frac{(a+1)x^2}{2} - \frac{x^3}{3} \Big|_\alpha^\beta$$

$$= 3(\beta - \alpha) + \frac{(a+1)}{2}(\beta^2 - \alpha^2) - \frac{1}{3}(\beta^3 - \alpha^3)$$

$$= \frac{\beta - \alpha}{6}\left[18 + 3(a+1)(\beta + \alpha) - 2(\beta^2 + \alpha\beta + \alpha^2)\right]$$

$$= \frac{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}{6}\left[18 + 3(a+1)(\alpha+\beta) - 2\left((\alpha+\beta)^2 - \alpha\beta\right)\right]$$

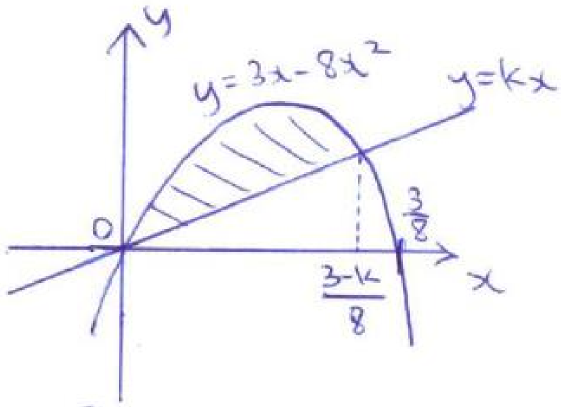
$$= \frac{\sqrt{(a+1)^2 + 12}}{6}\left[18 + 3(a+1)^2 - 2(a+1)^2 - 6\right]$$

$$= \frac{1}{6}\left[(a+1)^2 + 12\right]^{\frac{3}{2}} \geq \frac{1}{6}(12)^{\frac{3}{2}}$$

$\therefore$ Area is minimum when $a = -1$.

Example 3 (Computation of Area)
Find k such that the straight line $y = kx$ divides the region bounded by the parabola $y = 3x - 8x^2$ and the $x-$axis into two regions with equal area.

Solve $\begin{cases} y = kx \\ y = 3x - 8x^2 \end{cases}$



$$8x^2 + (k-3)x = 0$$

$$x(8x + k - 3) = 0$$

$$x = 0 \text{ or } \frac{3-k}{8}$$

$$\therefore\ 2\int_0^{\frac{3-k}{8}} (3x - 8x^2) - kx \, dx = \int_0^{\frac{3}{8}} 3x - 8x^2 \, dx$$

$$2\left[\frac{(3-k)x^2}{2} - \frac{8x^3}{3}\right]\Bigg|_0^{\frac{3-k}{8}} = \frac{3x^2}{2} - \frac{8x^3}{3}\Bigg|_0^{\frac{3}{8}}$$

$$\frac{(3-k)^3}{64} - \frac{(3-k)^3}{96} = \frac{9}{128}$$

$$\frac{(3-k)^3}{6} = \frac{9}{4}$$

$$3 - k = \frac{3}{\sqrt[3]{2}}$$

$$k = 3 - \frac{3}{\sqrt[3]{2}}$$

Example 4 (Volume by Disk Method)
Find the volume of the solid of revolution od the region bounded by $y = \frac{\ln x}{\sqrt{x}}$, $y = 0$ and $x = 2$ revolved about the $x-$axis.

$$\begin{aligned}
\text{Volume} &= \int_1^2 \pi y^2 \, dx \\
&= \int_1^2 \pi \cdot \frac{(\ln x)^2}{x} \, dx \\
&= \pi \int_1^2 (\ln x)^2 \, d \ln x \\
&= \pi \cdot \frac{(\ln x)^3}{3} \Big|_1^2 \\
&= \frac{\pi}{3} (\ln 2)^3
\end{aligned}$$