1. Method of Substitution (including trigonometric integrals)

2. Integration by Parts

3. Integrals of Rational Functions $\quad R(x) = \frac{P(x)}{Q(x)}$ $\quad P, Q$ polinomials

4. Integrals of Rational Trigonometric Functions $\quad \int \frac{1}{\sin x + 2\cos x + 1}\,dx$

Method of Substitution

interval I, J $\quad I \subseteq J$

$u = \varphi(x) \quad x \in I \qquad f(u) \quad u \in J$

$$\int f(u)\,du = \int f(\varphi(x))\,d\varphi(x) = \int f(\varphi(x))\varphi'(x)\,dx$$

$$\int \sin^m x \cos^n x \,dx$$

(a) $n = 2k+1$

$\cos^2 x = 1 - \sin^2 x$

$\int \sin^m x \cos^n x\,dx = \int \sin^m x (\cos^2 x)^k \cos x\,dx \qquad u = \sin x$

(b) $m = 2k+1$

$\sin^2 x = 1 - \cos^2 x$

$\int \sin^{2k+1} x \cos^n x\,dx = \int (\sin^2 x)^k \cos^n x \sin x\,dx \qquad u = \cos x$

(c) n, m are even

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

$\sin x \cos x = \frac{1}{2}\sin 2x$

e.g.1 $\quad \int \frac{e^{2\sqrt{x}+1}}{\sqrt{x}}\,dx$

$= \int e^{2\sqrt{x}+1}\,d(\underbrace{2\sqrt{x}+1}) \qquad d(2\sqrt{x}) = \frac{1}{\sqrt{x}}$

$$= \int e^{2\sqrt{x}+1}\, d(2\sqrt{x}+1)$$

$u = 2\sqrt{x}+1$ $$= \int e^u\, du$$

$$= e^{2\sqrt{x}+1} + C \qquad C \text{ is const.}$$

$\square$

2. Integration by Parts

u, v differentiable w.r.t. x

$$\int u(x)\, dv(x) = \underline{u(x)v(x) - \int v(x)\, du(x)}$$

$\int v(x)\, du(x)$ $\qquad I'$

$$\int u\, dv = uv - \int v\, du$$

$$\int_a^b u(x)\, dv(x) = u(x)v(x)\Big|_a^b - \int_a^b v(x)\, du(x)$$

Remark

1. Procedures
2. ① Repeat Itself

$$aI = F(u,v) - bI$$

$$(a+b)I = F(u,v)$$

② Reduction Formula I_n

$$I_n = a I_{n-1}$$
$$= a^2 I_{n-2}$$
$$= \cdots$$
$$= a^n I_0$$

e.g.2 $\int \ln x\, dx = x\ln x - \int x\, d(\ln x)$

$$= x\ln x - \int x \cdot \frac{1}{x}\, dx$$

$$= x\ln x - x + C \qquad C \text{ is const.} \ \square$$

Examples

Example 1 (Integration by Parts)**.** Evaluate the following integrals.

(a) **(Famous Formula)** $\int \ln x \, dx$

(b) **(Integration of Inverse Trigonometric Function)** $\int \sin^{-1} x \, dx$

(c) **(Repeat Itself)** $\int e^x \cos x \, dx$

Example 2 (Reduction Formula)**.** Let $I_n = \int x^n e^{ax} \, dx$ for $a \neq 0$.

(a) Use integration by parts to derive the reduction formulas

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

(b) Use the reduction formulas in (a) to evaluate $\int x^2 e^{3x} \, dx$.

Example 3 (Harder Example of Integration by Parts)**.** Show that if f is a function satisfying $f(1) = 3$, $f'(1) = 2$ and f'' is continuous on the interval $[0, 1]$ with $|f''(x)| \leq 4$ for $0 \leq x \leq 1$, then

$$\left| \int_0^1 f(x) dx \right| \leq \frac{8}{3}.$$