1. Trigonometric Integral

1.1 Integrals of $\tan x$, $\cot x$, $\sec x$, $\csc x$

$$\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C = \ln|\csc x - \cot x| + C$$

1.2 Strategy for evaluating $\int \sin^m x \cos^n x \, dx$

(a) $n = 2k+1 \quad k \in \mathbb{N}$

$$\cos^2 x = 1 - \sin^2 x$$

$$I = \int \sin^m x (\cos^2 x)^k \underline{\cos x} \, dx$$

$$= \int \sin^m x (1 - \sin^2 x)^k \underline{\cos x \, dx} = d \sin x$$

$u = \sin x$

(b) $m = 2k+1 \quad k \in \mathbb{N}$

$$\sin x^2 = 1 - \cos^2 x$$

$$I = \int (\sin^2 x)^k \cos^n x \, \underline{\sin x} \, dx$$

$$= -\int (1 - \cos^2 x)^k \cos^n x \, d\cos x$$

$u = \cos x$

(c) $m.n$ both even

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\sin x \cdot \cos x = \frac{1}{2}\sin 2x$$

1.3 $\int \tan^m x \sec^n x \, dx$

(a) If $n = 2k$ $k \in \mathbb{N}$

$$\sec^2 x = 1 + \tan^2 x$$

$$I = \int \tan^m x \sec^{2(k-1)} x \cdot \sec^2 x \, dx$$
$$= \int \tan^m x \sec^{2(k-1)} x \, d(\tan x)$$
$$= \int \tan^m x (1 + \tan^2 x)^{k-1} \, d(\tan x)$$

$u = \tan x$

(b) $m = 2k+1$ $k \in \mathbb{N}$

$$\tan^2 x = \sec^2 x - 1$$

$$I = \int (\tan^2 x)^k \sec^{n-1} x \cdot (\sec x \tan x) dx$$
$$= \int (\tan^2 x)^k \sec^{n-1} x \cdot d(\sec x)$$
$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, d(\sec x)$$

$u = \sec x$

1.4 $\int \sin mx \cos nx \, dx$

$$\sin A \cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$
$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$
$$\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

3. Trigonometric Substitution

$\sin^2 x + \cos^2 x = 1$

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$	$1 + \tan^2\theta = \sec^2\theta$

$\sqrt{x^2-a^2}$ | $x = a\sec\theta$, $\theta \in [0,\frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$ | $\sec^2\theta - 1 = \tan^2\theta$.

4. Weierstrass Method

e.g. $\int \frac{1}{\sin x + 2\cos x + 3} dx$

$$t = \tan\frac{x}{2}$$

$$\Rightarrow x = 2\arctan t \qquad dx = \frac{2}{1+t^2}dt$$

$$\Rightarrow \sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

$$\tan x = \frac{\sin x}{\cos x} = \frac{2t}{1+t^2}$$

e.g.1 $\int \sin^3 x \cos^2 x\, dx$

$= \int \sin^2 x \cos^2 x \cdot \sin x\, dx$

$= -\int \sin^2 x \cos^2 x\, d(\cos x)$

$= -\int (1-\cos^2 x)\cos^2 x\, d(\cos x)$

$(u = \cos x)$

$= -\int (1-u^2)u^2\, du$

$= \int (u^4 - u^2)\, du$

$= \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C \qquad C = \text{const}$

e.g.2 $\int \sin^2 x \cos^2 x\, dx \qquad \sin 2x = 2\sin x \cdot \cos x$

$= \int (\frac{1}{2}\sin 2x)^2\, dx$

$$= \frac{1}{4}\int \frac{1-\cos 4x}{2}\,dx$$

$$= \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C \qquad C = \text{const}$$

e.g.3 $\int \tan x \sec^2 x \, dx$ $\qquad d\tan x = \frac{1}{\cos^2 x} = \sec^2 x$

$$= \int \tan x \, d\tan x$$

$$= \frac{1}{2}\tan^2 x + C \qquad C = \text{const}$$

e.g.4 $\int \frac{1-r^2}{1-2r\cos x + r^2}\,dx \quad (0<r<1)$

$t = \tan\frac{x}{2}$

$$I = \int \frac{1-r^2}{1-2r\frac{1-t^2}{1+t^2}+r^2}\,\frac{2}{1+t^2}\,dt$$

$$= 2\int \frac{1-r^2}{(1+r^2)(1+t^2)-2r(1-t^2)}\,dt$$

$$= 2(1-r^2)\int \frac{dt}{(1-r)^2+(1+r)^2t^2}$$

$$= 2\arctan\left(\frac{1+r}{1-r}\tan\frac{x}{2}\right) + C \qquad C = \text{const}$$

e.g.5 $\int \frac{\sin^5 x}{\cos^3 x}\,dx$ $\quad \sin^4 x \sin x \quad \sin^2 x + \cos^2 x = 1$

$t = \cos x$

$$I = \int \frac{\sin^4 x}{\cos^3 x}\sin x \, dx$$

$$= -\int \frac{\sin^4 x}{\cos^3 x}\,d\cos x$$

$$\int \quad \cos^3 x \quad d\cos x$$

$$= -\int \frac{(1-\cos^2 x)^2}{\cos^3 x}\, d\cos x$$

$$= -\int \frac{(1-t^2)^2}{t^3}\, dt$$

$$= -\int \left(\frac{1}{t^3} - \frac{2}{t} + t\right) dt$$

$$= \frac{1}{2t^2} + 2\ln|t| - \frac{1}{2}t^2 + C$$

$$= \frac{1}{2\cos^2 x} + 2\ln|\cos x| - \frac{1}{2}\cos^2 x + C \qquad C = \text{cons}$$

e.g. 6 $\displaystyle\int \frac{dx}{a^2\sin^2 x + b^2\cos^2 x}$ $\qquad \tan x = \dfrac{\sin x}{\cos x}$

$$= \int \frac{\overbrace{\frac{1}{\cos^2 x}}^{d\tan x}}{a^2 \underbrace{\frac{\sin^2 x}{\cos^2 x}}_{\tan^2 x} + b^2}\, dx$$

$$= \int \frac{1}{a^2\tan^2 x + b^2}\, d\tan x$$

$t = \tan x$

$$= \int \frac{1}{a^2t^2 + b^2}\, dt$$

$$= \frac{1}{ab}\arctan\left(\frac{a}{b}\tan x\right) + C \qquad C = \text{const}$$