1. Power series

Def general form :

$$\sum_{k=0}^{\infty} \underbrace{C_k}_{\text{coefficient}}(x-\underbrace{a}_{\text{center}})^k \qquad C_k \in \mathbb{R}$$

Radius of convergence R

Thm1 (Convergence of Power Series)

$\sum_{k=0}^{\infty} C_k(x-a)^k$ converges if

(a) the series converges absolutely for all x.

$\Rightarrow$ interval of convergence : $(-\infty, \infty)$

radius of convergence : $R=\infty$

(b) there is $R \in \mathbb{R}$, $R>0$. s.t. the series converges absolutely for $|x-a|<R$ .and diverges $|x-a|>R$

$\Rightarrow$ radius of convergence $=R$

(c) the series converges only at a.

$\Rightarrow \cdots R=0$

2. Representations of Functions as Power Series

Geometric Formula when $|x|<1$, we have

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1+x+x^2+x^3+\cdots$$

Thm 2 (Combining Power Series)

$\sum c_k x^k$ converges absolutely to $f(x)$

$\sum d_k x^k$ ... $g(x)$

(1) Sum & difference:

$\sum (c_k x^k \pm d_k x^k)$ converges abs to $f(x) \pm g(x)$

(2) Multiplication by a power

$x^m \cdot \sum c_k x^k = \sum c_k x^{k+m}$ converges abs to $x^m \cdot f(x)$

(3) Composition

If $h(x) = c\, x^m$ $m \in \mathbb{N}^+$, $c \in \mathbb{R}$

$\Rightarrow \sum c_k (h(x))^k$ converges abs to $f(h(x))$

Thm 3 (Differentiating & Integrating Power Series)

$f = \sum a_k (x-a)^k$ $x \in I$

(a) f is a continuous function on I

(b) The power series can be differentiated and integrated term by term.

$\downarrow$ $f'(x)$ $\downarrow$ $\int f(x)\,dx + C$

3. Talor Series

Def (Taylor Polynomials)

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

c between x and a

Def (Remainder)

$$R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Taylor Expansion of f

$$f(x) = P_n(x) + R_n(x)$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Maclaurin Series Taylor series centered at 0 ($a=0$)

1. Find radius and interval of convergence

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} x^n$ $\quad y_n \qquad \lim_{n\to\infty} \frac{|y_{n+1}|}{|y_n|}$

Since $\lim_{n\to\infty} \frac{\left|(-1)^{n+1} \frac{3^{n+1}}{n+1} x^{n+1}\right|}{\left|(-1)^n \frac{3^n}{n} x^n\right|}$

$= \lim_{n\to\infty} 3 \cdot \frac{n}{n+1} |x|$

$= 3|x|$

$3|x| < 1$ when $|x| < \frac{1}{3}$ i.e. $-\frac{1}{3} < x < \frac{1}{3}$

By Ratio test. it converges when $-\frac{1}{3} < x < \frac{1}{3}$

When $x = -\frac{1}{3}$

$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} \left(-\frac{1}{3}\right)^n = \sum \frac{1}{n}$ diverges (by p-test)

When $x = \frac{1}{3}$

$\sum_{n=1}^{\infty} (-1)^n \frac{3^n}{n} \left(\frac{1}{3}\right)^n = \sum (-1)^n \frac{1}{n}$ converges (by alternating ser

$\Rightarrow R = \frac{1}{3} \qquad \left(-\frac{1}{3}, \frac{1}{3}\right]$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + (-1)^n}{n^2} (x-3)^n$

$$\lim_{n \to \infty} \frac{\left| \frac{2^{n+1} + (-1)^{n+1}}{(n+1)^2} (x-3)^{n+1} \right|}{\left| \frac{2^n + (-1)^n}{n^2} (x-3)^n \right|} = 2|x-3| < 1$$

$\Rightarrow |x-3| < \frac{1}{2}$ i.e. $\frac{5}{2} < x < \frac{7}{2}$

When $x = \frac{5}{2}$

$$\boxed{\quad} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + \sum_{n=1}^{\infty} \frac{1}{2^n n^2} \quad \text{converges}$$

When $x = \frac{7}{2}$

$$\boxed{\quad} = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^n n^2} \quad \text{converges}$$

Thus. $R = \frac{1}{2}$

$\left[\frac{5}{2}, \frac{7}{2}\right]$

(c) $\sum_{n=1}^{\infty} \frac{x^n}{2^{n^2}}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{x^n}{2^{n^2}}\right|} = 0 < 1$$

By root test. it converges for all $x \in \mathbb{R}$

$\Rightarrow R = \infty$

$(-\infty, \infty)$

(d) $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$

$$\lim_{n\to\infty} \frac{\left|\frac{(n+1)!}{(n+1)^{n+1}} x^{n+1}\right|}{\left|\frac{n!}{n^n} x^n\right|} = \frac{|x|}{e} < 1$$

$\Rightarrow \; -e < x < e$

When $x = e$

since $\lim \frac{n!}{n^n} e^n = \infty$ diverges (by divergence test)

When $x = -e$, diverges.

$R = e \qquad (-e, e)$

2. Find the power series representation of

$$f(x) = \ln\sqrt{4-x^2}$$

Compute its radius of convergence.

$$f(x) = \ln(4-x^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\ln(4-x^2)$$

$$= \frac{1}{2}\left[\ln 4 + \ln\left(1+\frac{x^2}{4}\right)\right]$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$= \ln 2 - \frac{1}{2}\left[\frac{x^2}{4} + \frac{1}{2}\left(\frac{x^2}{4}\right)^2 + \frac{1}{3}\left(\frac{x^2}{4}\right)^2 + \cdots\right]$$

$$= \ln 2 - \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x^2}{4}\right)^k$$

$$= \ln 2 - \sum_{k=1}^{\infty}\frac{1}{2k\cdot 4^k}x^{2k}$$

It converges when $\left|\frac{x^2}{4}\right| < 1$ i.e. $|x| < 2$

$R = 2$

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