Math1014 Sample Final Exam

Part I: MC Questions.

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Answer	E	B	D	A	C	B	B	C	E	E	D	A	D	C	E	A

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1. What is the colour version of your exam paper? (Read the top left corner of the cover page!) **Make sure that you have also written and marked your ID number correctly in the I.D. No. Box in the MC answer sheet. If you do not do both correctly, you lose the points of this question.**

   (a) Green (b) Orange (c) White (d) Yellow (e) Sample

2.
$$\int_0^1 4\cos(2\pi x)\sin^2(\pi x)dx = \int_0^1 2\cos(2\pi x)(1-\cos(2\pi x))dx$$
$$= \int_0^1 (2\cos(2\pi x) - 1 - \cos(4\pi x))dx = \Big[\frac{1}{\pi}\sin(2\pi x) - x - \frac{1}{4\pi}\sin(4\pi x)\Big]_0^1 = -1$$

3.
$$\int_0^2 \frac{4x}{\sqrt{4-x^2}}dx = \lim_{b\to 2^-}\int_0^b \frac{4x}{\sqrt{4-x^2}}dx$$
$$= \lim_{b\to 2^-}\Big[-4(4-x^2)^{1/2}\Big]_0^b = \lim_{b\to 2^-} -4(4-b^2)^{1/2} + 8 = 8$$

4.
$$\int_0^\infty \frac{2x}{(x^2+1)^2}dx = \lim_{L\to\infty}\int_0^L 2x(x^2+1)^{-2}dx$$
$$= \lim_{L\to\infty}\Big[-(x^2+1)^{-1}\Big]_0^L$$
$$= \lim_{L\to\infty} -\frac{1}{L^2+1} + 1 = 1$$

5. The area is
$$\int_0^4 (x+1-\frac{2}{(x+1)(x+2)})dx = \int_0^4 (x+1-\frac{2}{x+1}+\frac{2}{x+2})dx$$
$$= \Big[\frac{x^2}{2} + x + 2\ln\frac{x+2}{x+1}\Big]_0^4 = 12 + 2\ln\frac{3}{5}$$

6. The area of the surface is

$$\int_0^{\sqrt{12}} 2\pi x ds = \int_0^{\sqrt{12}} 2\pi x\sqrt{1+4x^2}\,dx$$
$$= \left[\frac{\pi}{6}(1+4x^2)^{3/2}\right]_0^{\sqrt{12}} = \frac{\pi}{6}(7^3-1) = 57\pi$$

7. The area is

$$2\int_0^{\frac{\pi}{2}} \frac{1}{2}\cdot 4\sin(2t)e^{2\cos(2t)}dt$$
$$= -e^{2\cos(2t)}\Big|_0^{\frac{\pi}{2}} = e^2 - e^{-2}$$

8. The work integral is

$$\int_0^4 \rho g\pi y(4-y)dy$$

Using the trapezoidal rule with $n=4$, we have

$$T_4 = \frac{4\rho g\pi}{2\cdot 4}\Big[2\cdot 1\cdot(4-1) + 2\cdot 2(4-2) + 2\cdot 3(4-3)\Big] = 10\pi\rho g$$

9.

$$\int_1^{\infty} \frac{x}{e^{-x}+x^2}\,dx \geq \int_1^{\infty}\frac{x}{x^2+x^2}\,dx$$
$$= \int_1^{\infty}\frac{1}{2x}\,dx = \frac{1}{2}\ln x\Big|_1^{\infty} = \infty$$

10.

$$\sum_{n=1}^{\infty}\left(\frac{3^{n+3}-2^n}{4^{n+1}}\right) = \frac{3^4}{4^2}(1+\frac{3}{4}+\frac{3^2}{4^2}+\cdots) - \frac{2}{4^2}(1+\frac{2}{4}+\frac{2^2}{4^2}+\cdots)$$
$$= \frac{81}{16}\cdot\frac{1}{1-\frac{3}{4}} - \frac{2}{16}\cdot\frac{1}{1-\frac{2}{4}} = \frac{81}{4} - \frac{2}{8} = 20$$

11.

$$\sum_{n=1}^{\infty}\left(9^{1/n} - 9^{1/(n+2)}\right) = \lim_{N\to\infty}\sum_{n=1}^{N}\left(9^{1/n} - 9^{1/(n+2)}\right)$$
$$= \lim_{N\to\infty}\left[\begin{array}{c}(9+9^{1/2}+9^{1/3}+\cdots+9^{1/N}) \\ - \\ (9^{\frac{1}{3}}+9^{\frac{1}{4}}+\cdots+9^{\frac{1}{N+1}}+9^{\frac{1}{N+2}})\end{array}\right]$$
$$= \lim_{N\to\infty}(9+9^{1/2}-9^{1/(N+1)}-9^{1/(N+2)}) = 9+3-1-1 = 10$$

12. (ii) and (iii) are divergent by the Divergence Test.

    (i) is convergent by the Ratio Test, and (iv) converges absolutely:

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{2^{n+1}(n+1)!}{(2n+2)!}}{\frac{2^n n!}{(2n)!}} &= \lim_{n\to\infty}\frac{4(n+1)}{(2n+2)(2n+1)}\\ &= 0<1\end{aligned}$$

$$\sum_{n=0}^{\infty}\frac{|\sin(n!)|}{n^2}\le\sum_{n=0}^{\infty}\frac{1}{n^2}<\infty$$

13. The series is divergent when $p=-1$ by the Limit Comparison Test:

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n^{-1+2}}(\ln n)^2}}{\frac{1}{n^{3/4}}}=\lim_{n\to\infty}\frac{n^{1/4}}{(\ln n)^2}=\infty$$

while $\sum_{n=2}^{\infty}\frac{1}{n^{3/4}}$ is divergent. The series is convergent when $p\ge 1$, which follows easily from basic comparison test: for $n\ge 2$,

$$\frac{1}{n^{\frac{p+2}{2}}(\ln n)^2}\le\frac{1}{n^{\frac{3}{2}}(\ln n)^2}<\frac{1}{n^{\frac{3}{2}}(\ln 2)^2}$$

while $\sum\frac{1}{n^{3/2}}$ is convergent.

14. Just apply the Binomial Series:

$$f(x)=1+\frac{3}{2}(8x^2)+\frac{\frac{3}{2}(\frac{3}{2}-1)}{2!}(8x^2)^2+\cdots$$

and hence the coefficient of the x^4 term is

$$\frac{\frac{3}{2}(\frac{3}{2}-1)}{2!}(64)=24$$

15. The projection is given by

$$\begin{aligned}&\mathbf{Proj}_{\langle 1,3,2\rangle}\langle -4,12,-2\rangle\\ &=\frac{\langle -4,12,-2\rangle\cdot\langle 1,3,2\rangle}{\|\langle 1,3,2\rangle\|^2}\langle 1,3,2\rangle\\ &=\frac{28}{14}\langle 1,3,2\rangle=\langle 2,6,4\rangle\end{aligned}$$

16. The area of the parallelogram generated by $\langle 1,2,0\rangle$ and $\langle 1,0,-1\rangle$ is

$$\|\langle 1,2,0\rangle\times\langle 1,0,-1\rangle\|=\|\langle -2,1,2\rangle\|=3$$

The volume of the parallelopiped generated by the three vectors is then

$$\begin{aligned}3\cdot\text{height}&=|\langle 3,0,0\rangle\cdot(\langle 1,2,0\rangle\times\langle 1,0,-1\rangle)|\\ &=|\langle 3,0,0\rangle\cdot\langle -2,1,2\rangle|=|-6|\\ \text{height}&=\frac{6}{3}=2\end{aligned}$$

Part II: Long Questions.

17. [*12 pts*]

(a) Note that $\dfrac{d}{dx}2\sqrt{x+1} = \dfrac{1}{\sqrt{x+1}}$. [7 pts]

Using integration by parts with

$$u = 2x^n, \quad v = \sqrt{x+1},$$

we have

$$\int \frac{x^n}{\sqrt{x+1}}dx = \int 2x^n d\sqrt{x+1} = 2x^n\sqrt{x+1} - \int 2nx^{n-1}\sqrt{x+1}dx$$

$$= 2x^n\sqrt{x+1} - 2n\int \frac{x^{n-1}(x+1)}{\sqrt{x+1}}dx$$

i.e.

$$I_n = 2x^n\sqrt{x+1} - 2nI_n - 2nI_{n-1}$$

$$I_n = \frac{2}{2n+1}x^n\sqrt{x+1} - \frac{2n}{2n+1}I_{n-1}$$

Hence $A_n = \dfrac{2}{2n+1}$, $B = -\dfrac{2n}{2n+1}$.

(b) Using the reduction formula in part (a), we have [5 pts]

$$\int_0^3 \frac{15x^2}{\sqrt{x+1}}dx = 15 \cdot \left[\frac{2}{5}x^2\sqrt{x+1}\Big|_0^3 - \frac{4}{5}\int_0^3 \frac{x}{\sqrt{x+1}}dx\right]$$

$$= 108 - 12\int_0^3 \frac{x}{\sqrt{x+1}}dx$$

$$= 108 - 12\left[\frac{2}{3}x\sqrt{x+1}\Big|_0^3 - \frac{2}{3}\int_0^3 \frac{1}{\sqrt{x+1}}dx\right]$$

$$= 108 - 48 + 16\sqrt{x+1}\Big|_0^3 = 76$$

18. [*14 pts*]

(a) The volume is [8 pts]

$$V = \int_0^\infty \pi y^2 dx = \int_0^\infty \frac{64\pi}{e^{2x}+4}dx$$

Let $u = e^x$, such that $du = e^x dx = udx$. Then we have

$$\int_0^\infty \frac{64\pi}{e^{2x}+4}dx = 64\pi\int_1^\infty \frac{1}{u(u^2+4)}du = 64\pi\int_1^\infty \frac{1}{4}\left(\frac{1}{u} - \frac{u}{u^2+4}\right)du$$

$$= 16\pi\Big[\ln|u| - \frac{1}{2}\ln|u^2+4|\Big]_1^\infty = 16\pi\Big[\lim_{u\to\infty}\ln\frac{u}{\sqrt{u^2+4}} + \frac{1}{2}\ln 5\Big] = 8\pi\ln 5$$

(b) The volume of the solid is [6 pts]

$$\int_0^\infty 2\pi xy dx = \int_0^\infty \frac{16\pi x}{\sqrt{e^{2x}+4}} dx$$

The volume is finite since

$$\int_0^\infty \frac{16\pi x}{\sqrt{e^{2x}+4}} dx < \int_0^\infty 16\pi x e^{-x} dx = -16\pi x e^{-x}\Big|_0^\infty + 16\pi \int_0^\infty e^{-x} dx$$

$$= -16\pi e^{-x}\Big|_0^\infty = 16\pi < \infty$$

19. [*12 pts*]

(a) Divergent; by the *Divergence Test* since [3 pts]

$$\lim_{n\to\infty} \frac{(-1)^n n}{\ln n} \text{ does not exist}$$

(b) Divergent; by applying the *Limit Comparison Test* with respect to the divergent p-series $\sum \frac{1}{\sqrt{n}}$: [3 pts]

$$\lim_{n\to\infty} \frac{\sqrt{\frac{n+2}{n^2+1}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n(n+2)}{n^2+1}} = 1$$

(c) Convergent; by applying the *Root Test*: [3 pts]

$$\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2} = \lim_{n\to\infty} e^{n^2 \ln \frac{n}{n+1}} = 0 < 1$$

since

$$\lim_{x\to\infty} x^2 \ln \frac{x}{x+1} = \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{x^{-2}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x+1}}{-2x^{-3}} = \lim_{x\to\infty} \frac{-2x^3}{x(x+1)} = -\infty$$

(d) Convergent, since it is absolutely convergent [3 pts]

$$\sum_{n=1}^\infty |e^{-n}\sin n| < \sum_{n=1}^\infty e^{-n} = \frac{e^{-1}}{1-e^{-1}} < \infty$$

20. [*14 pts*]

(a) Applying the Ratio Test, we have [7 pts]

$$\lim_{n\to\infty} \frac{\left|\frac{2(n+1)+3}{4^{2(n+1)}}(x-1)^{2(n+1)+1}\right|}{\left|\frac{2n+3}{4^{2n}}(x-1)^{2n+1}\right|} < 1$$

$$\lim_{n\to\infty} \frac{(2n+5)|x-1|^2}{16(2n+3)} < 1$$

$$\frac{1}{16}|x-1|^2 < 1$$

The open interval of convergence is given by $|x-1| < 4$, i.e., $-3 < x < 5$.

(b) Divergent at both endpoints $x = 5$, $x = -3$: [3 pts]

$$\sum_{n=0}^{\infty} \frac{2n+3}{4^{2n}}(5-1)^{2n+1} = \sum_{n=0}^{\infty} 4(2n+3) = \infty$$

$$\sum_{n=0}^{\infty} \frac{n+2}{4^{2n}}(-3-1)^{2n+1} = \sum_{n=0}^{\infty} (-4)(2n+3) = -\infty$$

(c) [4 pts]

$$\int_1^2 H'(x)dx = \sum_{n=0}^{\infty} \int_1^2 \frac{2n+3}{4^{2n}}(x-1)^{2n+2}dx$$

$$H(2) - H(1) = \sum_{n=0}^{\infty} \frac{1}{4^{2n}}(x-1)^{2n+3}\Big|_1^2$$

$$H(2) - (-1) = \sum_{n=0}^{\infty} \frac{1}{4^{2n}} = \frac{16}{15}$$

$$H(2) = \frac{1}{15}$$