Integration of Rational Functions

Step 1

$$\frac{P(x)}{Q(x)} = F(x) + \underbrace{\frac{R(x)}{Q(x)}}$$

improper

proper (degree of $R(x)$ < degree of $Q(x)$)

Step 2 Apply method of partial fraction

Decompose proper rational function into the sum of

(a) $\frac{A}{(x-a)^k}$ $(k \geq 1)$

(b) $\frac{Ax+B}{(x^2+px+q)^k}$ $(k \geq 1)$

① make sure the fraction is a proper rational function

② factor the polynomial $Q(x)$ into linear or/and quadratic factor (method of undetermined coefficients)

③ decompose the fraction into a sum of partial functions

Step 3

$\frac{A}{(x-a)^k}$ $\qquad$ $\frac{Ax+B}{(x^2+px+q)^k}$

- $k=1$ $\quad \int \frac{A}{x-a}\,dx = A \overset{(\ln)}{\log}|x-a| + C$

$k>1$ $\quad \int \frac{A}{(x-a)^k}\,dx = A \frac{(x-a)^{1-k}}{1-k} + C$

$$\frac{Ax+B}{(x^2+px+q)^k}$$

$2x+p = (x^2+px+q)'$.

$k=1$ $$\int \frac{Ax+B}{x^2+px+q} = \int \frac{\frac{A}{2}(2x+p)+(B-\frac{Ap}{2})}{x^2+px+q}dx$$

$$= \frac{A}{2}\int \frac{2x+p}{x^2+px+q}dx + (B-\frac{Ap}{2})\int \frac{1}{x^2+px+q}dx$$

$$= \frac{A}{2}\int \frac{d(x^2+px+q)}{x^2+px+q} + (B-\frac{Ap}{2})\int \frac{dx}{(x+\frac{p}{2})^2+(q-\frac{p^2}{4})}$$

$$= \frac{A}{2}\ln(x^2+px+q) + \frac{2B-Ap}{\sqrt{4q-p^2}}\arctan\frac{2x+p}{\sqrt{4q-p^2}} + C$$

$k>1$ $$\int \frac{Ax+B}{(x^2+px+q)^k}dx = \int \frac{\frac{A}{2}(2x+p)+(B-\frac{Ap}{2})}{(x^2+px+q)^k}dx$$

$$= \frac{A}{2}\int \frac{d(x^2+px+q)}{(x^2+px+q)^k}dx + (B-\frac{Ap}{2})\int \frac{dx}{(x^2+px+q)^k}$$

$$= \frac{A}{2}\frac{(x^2+px+q)^{1-k}}{1-k} + (B-\frac{Ap}{2})\int \frac{dx}{[\underbrace{(x+\frac{p}{2})}_{t}^2+\underbrace{(q-\frac{p^2}{4})}_{a}]^k}$$

$t = x+\frac{p}{2}$ $a = \frac{1}{2}\sqrt{4q-p^2}$

$$I_k = \int \frac{dt}{(t^2+a^2)^k}$$

$$I_{k+1} = \frac{2k+1}{2ka^2}I_k + \frac{1}{2ka^2}\frac{t}{(t^2+a^2)^k}$$

e.g. $\int \frac{1}{x^3+1}\,dx$

Step 2.

$$x^3+1 = (x+1)(x^2-x+1)$$

$$\frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)}$$

$$= \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

$$1 = A(x^2-x+1) + (Bx+C)(x+1)$$

$$= Ax^2 - Ax + A + Bx^2 + Bx + Cx + C$$

$$\left.\begin{array}{ll} x^2 \text{ terms}: & A+B=0 \\ x^1 \text{ terms}: & -A+B+C=0 \\ x^0 \text{ terms}: & A+C=1 \end{array}\right\} \Rightarrow \begin{cases} A=\frac{1}{3} \\ B=-\frac{1}{3} \\ C=\frac{2}{3} \end{cases}$$

$$\frac{1}{x^3+1} = \frac{1}{3}\cdot\frac{1}{x+1} + \frac{1}{3}\cdot\frac{-x+2}{x^2-x+1}$$

e.g.

$$\int \frac{1}{x^3+1} = \frac{1}{3}\int \frac{1}{x+1}\,dx + \frac{1}{3}\int \frac{-x+2}{x^2-x+1}\,dx$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+x)$$

$$+ \underbrace{\frac{1}{2}\int \frac{dx}{x^2-x+1}}$$

$$= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+x)$$

$$+ \frac{1}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right) + C$$

1. $\int \frac{1}{(x-6)(x+2)} dx$

$$\frac{1}{(x+6)(x-2)} = \frac{A}{x+6} + \frac{B}{x-2} = \frac{A(x-2)+B(x+6)}{(x+6)(x-2)}$$

$$\Rightarrow 1 = (A+B)x + (6B-2A)$$

$$\Rightarrow \begin{cases} A+B=0 \\ 6B-2A=1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{8} \\ B = \frac{1}{8} \end{cases}$$

$$I = \int \left(-\frac{1}{8(x+6)} + \frac{1}{8(x-2)} \right) dx$$

$$= -\frac{1}{8}\ln|x+6| + \frac{1}{8}\ln|x-2| + C$$

$$= -\frac{1}{8}\ln\left|\frac{x-2}{x+6}\right| + C \qquad \square$$

2. $\displaystyle\int \frac{5x^3-12x^2+5x-4}{(2x+1)(x-1)^3}\,dx$

$$\frac{5x^3-12x^2+5x-4}{(2x+1)(x-1)^3} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3}$$

$$5x^3-12x^2+5x-4 = A\underline{(x-1)^3} + B\underline{(2x+1)}(x-1)^2 + C(2x+1)(x-1)$$

Let $x=1$ $\quad 3D = -6 \Rightarrow D = -2$

$\quad x = -\frac{1}{2}$ $\quad -\frac{27}{8}A = -\frac{5}{8} - 3 - \frac{5}{2} - 4 \Rightarrow A = 3$

x^3 terms : $\quad 3 + 2B = 5 \Rightarrow B = 1$

x^0 : $\quad -3 + 1 - C - 2 = -4 \Rightarrow C = 0$

$$I = \int\left(\frac{3}{2x+1} + \frac{1}{x-1} - \frac{2}{(x-1)^3}\right)dx$$

$$= \frac{3}{2}\ln|2x+1| + \ln|x-1| + \frac{1}{(x-1)^2} + C$$

$\square$

3. $\displaystyle\int_0^1 \frac{x-1}{x^4-2x^3+4x-4}\,dx$ $\quad x_1=\sqrt{2} \quad x_2=-\sqrt{2}$

$(x-\sqrt{2})(x+\sqrt{2})$

$$x^4-2x^3+4x-4=(x^2-2)(x^2-2x+2)$$

$$=(x+\sqrt{2})(x-\sqrt{2})(x^2-2x+2)$$

Let $\displaystyle\frac{x-1}{x^4-2x^3+4x-4}=\frac{A}{x+\sqrt{2}}+\frac{B}{x-\sqrt{2}}+\frac{Cx+D}{x^2-2x+2}$

$A=\frac{1}{8} \quad B=\frac{1}{8} \quad C=-\frac{1}{4} \quad D=\frac{1}{2}$

$$I=\int_0^1\left(\frac{1}{8(x+\sqrt{2})}+\frac{1}{8(x-\sqrt{2})}-\frac{x-2}{4(x^2-2x+2)}\right)dx$$

$$=\left(-\frac{\ln 2}{8}-\frac{\ln(x^2-2x+2)}{8}+\frac{1}{4}\arctan(x-1)\right)\Big|_0^1$$

$$=\frac{\pi}{16}$$

$\square$