Infinite Series

Definition 5 (Infinite Series) Given a set of numbers $\{a_1, a_2, a_3, \ldots\}$, the sum

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

is called an **infinite series**. Its **sequence of partial sums** $\{S_n\}$ has the terms

$$\begin{aligned}
&S_1 = a_1\\
&S_2 = a_1 + a_2\\
&S_3 = a_1 + a_2 + a_3\\
&\vdots\\
&S_n = a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \text{ for } n = 1, 2, 3, \ldots
\end{aligned}$$

If the sequence of partial sums $\{S_n\}$ has a limit L, the infinite series converges to that limit, and we write

$$\sum_{k=1}^{\infty} a_k = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = L.$$

If the sequence of partial sums diverges, the infinite series also diverges.

Theorem 5 (Geometric Series) *Let $a \neq 0$ and r be real numbers. If $|r| < 1$, then*

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

If $|r| \geq 1$, then the series diverges.

Definition 6 (Telescoping Series) A series of the form

$$\sum_{k=1}^{\infty} (a_n - a_{n+1})$$

is called a **telescoping series** and its partial sums can be expressed as

$$S_n = (a_1 - a_2) + (a_2 - a_3) + \ldots + (a_n - a_{n+1}) = a_1 - a_{n+1}$$

therefore we have

$$\sum_{k=1}^{\infty} (a_n - a_{n+1}) = \lim_{n\to\infty} (a_1 - a_{n+1})$$

Divergence and Integral Tests

Test 1 (Divergence Test) If $\sum a_k$ converges, then $\lim_{k\to\infty} a_k = 0$.
Equivalently, if $\lim_{k\to\infty} a_k \neq 0$, then the series diverges.

Test 2 (Integral Test) Suppose f is a continuous, positive, decreasing function for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} f(x)\,dx$$

either both converge or both diverge.
In the case of convergence, the value of the integral is *not*, in general, equal to the value of the series.

Test 3 (p-Test) The p-series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$, and diverges for $p \leq 1$.

Theorem 1 (Estimating Series with Positive Terms) *Let f be a continuous, positive, decreasing function for $x \geq 1$, and let $a_k = f(k)$, for $k = 1, 2, 3, \ldots$. Suppose $\sum_{k=1}^{\infty} a_k$ converges and the remainder is $R_n = \sum_{k=n+1}^{\infty} a_k = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k$, then*

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$

Furthermore, the exact value of the series is bounded as follows:

$$\sum_{k=1}^{n} a_k + \int_{n+1}^{\infty} f(x)\,dx \leq \sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{n} a_k + \int_n^{\infty} f(x)\,dx$$

Theorem 2 (Properties of Convergent Series)

(a) *Suppose $\sum a_k$ converges to A and let c be a real number. The series $\sum c a_k$ converges and $\sum c a_k = c \sum a_k = cA$.*

(b) *Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum (a_k \pm b_k)$ converges and $\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B$.*

(c) ***Whether** a series converges does not depend on a finite number of terms added to or removed from the series. Specifically, if M is a positive integer, then $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=M}^{\infty} a_k$ both converge or both diverge. However, the **value** of a convergent series does change if nonzero terms are added or deleted.*

Exercise 1 (Limits of Sequences from Limits of Functions)
Determine if the limit of the sequence

$$\left\{\frac{3^{n+1}+3}{3^n}\right\} = a_n$$

exists. If it exists, find its limit.

$$f(x) = \frac{3^{x+1}+3}{3^x} \qquad f(n) = a_n$$

$$\lim_{x\to\infty} \frac{3^{x+1}+3}{3^x} = \lim_{x\to\infty}\left(3+\frac{3}{3^x}\right) = 3 \Rightarrow \frac{3^{n+1}+3}{3^n} \to 3 \ (n\to\infty)$$

Exercise 2 (Limits of Sequences from Limits of Functions)
Determine if the limit of the sequence

$$\left\{\left(\frac{1}{n}\right)^{\frac{1}{n}}\right\}$$

exists. If it exists, find its limit.

$$y = f(x) := \left(\frac{1}{x}\right)^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln\frac{1}{x}$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\frac{1}{x}}{x} \quad \frac{\infty}{\infty}$$

$$= \lim \frac{x\cdot\left(-\frac{1}{x^2}\right)}{1}$$

$$= \lim_{x\to\infty} -\frac{1}{x}$$

$$= 0$$

$\ln y \to 0$

$y \to e^0 = 1$

$\left(\frac{1}{n}\right)^{\frac{1}{n}} \to 1$ as $n\to\infty$

Exercise 3 (Squeeze/Sandwich Theorem)
Evaluate the limit of the sequence

(a) $\left\{\frac{\cos n}{n}\right\}$ $\quad -1 \le \cos n \le 1$

$$-\frac{1}{n} \le \frac{\cos n}{n} \le \frac{1}{n}$$

$-\frac{1}{n} \to 0$, $\frac{1}{n} \to 0$

By Squeezing Thm

$$\lim_{n\to\infty} \frac{\cos n}{n} = 0$$

(b) $\left\{\frac{\sin 6n}{5n}\right\}$ $\quad -1 \le \sin 6n \le 1$

$$-\frac{1}{5n} \le \frac{\sin 6n}{5n} \le \frac{1}{5n}$$

$-\frac{1}{5n} \to 0$, $\frac{1}{5n} \to 0$

$$\lim_{n\to\infty} \frac{\sin 6n}{5n} = 0$$

Exercise 4 (Monotone Convergence Theorem)
Define $a_0 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for $n = 0, 1, 2, \ldots$. Find the value for $L = \lim_{n\to\infty} a_n$ provided that the limit does exist.

Assume $\lim_{n\to\infty} a_n = L$

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(1+\frac{1}{a_n}\right)$$

$$L = 1 + \frac{1}{L}$$

$$L^2 - L - 1 = 0$$

$$L = \frac{1\pm\sqrt{5}}{2}$$

since $L \geq 0$

$$L = \frac{1+\sqrt{5}}{2}$$

Exercise 5 (Geometric Series)
Evaluate

$$\sum_{k=0}^{\infty} \left(\frac{e}{\pi}\right)^k$$

$$\sum_{k=0}^{\infty} \left(\frac{e}{\pi}\right)^k = \frac{1}{1-\frac{e}{\pi}}$$

$$= \frac{\pi}{\pi - e}$$

Exercise 6 (Telescoping Technique)
Evaluate

$$\sum_{k=0}^{\infty} \frac{1}{16k^2 + 8k - 3}$$

$$\frac{1}{16k^2+8k-3} = \frac{1}{(4k-1)(4k+3)}$$

$$= \frac{1}{4}\left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$$

$$= \lim_{n\to\infty} \frac{1}{4} \sum_{k=0}^{n} \left(\frac{1}{4k-1} - \frac{1}{4k+3}\right)$$

$$= \lim_{n\to\infty} \frac{1}{4}\left[\left(-1-\cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{7}}\right) + \cdots + \left(\cancel{\frac{1}{4n-1}} - \frac{1}{4n+3}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(-1 - \frac{1}{4n+3}\right)$$

$$= -\frac{1}{4} \qquad \square$$