MATH1014 Calculus II (L11) - Tutorial Note 4

1 Trigonometric Integral

1.1 Integrals of $\tan x, \cot x, \sec x, \csc x$

- $\int \tan x \mathrm{d}x = -\ln|\cos x| + C = \ln|\sec x| + C$
- $\int \cot x \mathrm{d}x = \ln|\sin x| + C$
- $\int \sec x \mathrm{d}x = \ln|\sec x + \tan x| + C$
- $\int \csc \mathrm{d}x = -\ln|\csc x + \cot x| + C = \ln|\csc x - \cot x| + C$

1.2 Strategy for evaluating $\int \sin^m x \cos^n x \mathrm{d}x$

(a) If n is odd, i.e. $n = 2k+1, k \in \mathbb{N}$, then save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$:

$$\begin{aligned}\int \sin^m x \cos^n x \mathrm{d}x &= \int \sin^m x (\cos^2 x)^k \cos x \mathrm{d}x \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \mathrm{d}x\end{aligned}$$

Then substitute $u = \sin x$.

(b) If m is odd, i.e. $m = 2k+1, k \in \mathbb{N}$, then save one sine factor and use $\sin^2 x = 1 - \cos^2 x$:

$$\begin{aligned}\int \sin^m x \cos^n x \mathrm{d}x &= \int (\sin^2 x)^k \cos^n x \sin x \mathrm{d}x \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \mathrm{d}x\end{aligned}$$

Then substitute $u = \cos x$.

(c) If m, n are both even, then use the half-angle identities: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$, $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$, $\sin x \cos x = \frac{1}{2}\sin 2x$

1.3 Strategy for evaluating $\int \tan^m x \sec^n x \mathrm{d}x$

(a) If n is even, i.e. $n = 2k, k \in \mathbb{N}$, save a factor of $\sec^2 x$ and use $\sec^2 x = 1+\tan^2$:

$$\begin{aligned}\int \tan^m x \sec^{2k} x \mathrm{d}x &= \int \tan^m x \sec^{2(k-1)} x \sec^2 x \mathrm{d}x \\ &= \int \tan^m x \sec^{2(k-1)} x \mathrm{d} \tan x \\ &= \int \tan^m x (1+\tan^2 x)^{k-1} \mathrm{d} \tan x\end{aligned}$$

Then substitute $u = \tan x$.

(b) If m is odd, i.e. $m = 2k+1, k \in \mathbb{N}$, and $n \geq 1$, then save $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$:

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \mathrm{d}x &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \mathrm{d}x \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \mathrm{d} \sec x\end{aligned}$$

Then substitute $u = \sec x$.

1.4 Strategy for evaluating $\int \sin mx \cos nx \mathrm{d}x$ etc.

$$\begin{aligned}\sin A \cos B &= \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right] \\ \sin A \sin B &= \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] \\ \cos A \cos B &= \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]\end{aligned}$$

2 Trigonometric Substitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$	$\sec^2\theta - 1 = \tan^2\theta$

3 Examples

1. $\int \sin^3 x \cos^2 x \mathrm{d}x$

2. $\int \sin^2 x \cos^2 x \mathrm{d}x$

3. $\int \tan x \sec^2 x \mathrm{d}x$

4. $\int \tan^3 4x \mathrm{d}x$

5. $\int \sin 5x \cos 3x \mathrm{d}x$

6. $\int_0^1 \sqrt{1-x^2}\mathrm{d}x$

7. $\int \frac{\mathrm{d}x}{x^2+4x+8}$