MATH 2011 Introduction to Multivariable Calculus - Tutorial Note 4

1 Calculus of Vector-Valued Functions

1.1 Derivative and Tangent Vector

Definition: Given $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$,$f, g, h$ are differentiable on $t \in (a, b)$, then

$$\vec{r}^{'}(t) = f^{'}(t)\vec{i} + g^{'}(t)\vec{j} + h^{'}(t)\vec{k}$$

Unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r}^{'}(t)}{|\vec{r}^{'}(t)|}$$

Ex 3.1.1 Calcute the derivate of the functions:
a. $\vec{r}(t) = e^t sint\vec{i} + e^t cost\vec{j} - e^{2t}\vec{k}$
b. $\vec{r}(t) = (tlnt)\vec{i} + (5e^t)\vec{j} + (cost - sint)\vec{k}$

Solution: **a.** $\vec{r}^{'}(t) = e^t(sint + cost)\vec{i} + e^t(cost - sint)\vec{j} - 2e^{2t}\vec{k}$
b. $\vec{r}^{'}(t) = (1 + lnt)\vec{i} + 5e^t\vec{j} - (sint + cost)\vec{k}$.

Ex 3.1.2 Calcute the unit tangent vector of $\vec{r}(t) = t\vec{i} + e^t\vec{j} - 3t^2\vec{k}$

Solution: $\vec{T}(t) = \frac{\langle 1, e^t, -6t \rangle}{\sqrt{1+e^{2t}+36t^2}}$

1.2 Properties of Derivative

The derivative has the following properties:
(a) $\frac{d}{dt}(\vec{c}) = \vec{0}$
(b) $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}'(t) + \vec{v}'(t)$
(c) $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
(d) $\frac{d}{dt}(\vec{u}f(t)) = \vec{u}'(f(t))f'(t)$
(e) $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}'(t) \cdot \vec{v}(t) + \vec{u}(t) \cdot \vec{v}'(t)$
(f) $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t)$

Ex 3.2.1 Given the vector-valued functions

$$\vec{r}(t) = (6t + 8)\vec{i} + (4t^2 + 2t - 3)\vec{j} + 5t\vec{k}$$

$$\vec{u}(t) = (t^2 - 3)\vec{i} + (2t + 4)\vec{j} + (t^3 - 3t)\vec{k}$$

calculate each of the following derivatives:
a. $\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{u}(t)\right]$
b. $\frac{d}{dt}\left[\vec{u}(t) \times \vec{u}'(t)\right]$

Solution:

a. $\frac{d}{dt}\left[\vec{r}(t) \cdot \vec{u}(t)\right] = 20t^3 + 42t^2 + 26t - 16$
b. $\frac{d}{dt}\left[\vec{u}(t) \times \vec{u}'(t)\right] = (12t^2 + 24t)\vec{i} + (12t - 4t^3)\vec{j}\ (4t + 8)\vec{k}$

1.3 Integrals of Vector-Valued Functions

Definition: If the antiderivate of $\vec{r}(t)$ is $\vec{R}(t)$, then the indefinite integral of $\vec{r}(t)$ is

$$\int \vec{r}(t)dt = \vec{R}(t) + \vec{C}$$

where $\vec{C}$ is an arbitrary constant vector.

The definite integral of $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$ is

$$\int_a^b \vec{r}(t)dt = \int_a^b f(t)dt\vec{i} + \int_a^b g(t)dt\vec{j} + \int_a^b h(t)dt\vec{k}$$

where f, g, h are integrable on the interval $[a, b]$.

Ex 3.3.1 Calculate the following indefinite integrals:
a. $\int \left[(3t^2+2t)\vec{i}+(3t-6)\vec{j}+(6t^3+5t^2-4)\vec{k}\right] dt$
b. $\int [\langle t,t^2,t^3\rangle \times \langle t^3,t^2,t\rangle]\, dt$

Solution: **a.** $(t^3+t^2)\vec{i}+(\frac{3}{2}t^2-6t)\vec{j}+(\frac{3}{2}t^4+\frac{5}{3}t^3-4t)\vec{k}+\vec{C}$
b. $(\frac{t^4}{4}-\frac{t^6}{6})\vec{i}+(\frac{t^7}{7}-\frac{t^3}{3})\vec{j}+(\frac{t^4}{4}-\frac{t^6}{6})\vec{k}+\vec{C}$

Ex 3.3.1 Calculate the following definite integrals:
a. $\int_0^{\frac{\pi}{3}}(sin2t\vec{i}+tant\vec{j}+e^{-2t}\vec{k})dt$
b. $\int_1^3 \left[(2t+4)\vec{i}+(3t^2-4t)\vec{j}\right] dt$

Solution: **a.** $\frac{3}{4}\vec{i}+(ln2)\vec{j}+(\frac{1}{2}-\frac{1}{2}e^{-2\pi/3})\vec{k}$
b. $16\vec{i}+10\vec{j}$

2 Motion in Space: Velocity and Acceleration

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$, then the **velocity vector** $\mathbf{v}(t)$ at time t is:

$$\mathbf{v}(t)=\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}=\mathbf{r}'(t) \tag{1}$$

the velocity vector is also the tangent vector and points in the direction of the tangent line. The **speed** of the particle at time t is the magnitude of the velocity vector, that is $|\mathbf{v}(t)|$. The **acceleration** of the particle is defined as the derivative of the velocity:

$$\mathbf{a}(t)=\mathbf{v}'(t)=\mathbf{r}''(t). \tag{2}$$

In general, vector integrals allow us to recover velocity when acceleration is known and position when velocity is known:

$$\mathbf{v}(t)=\mathbf{v}(t_0)+\int_{t_0}^{t}\mathbf{a}(u)du \quad \mathbf{r}(t)=\mathbf{r}(t_0)+\int_{t_0}^{t}\mathbf{v}(u)du \tag{3}$$

Example 1. Find the velocity, acceleration, and speed of a particle with position vector.

1. $\mathbf{r}(t)=\sqrt{2}t\mathbf{i}+e^t\mathbf{j}+e^{-t}\mathbf{k}$
2. $\mathbf{r}(t)=<t^2,\sin t-t\cos t,\cos t+t\sin t>$

Solution:

1. velocity $\mathbf{v}(t) = \mathbf{r}'(t) =< \sqrt{2}, e^t, -e^{-t} >$; acceleration $\mathbf{a}(t) = \mathbf{v}'(t) =< 0, e^t, e^{-t} >$; speed $|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}}$.

2. velocity $\mathbf{v}(t) =< 2t, t\sin t, t\cos t >$; acceleration $\mathbf{a}(t) =< 2, \sin t + t\cos t, \cos t - t\sin t >$; speed $|\mathbf{v}(t)| = \sqrt{5}t$.

Example 2. Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

1. $\mathbf{a}(t) = 2\mathbf{i} + 2t\mathbf{k}, \quad \mathbf{v}(0) = 3\mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$

2. $\mathbf{a}(t) =< \sin t, 2\cos t, 6t >, \quad \mathbf{v}(0) =< 0, 0, -1 >, \mathbf{r}(0) =< 0, 1, -4 >$

Solution:

1. $\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{a}(u)du =< 3, -1, 0 > + \int_0^t < 2, 0, 2u > du =< 3+2t, -1, t^2 >$; $\mathbf{r}(t) = \mathbf{r}(0) + \int_0^t \mathbf{v}(u)du =< 0, 1, 1 > + \int_0^t < 3+2u, -1, u^2 > du =< 3t + t^2, 1 - t, 1 + 1/3t^3 >$;

2. $\mathbf{v}(t) =< 1 - \cos t, 2\sin t, 3t^2 - 1 >$; $\mathbf{r}(t) =< t - \sin t, 3 - 2\cos t, t^3 - t - 4 >$.

Example 3. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

Solution: Now we have $|\mathbf{v}(t)| = C$, where C is a constant.

1. $|\mathbf{v}(t)|^2 = \mathbf{v}(t) \cdot \mathbf{v}(t) = C^2$,

2. take derivative on both sides: $\frac{d}{dt}\{\mathbf{v}(t) \cdot \mathbf{v}(t)\} = 2\mathbf{v}'(t) \cdot \mathbf{v}(t) = 2\mathbf{a}(t) \cdot \mathbf{v}(t) = 0$. Thus in this case, the velocity and acceleration vectors are orthogonal.

3 Length of a Curve

The length of a plane curve with parametric equations $x = f(t), y = g(t), a \leq t \leq b$, for the case where f' and g' are continuous, is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt = \int_a^b |\mathbf{r}'(t)|dt \qquad (4)$$

the length of a plane curve with parametric equations $x = f(t), y = g(t), z = h(t), a \leq t \leq b$, for the case where f', g' and h' are continuous, is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b |\mathbf{r}'(t)| dt \tag{5}$$

For a polar curve $r = f(\theta)$, we have $x = r\cos\theta = f(\theta)\cos\theta, y = r\sin\theta = f(\theta)\sin\theta$. Then in the interval $\theta \in [\alpha, \beta]$, the length is

$$L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2} d\theta.$$

Example 4. Find the length of the curves.

1. $\mathbf{r}_1(t) =< t, 3\cos t, 3\sin t >, \quad -4 \leq t \leq 4,$
2. $\mathbf{r}_2(u) =< 2u, 3\cos(2u), 3\sin(2u) >, \quad -2 \leq u \leq 2.$

Solution:

1. $L = \int_{-4}^4 |\mathbf{r}'(t)| dt = \int_{-4}^4 |< 1, -3\sin t, 3\cos t >| = \int_{-4}^4 \sqrt{10} dt = 8\sqrt{10},$
2. $L = \int_{-2}^2 |\mathbf{r}'(u)| du = \int_{-2}^2 |< 2, -6\sin(2u), 6\cos(2u) >| = \int_{-2}^2 2\sqrt{10} dt = 8\sqrt{10},$

Remark 1. Where the connection between the parameters t and u is given by $t = 2u$. If we were to use these two ways to compute the length of C, we would get the same answer. In general, it can be shown that when (4) or (5) is used to compute arc length, the answer is independent of the parametrization that is used.

4 The Arc Length Function

Suppose that C is a curve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}, \quad a \leq t \leq b$$

where $\mathbf{r}'(t)$ is continuous and C is traversed exactly once as t increases from a to b. We define its **arc length function** s by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du \tag{6}$$

Thus $s(t)$ is the length of the part of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. If we differentiate both sides of (6) using the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)|$$

It is often useful to parametrize a curve with respect to arc length because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system.

Example 5. (a) Find the arc length function for the curve measured from the point P in the direction of increasing t and then reparametrize the curve with respect to arc length starting from P.

(b) find the point 4 units along the curve (in the direction of increasing t) from P.

1. $\mathbf{r}(t) = (5-t)\mathbf{i} + (4t-3)\mathbf{j} + 3t\mathbf{k}, \quad P(4,1,3),$
2. $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} + \sqrt{2}e^t\mathbf{k}, \quad P(0,1,\sqrt{2}).$

Solution:

1. $P(4,1,3)$ corresponds to the parameter $t = 1$.
   $s(t) = \int_1^t |\mathbf{r}'(u)|du = \int_1^t \sqrt{(-1)^2+4^2+3^2}du = \sqrt{26}t - \sqrt{26}.$
   Solve t from the above equation: $t = \frac{s}{\sqrt{26}} + 1$, then substitute into the original equation, we reparametrize the curve with respect to arc length starting from P:
   $\mathbf{r}(s) = (4 - \frac{s}{\sqrt{26}})\mathbf{i} + (1 + \frac{4s}{\sqrt{26}})\mathbf{j} + (\frac{3s}{\sqrt{26}} + 3)\mathbf{k}.$
   Set $s = 4$, we obtain the point that 4 units along the curve from P:$(4 - \frac{4}{\sqrt{26}}, 1 + \frac{16}{\sqrt{26}}, \frac{12}{\sqrt{26}} + 3)$.
2. $P(0,1,\sqrt{2})$ corresponds to the parameter $t = 0$.
   $s(t) = \int_0^t 2e^u du = 2e^t - 2.$
   $t = \ln\frac{s+2}{2}$, $\mathbf{r}(s) =< \frac{s+2}{2}\sin(\ln\frac{s+2}{2}), \frac{s+2}{2}\cos(\ln\frac{s+2}{2}), \sqrt{2}\frac{s+2}{2} >$. Set $s = 4$, we obtain a point:$(3\sin(\ln 3), 3\cos(\ln 3), 3\sqrt{2})$.