Tutorial 02: Vectors

Def 1: Vectors are defined by magnitude and direction.

$u = \langle u_1, u_2 \rangle$ in the plane

$u = \langle u_1, u_2, u_3 \rangle$ in the space.

Def 2: $u.v$ are parallel if $\exists$ const k s.t. $u = kv$.

$u.v$ in same direction if $k > 0$

opposite direction if $k < 0$

Formula 3: magnitude of u $= \sqrt{u_1^2 + u_2^2 + u_3^2}$

Formula 4: The unit vector in the direction of u is $\frac{u}{|u|}$

Def 5: A plane Γ in space is defined by a point $P = (p_1, p_2, p_3)$ on Γ and its normal vector $n = (n_1, n_2, n_3)$.

$\Gamma: \; n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) = 0$

Def 6: A line ℓ in space is defined by a point $P = (p_1, p_2, p_3)$ on ℓ. and a vector $v = \langle v_1, v_2, v_3 \rangle$ which shows its direction.

$\ell: \; r = (x, y, z) = (p_1 + v_1 t, \, p_2 + v_2 t, \, p_3 + v_3 t)$

Dot Product & Cross Product

Def 7: $u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$

Thm 8: $u \cdot v = |u| \cdot |v| \cdot \cos\theta$

u and v are orthogonal iff $u \cdot v = 0$

Formula 9: The orthogonal projection of u onto v is

$$\mathrm{proj}_v u = \frac{u \cdot v}{|v|^2} v$$

Formula 10: The orthogonal projection of u onto a plane span$\{v, w\}$ is $\frac{u \cdot v}{|v|^2} \cdot v + \frac{u \cdot w}{|w|^2} \cdot w$

Def 11:

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2) i + (u_3 v_1 - u_1 v_3) j + (u_1 v_2 - u_2 v_1) k$$

Note $u \cdot v$ is a scalar, $u \times v$ is a vector

Formula 12 $|u \times v| = |u| \cdot |v| \cdot \sin\theta$

$u \times v$ is orthogonal to both u and v.

Examples:

Ex 1 Let $v = \langle 1, 2, 3 \rangle$, $w = \langle 2, -1, -2 \rangle$.

Find the orthogonal proj of v on w, and v on $v \times w$

① $\mathrm{proj}_w v = \frac{v \cdot w}{|w|^2} \cdot v$

② $r = v \times w$

$\mathrm{proj}_r v = \frac{v \cdot r}{|r|^2} \cdot v$

Ex 2 Show that if $u \cdot v = 0$, and $u \times v = 0$, then u or v is 0

$u \cdot v = |u| \cdot |v| \cdot \cos\theta = 0$

$u \times v = |u| \cdot |v| \cdot \sin\theta = 0$

Since $\cos\theta$, $\sin\theta$ cannot be 0 at the same time

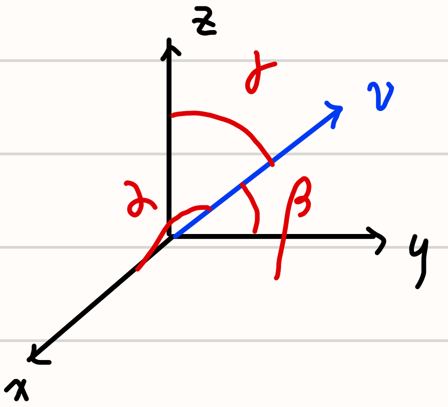
$\Rightarrow |u| \cdot |v| = 0$

$\Rightarrow |u| = 0$ or $|v| = 0$

$\Rightarrow u = 0$ or $v = 0$

Ex 3 The direction angles of a nonzero vector v are angles α, β, γ $(\in [0, \pi])$ that v makes with positive x, y, z-axis.

Find the direction angles of $v = \langle 1, 2, 3 \rangle$



e_x: unit vector on direction of x-axis

e_y, e_z: ....

$v = \langle x, y, z \rangle$

$|v| = \sqrt{x^2 + y^2 + z^2}$

$$\cos\alpha = \frac{v \cdot e_x}{|v|} = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$\cos\beta = \frac{v \cdot e_y}{|v|} = \frac{y}{\sqrt{x^2+y^2+z^2}}$$

$$\cos\gamma = \frac{v \cdot e_z}{|v|} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$\alpha = \arccos\left(\frac{x}{|v|}\right)$. $\beta = \arccos\left(\frac{y}{|v|}\right)$ $\gamma = \arccos\left(\frac{z}{|v|}\right)$

Note: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

Ex 4 Let Γ be the plane that goes through $P=(3,1,-1)$ with normal vector $n=(1,2,1)$

ℓ be line goes through $Q=(0,0,1)$ in direction of $v=(0,1,0)$

(a) Find min distance between Γ and Q.

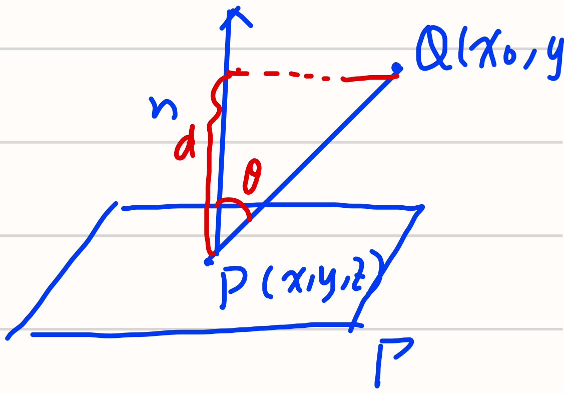
(b) Find min distance between Γ and ℓ.

(a) Given a plane $\Gamma: Ax+By+Cz+D=0$

$(\Gamma: (x-3)+2(y-1)+(z+1)=0 \Rightarrow x+2y+z-4=0)$

$Q: (x_0, y_0, z_0)$ outside Γ

$P: (x,y,z)$ on Γ



$d = |PQ| \cdot \cos\theta$

$= \dfrac{|n|}{|n|} \cdot |PQ| \cdot \cos\theta$ $= PQ \cdot n$

$= \dfrac{PQ \cdot n}{|n|}$

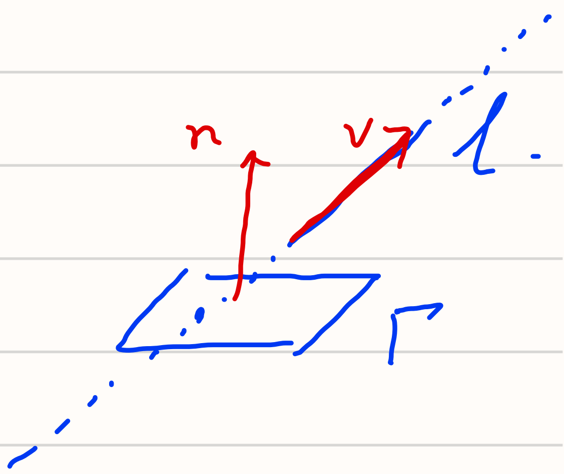
$= \left| \dfrac{A(x_0-x)+B(y_0-y)+C(z_0-z)}{\sqrt{A^2+B^2+C^2}} \right|$

$= \left| \dfrac{Ax_0+By_0+Cz_0-(Ax+By+Cz)}{\sqrt{A^2+B^2+C^2}} \right|$

P on Γ

$Ax+By+Cz+D=0$

$= \left| \dfrac{Ax_0+By_0+Cz_0+D}{\sqrt{A^2+B^2+C^2}} \right|$



(b) If ℓ is not parallel to Γ

The ℓ will cross Γ

$\Rightarrow$ min distance is 0

If $\ell \parallel \Gamma$ Then $n \perp v$. $\Rightarrow n \cdot v = 0$

Here, $n = \langle 1,2,1 \rangle$ $v = \langle 0,1,0 \rangle$ $\Rightarrow n \cdot v \neq 0$

$\Rightarrow \ell$ not parallel to Γ.

$\Rightarrow \min(d) = 0$