Math 2011 Tutorial 01 (13/09 & 15/09)

TA: Liu Yonglin 劉泳麟
Office : Rm 4381
Zoom : 437 935 7440 (Password: 000000).

1. Parametric and Polar Curves.

1.1. Curves defined by Parametric Eq.

x, y given as functions of a third variable t (parameter)

$$x = f(t) \qquad y = g(t)$$

each t determines $(x,y) = (f(t), g(t))$

t varies $\rightarrow$ Curve C. (parameter curve)

Remark A particle whose position is given by the parameter eqs. moves along the curve in the direction of the arrow as t increases.

Ex 1 Eliminate param to find a Cartesian eq of curve

$$x = t^2 - 3 \qquad t \in [-3, 3]$$
$$y = t + 2$$

$t = y - 2$

$-3 \le t \le 3 \Rightarrow -1 \le y \le 5$

$x = (y-2)^2 - 3$

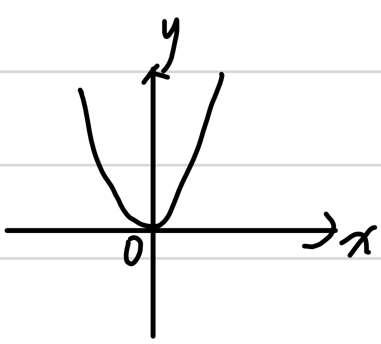
Examples of parametric curve in 2-D

① Circle $\quad x^2 + y^2 = r^2$

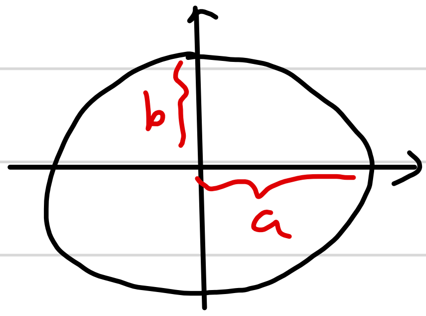
$$x = r\cos t$$
$$y = r\sin t \qquad t \in [0, 2\pi)$$

② Parabola $\quad y = x^2$

$$x = t$$
$$y = t^2 \qquad t \in (-\infty, +\infty)$$



③ Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

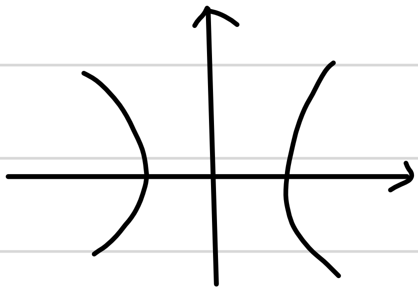


$x = a\cos t$

$y = b\sin t \qquad t \in [0, 2\pi)$

④ Hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$x = a\sec t$

$y = b\tan t \qquad t \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

Ex 2. Sketch the curve and indicate its direction.

$x = \frac{1}{2}\cos\theta$ $\qquad \underline{\theta \in [0, \pi]}$

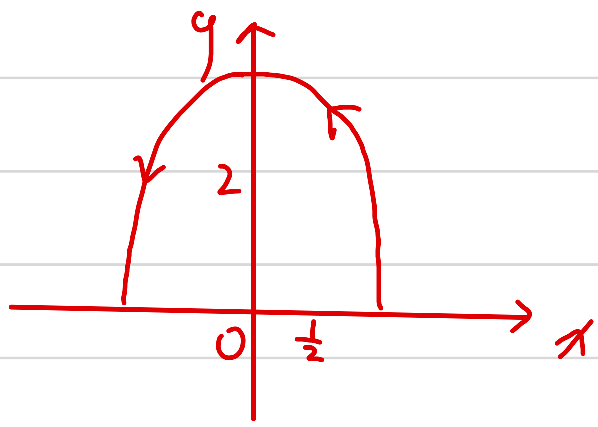
$y = 2\sin\theta$

$\cos^2\theta + \sin^2\theta = 1$

$2x = \cos\theta$

$\frac{1}{2}y = \sin\theta$

$\Rightarrow 4x^2 + \frac{1}{4}y^2 = 1 \qquad \Rightarrow \frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{2^2} = 1$



1.2 Derivatives for Parametric Curves

$f \cdot g$ differentiabe & continuous function

want to find tangent line of curve $(x, y) = (f(t), g(t))$.

(y is also a differentiable function of x)

By Chain Rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

If $\frac{dx}{dt} \neq 0$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$ (slope of tangent line)

If $\frac{dx}{dt} = 0$

The tangent line is a vertical line

Ex3. Find $\frac{dy}{dx}$ for $x = te^t$, $y = t + \sin t$

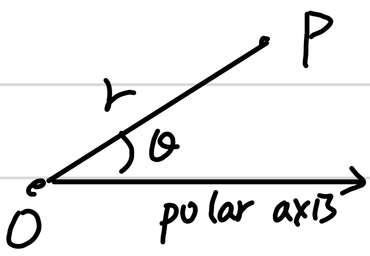
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+\cos t}{e^t + te^t}$$

1.3 Polar Coordinates

origin / pole: O

polar axis

r, θ.

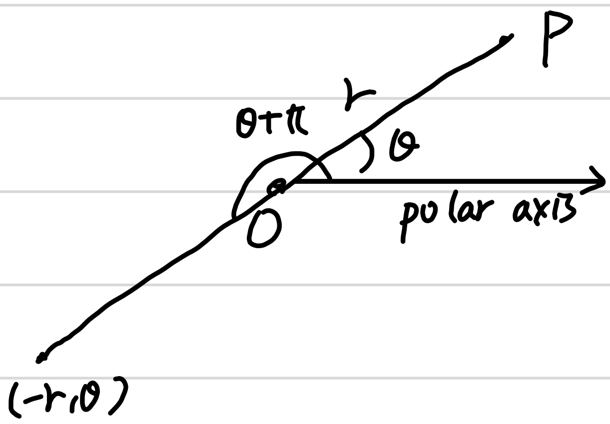


P is represented by (r, θ) polar coordinates of P.

$\theta > 0$ ↺

$\theta < 0$ ↻

$(-r, \theta) = (r, \theta + \pi)$



Polar & Cartesian

(r, θ) (x, y)

$x = r\cos\theta$

$y = r\sin\theta$

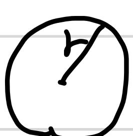
$\theta = \arccos \frac{x}{\sqrt{x^2+y^2}} = \arcsin \frac{y}{\sqrt{x^2+y^2}}$

$\cos\theta = \frac{x}{r}$ $r = \sqrt{x^2+y^2}$

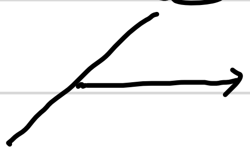
$\sin\theta = \frac{y}{r}$

Basic curves

① $r = a$



② $\theta = \theta_0$



Symmetry

① replace θ by $-\theta$ symmetric about polar axis

② r by $-r$ pole

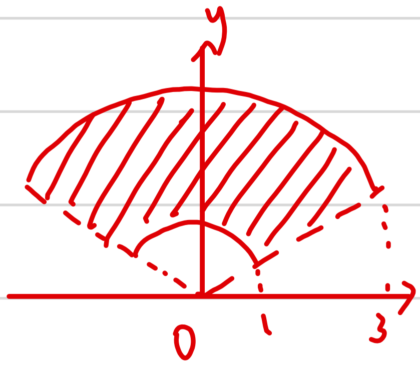
θ by $\theta+\pi$ pole

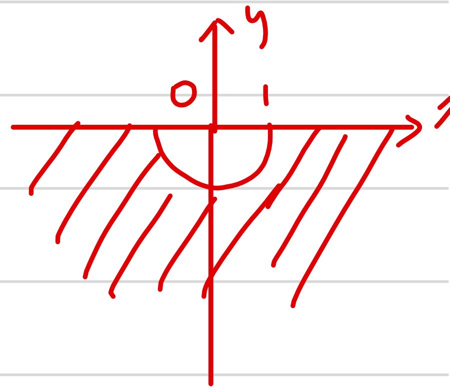
③ θ by $\pi-\theta$ $\theta = \frac{\pi}{2}$

Ex 4 Sketch region

(a) $1 \le r \le 3$. $\frac{\pi}{6} < \theta < \frac{5\pi}{6}$

(b) $r \ge 1$ $\pi \le \theta \le 2\pi$





Ex 5 Find Cartesian eq. of the curve

(a) $r = 5\cos\theta$

(b) $\theta = \frac{\pi}{3}$

(c) $r = a\sin\theta + b\cos\theta$ $ab \neq 0$

(a) $r^2 = 5r\cos\theta$ $\Rightarrow$ $x^2+y^2=5x$

(b) $y=\sqrt{3}x$

(c) $x^2+y^2=ay+bx$

2. Calculus in Polar Coordinates

2.1 Slope of tangent lines

$r=f(\theta)$

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

2.2 Area of Regions bounded by Polar Coordinates

R bounded by $r = f(\theta)$, $r = g(\theta)$ $\qquad f \geq g \geq 0$ in $[\alpha, \beta]$

$\theta = \alpha$, $\theta = \beta$

Area of R :

$$\int_\alpha^\beta \frac{1}{2} [f^2(\theta) - g^2(\theta)] \, d\theta$$