Math 2011 Final Exam

May 24, 2012

Time: 16:30-19:30

Suggested Solutions

Your Name ____________________

Student Number ____________________

Section Number ____________________

1. This is an open-book exam, but don't waste your time to flip the pages of your textbook.
2. Your calculator can be used, but probably is not helpful.
3. Answer Part I and II in the multiple choice answer sheet provided. Each question in Part I and II is worth 1.5 points. No point will be deducted for any wrong answer.
4. Provide all the details for Part III only. If your answer is too complicated, you must have made a mistake.

Number	Score
Part I	
Part II	
Part III	
Total	

Part I: True or False Questions (15 pts)

Determine whether each of the following statement is true or false (Option "A" = true. Option "B" = false).

1. An integral of one variable is an example of a line integral. A
2. An integral of two variables is an example of a surface integral. A
3. A line integral must be an integral over a straight line. B
4. A surface integral must be an integral over a surface. A
5. To evaluate a line integral, one needs to choose a parametrization for the curve and then turn the line integral into an integral of one variable. A
6. To evaluate a surface integral, one needs to choose a parametrization for the surface and then turn the surface integral into an integral of one variable. B
7. To evaluate an integral of two variables, one needs to compute two integrals of one variable. A
8. There are two types of surface integrals. A
9. An integral over a surface always depends on the orientation of the surface. B
10. An integral over a line always depends on the orientation of the curve. B

Part II: Multiple Choice Questions (15 pts)

11. Which of the following expressions do not make sense as a double integral? A

(I) $\int_0^1 \int_0^x f(x,y)\,dx\,dy$ (II) $\int_0^1 \int_0^y f(x,y)\,dx\,dy$

(III) $\int_0^1 \int_0^x f(x,y)\,dy\,dx$ (IV) $\int_0^y \int_0^1 f(x,y)\,dx\,dy$

a) (I) and (IV) only, b) (II) and (III) only, c) (I) and (II) only, d) (III) and (IV) only

12. Which of the following expressions make/makes sense as a triple integral? B

(I) $\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dx\,dy\,dz$ (II) $\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dy\,dx\,dz$

(III) $\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dx\,dz\,dy$ (IV) $\int_0^1 \int_0^z \int_0^y f(x,y,z)\,dx\,dy\,dz$

a) (II) only, b) (IV) only, c) (I) and (II) only, d) (III) and (IV) only

13. Let C be an unorieneted curve in $\mathbb{R}^3$ and $\vec{F}$ be a vector field on $\mathbb{R}^3$. Which of the following expressions make/makes sense as a line integral? C

(I) $\int_C \vec{F} \cdot d\vec{r}$ (II) $\int_C \vec{F} \cdot d\vec{S}$

(III) $\int_C (\vec{F} \cdot \vec{F})\,ds$ (IV) $\int_C (\vec{F} \cdot \vec{F})\,dS$

a) (I) only, b) (II), c) (III) only, d) (II) and (IV) only

14. Let Σ be an oriented surface in $\mathbb{R}^3$ and $\vec{F}$ be a vector field on $\mathbb{R}^3$. Which of the following expressions make sense as a surface integral? D

(I) $\iint_\Sigma \vec{F} \cdot d\vec{r}$ (II) $\iint_\Sigma \vec{F} \cdot d\vec{S}$

(III) $\iint_\Sigma (\vec{F} \cdot \vec{F})\, dV$ (IV) $\iint_\Sigma (\vec{F} \cdot \vec{F})\, dS$

a) (I) and (II) only, b) (II) and (III) only, c) (III) and (IV) only, d) (II) and (IV) only

15. Integral $\int_0^1 \int_0^y f(x,y)\, dx\, dy$ is equal to C

a) $\int_0^1 \int_0^y f(x,y)\, dy\, dx$, b) $\int_0^1 \int_0^{1-x} f(x,y)\, dy\, dx$, c) $\int_0^1 \int_x^1 f(x,y)\, dy\, dx$, d) $\int_0^1 \int_0^x f(x,y)\, dy\, dx$.

16. Let D be the unit disk centered at $(0,0)$. Integral $\iint_D f(x,y)\, dA$ is equal to C

a) $\int_0^1 \int_0^{2\pi} f(x,y)\, dx\, dy$, b) $\int_0^1 \int_0^{2\pi} f(r\cos\theta, r\sin\theta) d\theta\, dr$, c) $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\, dx\, dy$,
d) $\int_{-1}^1 \int_{-1}^1 f(x,y)\, dx\, dy$.

17. Let G be the solid region in $\mathbb{R}^3$ bounded by three coordinate planes and the plane $x+y+z=1$. Integral $\iiint_G f(x,y,z)\, dV$ is equal to B

a) $\int_0^1 \int_0^1 \int_0^1 f(x,y,z)\, dx\, dy\, dz$, b) $\int_0^1 \int_0^{1-z} \int_0^{1-y-z} f(x,y,z)\, dx\, dy\, dz$, c) $\int_0^1 \int_0^{1-z} \int_0^{1-y} f(x,y,z)\, dx\, dy\, dz$,
d) $\int_0^1 \int_0^1 \int_0^{1-y} f(x,y,z)\, dx\, dy\, dz$.

18. Let $\vec{F}$ be a vector field on $\mathbb{R}^3$. Then A

a) $\nabla \cdot (\nabla \times \vec{F}) = 0$, b) $\nabla \times (\nabla \times \vec{F}) = \vec{0}$, c) $\nabla(\nabla \cdot \vec{F}) = \vec{0}$, d) none of the above is correct.

19. Let ϕ be a function on $\mathbb{R}^3$. Then B

a) $\nabla \cdot (\nabla\phi) = 0$, b) $\nabla \times (\nabla\phi) = 0$, c) $\nabla\phi \cdot \nabla\phi = 0$, d) none of the above is correct.

20. Recall that $\vec{r} = \langle x, y, z\rangle$ and $r = \sqrt{x^2+y^2+z^2}$. Which of the following must be true? D

(I) $\nabla \cdot \vec{r} = 1$ (II) $\nabla \times \vec{r} = \vec{0}$

(III) $\nabla r = \frac{1}{r}\vec{r}$ (IV) $\nabla \cdot \vec{r} = 3$

a) (I) and (II) only, b) (II) and (III) only, c) (I), (II) and (III) only,
d) (II), (III) and (IV) only.

Part III: Long Questions (70 pts)

1. [13 pts] Let S_1 be the cylinder $(x-1)^2+y^2=1$ and S_2 be the cone $x=\sqrt{y^2+z^2}$. Let C be the part of the intersecting curve C of S_1 and S_2 that lies in the first octant i.e. the portion of the rectangular coordinate system in which all three variables are nonnegative.

   (a) Find a parametrization $r(t)=\langle x(t),y(t),z(t)\rangle$ of C. (Hint: Set $x=t$ and express y and z in terms of t. Then specify the range of values of t). [5 pts]

   (b) Using the parametrization in (a), or otherwise, find the parametric equation of the tangent line of C at the point $(\frac{3}{2},\frac{\sqrt{3}}{2},\sqrt{\frac{3}{2}})$. [4 pts]

   (c) Find two normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ of S_1 and S_2 respectively at the point $(\frac{3}{2},\frac{\sqrt{3}}{2},\sqrt{\frac{3}{2}})$. [4 pts]

(a) let $x=t$, $(t-1)^2+y^2=1$

$$\Rightarrow y=\sqrt{1-(t-1)^2}=\sqrt{2t-t^2}$$

$$t^2=y^2+z^2$$

$$\Rightarrow z=\sqrt{t^2-(2t-t^2)}=\sqrt{2t^2-2t}$$

$2t-t^2\geq 0 \Rightarrow t(2-t)\geq 0 \Rightarrow 0\leq t\leq 2$

$2t^2-2t\geq 0 \Rightarrow 2t(t-1)\geq 0 \Rightarrow t\leq 0$ or $t\geq 1$

$\therefore$ $r(t)=\langle t,\sqrt{2t-t^2},\sqrt{2t^2-2t}\rangle$ where $1\leq t\leq 2$

(b) $r'(t)=\langle 1,\dfrac{1-t}{\sqrt{2t-t^2}},\dfrac{2t-1}{\sqrt{2t^2-2t}}\rangle$

$r'(\frac{3}{2})=\langle 1,-\frac{\sqrt{3}}{3},\frac{2\sqrt{6}}{3}\rangle$

$\therefore$ The parametric equation of the required tangent line:

$$\begin{cases} x=\frac{3}{2}+s \\ y=\frac{\sqrt{3}}{2}+(-\frac{\sqrt{3}}{3})s \\ z=\sqrt{\frac{3}{2}}+\frac{2\sqrt{6}}{3}s \end{cases} \qquad s\in\mathbb{R}$$

(c) Let $F(x, y, z) = (x-1)^2 + y^2 - 1$

and $G(x, y, z) = x^2 - y^2 - z^2$

So S_1 is a level surface of F

S_2 is a level surface of G

$$\therefore \quad n_1 = \nabla F\left(\frac{3}{2}, \frac{\sqrt{3}}{2}, \sqrt{\frac{3}{2}}\right)$$

$$= \langle 2(x-1), 2y, 0 \rangle \Big|_{\left(\frac{3}{2}, \frac{\sqrt{3}}{2}, \sqrt{\frac{3}{2}}\right)}$$

$$= \langle 1, \sqrt{3}, 0 \rangle$$

$$n_2 = \nabla G\left(\frac{3}{2}, \frac{\sqrt{3}}{2}, \sqrt{\frac{3}{2}}\right)$$

$$= \langle 2x, -2y, -2z \rangle \Big|_{\left(\frac{3}{2}, \frac{\sqrt{3}}{2}, \sqrt{\frac{3}{2}}\right)}$$

$$= \langle 3, -\sqrt{3}, -\sqrt{6} \rangle$$

2. [14 pts] Let L be the part of the plane $x + y + z = 4$ that lies in the first octant i.e. the portion of the rectangular coordinate system in which all three variables are nonnegative. Given $P = (4, 3, 1)$. Let $f(x, y, z)$ be the square of the distance of any point (x, y, z) on L from P.

   (a) Show that f can be expressed in terms of x and y only as follows:

   $$f(x, y) = 2x^2 + 2xy - 14x + 2y^2 - 12y + 34.$$

   [3 pts]

   (b) Describe the domain D of $f(x, y)$. [2 pts]

   (c) Find the critical point(s) of $f(x, y)$ if exists. [3 pts]

   (d) Find the minimum value of $f(x, y)$ on the boundary of D. [4 pts]

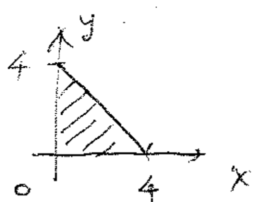
   (e) Using (c) and (d), or otherwise, find the minimum distance from L to P. [2 pts]

(a) $f(x,y) = (x-4)^2 + (y-3)^2 + (4-x-y-1)^2$

$= x^2 - 8x + 16 + y^2 - 6y + 9 + 9 + x^2 + y^2 - 6x - 6y + 2xy$

$= 2x^2 + 2xy - 14x + 2y^2 - 12y + 34$

(b)



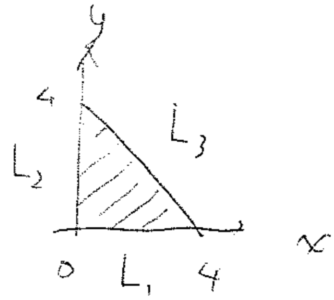
$D = \{ x \geq 0,\ y \geq 0,\ x + y \leq 4 \}$

(c) $\left.\begin{aligned} f_x &= 4x + 2y - 14 = 0 \\ f_y &= 2x + 4y - 12 = 0 \end{aligned}\right\} \Rightarrow x = \frac{8}{3},\ y = \frac{5}{3}$

But $\frac{8}{3} + \frac{5}{3} = \frac{13}{3} > 4 \quad \therefore \left(\frac{8}{3}, \frac{5}{3}\right) \notin D$

There is no critical point.

(d)



On L_1 : $(t, 0)$, $0 \le t \le 4$

$$f(t,0) = 2t^2 - 14t + 34$$
$$= 2\left(x - \frac{7}{2}\right)^2 + \frac{19}{2}$$

$\therefore$ Minimum Value $= \frac{19}{2}$.

On L_2 : $(0, t)$, $0 \le t \le 4$

$$f(0,t) = 2t^2 - 12t + 34$$
$$= 2(t-3)^2 + 16$$

$\therefore$ Minimum value $= 16$

On L_3 : $(t, 4-t)$, $0 \le t \le 4$

$$f(t, 4-t) = (t-4)^2 + (1-t)^2 + 1$$
$$= 2t^2 - 10t + 18$$
$$= 2\left(t - \frac{5}{2}\right)^2 + \frac{11}{2}$$

$\therefore$ Minimum Value $= \frac{11}{2}$.

Therefore, the minimum value of f on the boundary of D is $\frac{11}{2}$

(e) The minimum distance $= \sqrt{\frac{11}{2}} = \frac{\sqrt{22}}{2}$

3. [7 pts] Let S be the sphere $x^2 + y^2 + z^2 = 1$.

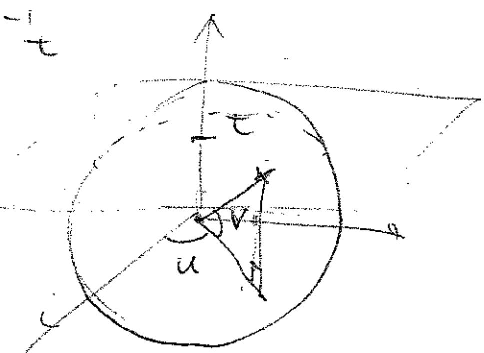
   (a) Show that the surface area of the part of the sphere below the plane $z = t$, where $-1 \leq t \leq 1$, is $2\pi(1+t)$. [5 pts]

   (b) Using (a), or otherwise, show that the surface area of the part of the sphere S between two plane $z = a$ and $z = b$, where $-1 \leq a < b \leq 1$, is $2\pi(b-a)$. (That is to say, the surface area depends only on the distance between two planes.) [2 pts]

(a) parametrize S :

$$r(u,v) = \langle \cos v \cos u, \cos v \sin u, \sin v \rangle$$

$$0 \leq u \leq 2\pi, \quad -\frac{\pi}{2} \leq v \leq \sin^{-1} t$$



$$\frac{\partial r}{\partial u} = \langle -\cos v \sin u, \cos v \cos u, 0 \rangle$$

$$\frac{\partial r}{\partial v} = \langle -\sin v \cos u, -\sin v \sin u, \cos v \rangle$$

$$\left| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right| = \cos v$$

$$\therefore \text{Surface area} = \int_{-\frac{\pi}{2}}^{\sin^{-1} t} \int_{0}^{2\pi} (\cos v)\, du\, dv$$

$$= 2\pi (\sin v) \Big|_{-\frac{\pi}{2}}^{\sin^{-1} t}$$

$$= \underline{\underline{2\pi(t+1)}}$$

(b) By (a)

The surface area of S between $z = a$ and $z = b$

$$= 2\pi(1+b) - 2\pi(1+a)$$

$$= \underline{\underline{2\pi(b-a)}}$$

4. [12 pts] Consider the following triple integrals:

(a) Let $I_1 = \iiint_{G_1} xz\, dV$, where G_1 is the region that lies inside the sphere $x^2 + y^2 + z^2 = 4$, above the plane $z = 0$ and below the cone $z = \sqrt{3x^2 + 3y^2}$. Express I_1 as an iterated integral in spherical coordinates. You are NOT required to evaluate the iterated integral.

[4 pts]

(b) Let $I_2 = \iiint_{G_2} xy\, dV$, where G_2 is the region that lies inside the cylinder $x^2 + y^2 = 9$ and is bounded by the plane $z = 0$ and the paraboloid $z = 4 - x^2 - y^2$. Express I_2 as an iterated integral in cylindrical coordinates. You are NOT required to evaluate the iterated integral.

[4 pts]

(c) Let $I_3 = \iiint_{G_3} yz\, dV$, where G_3 is the region that is bounded by the parabolic cylinder $y = x^2$, the planes $z = 0$, $x = y$ and the sphere $x^2 + y^2 + z^2 = 4$. Express I_3 as an iterated integral in rectangular coordinates. You are NOT required to evaluate the iterated integral.

[4 pts]

4. (a)

$$I_1 = \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^2 (\rho\sin\phi\cos\theta)(\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\int_0^2 \rho^4\sin^2\phi\cos\theta\cos\phi\, d\rho\, d\phi\, d\theta$$

(b)

$$I_2 = \int_0^{2\pi}\int_0^3\int_0^{16-r^2} (r\cos\theta)(r\sin\theta)\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3\int_0^{16-r^2} r^3\cos\theta\sin\theta\, dz\, dr\, d\theta$$

(c) CANCELLED

5. [12 pts] Let $\vec{F}(x,y,z) = \langle yz, zx, xy \rangle$ be a vector field on $\mathbb{R}^3$.

(a) Is $\vec{F}$ a conservative vector field? If yes, find a potential function for $\vec{F}$, i.e., a function ϕ such that $\vec{F} = \nabla\phi$ (i.e., $\vec{F} \cdot d\vec{r} = d\phi$).

[4pts]

(b) Compute the line integral

$$W = \int_C \vec{F} \cdot d\vec{r}\,,$$

here C is an oriented space curve with starting point P on the xy-coordinate plane and ending point Q on the yz-coordinate plane.

[4pts]

(c) Compute the surface integral

$$W = \iint_\Sigma \vec{F} \cdot d\vec{S}\,,$$

here Σ is a sphere centered at the origin of $\mathbb{R}^3$, oriented outward.

[4pts]

(a) $\text{Curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} = i(x-x) - j(y-y) + k(z-z) = \vec{0}$

$\therefore \vec{F}$ is a conservative vector field.

$\phi(x,y,z) = xyz$

(b) Assume $P = (p_1, p_2, 0)$ and $Q = (0, q_1, q_2)$

Since $\vec{F}$ is conservative, its line integral is path independent

$\therefore W = \int_C \vec{F} \cdot d\vec{r} = \phi(0, q_1, q_2) - \phi(p_1, p_2, 0) = 0$

(c) $\text{div } \vec{F} = \frac{\partial(yz)}{\partial x} + \frac{\partial(zx)}{\partial y} + \frac{\partial(xy)}{\partial z} = 0$

Divergence Thm: $W = \iint_\Sigma \vec{F} \cdot d\vec{S} = \iiint_G \text{div } \vec{F} \, dV = 0$

where G is the solid ball such that $\partial G = \Sigma$

6. [12 pts] Let $\vec{F}(x,y) = \langle f(x,y), g(x,y)\rangle = \langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}\rangle$ be a vector field on $U = \mathbb{R}^2 \setminus \{(0,0)\}$, C be one of the following three closed oriented curves on $\mathbb{R}^2$:

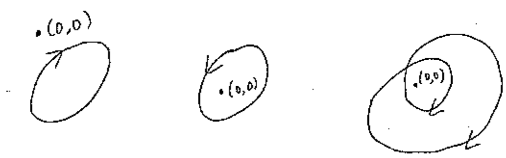


Fig. 1 (Clockwise) Fig. 2 Fig. 3

and $W = \int_C \vec{F}\cdot d\vec{r}$.

(a) Compute $\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}$. ~~[4pts]~~ 3 pts

(b) Compute W if C is the oriented curve in Fig. 1. ~~[4pts]~~ 3 pts

(c) Compute W if C is the oriented curve in Fig. 2. ~~[4pts]~~ 3 pts

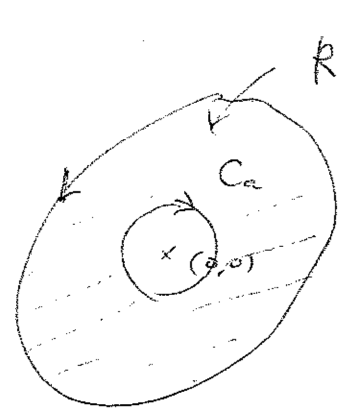
(d) Compute W if C is the oriented curve in Fig. 3. [3pts]

(a) $$\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x} = \frac{(x^2+y^2) - 2y\cdot y}{(x^2+y^2)^2} + \frac{(x^2+y^2) - 2\cdot x\cdot x}{(x^2+y^2)^2} = 0$$

(b) Green's Thm : $$\oint_{-C} \vec{F}\cdot d\vec{r} = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = 0$$

$$\therefore\ W = \oint_C \vec{F}\cdot d\vec{r} = -\oint_{-C} \vec{F}\cdot d\vec{r} = 0$$

(c) C_a is a circle centered at $(0,0)$ with radius a



$$\int_{C+C_a} \vec{F}\cdot d\vec{r} = \iint_{R'} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA = 0$$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = -\int_{C_a} \vec{F}\cdot d\vec{r} = \int_{-C_a} \vec{F}\cdot d\vec{r}$$

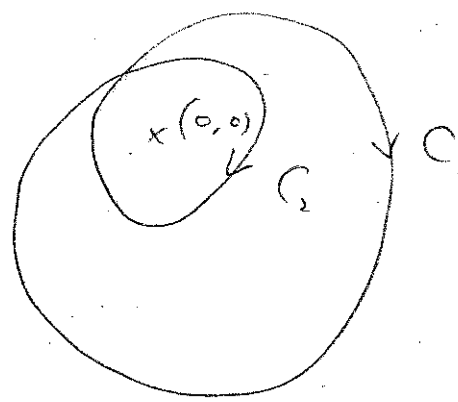
$-C_a : \langle a\cos t, a\sin t\rangle \quad 0 \le t \le 2\pi$

$$\therefore \int_{-C_a} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left(\frac{a\sin t}{a^2}(-a\sin t) + \frac{-a\cos t}{a^2}(a\cos t)\right) dt$$

$$= \int_0^{2\pi} -1\, dt = -2\pi$$

$$\therefore W = \int_C \vec{F}\cdot d\vec{r} = \underline{\underline{-2\pi}}$$

(d) $\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$



By part (c)

$$\int_{-C_1} \vec{F}\cdot d\vec{r} = \int_{-C_2} \vec{F}\cdot d\vec{r} = -2\pi$$

$$\therefore \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = 2\pi$$

Hence $W = \int_C \vec{F}\cdot d\vec{r} = 2\pi + 2\pi = \underline{\underline{4\pi}}$

*** END OF PAPER ***