1. (20 pts)

(a) Consider a point on the surface of $z = x^2 + \frac{1}{4}y^2 - 1$. The plane tangent to this surface at this point is parallel to the plane $2x + y + z = 0$. Find this point and give the equation of this tangent plane.

(b) Consider $f(x, y) = \ln(1+x+y)$. Estimate the value of $f(0.1, -0.2)$ using the linear approximation at $(0, 0)$.

2. (15 pts)

Find the absolute maximum and absolute minimum values of the function $f(x, y) = x^2 + y^2 - 4x + 5$ on the domain $R = \{(x, y) : x^2 + y^2 \leq 9\}$.

3. (15 pts)

   Find the area inside the rectangle enclosed by $x = 2$, $x = 0$, $y = 2$, and $y = 0$, but outside $x^2 + y^2 = 1$ in the first quadrant. (The calculation must be carried out using polar coordinates ONLY.)

4. (20 pts)

   Find the volume under the surface of $z = 4 - x^2 - y^2$ and above the surface of $z = -\sqrt{4 - x^2 - y^2}$.

5. (15 pts)

Determine whether $\mathbf{F}$ is a conservative vector field. If $\mathbf{F}$ is a conservative vector field, then find a function ϕ such that $\mathbf{F} = \nabla\phi$ and use ϕ to evaluate $\int_C \mathbf{F} \cdot \mathbf{dr}$ along the given curve C. If $\mathbf{F}$ is NOT a conservative vector field, then STOP.
(a) $\mathbf{F}(x, y) = (1+xy)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}$, with C given by $\mathbf{r}(t) = (t+\sin(\frac{\pi t}{2}))\mathbf{i} + (t+\cos(\frac{\pi t}{2}))\mathbf{j}$ for t changing from $t = 0$ to $t = 1$.
(b) $\mathbf{F}(x, y) = e^x \cos y\mathbf{i} + e^x \sin y\mathbf{j}$, with C given by $\mathbf{r}(t) = \cos t\mathbf{i} + 2\sin t\mathbf{j}$ for t changing from $t = 0$ to $t = \pi/2$.

6. (15 pts)

Use Green's Theorem to evaluate the line integral along the given positively oriented curve C:

$$\int_C (y + e^{\sqrt{x}})\, dx + (2x + \cos y^2)\, dy$$

where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Q1

(a) Point (x_0, y_0, z_0)

$$x_0^2 + \frac{1}{4} y_0^2 - 1 = z_0$$

$$F(x,y,z) = x^2 + y^2/4 - 1 - z = 0$$

$$\nabla F = \langle 2x, y/2, -1 \rangle$$

$\langle 2x_0, y_0/2, -1 \rangle$ parallel to $\langle 2, 1, 1 \rangle$

$$\frac{2x_0}{2} = \frac{y_0/2}{1} = \frac{-1}{1}$$

$x_0 = -1, \quad y_0 = -2, \quad z_0 = 1$

Point $(-1, -2, 1)$

Tangent plane $\quad 2(x+1) + (y+2) + (z-1) = 0$

$$2x + y + z + 3 = 0$$

(b) $f(0.1, -0.2) \approx f(0,0) + f_x(0,0)(0.1) + f_y(0,0)(-0.2)$

$f(0,0) = \ln 1 = 0$

$f_x = \dfrac{1}{1+x+y}, \quad f_y = \dfrac{1}{1+x+y}$

$f_x(0,0) = f_y(0,0) = 1$

$f(0.1, -0.2) \approx 0 + 0.1 - 0.2 = -0.1$

Q2.

$f_x = 2x - 4$, $f_y = 2y$

Critical point $(2, 0)$ where $f_x = f_y = 0$

$f(2,0) = 1$

$f(x,y) = x^2 + y^2 - 4x + 5$ becomes

$g(x) = 14 - 4x$ at the boundary of R

Min : $g(3) = 14 - 4(3) = 2$

Max : $g(-3) = 14 - 4(-3) = 26$

Abs. max : $f(-3, 0) = 26$

Abs. min : $f(2, 0) = 1$

Q3 $\displaystyle\int_{\alpha}^{\beta}\int_{r_1(\theta)}^{r_2(\theta)} r\,dr\,d\theta$

to be used

$$\int_0^{\frac{\pi}{4}}\int_1^{\frac{2}{\cos\theta}} r\,dr\,d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_1^{\frac{2}{\sin\theta}} r\,dr\,d\theta$$

$(x = r\cos\theta = 2$ and $y = r\sin\theta = 2)$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\left(\frac{2}{\cos\theta}\right)^2 - 1\right]d\theta + \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[\left(\frac{2}{\sin\theta}\right)^2 - 1\right]d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}(4\sec^2\theta - 1)\,d\theta$$

$$+ \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(4\csc^2\theta - 1)\,d\theta$$

$$= \frac{1}{2}\left(4\tan\theta\Big|_0^{\frac{\pi}{4}} - \frac{\pi}{4}\right) + \frac{1}{2}\left(-4\cot\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left(4 - \frac{\pi}{4}\right) + \frac{1}{2}\left(-4(0-1) - \frac{\pi}{4}\right)$$

$$= 4 - \frac{\pi}{4}$$

Q4

$$z = 4 - r^2$$
$$z = -\sqrt{4 - r^2}$$

Intersection $4 - r^2 = -\sqrt{4 - r^2}$

$4 - r^2 = 0$, $r = 2$, $z = 0$

$$\text{Volume} = \int_0^{2\pi} \int_0^2 \left(\int_{-\sqrt{4-r^2}}^{4-r^2} dz \right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left(4 - r^2 + \sqrt{4-r^2} \right) r\,dr\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^2 \left(4 - r^2 + \sqrt{4-r^2} \right) dr^2\,d\theta$$

$$= \frac{1}{2}(-1) \int_0^{2\pi} \int_4^0 \left(u + \sqrt{u} \right) du\,d\theta$$

$$= \frac{1}{2} \int_0^{2\pi} \int_0^4 \left(u + \sqrt{u} \right) du\,d\theta$$

$$= \pi \left(\frac{1}{2} u^2 \Big|_0^4 + \frac{2}{3} u^{\frac{3}{2}} \Big|_0^4 \right)$$

$$= \pi \left(8 + \frac{2}{3} \times 8 \right) = \frac{40}{3}\pi$$

Q 5

(a) $f = (1 + xy)\, e^{xy}$

$g = x^2 e^{xy}$

$\partial_x g = 2x\, e^{xy} + x^2 y\, e^{xy} = (2x + x^2 y)\, e^{xy}$

$\partial_y f = x\, e^{xy} + (1 + xy)\, x\, e^{xy} = (2x + x^2 y)\, e^{xy}$

$\partial_x g = \partial_y f$: Conservative vec field.

$\vec{r}(0) = 0\vec{i} + 1\vec{j} = \langle 0, 1 \rangle$

$\vec{r}(1) = 2\vec{i} + 1\vec{j} = \langle 2, 1 \rangle$

$\partial_y \phi = g = x^2 e^{xy}$

$\phi = \int g\, dy = \int x^2 e^{xy}\, dy = x\, e^{xy} + h(x)$

$h(x)$ to be determined.

$\partial_x \phi = e^{xy} + xy\, e^{xy} + h'(x) = f = (1 + xy)\, e^{xy}$

$h'(x) = 0$, $h(x) = \text{Const.}$

$\phi = x\, e^{xy} + \text{Const.}$

$$\int_C \vec{F} \cdot d\vec{r} = \phi(2,1) - \phi(0,1) = 2e^2 - 0 = \underline{\underline{2e^2}}$$

(b)

$f = e^x \cos y$, $g = e^x \sin y$

$\partial_y f = -e^x \sin y$, $\partial_x g = e^x \sin y$

$\partial_y f \neq \partial_x g$, Not a conservative vec. field.

Q 6

$$f = y + e^{\sqrt{x}}$$

$$g = 2x + \cos y^2$$

$$\partial_x g - \partial_y f = 2 - 1 = 1$$

$$\oint_C f\,dx + g\,dy$$

$$= \iint_R (\partial_x g - \partial_y f)\,dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} dy\,dx$$

$$= \int_0^1 (\sqrt{x} - x^2)\,dx$$

$$= \frac{2}{3} - \frac{1}{3} = \underline{\underline{\frac{1}{3}}}$$