MATH2011 Intro to Multivariable Calculus
Tutorial 1: Vectors

Definitions and Results

Defintion 1. Vectors in the plane and in space are arrows between two points that are defined by its magnitude and direction, expressed as $\langle u_1, u_2\rangle$ in the plane or $\langle u_1, u_2, u_3\rangle$ in space. (I will generally use a vector in space)

Definition 2. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if there is a constant k such that $\mathbf{u}=k\mathbf{v}$. We say that $\mathbf{u}$ and $\mathbf{v}$ are in the **same** direction if $k > 0$ and the **opposite** direction if $k < 0$.

Formula 3. The magnitude of $\mathbf{u}$ is defined by $\sqrt{u_1^2 + u_2^2 + u_3^2}$.

Formula 4. The unit vector in the direction of $\mathbf{u}$ is calculated by $\dfrac{\mathbf{u}}{\| u \|}$.

Definition 5. A plane Γ in space is defined by a point $P = (p_1, p_2, p_3)$ on Γ and the normal vector $\mathbf{n} = (n_1, n_2, n_3)$. Γ: $n_1(x - p_1) + n_2(y - p_2) + n_3(z - p_3) = 0$.

Definition 6. A line ℓ in space is defined by a vector $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ that shows its direction and a point $P = (p_1, p_2, p_3)$ that lies on ℓ. Equation of ℓ:
$\mathbf{r} = (x, y, z) = (p_1 + v_1t, p_2 + v_2t, p_3 + v_3t)$.

Definition 7. Dot product of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \cdot \mathbf{v}$, is $u_1v_1 + u_2v_2 + u_3v_3$.

Theorem 8. $\mathbf{u} \cdot \mathbf{v} =\| u \|\| v \| \cos\theta$. $\mathbf{u}$ and $\mathbf{v}$ are orthogonal iff $\mathbf{u} \cdot \mathbf{v} = 0$.

Formula 9. The orthogonal projection of $\mathbf{u}$ onto a vector $\mathbf{v}$ is evaluated by
$proj_{\mathbf{v}}\mathbf{u} = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{v} \|^2}\mathbf{v}$.

Formula 10. The orthogonal projection of $\mathbf{u}$ onto a plane spanned by $\mathbf{v}$ and $\mathbf{w}$ is defined by $\dfrac{\mathbf{u} \cdot \mathbf{v}}{\| \mathbf{v} \|^2}\mathbf{v} + \dfrac{\mathbf{u} \cdot \mathbf{w}}{\| \mathbf{w} \|^2}\mathbf{w}$.

Defintion 11. Cross product of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is a vector calculated by the determinant of the matrix

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$$

which reduces to $(u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_3 - u_3v_1)\mathbf{k}$.

Theorem 12. The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ Also,
$\| \mathbf{u} \times \mathbf{v} \|=\| \mathbf{u} \|\| \mathbf{v} \| \sin\theta$.

Example 1

Let $\mathbf{v} = \langle 1, 2, 3\rangle$ and $\mathbf{w} = \langle 2, -1, -2\rangle$. Find the orthogonal projection of $\mathbf{v}$ on $\mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$.

Solution

Example 2

Show that if $\mathbf{u} \cdot \mathbf{v} = 0$ and $\mathbf{u} \times \mathbf{v} = 0$ then $\mathbf{u}$ or $\mathbf{v}$ is 0.

Solution

Example 3

Let Γ be the plane that goes through point $P = (3, 1, -1)$ with normal vector $\mathbf{n} = \langle 1, 2, 1 \rangle$ and let ℓ be the line that goes through $Q = (0, 0, 1)$ in the direction of $\mathbf{v} = \langle 0, 1, 0 \rangle$.

(A) Find minimum distance between Γ and the origin. Also find the minimum distance from ℓ to the origin.

(B) Find the minimum distance from Γ to ℓ.