Tutorial 03: Vectors (II)

Formula 1: The distance between points $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_2)$ is the length of vector $\overrightarrow{PQ}$.

Def 2: A vector-valued function is a fuction of the form

$$r(t) = \langle f(t), g(t), h(t) \rangle$$

Def 3: The positive direction of curve $r(t)$ is the direction this curve forms when t increases.

Formula 4: Let $r(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$. $L = \langle L_1, L_2, L_3 \rangle$

$r(t)$ approaches L as t approaches x_0 if

$$\| r(t) - L \| \to 0 \quad \text{as} \quad t \to x_0$$

or

$$f_i(t) \to L_i \quad \text{as} \quad t \to x_0 \;, \quad i = 1, 2, 3$$

Def 5: $r(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$, f_i are differentiable functions.

$r'(t) = \langle f_1'(t), f_2'(t), f_3'(t) \rangle$ is the derivative of $r(t)$

Formula 6: The unit tangent vector of $r(t)$ is $\dfrac{r'(t)}{\| r(t) \|}$

Def 7: Let $r(t) = \langle f_1(t), f_2(t), f_3(t) \rangle$.

$F_1(t)$, $F_2(t)$, $F_3(t)$ are the respective antiderivatives.

The indefinite integral of r is

$$\int r(t)\, dt = \langle F_1(t), F_2(t), F_3(t) \rangle$$

Examples:

Ex 1 For r(t) and R(s). find a line prependicular to them both which passes through their intersection.

(1) $r(t) = \langle -2+3t, 2t, 3t \rangle$, $R(s) = \langle -6+s, -8+2s, -12+3s \rangle$

(2) $r(t) = \langle 4t, 1+2t, 3t \rangle$, $R(s) = \langle -1+s, -7+2s, -12+3s \rangle$

Ex 2 Let $u(0) = \langle 0,1,1 \rangle$, $u'(0) = \langle 0,7,1 \rangle$, $v(0) = \langle 0,1,1 \rangle$, $v'(0) = \langle 1,1,2$

(1) $\frac{d}{dt}(u \cdot v)$

(2) $\frac{d}{dt}(\cos t \, u(t))$

Ex 3 Find the points t at which r(t) is orthogonal to r'(t):

(1) $r(t) = \langle a\cos t, a\sin t \rangle$

(2) $r(t) = \langle at^2+1, t \rangle$

(3) $r(t) = \langle \cos t, \sin t, t \rangle$

Ex 4 Calculate $\lim_{t\to 0} r(t)$, $r(t) = \langle \frac{\sin t}{t}, t^2-3t+3, \cos t \rangle$

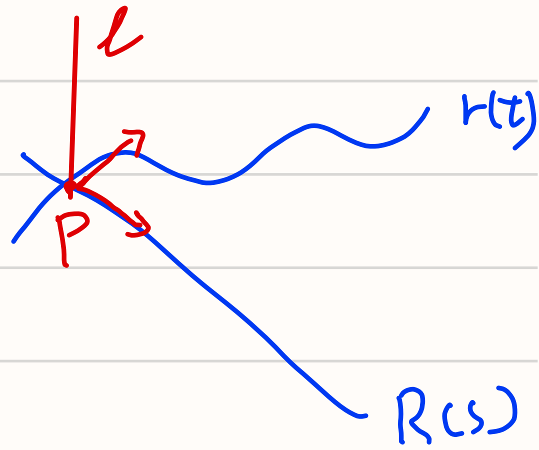
Ex 1 For $r(t)$ and $R(s)$. find a line prependicular to them both which passes through their intersection.

(1) $r(t) = \langle -2+3t, 2t, 3t \rangle$, $R(s) = \langle -6+s, -8+2s, -12+3s \rangle$

(2) $r(t) = \langle 4t, 1+2t, 3t \rangle$, $R(s) = \langle -1+s, -7+2s, -12+3s \rangle$

(1) intersection:

$$\begin{cases} -2+3t = -6+s \\ 2t = -8+2s \\ 3t = -12+3s \end{cases} \Rightarrow \begin{cases} t=0 \\ s=4 \end{cases}$$



$P = (-2, 0, 0)$

direction:

$r'(t) = (3, 2, 3)$

$R'(s) = (1, 2, 3)$

$d = r'(t) \times R'(s) = (0, -6, 4)$

Eq of l:

$P + xd = (-2, 0, 0) + (0, -6, 4)x \qquad x \in \mathbb{R}$

Ex 2 Let $u(0) = \langle 0,1,1 \rangle$, $u'(0) = \langle 0,7,1 \rangle$, $v(0) = \langle 0,1,1 \rangle$, $v'(0) = \langle 1,1,2 \rangle$

(1) $\frac{d}{dt}(u \cdot v)$

(2) $\frac{d}{dt}(\cos t \, u(t))$

(1) $$\frac{d}{dt}(u \cdot v)\Big|_{t=0} = \frac{du}{dt} \cdot v \Big|_{t=0} + \frac{dv}{dt} \cdot u \Big|_{t=0}$$

(2) $$\frac{d}{dt}(\cos t \; u(t))\Big|_{t=0} = \left(\frac{d}{dt} \cos t\right) \cdot u(t) \Big|_{t=0} + \cos t \cdot \frac{du}{dt} \Big|_{t=0}$$

$$= -\sin t \cdot u(t) \Big|_{t=0} + \cos t \cdot u'(t) \Big|_{t=0}$$

Ex 3 Find the points t at which $r(t)$ is orthogonal to $r'(t)$:

(1) $r(t) = \langle a\cos t, a\sin t \rangle$ $\qquad r(t)\cdot r'(t) = 0$

(2) $r(t) = \langle at^2+1, t \rangle$

(3) $r(t) = \langle \cos t, \sin t, t \rangle$

(1) $r'(t) = \langle -a\sin t, a\cos t \rangle$.

$r(t)\cdot r'(t) = -a^2 \sin t\cos t + a^2 \cos t \sin t = 0$

(2) $r'(t) = \langle 2at, 1 \rangle$.

(3) $r'(t) = \langle -\sin t, \cos t, 1 \rangle$

Ex 4 Calculate $\lim_{t\to 0} r(t)$, $r(t) = \langle \frac{\sin t}{t}, t^2-3t+3, \cos t \rangle$

$\lim_{t\to 0} \frac{\sin t}{t} = \lim_{t\to 0} \frac{(\sin t)'}{(t)'} = \lim_{t\to 0} \cos t = 1$

$\lim_{t\to 0} t^2-3t+3 = 3$

$\lim_{t\to 0} \cos t = 1$

$\Rightarrow \lim_{t\to 0} r(t) = \langle 1, 3, 1 \rangle$