MATH 2011 Introduction to Multivariable calculus - Tutorial Note 1

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1 Parametric and Polar curves

Key points: **Parametric Equations**; **Derivatives**; **Polar Coordinates**

1.1 Curves Defined by Parametric Equations

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t), \quad y = g(t), \tag{1}$$

each value of t determines a point (x, y). As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve**. we can interpret $(x, y) = (f(t), g(t))$ as the position of a particle at time t.

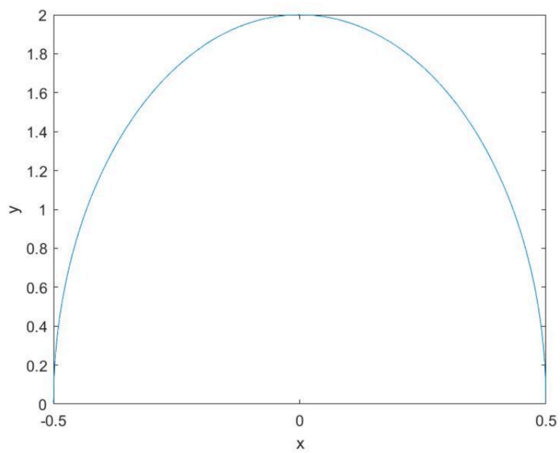
Remark 1. A particle whose position is given by the parametric equations moves along the curve in the direction of the arrows as t **increases**. If $a \leq t \leq b$, think about the initial point and terminal point for (1)?

Example 1. Eliminate the parameter to find a Cartesian equation of the curve $x = t^2 - 3, y = t + 2, -3 \leq t \leq 3$.

Solution: $x = (y-2)^2 - 3, -1 \leq y \leq 5$.

Example 2. Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases: $x = \dfrac{1}{2}\cos\theta, y = 2\sin\theta, 0 \leq \theta \leq \pi$.

Solution: See figure above.



1.2 Derivatives for Parametric Curves

Suppose f and g are differentiable functions and we want to find the tangent line at a point on the parametric curve $x = f(t), y = g(t)$, where y is also a differentiable function of x. Then the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

If $dx/dt \neq 0$, we can solve for dx/dt:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad dx/dt \neq 0. \tag{2}$$

Equation (2) (which you can remember by thinking of canceling the dts) enables us to find the slope dy/dx of the tangent to a parametric curve without having to eliminate the parameter t.

Remark 2. If $dx/dt = 0$, this is vertical line.

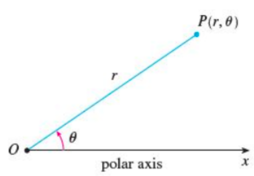
Example 3. Find dy/dx for $x = te^t, y = t + \sin t$.

Solution: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+\cos t}{e^t + te^t}$.

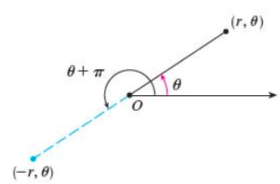
1.3 Polar Coordinates

Polar coordinate system: We choose a point in the plane that is called the pole (or origin) and is labeled O, If P is any other point in the plane, let r be the distance from O to P and let θ be the angle (usually measured in radians) between the polar axis and the line OP as in aaaaaaa. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P. Conventionally, an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction.

We extend the meaning of polar coordinates to the case in which r is negative as Figure 2 shows, the points $(-r, \theta)$ and (r, θ) lie on the same line through origin and at the same distance $|v|$ from origin, but on opposite sides. Notice that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$.



(a) Figure 1



(b) Figure 2

connection between polar and Cartesian coordinates: If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then

$$\begin{aligned} \cos\theta &= \frac{x}{r}, \\ \sin\theta &= \frac{y}{r}, \\ x &= r\cos\theta, \\ y &= r\sin\theta. \end{aligned}$$

Basic Curves:

- $r = a$, circle of radius $|a|$.
- $\theta = \theta_0$, line through the origin.

Symmetry

- If a polar equation is unchanged when θ is replaced by $-\theta$, the curve is symmetric about the polar axis (x axis in Cartesian).
- If the equation is unchanged when (1)r is replaced by $-r$, or when (2)θ is replaced by $\theta + \pi$, the curve is symmetric about the pole.
- If the equation is unchanged when θ is replaced by $\pi - \theta$ the curve is symmetric about the vertical line $\theta = \pi/2$.

Example 4. Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

- $1 \le r \le 3, \pi/6 < \theta < 5\pi/6,$
- $r \ge 1, \pi \le \theta \le 2\pi.$

Solution:

Example 5. Identify the curve by finding a Cartesian equation for the curve.

- $r = 5\cos\theta \Longleftrightarrow x^2 + y^2 = 5x$
- $\theta = \pi/3 \Longleftrightarrow y = \sqrt{3}x$
- $r = a\sin\theta + b\cos\theta, \quad ab \neq 0 \Longleftrightarrow x^2 + y^2 = ay + bx$

2 Calculus in Polar Coordinates

2.1 Slope of tangent lines

Given curve $r = f(\theta)$ in polar coordinates, the parametric equations with θ acting as the parameter are:

$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

and

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

2.2 Area of Regions Bounded by Polar Curves

Let $\mathcal{R}$ be the region bounded by $r = f(\theta)$ and $r = g(\theta)$, between $\theta = \alpha$ and $\theta = \beta$, where f and g are continuous and $f(\theta) \geq g(\theta) \geq 0$ on $[\alpha, \beta]$, the area of $\mathcal{R}$ is $\int_\alpha^\beta \frac{1}{2}\left(f(\theta)^2 - g(\theta)^2\right) d\theta$, see Figure 1: