Tutorial 04

1. Calculus of Vector-Valued Functions

1.1. Derivative and Tangent Vector

Def: $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$. f, g, h differentiable of $t \in (a, b)$

$$\vec{r}\,'(t) = f'(t)\vec{i} + g'(t)\vec{j} + h'(t)\vec{k}$$

The unit tangent vector is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

1.2 Properties of Derivative

(a) $\frac{d}{dt}(\vec{c}) = 0$

(b) $\frac{d}{dt}(\vec{u}(t) + \vec{v}(t)) = \vec{u}\,'(t) + \vec{v}\,'(t)$

(c) $\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\cdot\vec{u}(t) + f(t)\cdot\vec{u}\,'(t)$

d) $\frac{d}{dt}[\vec{u}(f(t))] = \vec{u}\,'(f(t))\cdot f'(t)$

e) $\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$

f) $\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$

1.3 Integrals of Vector-Valued Functions

Def $\vec{r}(t)$. $\vec{R}(t)$ is the antiderivative of $\vec{r}(t)$

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C} \qquad \vec{C}: \text{const vector}$$

If $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$.

$$\int_a^b \vec{r}(t)\,dt = \int_a^b f(t)\,dt\,\vec{i} + \int_a^b g(t)\,dt\,\vec{j} + \int_a^b h(t)\,dt\,\vec{k}$$

(f, g, h integrable)

2. Velocity & Acceleration

position vector at time t $\quad \vec{r}(t)$

velocity vector $\quad \vec{v}(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h} = \vec{r}'(t)$

speed $\quad |\vec{v}(t)|$

acceleration $\quad \vec{a}(t) = v'(t) = r''(t)$

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} a(u)\,du$$

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} v(u)\,du$$

3. Length of a Curve

Parametric eqs of curve : $x = f(t),\ y = g(t),\ z = h(t) \quad t \in [a,b]$

f', g', h' are continuous.

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_a^b |r'(t)|\,dt$$

For a polar curve $r = f(\theta) \qquad \theta \in [\alpha, \beta]$

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta$

$$L = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\,d\theta$$

4. The Arc Length Function

A curve is given by $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k} \quad t \in [a,b]$

$r'(t)$ is continuous.

The arc length function is

$$s(t) = \int_a^t |r'(u)| du$$

$$= \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du.$$

$s(t)$ is the length of the part of curve from $r(a)$ to $r(t)$

$$\frac{ds}{dt} = |\vec{r}'(t)|$$

Ex 3.1.1 Calcute the derivate of the functions:

a. $\vec{r}(t) = e^t sint\vec{i} + e^t cost\vec{j} - e^{2t}\vec{k}$

b. $\vec{r}(t) = (tlnt)\vec{i} + (5e^t)\vec{j} + (cost - sint)\vec{k}$

(a) $r'(t) = (e^t sint)' i + (e^t cost)' j + (-e^{2t})' k$

$= e^t(sint + cost) i + e^t(cost - sint) j - 2e^{2t} k.$

(b) $r'(t) = (1 + lnt) i + 5e^t j - (sint + cos t) k$

Ex 3.1.2 Calcute the unit tangent vector of $\vec{r}(t) = t\vec{i} + e^t\vec{j} - 3t^2\vec{k}$

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$= \frac{\langle 1, e^t, -6t \rangle}{\sqrt{1 + e^{2t} + 36t^2}}$$

Ex 3.2.1 Given the vector-valued functions

$$\vec{r}(t) = (6t+8)\vec{i} + (4t^2+2t-3)\vec{j} + 5t\vec{k}$$

$$\vec{u}(t) = (t^2-3)\vec{i} + (2t+4)\vec{j} + (t^3-3t)\vec{k}$$

calculate each of the following derivatives:

a. $\frac{d}{dt}\left[\vec{r}(t)\cdot\vec{u}(t)\right]$

b. $\frac{d}{dt}\left[\vec{u}(t)\times\vec{u}'(t)\right]$

(a) $\frac{d}{dt}[r(t)\cdot u(t)] = r'(t)\cdot u(t) + r(t)\cdot u'(t)$

$= 20t^3 + 42t^2 + 26t - 16$

(b) $\frac{d}{dt}[u(t)\times u'(t)] = u'(t)\times u'(t) + u(t)\times u''(t)$

$= (12t^2+24t)i + (12t-4t^3)j + (4t+8)k$

Ex 3.3.1 Calculate the following indefinite integrals:

a. $\int\left[(3t^2+2t)\vec{i} + (3t-6)\vec{j} + (6t^3+5t^2-4)\vec{k}\right]dt$

b. $\int\left[\langle t, t^2, t^3\rangle \times \langle t^3, t^2, t\rangle\right]dt$

(a) $= (t^3+t^2)i + (\frac{3}{2}t^2-6t)j + (\frac{3}{2}t^4 + \frac{5}{3}t^3 - 4t)k + C$ C is const vector

(b) $= (\frac{t^4}{4} - \frac{t^6}{6})i + (\frac{t^7}{7} - \frac{t^3}{3})j + (\frac{t^4}{4} - \frac{t^6}{6})k + C.$

Ex 3.3.1 Calculate the following definite integrals:

a. $\int_0^{\frac{\pi}{3}}(sin2t\vec{i}+tant\vec{j}+e^{-2t}\vec{k})dt$

b. $\int_1^3\left[(2t+4)\vec{i}+(3t^2-4t)\vec{j}\right]dt$

(a) $= \left(\int_0^{\frac{\pi}{3}} \sin 2t\,dt\right) i + \left(\int_0^{\frac{\pi}{3}} \tan t\,dt\right) j + \left(\int_0^{\frac{\pi}{3}} e^{-2t}\,dt\right) k$

$= \frac{3}{4} i + (\ln 2) j + \left(\frac{1}{2} - \frac{1}{2} e^{-\frac{2\pi}{3}}\right) k$

(b) $= 16 i + 10 j$

Example 1. Find the velocity, acceleration, and speed of a particle with position vector.

1. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$
2. $\mathbf{r}(t) =< t^2, \sin t - t\cos t, \cos t + t\sin t >$

(1) $v(t) = r'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$

$a(t) = v'(t) = \langle 0, e^t, e^{-t} \rangle$

speed: $|v(t)| = \sqrt{2 + e^{2t} + e^{-2t}}$

(2) $v(t) = \langle 2t, t\sin t, t\cos t \rangle$

$a(t) = \langle 2, \sin t + t\cos t, \cos t - t\sin t \rangle$

$|v(t)| = \sqrt{4t^2 + t^2} = \sqrt{5}t.$

Example 2. Find the velocity and position vectors of a particle that has the given acceleration and the given initial velocity and position.

1. $\mathbf{a}(t) = 2\mathbf{i} + 2t\mathbf{k}, \quad \mathbf{v}(0) = 3\mathbf{i} - \mathbf{j}, \quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$
2. $\mathbf{a}(t) =< \sin t, 2\cos t, 6t >, \quad \mathbf{v}(0) =< 0, 0, -1 >, \mathbf{r}(0) =< 0, 1, -4 >$

(1) $v(t) = v(0) + \int_0^t a(u)\,du$

$= \langle 3, -1, 0 \rangle + \int_0^t \langle 2, 0, 2u \rangle\,du$

$= \langle 3+2t, -1, t^2 \rangle$

$r(t) = r(0) + \int_0^t v(u)\,du$

$= \langle 0, 1, 1 \rangle + \int_0^t \langle 3+2u, -1, u^2 \rangle\,du$

$= \langle 3t + t^2, 1-t, 1 + \frac{1}{3}t^3 \rangle$

(2) $v(t) = \cdots = \langle 1 - \cos t, 2\sin t, 3t^2 - 1 \rangle$

$r(t) = \cdots = \langle t - \sin t, 3 - 2\cos t, t^3 - t - 4 \rangle$

Ex 3. Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

Since speed is const.

$\Rightarrow |v(t)| = C \qquad C = \text{const.}$

$\Rightarrow |v(t)|^2 = C^2$

Take derivative on both sides

$$\frac{d}{dt}(v(t)\cdot v(t)) = v'(t)\cdot v(t) + v(t)\cdot v'(t)$$

$$= 2v'(t)\cdot v(t) = 2a(t)\cdot v(t) = 0$$

Ex 4 Find the arc length of $r(t) = \langle t, 3\cos t, 3\sin t\rangle \quad t \in [-4, 4]$

$$L = \int_{-4}^{4} \sqrt{t^2 + (3\cos t)^2 + (3\sin t)^2}\, dt = 8\sqrt{10}$$

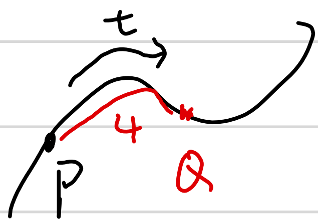
Ex 5 Find the arc length function of the curve, which is measured from the point P in the direction of t ↑.

(1) Reparametrize the curve w.r.t. arc length starting from P.

(2) Find the point 4 units along the curve from P.

$\vec{r}(t) = (5-t)\vec{i} + (4t-3)\vec{j} + 3t\vec{k}$ P

P(4, 1, 3)



(1) P correspond to $t=1$

$$s(t) = \int_1^t |r'(u)|\,du$$

$$= \int_1^t \sqrt{(-1)^2+4^2+3^2}\,du$$

$$= \sqrt{26}\,t - \sqrt{26} \quad = \sqrt{26}(t-1)$$

$$\Rightarrow t = \frac{s}{\sqrt{26}} + 1$$

Substitude t into eq.

$$\vec{r}(s) = \left(4 - \frac{s}{\sqrt{26}}\right)\vec{i} + \left(1 + \frac{4s}{\sqrt{26}}\right)\vec{j} + \left(\frac{3s}{\sqrt{26}} + 3\right)\vec{k}$$

(2) arc length between P, Q is 4

$s = 4.$

$$Q : \left(4 - \frac{4}{\sqrt{26}},\ 1 + \frac{16}{\sqrt{26}},\ \frac{12}{\sqrt{26}} + 3\right)$$