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1. Use the method of characteristics to determine two different solutions to the initial value problem

$$u = u_x^2 - 3u_t^2, \quad x \in \mathbf{R}, \ t > 0.$$

$$u(x,0) = x^2, \quad x \in \mathbf{R}.$$

Denote $p = u_x$, $q = u_t$.

$H(x,t,u,p,q) = u - p^2 + 3q^2 = 0$

$H_x = 0$. $H_t = 0$. $H_u = 1$. $H_p = -2p$. $H_q = 6q$

The characteristic system is:

$$\begin{cases} x' = H_p = -2p \\ t' = H_q = 6q \\ u' = pH_p + qH_q = -2p^2 + 6q^2 \\ p' = -H_x - H_u p = -p \\ q' = -H_t - H_u q = -q \end{cases}$$

The initial conditions when $s=0$ are:

$$\begin{cases} x = \xi \\ t = 0 \\ u = \xi^2 \\ p = u_0'(\xi) = 2\xi \\ q = -3p^2 + 3q^2 = 0 \Rightarrow q = \pm \xi \end{cases}$$

- When $q = \xi$ (at $s=0$)

$$\begin{cases} t = 6\xi e^{-s} - 6\xi \\ x = -4\xi e^{-s} + 3\xi \\ p = 2\xi e^{-s} \\ q = \xi e^{-s} \\ u = p^2 - 3q^2 = 4\xi^2 e^{-2s} - 3\xi^2 e^{-2s} = \frac{1}{4}(2x+t)^2 \end{cases}$$

- When $q = -\xi$ (at $s=0$)

Similarly, $u = \frac{1}{4}(2x-t)^2$

-1.5

2. Find the solution that satisfies the entropy condition, and plot the characteristic diagram:

(1) $\begin{cases} u_t + (e^{6u})_x = 0, & x \in \mathbf{R},\ t > 0. \\ u(x,0) = 2 \text{ if } x < 0; & u(x,0) = 4 \text{ if } x > 0. \end{cases}$

(2) $\begin{cases} u_t + (e^{6u})_x = 0, & x \in \mathbf{R},\ t > 0. \\ u(x,0) = 4 \text{ if } x < 0; & u(x,0) = 2 \text{ if } x > 0. \end{cases}$

(1) $\because\ 2 = u^- \le u^+ = 4$

$\therefore$ The rarefaction solution satisfy the entropy condition.

From the conservation law, $u_t + (\varphi(u))_x = 0$

$\Rightarrow \varphi(u) = e^{6u}$

In the rarefaction fan starting from 0,

$\varphi'(u) = \frac{dx}{dt} = \frac{x-0}{t}$, i.e. $6e^{6u} = \frac{x}{t}$

$\Rightarrow u = \frac{1}{6}\ln\left(\frac{x}{6t}\right)$

The solution is

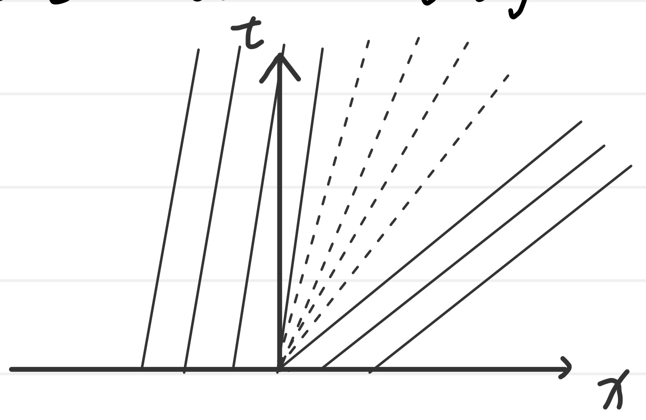
$$u(x,t) = \begin{cases} 2 & x < 2t \\ \frac{1}{6}\ln\left(\frac{x}{6t}\right) & 2t \le x \le 4t \\ 4 & x > 4t \end{cases}$$

$x < 6e^{12}t$

$x > 6e^{24}t$

-3

The characteristic diagram is



(2) $\because\ 4 = u_- > u_+ = 2$

$\therefore$ The jump solution satisfies the entropy condition.

From the conservation law, $u_t + (\varphi(u))_x = 0$

$\Rightarrow \varphi(u) = e^{6u}$. $\varphi'(u) = 6e^{6u}$

The form of solution is

$$u(x,t) = \begin{cases} 4 & x < s(t) \\ 2 & x > s(t) \end{cases}$$

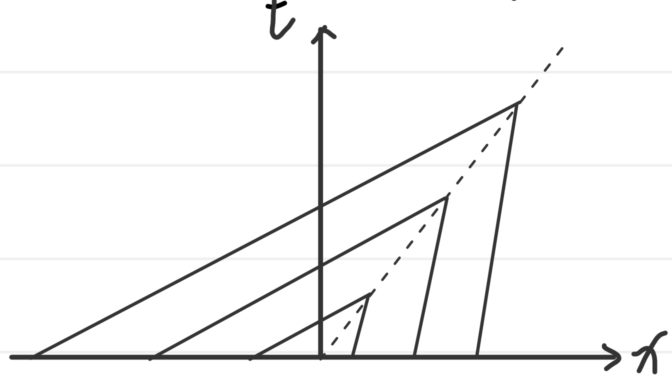
The jump condition gives

$$s'(t) = \frac{e^{6u_-} - e^{6u_+}}{u_- - u_+} = \frac{e^{24} - e^{12}}{2}$$

$$s(0)=0 \,. \quad s(t) = \frac{e^{24}-e^{12}}{2} t$$

$$\therefore u(x,t) = \begin{cases} 4 & x < \frac{e^{24}-e^{12}}{2} t \\ 2 & x > \frac{e^{24}-e^{12}}{2} t \end{cases}$$

The characteristic diagram is:



3. Consider the initial value problem

$$u_t + uu_x = 0, \quad x \in \mathbf{R}, \ t > 0.$$

$$u(x,0) = \frac{1}{1+x^2}, \quad x \in \mathbf{R}.$$

Find the breaking time t_b of the wave and write the solution for $0 < t < t_b$ in implicit form.

$C(u) = u$

$F(\xi) = c(u_0(\xi)) = \dfrac{1}{1+\xi^2}$

$u_0'(x) = \dfrac{-2x}{(1+x^2)^2} < 0$ for $x > 0$.

$F'(\xi) = \dfrac{-2\xi}{(1+\xi^2)^2}$

$F''(\xi) = \dfrac{6\xi^2 - 2}{(1+\xi^2)^3} = 0 \Rightarrow \xi = \pm\frac{\sqrt{3}}{3}$

$F'(\frac{\sqrt{3}}{3}) = -\frac{3\sqrt{3}}{8}$ $\quad F'(-\frac{\sqrt{3}}{3}) = \frac{3\sqrt{3}}{8}$

$F'(+\infty) = F'(-\infty) = 0$

$\therefore F'(\xi)$ attains its minimum at $\xi = \frac{\sqrt{3}}{3}$

The breaking time $t_b = \dfrac{1}{\min\limits_{-\infty < \xi < +\infty} F'(\xi)} = \dfrac{8\sqrt{3}}{9}$

When $0 < t < t_b$, the solution in implicit form is

$$\begin{cases} x - \xi = \dfrac{1}{1+\xi^2} t \\ u = \dfrac{1}{1+\xi^2} t \end{cases}$$

characteristic $\quad x - \xi = \dfrac{1}{1+\xi^2} t$

$u(x,t) = u(\xi, 0) = u_0(\xi) = \dfrac{1}{1+\xi^2}$

$\therefore x - \xi = ut \quad \Rightarrow \xi = x - ut$

$\therefore u(x,t) = \dfrac{1}{1+\xi^2} = \dfrac{1}{1+(x-ut)^2}$

4. Suppose $u(x,t)$ is a weak solution of the initial value problem

$$u_t + c(u)u_x = 0, \quad x \in \mathbf{R}, \ t > 0.$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}.$$

Let $a < b$ be chosen such that $u_0(a) = u_0(b)$, where u_0 satisfies properties (P) (see the textbook), and assume that the shock path is given by $x = s(t)$ with $a + tF(a) < s(t) < b + tF(b)$ for t in some open interval I. Show that

$$\int_{a+tF(a)}^{b+tF(b)} u(x,t)dx = \text{constant}, \quad t \in I,$$

where $F(x) = c(u_0(x))$. Hint: Show the derivative of the integral is zero.

Remark: This is a lemma for the proof of the equal area principle.

Take derivation on the left hand side.

$$\frac{d}{dt}\int_{a+tF(a)}^{b+tF(b)} u(x,t)\,dx$$

$$= \frac{d}{dt}\left(\int_{a+tF(a)}^{s(t)} u(x,t)\,dx + \int_{s(t)}^{b+tF(b)} u(x,t)\,dx\right)$$

$$= \int_{a+tF(a)}^{s(t)} u_t(x,t)\,dt + u(s(t)^-,t)\cdot s'(t) - u(a+tF(a),t)\cdot F(a)$$

$$+ \int_{s(t)}^{b+tF(b)} u_t(x,t)\,dt + u(b+tF(b),t)F(b) - u(s(t)^+,t)\cdot s'(t)$$

$$= u(s(t)^-,t)\cdot s'(t) - u(a+tF(a),t)F(a) + \varphi(u(a+tF(a),t)) - \varphi(u(s(t^-),t))$$

$$+ u(b+tF(b),t)\cdot F(b) - u(s(t)^+,t)\cdot s'(t) + \varphi(u(s(t)^+,t)) - \varphi(u(b+tF(b),t))$$

$\because u_0(a) = u_0(b) \quad \therefore F(a) = F(b)$

$\because u(a+tF(a),t) = u(a+tc(u_0(a),t)) = u_0(a)$

$\quad u(b+tF(b),t) = u(b+tc(u_0(b),t)) = u_0(b)$

$\therefore u(b+tF(b),t) = u(a+tF(a),t)$

$$\therefore \frac{d}{dt}\int_{a+tF(a)}^{b+tF(b)} u(x,t)\,dx = u(s(t)^-,t)\,s'(t) - u(s(t)^+,t)\,s'(t)$$

$$-[\varphi(u(s(t)^-,t)) - \varphi(u(s(t)^+,t))]$$

$$= s'(u) - \varphi(u)$$

$$= 0$$

$$\therefore \int_{a+tF(a)}^{b+tF(b)} u(x,t)\,dx = \text{const.}$$

5. Consider the barotropic flow of a gas, which is governed by equations

$$\rho_t + (\rho u)_x = 0,$$

$$\rho(u_t + uu_x) + p_x = 0,$$

$$p = f(\rho), \;\; f', f'' > 0,$$

where ρ is the density, u is the velocity, p is the pressure, and f is a given function. Let $c^2 = f'(\rho)$.

(1) Verify that $u = 0$, $\rho = \rho_0$, $p = p_0$ is a constant state for the gas.

(2) Let

$$u = 0 + \tilde{u}(x,t), \quad \rho = \rho_0 + \tilde{\rho}(x,t),$$

where $\tilde{u}$ is a small velocity perturbation, and $\tilde{\rho}$ is a small (compared to ρ_0) density perturbation. Let $c_0^2 = f'(\rho_0)$. Show that in the linear approximation, $\tilde{u}$ and $\tilde{\rho}$ satisfy the wave equation

$$\psi_{tt} - c_0^2 \psi_{xx} = 0,$$

and therefore acoustic signals travel at speed c_0 (speed of sound).

(1) $\because$ $\rho_t + (\rho u)_x = (\rho_0)_t + 0 = 0$

$\rho(u_t + uu_x) + p_x = \rho_0 \cdot 0 + (p_0)_x = 0$

$\therefore$ $u=0$. $p=p_0$. $\rho=\rho_0$ is constant stage for the gas.

(2) Let $u = 0 + \tilde{u}(x,t)$. $\rho = \rho_0 + \tilde{\rho}(x,t)$

$$\rho_t + (\rho u)_x = (\rho_0 + \tilde{\rho})_t + [(\rho_0 + \tilde{\rho})\tilde{u}]_x$$
$$= \tilde{\rho}_t + \rho_0 \tilde{u}_x + \tilde{\rho}_x \tilde{u} + \tilde{\rho}\tilde{u}_x$$
$$= 0 \qquad (1)$$

$$\rho(u_t + uu_x) + p_x = (\rho_0 + \tilde{\rho})(\tilde{u}_t + \tilde{u}\tilde{u}_x) + f'(\rho)\rho_x$$
$$= \rho_0 \tilde{u}_t + \rho_0 \tilde{u}\tilde{u}_x + \tilde{\rho}\tilde{u}_t + \tilde{\rho}\tilde{u}\tilde{u}_x + f'(\rho_0 + \tilde{\rho})(\rho_0 + \tilde{\rho})_x$$
$$= \rho_0 \tilde{u}_t + \rho_0 \tilde{u}\tilde{u}_x + \tilde{\rho}\tilde{u}_t + \tilde{\rho}\tilde{u}\tilde{u}_x + f'(\rho_0)(\rho_0 + \tilde{\rho})_x$$
$$+ f''(\xi)\tilde{\rho}(\rho_0 + \tilde{\rho})_x$$
$$= 0 \qquad (2)$$

Retain only linear terms in (1). (2)

$\tilde{\rho}_t + \rho_0 \tilde{u}_x = 0$

$\rho_0 \tilde{u}_t + c_0^2 \tilde{\rho}_x = 0$

Then we have

$$(\tilde{\rho}_t + \rho_0 \tilde{u}_x)_t = \tilde{\rho}_{tt} + \rho_0 \tilde{u}_{xt} = 0 \qquad (3)$$

$$(\tilde{\rho}_t + \rho_0 \tilde{u}_x)_x = \tilde{\rho}_{tx} + \rho_0 \tilde{u}_{xx} = 0 \qquad (4)$$

$$(\rho_0 \tilde{u}_t + c_0^2 \tilde{\rho}_x)_t = \rho_0 \tilde{u}_{tt} + c_0^2 \tilde{\rho}_{xt} = 0 \qquad (5)$$

$$(\rho_0 \tilde{u}_t + c_0^2 \rho_x)_x = \rho_0 \tilde{u}_{tx} + c_0^2 \tilde{\rho}_{xx} = 0 \qquad (6)$$

(3) + (6) $\Rightarrow$ $\tilde{\rho}_{tt} + \rho_0 \tilde{u}_{xt} = \rho_0 \tilde{u}_{tx} + c_0^2 \tilde{\rho}_{xx}$

$$\tilde{\rho}_{tt} - c_0^2 \tilde{\rho}_{xx} = 0$$

(4) + (5) $\Rightarrow$ $c_0^2 \tilde{\rho}_{tx} + c_0^2 \rho_0 \tilde{u}_{xx} = \rho_0 \tilde{u}_{tt} + c_0^2 \tilde{\rho}_{xt}$

$$\tilde{u}_{tt} - c_0^2 \tilde{u}_{xx} = 0 \quad (\rho_0 \neq 0)$$

$\therefore$ $\tilde{u}$ and $\tilde{p}$ satisfy the wave equation $\psi_{tt} - c_0^2 \psi_{xx} = 0$

$\therefore$ Acoustic signals travels at speed c_0.