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HW 2-1

1. Find a traveling wave solution of the equation

$$u_{tt} - u_{xx} = -\sin u$$

with the conditions $u(+\infty, t) = 2\pi$, $u(-\infty, t) = 0$, $u_x \geq 0$, and all the partial derivatives of u with respect to x at $x = \pm\infty$ are 0.

The form of solution is $u(x,t) = f(x-ct)$, c is constant

$u_t = -cf'(x-ct)$, $u_{tt} = c^2 f''(x-ct)$

$u_x = f'(x-ct)$, $u_{xx} = f''(x-ct)$

$\because u_{tt} - u_{xx} = -\sin u$

$\therefore c^2 f'' - f'' = -\sin f \Rightarrow (c^2-1) f'' = -\sin f$

Multiply by f' on both sides, we have

$(c^2-1) f' f'' = -\sin f \cdot f'$

Let $f = f(z)$, we have

$$\frac{d}{dz}\left[(c^2-1)\frac{f'^2}{2}\right] = \frac{d}{dz}(\cos f)$$

$\Rightarrow \frac{1}{2}(c^2-1) f'^2 = \cos f + \alpha$, α is constant

The solution is

$$f(z) = 4\arctan e^{-\frac{z}{\sqrt{c^2-1}}}$$

$$\therefore u(x,t) = 4\arctan e^{-\frac{x-ct}{\sqrt{c^2-1}}}$$

And $u(x,t)$ satisfies the boundary conditions.

HW 2-2

2. Find the dispersion relation of the Klein-Gordon equation

$$u_{tt} - c^2 u_{xx} + bu = 0,$$

where c and b are constants.

The form of solution is $u(x,t) = e^{i(kx-\omega t)}$

$u_x = ike^{i(kx-\omega t)}$, $u_{xx} = -k^2 e^{i(kx-\omega t)}$

$u_t = -i\omega e^{i(kx-\omega t)}$, $u_{tt} = -\omega^2 e^{i(kx-\omega t)}$

$\because u_{tt} - c^2 u_{xx} + bu = 0$

$\therefore -\omega^2 e^{i(kx-\omega t)} + c^2k^2 e^{i(kx-\omega t)} + be^{i(kx-\omega t)} = 0$

$\therefore \omega^2 = c^2k^2 + b$

$\therefore$ The dispersion relation is $\omega = \pm\sqrt{c^2k^2+b}$

3. The nonlinear Schrodinger equation occurs in the description of water waves, nonlinear optics, and plasma physics. It is given by

$$iu_t + u_{xx} + \gamma |u^2| u = 0,$$

where $\gamma > 0$ and $u = u(x,t)$ is complex-valued.

(a) If $u = U(z)e^{i(kz-\omega t)}$, where $z = x - ct$, show that

$$\frac{d^2U}{dz^2} + i(2k - c)\frac{dU}{dz} + (kc + \omega - k^2)U + \gamma U^3 = 0.$$

(b) If $c = 2k$, show that

$$\left(\frac{dU}{dz}\right)^2 = aU^2 - \frac{\gamma}{2}U^4 + C$$

for some appropriately chosen constant a, where C is an arbitrary constant.

(c) Taking $C = 0$ and $a > 0$, show that

$$U(z) = \sqrt{\frac{2a}{\gamma}}\operatorname{sech}\left(\sqrt{a}z\right),$$

and write down the solution $u(x,t)$ of the original PDE.

(a) $u_t = -c\dfrac{dU}{dz}e^{i(kz-wt)} - i(ck+w)U(z)e^{i(kz-wt)}$

$u_x = \dfrac{dU}{dz}e^{i(kz-wt)} + ik\,U(z)e^{i(kz-wt)}$

$u_{xx} = \dfrac{d^2U}{dz^2}e^{i(kz-wt)} + 2\dfrac{dU}{dz}ike^{i(kz-wt)} - k^2U(z)e^{i(kz-wt)}$

$\because\ iu_t + u_{xx} + \gamma|u^2|u = 0$

$\therefore\ -ic\dfrac{dU}{dz}e^{i(kz-wt)} + (ck+w)U(z)e^{i(kz-wt)} + \dfrac{d^2U}{dz^2}e^{i(kz-wt)}$

$+2ik\dfrac{dU}{dz}e^{i(kz-wt)} - k^2U(z)e^{i(kz-wt)} + \gamma U^3e^{i(kz-wt)} = 0$

$\therefore\ \dfrac{d^2U}{dz^2} + i(2k-c)\dfrac{dU}{dz} + (kc+w-k^2)U + \gamma U^3 = 0$

(b) Given $c=2k$, from (a) we have

$$\frac{d^2U}{dz^2} + (k^2+w)U + \gamma U^3 = 0$$

$\Rightarrow\ u'' + (k^2+w)u + \gamma u^3 = 0$

Multiply u' on both sides,

$u''u' + (k^2+w)uu' + \gamma u^3u' = 0$

$\Rightarrow\ [(u')^2]' + (k^2+w)(u^2)' + \frac{1}{2}\gamma(u^4)' = 0$

$\Rightarrow\ (u')^2 + (k^2+w)u^2 + \frac{1}{2}\gamma u^4 = C$, C is a constant

Choose $a = -(k^2+w)$, we have

$$\left(\frac{dU}{dz}\right)^2 = aU^2 - \frac{\gamma}{2}U^4 + C$$

(c) $C=0$, $a>0$. from (b) we have

$$\left(\frac{du}{dz}\right)^2 = au^2 - \frac{\gamma}{2}u^4$$

$\Rightarrow\ u' = \pm u\sqrt{a - \frac{\gamma}{2}u^2}$

$\Rightarrow\ \dfrac{du}{u\sqrt{a-\frac{\gamma}{2}u^2}} = \pm dz$

Let $u = \operatorname{sech}^2(w)$

The ODE becomes ...

$$\frac{-2\operatorname{sech}^2(w)\cdot\tanh(w)}{\operatorname{sech}(w)\sqrt{a-\frac{r}{2}\operatorname{sech}^2(w)}} = \pm dz$$

$$\therefore\ u(z) = \sqrt{\frac{2a}{r}}\operatorname{sech}(\sqrt{a}\,z)$$

$$\because\ z = x - ct$$

$$\therefore\ u(x,t) = \sqrt{\frac{2a}{r}}\operatorname{sech}(\sqrt{a}(x-ct))$$

HW 2-3

4. Solve the initial value problem

$$u_t + 2u_x = 0, \quad x \in \mathbf{R},\ t > 0.$$

$$u(x,0) = \frac{1}{1+x^2}, \quad x \in \mathbf{R}.$$

Sketch the characteristics. Sketch the solution profiles at $t = 0$, $t = 1$, and $t = 2$.

The characteristic system is

$$\begin{cases} \dfrac{dx}{dt} = 2 \\ \dfrac{du}{dt} = 0 \end{cases}$$

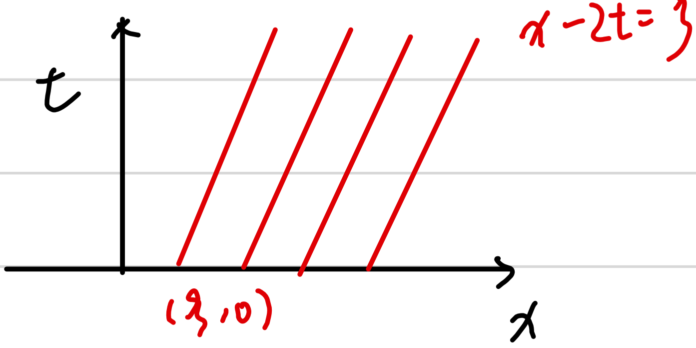
From $\frac{dx}{dt} = 2$, we have $x - 2t = \xi$ (ξ is constant)

Along the characteristic, u is constant

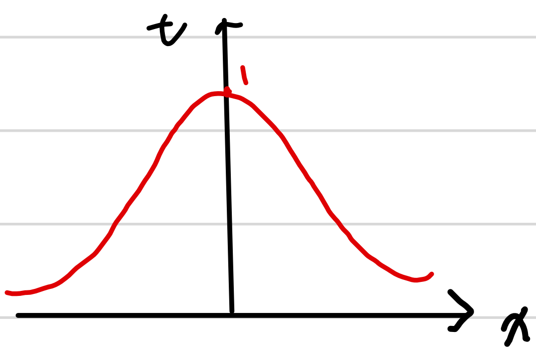
$$u(x,t) = u(\xi,0) = u_0(\xi) = \frac{1}{1+\xi^2}$$

$$= \frac{1}{1+(2t-x)^2}$$

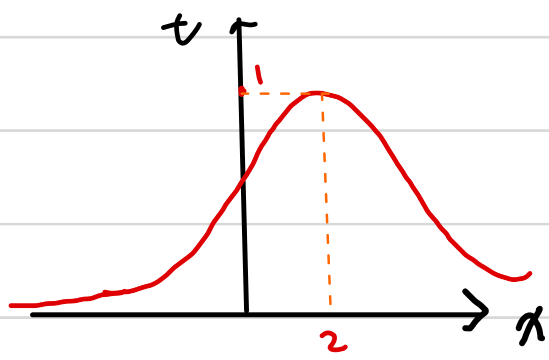
characteristic



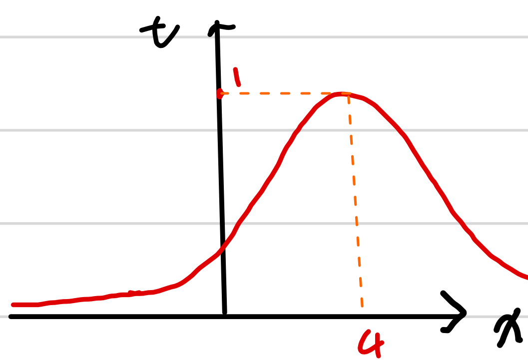
solution profile:



$t=0$ $u = \frac{1}{1+x^2}$



$t=1$ $u = \frac{1}{1+(2-x)^2}$



$t=2$ $u = \frac{1}{1+(4-x)^2}$

5. Solve the initial value problem

$$u_t - x^2 u_x = 0, \quad x \in \mathbf{R},\ t > 0.$$

$$u(x,0) = x - 1, \quad x \in \mathbf{R}.$$

The characteristic system is

$$\begin{cases} \dfrac{dx}{dt} = x^2 \\[2ex] \dfrac{du}{dt} = 0 \end{cases}$$

From $\dfrac{dx}{dt} = x^2$, we have $x = x^2 t + \xi$ (ξ is constant)

Along the characteristic, u is constant

$u(x,t) = u(\xi, 0) = u_0(\xi) = \xi - 1$

$= x - x^2 t - 1$

6. Consider the initial value problem

$$u_t + uu_x = 0, \quad x \in \mathbf{R}, \ t > 0.$$

$$u(x,0) = -x, \quad x \in \mathbf{R}.$$

Sketch the characteristic diagram and find the solution when $t < 1$.

The characteristic system is

$$\begin{cases} \frac{dx}{dt} = u \\ \frac{du}{dt} = 0 \end{cases}$$

From $\frac{dx}{dt} = u$, we have $x = ut + \xi$ (ξ is constant)

Along the characteristic, u is constant

$$u(x,t) = u(\xi,0) = u_0(\xi) = -\xi$$

$$= ut - x$$

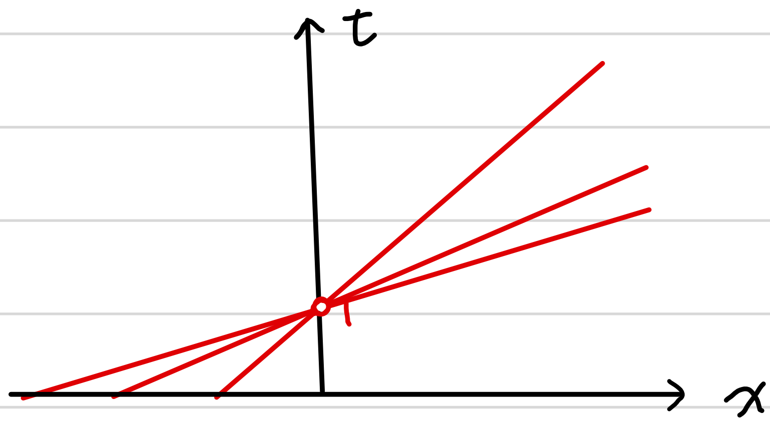
$\therefore u = ut - x$

$\therefore u = \frac{x}{t-1} \quad (t > 1)$

Characteristics:

$x = \left(\frac{x}{t-1}\right)t - \xi$

$\Rightarrow t = \frac{x}{\xi} + 1$



HW2-4

7. Solve the initial value problem

$$u_t + uu_x = -ku^2, \quad x \in \mathbf{R},\ t > 0.$$

$$u(x,0) = 1, \quad x \in \mathbf{R}.$$

where k is a positive constant. Sketch the characteristics.

The characteristic system is

$$\begin{cases} \frac{dx}{dt} = u \\ \frac{du}{dt} = -ku^2 \end{cases}$$

The initial conditions are $t=0$. $x=\xi$. $u=1$

From $\frac{dx}{dt} = u$. $t=0$. $x=\xi$, we have $x = ut+\xi$

$\frac{du}{dt} = -ku^2 \Rightarrow u = \frac{1}{kt+c}$

$\because u(x,0)=1 \quad \therefore c=1$

$\therefore u = \frac{1}{kt+1}$

Characteristics:

$x = \frac{1}{kt+1} + \xi$