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1. Consider the system

$$\begin{cases} u_t + 4u_x - 6v_x = 0 \\ v_t + u_x - 3v_x = 0 \end{cases}$$

(1) Find the eigenvalues and left eigenvectors of the coefficient matrix, and show that the system is strictly hyperbolic.

(2) Find the characteristics and the Riemann invariants.

(3) Find the general solution.

(4) Find the solution of the initial value problem of this system in $x \in \mathbf{R}$, $t > 0$, with the initial condition

$$u(x,0) = \cos x, \quad v(x,0) = e^{2x}, \quad x \in \mathbf{R}.$$

(1) The system can be written as

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 4 & -6 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The coefficient matrix is

$$A = \begin{pmatrix} 4 & -6 \\ 1 & -3 \end{pmatrix}$$

$$|\lambda I - A| = \begin{pmatrix} \lambda-4 & 6 \\ -1 & \lambda+3 \end{pmatrix} = (\lambda-3)(\lambda+2)$$

The eigenvalues are $\lambda=3$, $\lambda=-2$

- For $\lambda=3$, assume the eigenvectors are (ξ_1, ξ_2)

$$(\xi_1, \xi_2) \begin{pmatrix} -1 & 6 \\ -1 & 6 \end{pmatrix} = 0$$

A non-zero solution is $(1,-1)$

- For $\lambda=-2$, assume the eigenvectors are (ξ_1, ξ_2)

$$(\xi_1, \xi_2) \begin{pmatrix} -6 & 6 \\ -1 & 1 \end{pmatrix} = 0$$

A non-zero solution is $(1,-6)$

$\therefore$ One of the left eigenvectors of $\lambda=3$, $\lambda=-2$ are $(1,-1)$, $(1,-6)$

$\because$ A has 2 distinct eigenvalues and 2 linearly independent eigenvectors
$\therefore$ The system is strictly hyperbolic

(2) The characteristics are

$$\frac{dx}{dt} = 3$$

$$\frac{dx}{dt} = -2$$

$$\therefore \quad \frac{du}{dt} - \frac{dv}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = 3 \qquad (1)$$

$$\frac{du}{dt} - 6\frac{dv}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = -2 \qquad (2)$$

Integrating (1).(2), we have The Riemann invariants

$u - v = \text{const}$ along $x - 3t = \text{const}$

$u - 6v = \text{const}$ along $x + 2t = \text{const}$

(3) The general solution are

$$\begin{cases} u - v = f(x-3t) & (3) \\ u - 6v = g(x+2t) & (4) \end{cases} \Rightarrow \begin{cases} u = \frac{1}{5}\left(6f(x-3t) - g(x+2t)\right) \\ v = \frac{1}{5}\left(f(x-3t) - g(x+2t)\right) \end{cases}$$

f, g are arbitary functions

(4) Substituting $u(x,0) = \cos x$, $v(x,0) = e^{2x}$ into (3), (4). we have

$$f(x) = \cos x - e^{2x}$$

$$g(x) = \cos x - 6e^{2x}$$

$\therefore$ The solution of the initial value problem is

$$\begin{cases} u - v = \cos(x-3t) - e^{2(x-3t)} \\ u - 6v = \cos(x+2t) - 6e^{2(x+2t)} \end{cases}$$

$$\Rightarrow \begin{cases} u = \frac{1}{5}\left(6\cos(x-3t) - 6e^{2(x-3t)} - \cos(x+2t) + 6e^{2(x+2t)}\right) \\ v = \frac{1}{5}\left(\cos(x-3t) - e^{2(x-3t)} - \cos(x+2t) + 6e^{2(x+2t)}\right) \end{cases}$$

2. The system for the barotropic flow of gas can be written as

$$\begin{cases} \rho_t + \rho_x u + \rho u_x = 0 \\ u_t + u u_x + \frac{f'(\rho)}{\rho}\rho_x = 0. \end{cases}$$

Let $f(\rho) = k\rho^\gamma$, where constants $k > 0$ and $\gamma > 1$.

(1) Find the eigenvalues and left eigenvectors of the coefficient matrix, and show that the system is strictly hyperbolic.

(2) Find the characteristic form of the system and the Riemann invariants.

(1) The system can be written as

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ k\gamma\rho^{\gamma-2} & u \end{pmatrix}\begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The coefficient matrix is

$$A = \begin{pmatrix} u & \rho \\ k\gamma\rho^{\gamma-2} & u \end{pmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda - u & -\rho \\ -k\gamma\rho^{\gamma-2} & \lambda - u \end{vmatrix} = (\lambda-u)^2 - k\gamma\rho^{\gamma-1} = 0$$

The eigenvalues are $\lambda = u + \sqrt{k\gamma\rho^{\gamma-1}}$, $\lambda = u - \sqrt{k\gamma\rho^{\gamma-1}}$

- For $\lambda = u + \sqrt{k\gamma\rho^{\gamma-1}}$, assume the eigenvectors are (ξ_1, ξ_2)

$$(\xi_1, \xi_2)\begin{pmatrix} \sqrt{k\gamma\rho^{\gamma-1}} & -\rho \\ -k\gamma\rho^{\gamma-2} & \sqrt{k\gamma\rho^{\gamma-1}} \end{pmatrix} = 0$$

A non-zero solution is $(\sqrt{k\gamma\rho^{\gamma-1}}, \rho)$

- For $\lambda = u - \sqrt{k\gamma\rho^{\gamma-1}}$, assume the eigenvectors are (ξ_1, ξ_2)

$$(\xi_1, \xi_2)\begin{pmatrix} -\sqrt{k\gamma\rho^{\gamma-1}} & -\rho \\ -k\gamma\rho^{\gamma-2} & -\sqrt{k\gamma\rho^{\gamma-1}} \end{pmatrix} = 0$$

A non-zero solution is $(\sqrt{k\gamma\rho^{\gamma-1}}, -\rho)$

$\therefore$ The left eigenvectors are $(\sqrt{k\gamma\rho^{\gamma-1}}, \rho)$ $(\sqrt{k\gamma\rho^{\gamma-1}}, -\rho)$

$\because$ A has 2 distinct eigenvalues and 2 linearly independent eigenvectors

$\therefore$ The system is strictly hyperbolic

(2) The characterics are

$$\frac{dx}{dt} = u + \sqrt{k\gamma\rho^{\gamma-1}}$$

$$\frac{dx}{dt} = u - \sqrt{k\gamma\rho^{\gamma-1}}$$

$$\therefore \sqrt{k\gamma\rho^{\gamma-1}}\,\frac{d\rho}{dt} + \rho\,\frac{du}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = u + \sqrt{k\gamma\rho^{\gamma-1}} \qquad (1)$$

$$\sqrt{k\gamma\rho^{\gamma-1}}\,\frac{d\rho}{dt} - \rho\,\frac{du}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = u - \sqrt{k\gamma\rho^{\gamma-1}} \qquad (2)$$

Multiply ρ^{-1} on both sides, we have

$$\sqrt{k\gamma\rho^{\gamma-3}}\,\frac{d\rho}{dt} + \frac{du}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = u + \sqrt{k\gamma\rho^{\gamma-1}} \qquad (3)$$

$$\sqrt{k\gamma\rho^{\gamma-3}}\,\frac{d\rho}{dt} - \frac{du}{dt} = 0 \quad \text{along} \quad \frac{dx}{dt} = u - \sqrt{k\gamma\rho^{\gamma-1}} \qquad (4)$$

Integrating (3).(4), we have the Riemann invariants.

$$\frac{2\sqrt{k\gamma\rho^{\gamma-1}}}{\gamma-1} + u = \text{const}$$

$$\frac{2\sqrt{k\gamma\rho^{\gamma-1}}}{\gamma-1} - u = \text{const}$$

3. The shallow water equations

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + (hu^2 + \frac{g}{2}h^2)_x = 0 \end{cases}$$

are conservation laws in dimensioned variables. Select new dimensionless independent and dependent variables according to

$$\bar{x} = \frac{x}{L}, \ \bar{t} = \frac{t}{T}, \ \bar{h} = \frac{h}{H}, \ \bar{u} = \frac{u}{\sqrt{gH}}$$

where $T = L/\sqrt{gH}$, and where H is the undisturbed water height and L is a typical wavelength. Rewrite the shallow water equations in terms of the dimensionless variables.

$$\bar{h} = \frac{h}{H} \Rightarrow \frac{\partial h}{\partial \bar{h}} = H, \qquad \bar{t} = \frac{t}{T} \Rightarrow \frac{\partial t}{\partial \bar{t}} = T$$

$$\bar{x} = \frac{x}{L} \Rightarrow \frac{\partial x}{\partial \bar{x}} = L, \qquad \bar{u} = \frac{u}{\sqrt{gH}} \Rightarrow \frac{\partial u}{\partial \bar{u}} = \sqrt{gH}$$

$$h_t = \frac{\partial h}{\partial t} = \frac{\partial h}{\partial \bar{h}} \cdot \frac{\partial \bar{h}}{\partial \bar{t}} \cdot \frac{\partial \bar{t}}{\partial t}$$

$$= H \cdot \frac{\partial \bar{h}}{\partial \bar{t}} \cdot \frac{1}{T}$$

$$= \frac{H}{T} \bar{h}_{\bar{t}}$$

$$\because T = \frac{L}{\sqrt{gH}}$$

$$\therefore h_t = \frac{H\sqrt{gH}}{L} \bar{h}_{\bar{t}} \qquad (1)$$

$$(hu)_x = (H\sqrt{gH}\,\bar{h}\bar{u})_x$$

$$= H\sqrt{gH} \frac{\partial \bar{h}\bar{u}}{\partial x}$$

$$= H\sqrt{gH} \frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} \frac{\partial \bar{x}}{\partial x}$$

$$= H\sqrt{gH} \frac{\partial \bar{h}\bar{u}}{\partial \bar{x}} \cdot \frac{1}{L}$$

$$= \frac{H\sqrt{gH}}{L} (\bar{h}\bar{u})_{\bar{x}} \qquad (2)$$

$$(hu)_t = (H\sqrt{gH}\,\bar{h}\bar{u})_t$$

$$= H\sqrt{gH} \frac{\partial \bar{h}\bar{u}}{\partial t}$$

$$= H\sqrt{gH} \frac{\partial \bar{h}\bar{u}}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t}$$

$$= \frac{H\sqrt{gH}}{T} (\bar{h}\bar{u})_{\bar{t}} \qquad (3)$$

$$(hu^2 + \tfrac{g}{2}h^2)_x = (gH^2\bar{h}\bar{u}^2 + \tfrac{g}{2}H^2\bar{h}^2)_x$$

$$= gH^2 \frac{\partial \bar{h}\bar{u}^2}{\partial x} + \frac{gH^2}{2} \frac{\partial \bar{h}^2}{\partial x}$$

$$= gH^2 \frac{\partial \bar{h}\bar{u}^2}{\partial \bar{x}} \frac{d\bar{x}}{dx} + \frac{gH^2}{2} \frac{\partial \bar{h}^2}{\partial \bar{x}} \cdot \frac{d\bar{x}}{dx}$$

$$= \frac{gH^2}{L} (\bar{h}\bar{u}^2)_x + \frac{gH^2}{2L} (\bar{h}^2)_{\bar{x}} \qquad (4)$$

Substitute (1) ~ (4) into the given equations, we have

$$\begin{cases} \frac{H\sqrt{gH}}{L} \bar{h}_{\bar{t}} + \frac{H\sqrt{gH}}{L} (\bar{h}\bar{u})_{\bar{x}} = 0 \\ \frac{gH^2}{L} (\bar{h}\bar{u})_{\bar{t}} + \frac{gH^2}{L} (\bar{h}\bar{u}^2)_x + \frac{gH^2}{2L} (\bar{h}^2)_{\bar{x}} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \bar{h}_{\bar{t}} + (\bar{h}\bar{u})_{\bar{x}} = 0 \\ (\bar{h}\bar{u})_{\bar{t}} + (\bar{h}\bar{u}^2)_{\bar{x}} + \frac{1}{2}(\bar{h}^2)_{\bar{x}} = 0 \end{cases}$$

4. The purpose of this problem is to perform a weakly nonlinear analysis on a simple nonlinear equations. Consider the scalar equation

$$u_t + \phi(u)_x = \mu u_{xx} \qquad (1)$$

where t, x and u are dimensionless variables of order 1, $\mu = \mu_1\epsilon^2 + O(\epsilon^3)$, $\epsilon << 1$, and $\phi(u)$ can be expanded in a Taylor series about $u = u_0 =$ constant. Let $x = X(t)$ be a representative location on a wave that is propagating into the uniform state u_0. Introduce the variable $\xi = (x - X(t))/\epsilon$ and assume that

$$u(\xi, t) = u_0 + \epsilon u_1(\xi, t) + \epsilon^2 u_2(\xi, t) + O(\epsilon^3) \qquad (2)$$

$$D(t) = D_0(t) + \epsilon D_1(t) + O(\epsilon^2) \qquad (3)$$

where $D(t) = X'(t)$. Show that u_1 satisfies the Burgers-like equation

$$u_{1\tau} + \left(\frac{\alpha}{2}u_1^2\right)_\eta = \mu_1 u_{1\eta\eta}$$

where η and τ are suitable spatial and time variables, respectively, and α is a constant.

Let $\tau = t$.

From the chain rule, we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{X'(t)}{\varepsilon}u_\xi + u_\tau = -\frac{D(t)}{\varepsilon}u_\xi + u_\tau$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{\varepsilon}\frac{\partial u}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x}\left(\frac{1}{\varepsilon}\frac{\partial u}{\partial \xi}\right) = \frac{1}{\varepsilon^2}u_{\xi\xi}$$

$$\frac{\partial \phi(u)}{\partial x} = \frac{\partial \phi(u)}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{\varepsilon}\phi_\xi(u)$$

Substituting them into (1), we have

$$-\frac{D(\tau)}{\varepsilon}u_\xi + u_\tau + \frac{1}{\varepsilon}(\phi(u))_\xi = \frac{1}{\varepsilon^2}\mu u_{\xi\xi} \qquad (4)$$

Substituting (2), (3) into (4), we have

$$-(D_0(\tau) + \varepsilon D_1(\tau) + O(\varepsilon^2))(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3))_\xi +$$

$$\varepsilon(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3))_\tau +$$

$$(\phi(u_0) + \phi'(u_0)(\varepsilon u_1 + \varepsilon^2 u_2) + \tfrac{1}{2}(\phi''(u_0)(\varepsilon u_1 + \varepsilon^2 u_2)^2)_\xi$$

$$= \frac{1}{\varepsilon}(\mu_1\varepsilon^2 + O(\varepsilon^3))(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + O(\varepsilon^3))_{\xi\xi}$$

Consider the coefficient of ε, we have

$$-D_1(\tau)u_{0\xi} - D_0(\tau)u_{1\xi} + u_{0t} + \phi'(u_0)u_{1\xi} = 0$$

$\because$ u_0 is constant $\therefore$ $u_{0\xi} = 0$

$\therefore$ $-D_0(\tau)u_{1\xi} + \phi'(u_0)u_{1\xi} = 0$

$\therefore$ $-D_0(\tau) + \phi'(u_0) = 0$

Consider the coefficient of ε^2

$\because$ u_0 is constant $\therefore$ $u_{0\xi} = 0$, $u_{0\xi} = 0$

we have

$$-D_0(\tau)u_{2\xi} - D_1(\tau)u_{1\xi} + u_{1\tau} + (\phi'(u_0)u_2 + \tfrac{1}{2}\phi''(u_0)u_1^2)_\xi = \mu_1 u_{1\xi\xi}$$

$\therefore$ $-D_1(\tau)u_{1\xi} + u_{1\tau} + \phi''(u_0)u_1u_{1\xi} = \mu_1 u_{1\xi\xi}$ (*)

Assume that $\eta = \xi + f(\tau)$

$$\frac{\partial u_1}{\partial \xi} = \frac{\partial u_1}{\partial \eta}\frac{\partial \eta}{\partial \xi} = u_{1\eta}$$

$$\frac{\partial^2 u_1}{\partial \xi^2} = \frac{\partial}{\partial \xi}\left(\frac{\partial u_1}{\partial \xi}\right) = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial \xi}\left(\frac{\partial u_1}{\partial \xi}\right) = u_{1\eta\eta}$$

$$\frac{\partial u_1}{\partial \tau} = \frac{\partial u_1}{\partial \eta}\frac{\partial \eta}{\partial \tau} + \frac{\partial u_1}{\partial \tau}\frac{\partial \tau}{\partial t} = u_{1\eta}f'(t) + u_{1\tau}$$

Substuting them into (*), we have

$$-D_1(\tau)u_{1\eta} + u_{1\eta}f'(\tau) + u_{1\tau} + \left(\frac{\phi''(u_0)}{2}u_1^2\right)_\eta = \mu_1 u_{1\eta\eta}$$

Let $f'(\tau) = D_1\tau$

then $f(\tau) = \int_0^\tau D_1(s)ds$

then $\eta = \xi + \int_0^\tau D_1(s)ds$

Therefore,

$$u_{1\tau} + \tfrac{1}{2}(f''(u_0)u_1^2)_\eta = \mu_1 u_{1\eta\eta}$$

Let $\alpha = f''(u_0)$. Then α is constant.

5. Find the Euler-Lagrange equation of the following variational problem:

$$\begin{cases} u \in K \\ J(u) = \min\limits_{w \in K} J(w) \end{cases}$$

where $K = \{w | w \in C^2[a,b] \text{ with periodic boundary condition}\}$, and

$$J(u) = \int_a^b \left[(u - u_0)^2 + (u_x)^2 + (u_{xx})^2\right] dx,$$

where u_0 is a given function.

Warning: No credit will be given if you use some formula that is not given in the course lectures.

Assume that u is the solution of $\min\limits_{w \in K} J(w)$

Then $\forall w \in K$. $\forall \varepsilon \in \mathbb{R}$, we have $u + \varepsilon w \in K$

Then $J(u) \le J(u+\varepsilon w)$

Denote $\phi(\varepsilon) = J(u+\varepsilon w)$. we have $\phi(0) \le \phi(\varepsilon)$

$\therefore$ $\phi(0)$ is the minimum of $\phi(\varepsilon)$ $\quad \therefore \phi'(0) = 0$

$$\phi(\varepsilon) = J(u+\varepsilon w)$$

$$= \int_a^b [(u+\varepsilon w - u_0)^2 + (u+\varepsilon w)_x^2 + (u+\varepsilon w)_{xx}^2] dx$$

$$= \int_a^b [(u-u_0)^2 + u_x^2 + u_{xx}^2] dx + 2\varepsilon \int_a^b (uw - u_0 w + u_x w_x + u_{xx} w_{xx}) dx$$

$$+ \varepsilon^2 \int_a^b (w^2 + w_x^2 + w_{xx}^2) dx$$

$$\phi'(\varepsilon) = 2\int_a^b (uw - u_0 w + u_x w_x + u_{xx} w_{xx}) dx + 2\varepsilon \int_a^b (w^2 + w_x^2 + w_{xx}^2) dx$$

$$\phi'(0) = 2\int_a^b (uw - u_0 w + u_x w_x + u_{xx} w_{xx}) dx \quad (*)$$

From integration by parts, we have

$$\int_a^b u_x w_x \, dx = u_x w \Big|_a^b - \int_a^b u_{xx} w \, dx$$

$$= 0 - \int_a^b u_{xx} w \, dx$$

$$= \int_a^b u_{xx} w \, dx$$

$$\int_a^b u_{xx} w_{xx} \, dx = u_{xx} w_x \Big|_a^b - \int_a^b u_{xxx} w_x \, dx$$

$$= 0 - \int_a^b u_{xxx} w_x \, dx$$

$$= -\left(u_{xxx} w \Big|_a^b - \int_a^b u_{xxxx} w \, dx\right)$$

$$= \int_a^b u_{xxxx} w \, dx$$

Substituting them into (*), we have

$$\int_a^b (u - u_0 - u_{xx} + u_{xxxx}) w \, dx$$

$\because$ w is arbitary

$\therefore$ $u - u_0 - u_{xx} + u_{xxxx} = 0$ — Euler-Lagrange Eq.

6. Consider the problem

$$xu_t - u_{xx} = 0, \quad x > 0, \ t > 0, \qquad (1)$$
$$u(x, 0) = 0, \ x > 0, \qquad (2)$$
$$u(\infty, t) = 0, \ t > 0, \qquad (3)$$
$$u_x(0, t) = -1, \ t > 0. \qquad (4)$$

Find (1) the form of a similarity solution and (2) the ODE satisfied by it.

(1) Let $\bar{u}(x,t) = \varepsilon^c u\left(\frac{x}{\varepsilon^a}, \frac{t}{\varepsilon^b}\right)$

We look for a, b, c s.t. when $u(x,t)$ is a solution, $\bar{u}(x,t)$ is also a solution.

Then,

$$\bar{u}_t = \varepsilon^{c-b} u_t$$
$$\bar{u}_x = \varepsilon^{c-a} u_x$$
$$\bar{u}_{xx} = \varepsilon^{c-2a} u_{xx}$$

For $\bar{u}$ to be a solution of (1), we have

$$x \cdot \varepsilon^{c-b} u_t - \varepsilon^{c-2a} u_{xx} = 0$$

i.e. $\varepsilon^{a-b+c} \cdot \frac{x}{\varepsilon^a} u_t - \varepsilon^{c-2a} u_{xx} = 0$

Since u is a solution, i.e. $xu_t - u_{xx} = 0$

$\bar{u}$ is also a solution if $b = 3a$

Therefore, when $b = 3a$, Eq. (1) has a similarity solution, which takes the form

$$u(x,t) = t^{\frac{c}{3a}} y(z)$$
$$z = xt^{-\frac{1}{3}}$$

(2) From (1), $u(x,t) = t^{\frac{c}{3a}} y(z) = t^{\frac{c}{3a}} y(xt^{-\frac{1}{3}})$

$$u_t = \frac{c}{3a} t^{\frac{c}{3a}-1} y(xt^{-\frac{1}{3}}) - \frac{1}{3} x t^{\frac{c}{3a}-\frac{4}{3}} y'(xt^{-\frac{1}{3}})$$

$$u_x = t^{\frac{c}{3a}-\frac{1}{3}} y'(xt^{-\frac{1}{3}})$$
$$u_{xx} = t^{\frac{c}{3a}-\frac{2}{3}} y''(xt^{-\frac{1}{3}})$$

Substitude them into (1), we have

$$\frac{c}{3a} x t^{\frac{c}{3a}-1} y(xt^{-\frac{1}{3}}) - \frac{1}{3} x^2 t^{\frac{c}{3a}-\frac{4}{3}} y'(xt^{-\frac{1}{3}}) - t^{\frac{c}{3a}-\frac{2}{3}} y''(xt^{-\frac{1}{3}}) = 0$$

$$\Rightarrow \frac{c}{3a} x t^{-\frac{1}{3}} y(xt^{-\frac{1}{3}}) - \frac{1}{3} x^2 t^{-\frac{2}{3}} y'(xt^{-\frac{1}{3}}) - y''(xt^{-\frac{1}{3}}) = 0$$

$\because z = xt^{-\frac{1}{3}}$

$\therefore \frac{c}{3a} z y(z) - \frac{1}{3} z^2 y'(z) - y''(z) = 0$

$\because u(x,t) = t^{\frac{c}{b}} y\left(\frac{x}{t^{\frac{c}{b}}}\right)$

From (2), (3), (4) we have $y(\infty) = 0$. $y'(0) = -1$

$\therefore$ The ODE is

$$\frac{c}{3a} z y(z) - \frac{1}{3} z^2 y'(z) - y''(z) = 0 \quad , z>0$$ -1.5

$y(\infty) = 0$. $y'(0) = -1$ X

boundary condition $a=c$

$u(x,0) = t^{\frac{c}{b}} y(\infty) = t^{\frac{c}{3a}} y(\infty) = 0$

$u_x(x,t) = [t^{\frac{c}{3a}} y(xt^{-\frac{1}{3}})]'$

$= t^{\frac{c}{3a}-\frac{1}{3}} y'(xt^{-\frac{1}{3}})$

$u_x(0,t) = t^{\frac{c}{3a}-\frac{1}{3}} y'(0) = -1$

7. Consider the evolution equation for a crystal surface $h(x,t)$:

$$h_t = -\frac{\partial}{\partial x}\left(\frac{h_x}{1+l_d h_x^2}\right) - \gamma h_{xxxx},$$

where γ and l_d are parameters (positive). Study the linear instability of a flat surface $h(x,t) = h_0$, where h_0 is a constant height.

(1) Find and sketch the dispersion relation for small perturbations.

(2) Discuss for perturbations with what wavenumers the flat surface is stable or unstable, and determine the most unstable wavenumber of the perturbations.

(1) Assume that v is a small pertubation, $v \ll 1$

$h = h_0 + v$

$\because \ h(t) = -\frac{\partial}{\partial x}\left(\frac{h_x}{1+l_d h_x^2}\right) - \gamma h_{xxxx}$

$\therefore \ (h_0+v)(t) = -\frac{\partial}{\partial x}\frac{(h_0+v)_x}{1+l_d(h_0+v)_x^2} - \gamma(h_0+v)_{xxxx}$

$\Rightarrow \ v_t = -\frac{\partial}{\partial x}\left(\frac{v_x}{1+l_d v_x^2}\right) - \gamma v_{xxxx}$

$\Rightarrow \ v_t \approx -v_{xx} - \gamma v_{xxxx}$ (1)

(eliminate v_x^2 term because it's rather small)

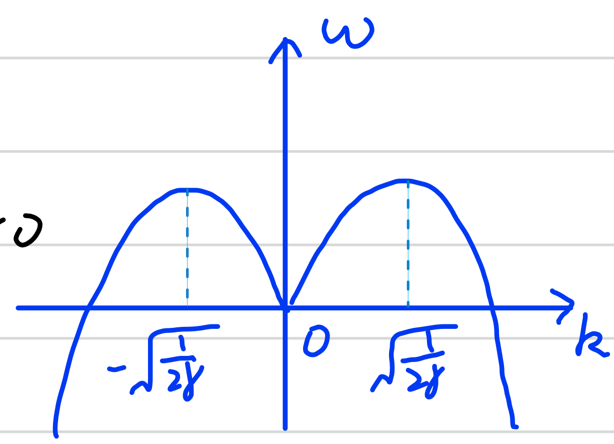
Let $v(x,t) = \varepsilon e^{ikx+wt}$

substitute it into (1), we have

$$w\varepsilon e^{ikx+wt} = k^2\varepsilon e^{ikx+wt} - \gamma k^4 \varepsilon e^{ikx+wt}$$

$\therefore$ The dispersion relation is

$$w = k^2 - \gamma k^4$$



(2) The solution is stable when $w(k) = k^2 - \gamma k^4 < 0$

$\Rightarrow \ |k| > \frac{\sqrt{\gamma}}{\gamma}$

It is unstable when $|k| < \frac{\sqrt{\gamma}}{\gamma}$

$\because$ $w(k)$ attains its positive maximum at $k = \pm\sqrt{\frac{1}{2\gamma}}$

$\therefore$ The most unstable wave number is $k = \pm\sqrt{\frac{1}{2\gamma}}$