Chap. 1 Introduction

§ 1.1 Partial Differential Equations

1. PDEs, order and solutions

PDE: an equation envolving an unknown function of 2 or more variables and its partial derivations

e.g. unknown function

① $u_t = u_{xx}$ $\quad \left(\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u_t(x,t) = u_{xx}(x,t) \right)$ $\quad u(x,t)$

② $u_t + u_{xx} + u_{xxx} = 0$ $\quad u(x,t)$

③ $u_{xx} + u_{yy} + u - u^3 = 0$ $\quad u(x,y)$

Compared with ODEs

ODE: an unknown function of 1 variable

e.g.

$y''(x) + x y'(x) = 0$ $\quad y(x)$

order of a PDE

order of the highest partial derivation

e.g.

① second order

② third order

③ second order

Solution of a PDE

A function that has all the partial derivatives in the PDE, and satisfies the PDE

e.g.

$u_t + u u_x = 0 \qquad (x,t) \in D$

Solution $u(x,t)$:

- classical solution

  u_t, u_x exist in D

  u satisfies the equation in D

- weak solution (generalized solution)

  At some points in D, u_t or u_x does not exist,

  and u satisfies a generalized form of the PDE

2. Linear and nonlinear PDEs

linear PDE: linear in the unknown function and its partial dervations

e.g.

$u_t = u_{xx}$ diffussion equation — 扩散Eq

$u_{tt} = u_{xx}$ wave equation

$u_{xx} + u_{yy} = 0$ Laplace equation

review

波动方程(Wave Eq) : $u_{tt} = a^2 u_{xx} + f(x,t)$

热传导方程(Heat Eq) : $u_t = a^2 u_{xx} + f(x,t)$

位势方程(Potential Eq) : $u_{xx} + u_{yy} = f(x,y)$

高维情形 $u_{xx} \to \Delta u$ $\left(\Delta u = \sum_{k=1}^{n} u_{kk}\right)$

nonlinear PDE: otherwise

Principle of superposition 叠加原理

u_1 : a solution of $u_t - u_{xx} = f_1(x,t)$

u_2 : a solution of $u_t - u_{xx} = f_2(x,t)$

$\Rightarrow$ $c_1u_1 + c_2u_2$, where c_1 and c_2 are constants,

is a solution of $u_t - u_{xx} = c_1 f_1(x,t) + c_2 f_2(x,t)$

For linear PDEs, most solution methods are based on principle of superposition.

e.g.

Fourier transform

Green's function

eigenfunction expansion (method of separation of variables)

For nonlinear PDEs, Principle of superposition does not hold.

e.g.

u_1 and u_2 are solutions of $u_t + uu_x = 0$

$\not\Rightarrow$ $u_1 + u_2$ is a solution of $u_t + uu_x = 0$

$\not\Rightarrow$ cu_1 (c is a constant)

Methods based of Principle of superposition do not apply to nonlinear PDEs.

Methods for nonlinear PDEs

Analytical methods

characteristic methods

similaritic methods

transform methods transform a nonlinear PDE to a linear PDE

travelling wave solutions

steady state solutions

... ...

perturbation methods MATH 5352

Numerical methods MATH 5351

Prerequisites

Multivarible calculus

Linear algebra

ODEs

Linear PDEs (preferred)

Examples of ODEs

① $\frac{df}{dx} + 2f = x$

$\times\ e^{2x}$ integration factor

$\Rightarrow e^{2x}\frac{df}{dx} + 2e^{2x}f = xe^{2x}$

$\Rightarrow \frac{d}{dx}(e^{2x}f) = e^{2x}$

$\Rightarrow e^{2x}f = \int xe^{2x}dx$

$= \int x\, d(\frac{1}{2}e^{2x})$

$= \frac{x}{2}e^{2x} - \int \frac{1}{2}e^{2x}dx$

$= \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C$

$f(x) = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$

§1.2 Fundamental types of PDEs   HW DDL: P-18

1. Three fundamental types

(1) Hyperbolic equations 双曲方程

e.g. wave equation $u_{tt} - u_{xx} = 0$

(2) Parabolic equations 抛物方程

e.g. diffusion equation $u_t - u_{xx} = 0$

$u_t - (u_{xx} + u_{yy}) = 0$

(3) Ellyptic equations 椭圆方程

e.g. Poison equation $u_{xx} + u_{yy} = f(x,y)$

There are other PDEs do not belong to these types.

e.g. Shrödinger equation $u_t = i u_{xx}$

只有上述三种分类

2. Classification of second order PDEs

Determined by terms of the highest derivatives.

Consider second order PDE of the form

$$a u_{xx} + 2b u_{xt} + c u_{tt} = d(x, t, u, u_x, u_t) \qquad (x,t) \in D \qquad (1)$$

where $a = a(x,t)$, $b = b(x,t)$, $c = c(x,t)$

Look for simpler form by change of variables.

$(x,t) \to (\xi, \eta)$

$$\begin{cases} \xi = \xi(x,t) \\ \eta = \eta(x,t) \end{cases}$$

Requirement: one to one (invertible)

$$\Leftrightarrow \quad J \overset{\frac{\partial(\xi,\eta)}{\partial(x,y)}}{=} \underbrace{\begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix}} = \xi_x \eta_t - \xi_t \eta_x \neq 0$$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_{xx} = (u_x)_x$

$= (u_\xi \xi_x + u_\eta \eta_x)_x$

$= \frac{\partial u_\xi}{\partial x}\xi_x + u_\xi \xi_{xx} + \frac{\partial u_\eta}{\partial x}\eta_x + u_\eta \eta_{xx}$

$= \underline{(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x} + u_\xi \xi_{xx} + \underline{(u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x)\eta_x} + u_\eta \eta_{xx}$

$= \xi_x^2 u_{\xi\xi} + 2\xi_x\eta_x u_{\xi\eta} + \eta_x^2 u_{\eta\eta} + \xi_{xx}u_\xi + \eta_{xx}u_\eta$

The equation becomes

$$A u_{\xi\xi} + 2B u_{\xi\eta} + C u_{\eta\eta} = D(\xi,\eta,u,u_\xi,u_\eta) \qquad (2)$$

where $A = a\xi_x^2 + 2b\xi_x\xi_t + c\xi_t^2$ ①

$B = a\xi_x\eta_x + b(\xi_x\eta_t + \xi_t\eta_x) + c\xi_t\eta_t$ ②

$C = a\eta_x^2 + 2b\eta_x\eta_t + c\eta_t^2$ ③

$\Delta(\xi,\eta) = B^2 - AC$

$= (\xi_x\eta_t - \xi_t\eta_x)^2(b^2-ac)$

$= J^2\Delta(x,t)$

$\because J>0 \quad \therefore \Delta$的符号为不变量

we want to have $A=0$, $C=0$

The two equations ① ③ have the same form

$$a\varphi_x^2 + 2b\varphi_x\varphi_t + c\varphi_t^2 = 0$$

or

$$a\left(\frac{\varphi_x}{\varphi_t}\right)^2 + 2b\frac{\varphi_x}{\varphi_t} + c = 0 \qquad (3)$$

(1) when $b^2 - ac > 0$. it has two solutions

$$\frac{\varphi_x}{\varphi_t} = -\frac{b \pm \sqrt{b^2-ac}}{a}.$$

we can choose

$$\frac{\xi_x}{\xi_t} = -\frac{b+\sqrt{b^2-ac}}{a}, \quad \frac{\eta_x}{\eta_t} = -\frac{b-\sqrt{b^2-ac}}{a}$$

In this case, the equation becomes $(A=C=0,\ B\neq 0)$

$$u_{\xi\eta} = D_1(\xi,\eta,u,u_\xi,u_\eta) \qquad (4)$$

and Eq. (1) is hyperbolic. 双曲形

Eq (4) is canonical form of Eq. (1) 标准形

e.g. $u_{tt} - u_{xx} = 0$

$a=-1$. $b=0$. $c=1$

$b^2 - ac = 1 > 0$

$\frac{\xi_x}{\xi_t} = 1$. $\frac{\eta_x}{\eta_t} = -1$ $\Rightarrow \begin{cases} \xi(x,t) = x-t \\ \eta(x,t) = x+t \end{cases}$

canonical form is $u_{\xi\eta} = 0$ $A=0$. $C=0$

(2) When $b^2 - ac = 0$. Eq(1) is parabolic. 抛物形

Eq(3) has one solution $\frac{\xi_x}{\xi_t} = -\frac{b}{a}$

Further choose $\eta(x,t)$ that

$$J = \begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} \neq 0$$

the canonical form 代入:

$$u_{\eta\eta} = D_2(\xi, \eta, u_\xi, u_\eta) \qquad (5)$$

($A=0$. $B=0$? $C\neq 0$?) from $J \neq 0$

from calculation (略)

e.g. $u_t = D u_{xx}$ $D>0$

(3) When $b^2 - ac < 0$. Eq.(1) is elliptic.

Eq.(3) has no real solution but 2 complex solutions

$$\frac{\xi_x}{\xi_t} = -\frac{b + i\sqrt{ac-b^2}}{a}, \quad \frac{\eta_x}{\eta_t} = -\frac{b - i\sqrt{ac-b^2}}{a}$$

Let real functions

$$\alpha(x,t) = \frac{\xi(x,t) + \eta(x,t)}{2}, \quad \beta(x,t) = \frac{\xi(x,t) - \eta(x,t)}{2i}$$

The canonical form is

$$u_{\alpha\alpha} + u_{\beta\beta} = D_3(\alpha, \beta, u, u_\alpha, u_\beta) \qquad (6)$$

e.g. $u_{xx} + u_{yy} = f(x,y)$ Poisson Eq.

§1.3 Conservation law

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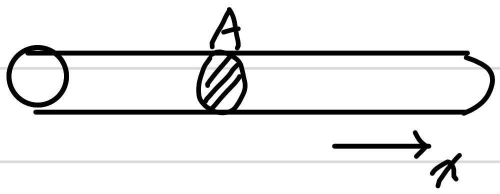
Some quantity is balanced through out a process

e.g. fluid. heat, amount of species...

1. Conservation law in one dimension

Physical background: quantity in a long tube

(uniform in crossection (area A))



motion of the quantity is only in x direction

density of the quantity : $u(x,t)$

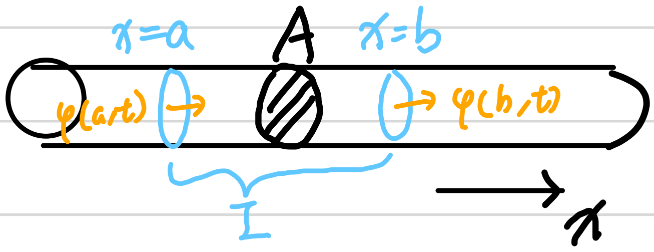
流量 flux : $\varphi(x,t)$

amount of the quantity passing through the crossection per unit area per unit time

$\varphi>0$ means the flux is in the $+x$ direction

source/sink : $f(x,t,u)$

quantity created or destroyed per unit volume per unit time



Consider an interval $I=[a,b]$ at time t

Total amount of the quantity in I : $\int_a^b u(x,t)A\,dx$

rate of change of the amount in I : $\frac{d}{dt}\int_a^b u(x,t)A\,dx$

On the other hand,

the amount that flows into I : $\varphi(a,t)A-\varphi(b,t)A$

in   out

amount that is produced in I: $\int_a^b f(x,t,u) A\, dx$

change of the amount in I

= amount that flows into I + amount produced in I

$$\frac{d}{dt}\int_a^b u(x,t) A\, dx = [\varphi(a,t)A - \varphi(b,t)A] + \int_a^b f(x,t,u) A\, dx$$

or

$$\frac{d}{dt}\int_a^b u(x,t)\, dx = [\varphi(a,t) - \varphi(b,t)] + \int_a^b f(x,t,u)\, dx$$

— Conservation law in integral form

Under some smoothness conditions

(e.g. u and φ are continuous differentiable, f continuous)

$$\frac{d}{dt}\int_a^b u(x,t)\, dx = \int_a^b u_t(x,t)\, dx$$

$$\varphi(a,t) - \varphi(b,t) = -\int_a^b \varphi_x(x,t)\, dx$$

The conservation law becomes

$$\int_a^b u_t(x,t)\, dx = -\int_a^b \varphi_x(x,t)\, dx + \int_a^b f(x,t,u)\, dx$$

Then for any inteval $[a,b]$ + continuity

$$u_t(x,t) = -\varphi_x(x,t) + f(x,t,u)$$

$$u_t + \varphi_x = f(x,t,u)$$

— A conservation law in differential form

Sometimes, we call

$u_t + \varphi_x = 0$ conservation law

$u_t + \varphi_x = f$ conservation law in source

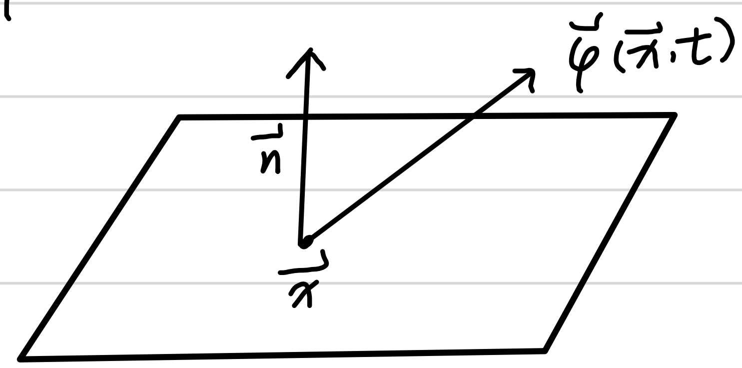
$\varphi = ?$ depends on physics

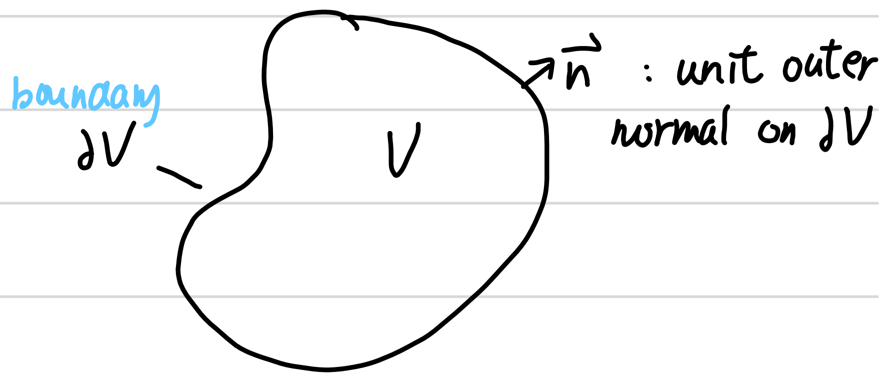
2. Conservation laws in higher dimensions

density of the quantity : $u(\vec{x},t)$. $\vec{x}=(x_1,x_2,x_3)\in\mathbb{R}^3$, $t>0$

flux : $\vec{\varphi}(\vec{x},t)$

Given a plane at $\vec{x}$ with normal $\vec{n}$, the amount passing through point $\vec{x}$ at time t per unit area per unit time in the direction of $\vec{n}$ is $\vec{\varphi}\cdot\vec{n}$.





Consider a volume V.

$$\frac{d}{dt}\int_V u(\vec{x},t)\,dV = -\int_{\partial V}\vec{\varphi}(\vec{x},t)\cdot\vec{n}\,dS + \int_V f(\vec{x},t,u)\,dV$$

? rate of change of total amount in V — quantity that flows in — source

— Conservation law in integral form

Using $\frac{d}{dt}\int_V(\vec{x},t)\,dV = \int_V u_t(\vec{x},t)\,dV$

$\int_{\partial V}\vec{\varphi}(\vec{x},t)\vec{n}\,dS = \int_V \nabla\cdot\vec{\varphi}\,dV$ — divergence Thm

with some smoothness conditions, we have

$$\int_V u_t\,dV = -\int_V \nabla\cdot\vec{\varphi}\,dV + \int_V f\,dV$$

for any V + continuity

$$u_t + \nabla\cdot\vec{\varphi} = f$$

— Conservation law in differential form

1.4 Constitutive relations

9.13

1. Constitutive relations

conservation law : $u_t + \varphi_x = 0$

unknown functions : u, φ

need another equation to close the system

constitutive relation :

relationship between φ and u

usually an approximation in some regime.

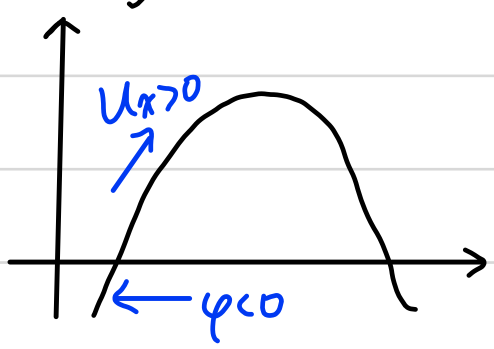
2. Diffusion Equation

$\varphi(x,t) = -D u_x(x,t)$ Fick's law linear relation

$D>0$: diffusion constant

u : density of the substance (temperature in heat flow)

meanings : substance moves from high density to low density



substitutive the consitutive relation to the conservation law

$u_t + (-Du_x)_x = 0$

$u_t - Du_{xx} = 0$ 化简为 one unknown function

3. Advection Equation 平流方程

$u(x,t)$ density

substance moves in x direction at constant speed c.

flux : $\varphi(x,t) = cu(x,t)$

substituting it into the conservation law

$\Rightarrow \quad u_t + c u_x = 0$

4. Burgers equation

$\varphi = \frac{1}{2}u^2 - D u_x$ nonlinear

$\Rightarrow u_t + \frac{1}{2}(u^2 - D u_x)_x = 0$

$\Rightarrow u_t + u u_x = D u_{xx}$

1.5 Initial and boundary value problems

Initial and/or boundary conditions are required to determine a unique physically relavant solution.

1. Initial value problems (Cauchy problem)

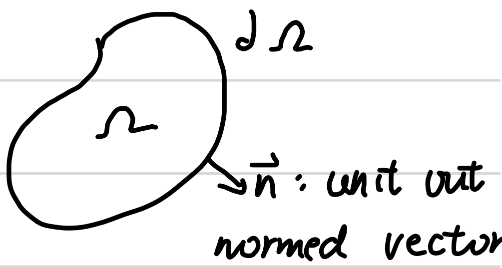
spatial domain : entire x-axis

e.g.

$$\begin{cases} u_t - Du_{xx} = 0 & -\infty < x < \infty,\ t > 0 \\ u(x,0) = u_0(x) & -\infty < x < \infty \end{cases}$$

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty,\ t > 0 \\ u(x,0) = u_0(x) & -\infty < x < \infty \\ u_t(x,0) = u_1(x) & -\infty < x < \infty \end{cases}$$

2. Boundary value problems



e.g. Dirichlet boundary condition

$$\begin{cases} u_{xx} + u_{yy} = f & (x,y) \in \Omega \\ u|_{\partial\Omega} = g(x,y) \end{cases}$$

Neumann boundary condition

$$\begin{cases} u_{xx} + u_{yy} = f & (x,y) \in \Omega \\ \frac{\partial u}{\partial n}\Big|_{\partial\Omega} = g(x,y) \end{cases}$$

Robin boundary condition

$$\begin{cases} u_{xx} + u_{yy} = f & (x,y) \in \Omega \\ \frac{\partial u}{\partial n} + \alpha u\Big|_{\partial\Omega} = g(x,y) & \alpha > 0 \end{cases}$$

3. Initial-boundary value problems

e.g.

$$\begin{cases} u_t - Du_{xx} = 0 & 0<x<l,\ t>0 \\ u(0,t) = g(t),\ u(l,t) = h(t) & t>0 \\ u(x,0) = u_0(x) & 0<x<l \end{cases}$$

Any of the three boundary conditions

4. Well-posed problem 适定问题

(1) The solution exists;

(2) The solution is unique;

(3) The solution depends continuously on the given data.

Initial, boundary conditions

inhomogeneous terms

$u_t - Du_{xx} = f$ ...

Condition (3) means that

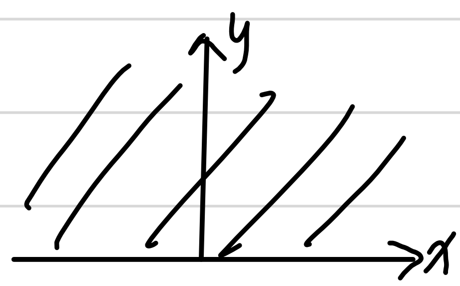
small change in data $\Rightarrow$ small change in the solution

The initial and/or boundary value problems reviewed above are well-posed.

Problem not well-posed: ill-posed 欠定的

e.g.

$$\begin{cases} u_{xx} + u_{yy} = 0 & -\infty<x<\infty,\ y>0 \\ u(x,0) = u_0(x) & -\infty<x<\infty \\ u_y(x,0) = u_1(x) & -\infty<x<\infty \end{cases}$$



not satisfying (3)

If $u_0(x) \equiv u_1(x) \equiv 0$, the solution is $u(x,y) \equiv 0$.

Consider $u_0(x) = \frac{1}{k}\sin kx$, $u_1(x) \equiv 0$

$u_0(x) \to 0$ as $k \to +\infty$ (small change in the initial conditions)

We look for solution of this form

$$u(x,y) = \sin kx \, Y(y)$$

From the equation

$$-k^2 \sin kx \, Y(y) + \sin kx \, Y''(y) = 0$$

$$\Rightarrow Y''(y) - k^2 Y(y) = 0$$

$$\Rightarrow Y(y) = c_1 e^{ky} + c_2 e^{-ky}$$

Solution of the PDE

$$u(x,y) = \sin kx \, (c_1 e^{ky} + c_2 e^{-ky})$$

From the initial conditions

① $u(x,0) = \sin kx \, (c_1 + c_2) = \frac{1}{k} \sin kx$

$\Rightarrow c_1 + c_2 = \frac{1}{k}$

② $u_y(x,y) = \sin kx \, (c_1 k e^{ky} - c_2 k e^{-ky})$

$u_y(x,0) = \sin kx \, (c_1 k - c_2 k) = 0$

$\Rightarrow c_1 - c_2 = 0$

Thus, we have $c_1 = c_2 = \frac{1}{2k}$. and the solution of the initial problem is

$$u(x,y) = \frac{1}{2k} \sin kx \, (e^{ky} + e^{-ky})$$

It is unbounded as $k \to +\infty$. $\left(\lim_{k\to\infty} \frac{e^k}{k} = +\infty \right)$

When k is large, we have a small change from 0 to $\frac{1}{k}\sin kx$ in the initial condition.

However, the change in the solution from 0 to $k \sin kx \, (e^{ky} + e^{-ky})$ is unbounded.

$\therefore$ The problem is ill-posed.

1.6 Cole–Hopf transformation

Burgers equation

$$u_t + uu_x = Du_{xx}$$ — Nonlinear PDE

$D>0$ constant

Try to convert it into a linear PDE

Solution:

Write the PDE as

$$u_t + \frac{1}{2}(u^2)_x = Du_{xx} \qquad (1)$$

Let $u = w_x \qquad (2)$

The PDE becomes

$$w_{xt} + \frac{1}{2}(w_x^2)_x = Dw_{xxx}$$

Intigrate it with respect to x

$$w_t + \frac{1}{2}w_x^2 = Dw_{xx} \qquad (3)$$

Let $w = g(v)$

$$w_t = g'(v)v_t. \quad w_x = g'(v)v_x$$

$$w_{xx} = (g'(v)v_x)_x = g''(v)v_x^2 + g'(v)v_{xx}$$

Substitude it into Eq.(3)

$$g'v_t + \left(\frac{1}{2}g'^2 - Dg''\right)v_x^2 = Dg'v_{xx}$$

If $\frac{1}{2}g'^2 = Dg'' \qquad (4)$

Then v satisfies

$$g'v_t = Dg'v_{xx}$$

or $$v_t = Dv_{xx} \qquad (5)$$

— Linear, diffusion equation

Look for $g(v)$ that satisfies Eq.(4)

$$\frac{1}{2}g'^2 = D\frac{dg'}{dv}$$ — ODE

$$\frac{dg'}{g'^2} = \frac{1}{2D}dv$$

$$-\frac{1}{g'} = \frac{v}{2D}, \quad g' = \frac{2D}{v}$$

$$g(v) = -2D\log v$$

$\therefore\ W = g(v) = -2D\log v$

By Eq. (2)

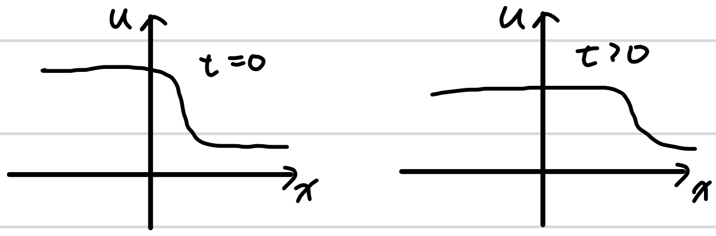
$$u = W_x = g(v)_x = -2D\frac{v_x}{v} \qquad (6)$$

Where v is the solution of the diffusion equation (5)

textbook 5.5.2

1.7 Waves

Wave: signal transferred with a recognizable speed



1. Traveling wave solutions

$u(x,t) = f(x-ct)$

c is the speed (of the profile in the xu plane)

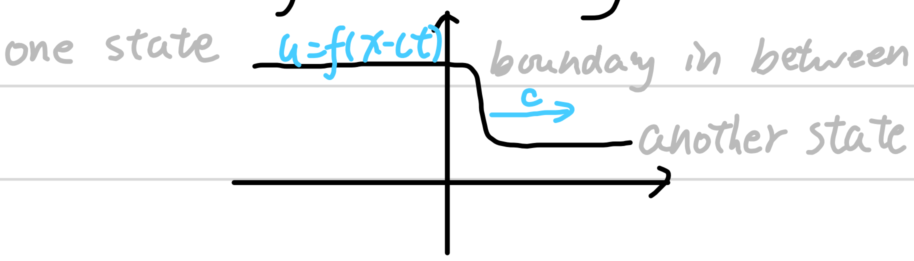
Wavefront solution

traveling wave solution with boundary conditions

$u(-\infty,t) = \text{constant} \qquad u(-\infty,t) = \text{constant}$

The above profile is a wavefront solution.

Physical meaning



e.g. 1

$u_t + a u_x = 0 \qquad a = \text{constant}$

traveling wave solution

$$u(x,t) = f(x-ct)$$

$$u_t = \frac{\partial f(x-ct)}{\partial t} = -cf'(x-ct)$$

$$u_x = \frac{\partial f(x-ct)}{\partial x} = f'(x-ct)$$

From the PDE, we have $c=a$ for any $f(\xi)$

The traveling wave solution is

$$u(x,t) = f(x-at)$$

If further, we know $u(x,0)=u_0(x)$

$$u(x,0)=f(x)=u_0(x)$$

Thus $$u(x,t)=u_0(x-at)$$

e.g.2 KdV equation

$$u_t+uu_x+u_{xxx}=0$$

Look for traveling wave solution $u(x,t)=f(x-ct)$

where function f and constant c are to be determined

$$u_t=-cf', \quad u_x=f', \quad u_{xxx}=f'''$$

The equation becomes

$$-cf'+ff'+f'''=0 \qquad \text{— ODE}$$

Integrating the equation

$$-cf+\frac{1}{2}f^2+f''=a \qquad (1)$$

a: constant

No "ξ". use "f" as the independent variable

$$f''=\frac{df'}{d\xi}=\frac{df'}{df}\frac{df}{d\xi}=f'\frac{df'}{df}$$

The equation becomes

$$-cf+\frac{1}{2}f^2+f'\frac{df'}{df}=a$$

$$f'df'=(cf-\frac{1}{2}f^2+a)df$$

Integrate it

$$\frac{1}{2}f'^2=\frac{1}{2}cf^2-\frac{1}{6}f^3+af+b \qquad (2)$$

b: constant

Look for solutions satisfying

$$f, f', f'' \to 0 \quad \text{as } |\xi|\to\infty \quad ?$$

Letting $|\xi|\to\infty$ in Eq.(1), we have $a=0$

Letting $|\xi|\to\infty$ in Eq.(2), we have $b=0$

Eq.(2) becomes

$$\frac{1}{2}f'^2=\frac{1}{2}cf^2-\frac{1}{6}f^3$$

$$3f'^2 = 3cf^2 - f^3$$

$$\sqrt{3}\, f' = \pm f\sqrt{3c-f}$$ (look for solution $0 \le f \le 3c$)

$$\frac{df}{f\sqrt{3c-f}} = \pm \frac{dz}{\sqrt{3}}$$

Let $f = 3c\,\mathrm{sech}^2(w)$ hyperbolic secant $\mathrm{sech}(x) = \frac{2}{e^x+e^{-x}}$

The ODE becomes

$$-\frac{2}{\sqrt{3c}}\, dw = \pm \frac{dz}{\sqrt{3}}$$

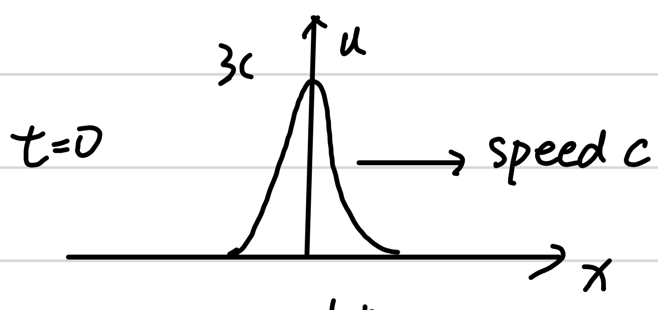
$$w = \pm \frac{\sqrt{c}}{2}(z+c_0)$$

$$f(z) = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(z+c_0)\right)$$

$$u(x,t) = f(x-ct) = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct+c_0)\right)$$

When $c_0 = 0$

$$u(x,t) = 3c\,\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$



soliton

2. Plane Waves

plane wave solution :

$$u(\vec{x},t) = f(\vec{k}\cdot\vec{x}, t)$$ $\vec{k}$: constant vector

($\vec{k}\cdot\vec{x}$ = constant, represents a plane)

traveling plane wave :

$$u(\vec{x},t) = f(\vec{k}\cdot\vec{x} - ct)$$

Consider solution with the form

$$u(\vec{x},t) = e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$ $\vec{k}$: constant vector

ω : constant

($\Leftrightarrow$ two real solutions : $\cos(kx-\omega t)$ and $\sin(kx-\omega t)$)

k : wave number — number of periods in 2π length unit

$\lambda = \frac{2\pi}{k}$ wave length

ω : frequency — number of periods in 2π time unit

$\frac{2\pi}{\omega}$: time unit

e.g.1 $u_t - Du_{xx} = 0$

Look for solution $u(x,t) = e^{i(kx-\omega t)}$

$$u_t = -i\omega e^{i(kx-\omega t)}$$

$$u_x = ike^{i(kx-\omega t)}$$

$$u_{xx} = -k^2 e^{i(kx-\omega t)}$$

Substitude them into the equation

$$-i\omega e^{i(kx-\omega t)} + Dk^2 e^{i(kx-\omega t)} = 0$$

$$i\omega = Dk^2$$

$$\omega = -iDk^2$$

Solution $u(x,t) = e^{i(kx+iDk^2t)} = e^{-Dk^2t}e^{ikx}$

amplitude $|u(x,t)| = e^{-Dk^2t}$ decays

In this case, the PDE is called diffusive (or dissipative)

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Generally, $u(x,t) = e^{i(kx-\omega t)}$

From the equation, we can obtain $\omega = \omega(k)$

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— dispersion relation

For $u_t - u_{xx} = 0$, the dispersion relation is $\omega = -iDk^2$

Sometimes, we write the plane wave solution as

$u(x,t) = e^{ikx+\omega x}$　　∵ final result is a real number.

and look for dispersion relation $\omega = \omega(k)$

For $u_t - u_{xx} = 0$, $\omega = -Dk^2$

e.g. 2　$u_t = i u_{xx}$　Schrödinger equation

$u(x,t) = e^{i(kx-\omega t)}$

$-i\omega = -ik^2$

$\omega = k^2$

Solution $u(x,t) = e^{i(kx-k^2t)} = e^{ik(x-kt)}$

speed k. depend on wave number

The PDE is dispersive　$\omega(k)$ is real and $\omega''(k) \neq 0$

Waves with different wavenumbers propagate at different speeds.

vs diffusive: $\omega(k)$ is complex

e.g. 3　$u_t + cu_x = 0$

$u(x,t) = e^{i(kx-\omega t)}$

$\omega = ck$

Solution $u(x,t) = e^{i(kx-ckt)} = e^{ik(x-ct)}$

speed c for all wavenumbers k

waves with different wave numbers propagate at the same speed.

The PDE is neither diffusive nor dispersive　∵ $\omega''(k) = 0$

Solution of the initial value problem

$$\begin{cases} u_t - D u_{xx} = 0 & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$ — linear

can be obtained by plane wave solutions $e^{i(kx-\omega t)}$ and principle of superposition (Fourier Transform Method) see textbook 1.5-3

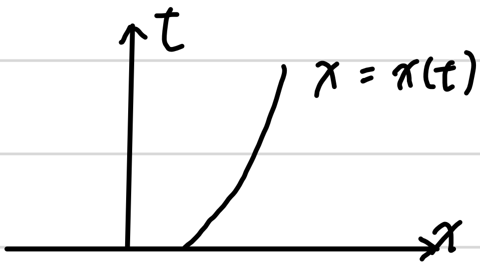
Chapter 2 First order equations and characteristic

A first order PDE is always hyperbolic.

— One family of characteristic exist

2.1 Linear first order equation

$$\begin{cases} u_t + C(x,t)u_x = 0 & x \in \mathbb{R},\ t>0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$



consider a curve in the x-t plane

$x = x(t)$

consider the solution of the PDE along this curve

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)\frac{dx}{dt}$$

If we choose $\frac{dx}{dt} = C(x,t)$, then alone the curve $x(t)$.

the PDE gives

$$\frac{d}{dt}u(x(t),t) = u_t(x(t),t) + u_x(x(t),t)\,c(x,t) = 0$$

Thus, along this curve

$$\frac{dx}{dt} = c(x,t) \qquad (1)$$

such a curve $x(t)$ is a characteristic, and $c(x,t)$ is the speed.

The PDE is simplified as

$$\frac{du}{dt} = 0 \qquad (2)$$

Eq.(1) and (2): charactistic system

— Method of characteristics

e.g. 1

$$\begin{cases} u_t + c u_x = 0 & x \in \mathbb{R},\ t>0 \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases} \qquad c: \text{constant}$$

characteristics $\frac{dx}{dt} = c$

along a characteristic $\frac{du}{dt} = 0$

From $\frac{dx}{dt} = c$. we have

$x - ct = \xi$, ξ: constant

From $\frac{du}{dt} = 0$, we have

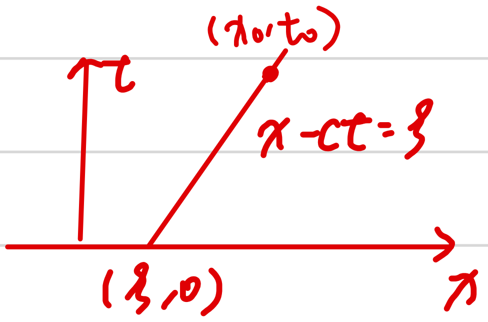
u is a constant along a characteristic

Thus the solution u is constant along a characteristic $x - ct = \xi$.

The characteristic $x - ct = \xi$ intersets with x axis at $(\xi, 0)$

For a point (x_0, t_0), the characteristic passing through it is

$x - ct = \xi$ with $\xi = x_0 - ct_0$



Thus, we have $u(x_0, t_0) = u(\xi, 0)$ u constant along a cha

$= u_0(\xi)$ initial condition

$= u_0(x_0 - ct_0)$ $\xi = x_0 - ct_0$ from the eq of the characteris

$u(x,t) = u(\xi, 0) = u_0(\xi) = u_0(x - ct)$