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HW 1-1

1. Let $u = u(x,t)$ be a solution of the nonlinear equation

$$a(u_x, u_t)u_{xx} + 2b(u_x, u_t)u_{xt} + c(u_x, u_t)u_{tt} = 0.$$

Introduce new independent variables $\xi = u_x(x,t)$ and $\eta = u_t(x,t)$, and a new function $\phi = \phi(\xi, \eta)$ defined by $\phi = xu_x + tu_t - u$. Prove that $\phi_\xi = x$, $\phi_\eta = t$, and ϕ satisfies the linear PDE $J \neq 0?$

$$a(\xi,\eta)\phi_{\eta\eta} - 2b(\xi,\eta)\phi_{\xi\eta} + c(\xi,\eta)\phi_{\xi\xi} = 0.$$

(Note that this transform, known as a hodograph transformation, or a Legendre transformation, transforms a nonlinear equation of the given form to a linear equation by reversing the roles of the dependent and independent variables.)

Prove:

Given $u(x,t)$. $\xi = \frac{\partial u}{\partial x}$. $\eta = \frac{\partial u}{\partial t}$

Then $u(x,t) = u(x(\xi,\eta), t(\xi,\eta))$

The partial derivative of u is

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \xi}$$

$$= \xi \frac{\partial x}{\partial \xi} + \eta \frac{\partial t}{\partial \xi}$$

$$= \frac{\partial(\xi x)}{\partial \xi} - x + \eta\frac{\partial t}{\partial \xi} \quad (1)$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial \eta}$$

$$= \xi\frac{\partial x}{\partial \eta} + \eta\frac{\partial t}{\partial \eta}$$

$$= \xi\frac{\partial x}{\partial \eta} + \frac{\partial(\eta t)}{\partial \eta} - t \quad (2)$$

From (1), we have $\frac{\partial u}{\partial \xi} = \frac{\partial(\xi x)}{\partial \xi} - x + \eta\frac{\partial t}{\partial \xi}$

$$\Rightarrow x = \frac{\partial(\xi x)}{\partial \xi} - \frac{\partial u}{\partial \xi} + \eta\frac{\partial t}{\partial \xi}$$

$$= \frac{\partial(\xi x + \eta t - u)}{\partial \xi} = \frac{\partial \phi}{\partial \xi}$$

Similarly, $t = \xi \frac{\partial x}{\partial \eta} + \frac{\partial(\eta t)}{\partial \eta} - \frac{\partial u}{\partial \eta}$

$$= \frac{\partial(\xi x + \eta t - u)}{\partial \eta} = \frac{\partial \phi}{\partial \eta}$$

$\therefore \phi_\xi = x.\ \phi_\eta = t$

Let Δ denote $\begin{vmatrix} \frac{\partial^2\phi}{\partial\xi^2} & \frac{\partial^2\phi}{\partial\xi\partial\eta} \\ \frac{\partial^2\phi}{\partial\xi\partial\eta} & \frac{\partial^2\phi}{\partial\eta^2} \end{vmatrix}$

Then $\frac{\partial^2 u}{\partial x^2} = \frac{1}{\Delta}\frac{\partial^2\phi}{\partial\eta^2}$, $\frac{\partial^2 u}{\partial x\partial t} = -\frac{1}{\Delta}\frac{\partial^2\phi}{\partial\xi\partial\eta}$. $\frac{\partial^2 u}{\partial t^2} = \frac{1}{\Delta}\frac{\partial^2\phi}{\partial\xi^2}$

Substitude $u_{xx}.\ u_{xt}.\ u_{tt}$ with $\frac{1}{\Delta}\phi_{\eta\eta}, -\frac{1}{\Delta}\phi_{\xi\eta}, \frac{1}{\Delta}\phi_{\xi\xi}$ in $(*)$

we have

$$a(\xi,\eta)\phi_{\eta\eta} - 2b(\xi,\eta)\phi_{\xi\eta} + c(\xi,\eta)\phi_{\xi\xi} = 0$$

$\square$

HW 1-2

2. In the derivation of the fundamental conservation law, we assumed that the cross-sectional area A of the tube was constant. Derive integral and differential forms of the conservation law in the case that the area is a slowly varying function of x, that is, $A = A(x)$. (Note that $A(x)$ cannot change significantly over small changes in x; otherwise, the one-dimensional assumption of the state functions u and ϕ being constant in any cross section would be violated.)

For $I = [a,b]$ in time t,

total amount of quantity in I: $\int_a^b u(x,t) A(x)\,dx$

rate of change of quantity in I: $\frac{d}{dt}\int_a^b u(x,t) A(x)\,dx$

Amount that flows in I: $\varphi(a,t)A(a) - \varphi(b,t)A(b)$

Amount that produced in I: $\int_a^b f(x,t,u)A(x)\,dx$

Then the conservation law in integral form is:

$$\frac{d}{dt}\int_a^b u(x,t)A(x)\,dx = [\varphi(a,t)A(a) - \varphi(b,t)A(b)] + \int_a^b f(x,t,u)A(x)\,dx$$

When u, φ, A are continuous differentiable & f is continuous

$$\frac{d}{dt}\int_a^b u(x,t)A(x)\,dx = \int_a^b u_t(x,t)A(x)\,dx$$

$$\varphi(a,t)A(a) - \varphi(b,t)A(b) = -\int_a^b (\varphi(x,t)A(x))'\,dx$$

The conservation law becomes

$$\int_a^b u_t(x,t)A(x)\,dx = -\left(\int_a^b \varphi_x(x,t)A(x)\,dx + \int_a^b \varphi(x,t)A'(x)\,dx\right)$$
$$+\int_a^b f(x,t,u)A(x)\,dx$$

For any interval $[a,b]$

$$u_t(x,t)A(x) = -(\varphi_x(x,t)A(x) + \varphi(x,t)A'(x)) + f(x,t,u)A(x)$$

$\square$

HW 1-3

3. Consider the initial value problem for the backward diffusion equation

$$\begin{cases} u_t + u_{xx} = 0, & x \in \mathbf{R}, \ t > 0, \\ u(x,0) = 1, & x \in \mathbf{R}. \end{cases}$$

Show that the solution does not depend continuously on the initial condition by considering the solution with the initial condition

$$u_n(x,0) = 1 + \frac{1}{n} \sin nx, \ x \in \mathbf{R}.$$

① Consider $\begin{cases} u_t + u_{xx} = 0 \\ u(x,0) = 1 \end{cases}$

The solution is $u(x,t) = 1$

② Consider $\begin{cases} u_t + u_{xx} = 0 \\ u(x,0) = 1 + \frac{1}{n}\sin nx \end{cases}$

The form of the solution is $u(x,t) = \underline{\sin nx \, T(t)}$

$1 + T(t)\sin nx.$

$\because u_t + u_{xx} = 0$

$\therefore \sin nx \, T'(t) - n^2 \sin nx \, T(t) = 0$

$\Rightarrow T'(t) - n^2 T(t) = 0$

$\Rightarrow T = Ce^{-n^2 t}$ (C is a constant) $e^{n^2 t + c'}$

$\Rightarrow u(x,t) = C \sin nx \, e^{-n^2 t}$

$\because u(x,0) = 1 + \frac{1}{n}\sin nx$

$\therefore C = \frac{1}{\sin nx} + \frac{1}{n}$

$u(x,t) = e^{-n^2 t} + \frac{\sin nx}{n} e^{-n^2 t} \longrightarrow 0$ as $n \to \infty$

$1 + \frac{1}{n} e^{n^2 t} \sin(nx)$

Thus, the solution does not depend continuously on the initial condition.

□