HW2.

Laplace's Method, Watson's lemma.

19/20

1. Consider the asymptotic behavior of the integral

what if φ  ?

$$I(x)=\int_1^3 e^{-x\left(\frac{9}{t}+t\right)}dt,\quad x\to+\infty.$$

(1) Find the leading asymptotic behavior using the Laplace method.
(2) Rewrite the integral using variable $u=9/t+t-6$ and find the first two leading terms in the asymptotic expansion (using Watson's lemma).

(1) $I(x)=\int_1^3 e^{x\varphi(t)}dt=\int_1^3 e^{-x\left(\frac{9}{t}+t\right)}dt \qquad x\to+\infty$

$\varphi(t)=-\left(\frac{9}{t}+t\right) \qquad t\in[1,3]$

$\varphi(t)$ reaches its maximum at $t=3$. $\varphi_{max}=\varphi(3)=-6$

The leading behavior of $I(x)$ is near $t=3$ as $x\to+\infty$.

- Step 1:

$$I(x)=\int_1^{3-\varepsilon} e^{-x\left(\frac{9}{t}+t\right)}dt+\int_{3-\varepsilon}^3 e^{-x\left(\frac{9}{t}+t\right)}dt \qquad \varepsilon\to0$$

要写过程吗? $\quad\sim\int_{3-\varepsilon}^{\varepsilon} e^{-x\left(\frac{9}{t}+t\right)}dt \quad$ as $x\to+\infty$

- Step 2:

When $t\in(3-\varepsilon,3]$, the Taylor expansion of $f(t)=\frac{9}{t}+t$ at $t=3$ is

$$f(t)=f(3)+f'(3)(t-3)+\sum_{n=2}^{+\infty}\frac{(-1)^n n!\,9\cdot3^{-(n+1)}}{n!}(t-t_0)^n$$

$$=6+\sum_{n=2}^{+\infty}(-1)^n 3^{-(n-1)}(t-3)^n$$

Thus, $\frac{9}{t}+t\sim 6-\frac{1}{3}(t-3)^2+o\left((t-3)^2\right)$

$$\int_{3-\varepsilon}^3 e^{-x\left(\frac{9}{t}+t\right)}dt\sim\int_{3-\varepsilon}^3 e^{-x\left(6+\frac{1}{3}(t-3)^2\right)}dt$$

$$=e^{-6x}\int_{3-\varepsilon}^3 e^{-\frac{1}{3}x(t-3)^2}dt$$

$$=e^{-6x}\int_{-\varepsilon}^0 e^{-\frac{1}{3}xt^2}dt$$

$s=\frac{1}{3}xt^2 \qquad =e^{-6x}\cdot\sqrt{\frac{3}{x}}\int_{-\varepsilon}^0 e^{-s^2}ds$

- Step 3: $\qquad\approx e^{-6x}\cdot\sqrt{\frac{3}{x}}\int_{-\infty}^0 e^{-s^2}ds$

$$=e^{-6x}\cdot\sqrt{\frac{3\pi}{4x}} \qquad x\to+\infty$$

(2) $I(x) = \int_1^3 e^{-x(\frac{9}{t}+t)}\,dt$

$= \int_1^3 e^{-x(\frac{9}{t}+t-6)} e^{-6x}\,dt$

$-1 \quad = e^{-6x}\int_1^3 e^{-ux}\,dt$

$u = \frac{9}{t}+t-6 \Rightarrow t = \frac{u+6 \pm \sqrt{u^2+12u}}{2} \qquad u \in [0,4)$

Since $t \in [1,3)$. $\quad t = \frac{u+6-\sqrt{u^2+12u}}{2}$

$I(x) = \int_0^4 e^{-ux}\,d\left(\frac{u+6-\sqrt{u^2+12u}}{2}\right)$

$= \int_0^4 \frac{1}{2}\left(1-\frac{u+6}{\sqrt{u^2+12u}}\right)e^{-ux}\,du$

$= \frac{1}{2}\int_0^4 e^{-ux}\,du - \int_0^4 \frac{u+6}{\sqrt{u^2+12u}}e^{-ux}\,du$

$= \frac{1}{2x}(e^{-4x}-1) - \int_0^4 \frac{u+6}{\sqrt{u^2+12u}}e^{-ux}\,du$

$\frac{u+6}{\sqrt{u^2+12u}} = \frac{u+6}{2\sqrt{3}\cdot\sqrt{u}\cdot\sqrt{\frac{u}{12}+1}}$

$= \frac{\sqrt{u}}{2\sqrt{3}\sqrt{1+\frac{u}{12}}} + \frac{\sqrt{3}}{\sqrt{u}\sqrt{1+\frac{u}{12}}}$

The Taylor expansion of $\frac{1}{\sqrt{1+\frac{u}{12}}} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{2^{n+1}\cdot 12^n\cdot n!}u^n$

$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{12^n}\frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}u^n$

Thus,

$\frac{u+6}{\sqrt{u^2+12u}} = \frac{1}{2\sqrt{3}}\cdot u^{\frac{1}{2}}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{12^n}\frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}u^n + \sqrt{3}\,u^{-\frac{1}{2}}\sum_{n=0}^{\infty}\cdots$

$= \underbrace{\frac{1}{2\sqrt{3}}\sum_{n=0}^{\infty}\underbrace{\frac{(-1)^n}{12^n}\frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}}_{a_n}u^{\overset{\beta}{n}+\overset{\alpha}{(\frac{1}{2})}}}_{①} + \underbrace{\sqrt{3}\sum_{n=0}^{\infty}\underbrace{\frac{(-1)^n}{12^n}\frac{\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}}_{a_n}u^{\overset{\beta}{n}\overset{\alpha}{(-\frac{1}{2})}}}_{②}$

扫描全能王 创建

By Watson's Lemma.

① $\alpha = \frac{1}{2}$, $\beta = 1$

$$\int_0^4 \frac{u+6}{\sqrt{u^2+12u}} e^{-ux} du \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n} \frac{\Gamma(n+\frac{1}{2})}{n!\Gamma(\frac{1}{2})} \cdot \frac{\Gamma(n+\frac{3}{2})}{x^{n+\frac{3}{2}}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n} \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{n!\Gamma(\frac{1}{2})\, x^{n+\frac{3}{2}}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n} \frac{(n+\frac{1}{2})(\Gamma(n+\frac{1}{2}))^2}{n!\Gamma(\frac{1}{2})\, x^{n+\frac{3}{2}}}$$

② $\alpha = -\frac{1}{2}$, $\beta = 1$

$$\int_0^4 \frac{u+6}{\sqrt{u^2+12u}} e^{-ux} du \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n} \frac{\Gamma(n+\frac{1}{2})}{n!\Gamma(\frac{1}{2})} \cdot \frac{\Gamma(n+\frac{1}{2})}{x^{n+\frac{1}{2}}}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n} \frac{(\Gamma(n+\frac{1}{2}))^2}{n!\Gamma(\frac{1}{2})\, x^{n+\frac{1}{2}}}$$

Therefore,

$$I(x) = \frac{1}{2x}(e^{-4x}-1) - \int_0^4 \frac{u+6}{\sqrt{u^2+12u}} e^{-ux} du$$

$$\sim \frac{1}{2x}(e^{-4x}-1) - \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n} \frac{(n+\frac{1}{2})(\Gamma(n+\frac{1}{2}))^2}{n!\Gamma(\frac{1}{2})\, x^{n+\frac{1}{2}}} \cdot x^{-1}$$

$$- \sum_{n=0}^{\infty} \frac{(-1)^n}{12^n} \frac{(\Gamma(n+\frac{1}{2}))^2}{n!\Gamma(\frac{1}{2})\, x^{n+\frac{1}{2}}}$$

$$= \frac{1}{2x}(e^{-4x}-1) - \left(\sum_{n=0}^{\infty} \left(\frac{n+\frac{1}{2}}{x}+1\right) \frac{(-1)^n}{12^n} \frac{(n+\frac{1}{2})(\Gamma(n+\frac{1}{2}))^2}{n!\Gamma(\frac{1}{2})\, x^{n+\frac{1}{2}}} \right)$$

The first two leading terms are:

$$\frac{1}{2x}(e^{-4x}-1) - \frac{(1+2x)\Gamma(\frac{1}{2})}{4x^{\frac{3}{2}}} = \frac{1}{2x}(\cancel{e^{-4x}}-1) - \frac{\sqrt{\pi}(1+2x)}{4x^{\frac{3}{2}}}$$

Stationary phase.

2. Use the stationary phase method to find the leading asymptotic behavior of the integral

10/10

$$I(x) = \int_0^{\pi} \cos(x\sin t - t)dt, \quad x \to +\infty.$$

Hint: (1) $\cos(x\sin t - t)$ is the real part of $e^{i(x\sin t-t)}$; (2) $\int_0^\infty e^{-iz^2}dz = \frac{\sqrt{\pi}}{2}e^{-i\pi/4}$.

Since $I(x) = \int_0^{\pi} \cos(x\sin t - t)$ is the real part of

$I_0(x) = \int_0^{\pi} e^{i(x\sin t - t)}dt = \int_0^{\pi} e^{-it}e^{ix\sin t}dt$

We consider $I_0(x)$ first.

$\varphi(t) = \sin t$, $f(t) = e^{-it}$

$\varphi'(t) = \cos t$ as $x \to \infty$

$\varphi'(\frac{\pi}{2}) = 0$ $\varphi''(\frac{\pi}{2}) \neq 0$

Thus $t = \frac{\pi}{2}$ is a stationary point

$$I_0(x) = \underbrace{\int_0^{\frac{\pi}{2}-\varepsilon} e^{i(x\sin t - t)}dt}_{I_1} + \underbrace{\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} e^{i(x\sin t - t)}dt}_{I_2} + \underbrace{\int_{\frac{\pi}{2}+\varepsilon}^{\pi} e^{i(x\sin t - t)}dt}_{I_3}$$

① $I_2 = \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} e^{-it}e^{ix\sin t}dt$

$$\sim \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} e^{-it}e^{ix\left(\varphi(\frac{\pi}{2}) + \frac{\varphi''(\frac{\pi}{2})}{2}(t-\frac{\pi}{2})^2\right)}dt$$

$$= \int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} e^{-it}e^{ix(1-\frac{1}{2}(t-\frac{\pi}{2})^2)}dt$$

$f(a)$

$$= e^{-\frac{\pi}{2}i}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} e^{ix}e^{-\frac{1}{2}ix(t-\frac{\pi}{2})^2}dt$$

$$= e^{i(x-\frac{\pi}{2})}\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} e^{-\frac{1}{2}ix(t-\frac{\pi}{2})^2}dt$$

$u = t - \frac{\pi}{2}$ $= e^{i(x-\frac{\pi}{2})}\int_{-\varepsilon}^{\varepsilon} e^{-\frac{1}{2}ixu^2}du$

$$\sim e^{i(x-\frac{\pi}{2})}\int_{-\infty}^{+\infty} e^{-\frac{1}{2}ixu^2}\,du$$

$$=2e^{i(x-\frac{\pi}{2})}\int_{0}^{+\infty} e^{-\frac{1}{2}ixu^2}\,du$$

$$z=\sqrt{\tfrac{1}{2}xu^2} \quad =2\sqrt{\frac{2}{x}}\,e^{i(x-\frac{\pi}{2})}\int_{0}^{+\infty} e^{-iz^2}\,dz$$

$$=\sqrt{\frac{2\pi}{x}}\,e^{i(x-\frac{3}{4}\pi)}$$

② By generalized Riemann-Lebesgue Theorem,

$$I_1=\int_0^{\frac{\pi}{2}-\varepsilon} e^{-it}e^{ix\sin t}\,dt \sim \frac{e^{-it}}{ix\cos t}e^{ix\sin t}\Big|_0^{\frac{\pi}{2}-\varepsilon} \qquad x\to+\infty$$

$$=O\left(\frac{1}{x}\right) \ll O\left(\frac{1}{\sqrt{x}}\right) \qquad x\to+\infty$$

$$I_3=\int_{\frac{\pi}{2}+\varepsilon}^{\pi} e^{-it}e^{ix\sin t}\,dt \sim \frac{e^{-it}}{ix\cos t}e^{ix\sin t}\Big|_{\frac{\pi}{2}+\varepsilon}^{\pi} \qquad x\to\infty$$

$$=O\left(\frac{1}{x}\right) \ll O\left(\frac{1}{\sqrt{x}}\right)$$

No need to prove

Therefore. $I_0(x)=\sqrt{\frac{2\pi}{x}}\,e^{i(x-\frac{3}{4}\pi)}$

$I(x)$ is the real part of $I_0(x)$

$I(x)=\sqrt{\frac{2\pi}{x}}\cos\left(x-\frac{3}{4}\pi\right)$

14.5/15 Perturbation methods.

3. (1) Compute the first four coefficients in the perturbation series (i.e., $O(1)$, $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$ terms) to the initial-value problem

$$y' = 2y + 4\epsilon x y, \quad y(0) = 1.$$

(2) Find the exact solution.

(3) Use some software, e.g., matlab, to plot and compare the exact solution and the n-term perturbation expansion for the solution, $n = 1, 2, 3, 4$ (i.e., y_0, $y_0 + \epsilon y_1$, $y_0 + \epsilon y_1 + \epsilon^2 y_2$ and $y_0 + \epsilon y_1 + \epsilon^2 y_2 + \epsilon^3 y_3$) on $x \in [0, 3]$ when $\epsilon = 0.1$.

(1) The problem is

$$\begin{cases} \dfrac{dy}{dx} = 2y + 4\varepsilon x y & (1) \\ y(0) = 1 \end{cases}$$

Look for a solution with the form $y(x) = \sum_{n=0}^{\infty} \varepsilon^n y_n(x)$ (2)

where $y_n(x)$ are to be determined.

Substituting (2) into (1), we have

$$\frac{dy_0}{dx} + \varepsilon \frac{dy_1}{dx} + \varepsilon^2 \frac{dy_2}{dx} + \cdots \sim 2y_0 + (4xy_0 + 2y_1)\varepsilon + (4xy_1 + 2y_2)\varepsilon^2 + (4xy_2 + 2y_3)\varepsilon^3 \cdots \quad \varepsilon \to 0$$

Compare coefficient functions:

ε^0 terms: $\dfrac{dy_0}{dx} = 2y_0$

ε^1 terms: $\dfrac{dy_1}{dx} = 4xy_0 + 2y_1$

ε^2 terms: $\dfrac{dy_2}{dx} = 4xy_1 + 2y_2$

...

The initial condition becomes:

$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \varepsilon^3 y_3(0) + \cdots \sim 1 \quad \varepsilon \to 0$

which implies

$y_0(0) = 1$

$y_1(0), y_2(0), y_3(0), \cdots = 0$

- Therefore, the zero order solution satisfies

$$\begin{cases} \frac{dy_0}{dx} = 2y_0 \\ y_0(0) = 1 \end{cases}$$

The solution is $y_0 = e^{2x}$

- The first-order solution satisfies

$$\begin{cases} \frac{dy_1}{dx} = 4xy_0 + 2y_1 = 4xe^{2x} + 2y_1 \\ y_1(0)=0 \end{cases}$$

The solution is $y_1 = 2x^2e^{2x}$

- The second-order solution satisfies

$$\begin{cases} \frac{dy_2}{dx} = 4xy_1 + 2y_2 = 8x^3e^{2x} + 2y_2 \\ y_2(0)=0 \end{cases}$$

The solution is $y_2 = 2x^4e^{2x}$

- The third-order solution satisfies

$$\begin{cases} \frac{dy_3}{dx} = 4xy_2 + 2y_3 = 8x^5e^{2x} + 2y_3 \\ y_3(0)=0 \end{cases}$$

The solution is $\frac{4}{3}x^6e^{2x}$ ✓

... ...

Therefore, the solution is

$$y \sim e^{2x} + 2\varepsilon x^2e^{2x} + 2\varepsilon^2x^4e^{2x} + \frac{4}{3}\varepsilon^3x^6e^{2x}\ldots \quad \varepsilon \to 0$$

which means $y^{(m)}(x) = \sum_{n=0}^{m} \varepsilon^n y_n(x)$ is an approximation to the exact solution.

(2) $\begin{cases} y' = 2y + 4\varepsilon xy \\ y(0) = 1 \end{cases}$

$\Leftrightarrow \frac{dy}{dx} = (2+4\varepsilon x)y$

$\Leftrightarrow \quad \frac{1}{y}\, dy = (2+4\varepsilon x)\, dx$

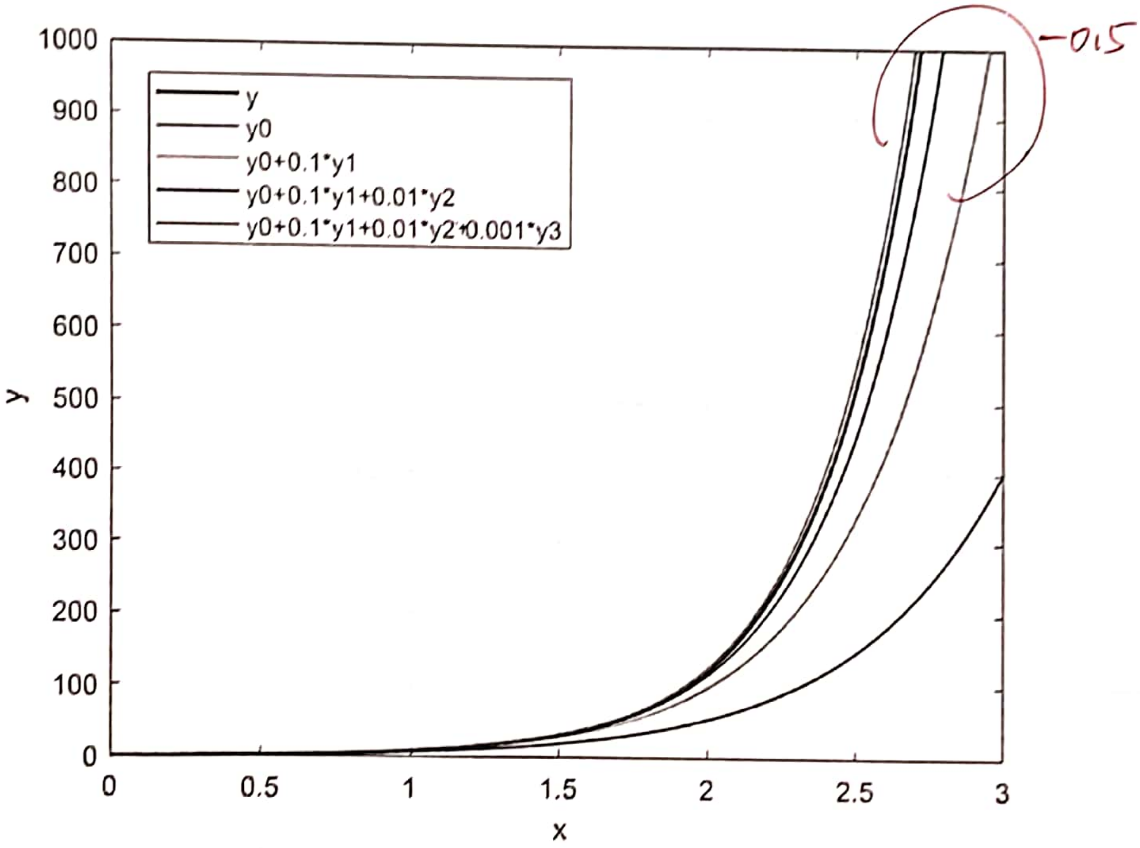
$\Leftrightarrow \quad \ln y = 2x + 2\varepsilon x^2 + C$

$\Leftrightarrow \quad y = e^{2x+2\varepsilon x^2} + C'$

Since $y(0)=1 \Rightarrow C'=0$

Thus, the exact solution is $y = e^{2x+2\varepsilon x^2}$, $\varepsilon \to 0$

(3)



扫描全能王 创建