HW1.

13/15 Frobenius Method

1. Find the series solutions about $x = 0$: 如果 $x=a$ $(a\neq 0)$ 又如何?

$$9y'' + \frac{18}{x}y' + y = 0.$$

$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\lambda}$ $(a_0 \neq 0)$

$y'(x) = \sum_{n=0}^{\infty} (n+\lambda) a_n x^{n+\lambda-1}$ a_0 is not zero. −1

$y''(x) = \sum_{n=0}^{\infty} (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-2}$

Substituting above terms into the equation:

$\sum_{n=0}^{\infty} 9(n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-2} + \sum_{n=0}^{\infty} 18(n+\lambda) a_n x^{n+\lambda-2} + \sum_{n=0}^{\infty} a_n x^{n+\lambda} = 0$

$\Rightarrow \sum_{n=0}^{\infty} 9(n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-2} + \sum_{n=0}^{\infty} 18(n+\lambda) a_n x^{n+\lambda-2} + \sum_{n=2}^{\infty} a_{n-2} x^{n+\lambda-2} = 0$

When $n=0$, consider the coefficient of $x^{\lambda-2}$ terms:

$$9\lambda(\lambda-1)a_0 + 18\lambda a_0 = 0$$

$$\Rightarrow 9\lambda(\lambda+1)a_0 = 0$$

Since $a_0 \neq 0$.

$$\lambda = 0 \text{ or } \lambda = -1$$

When $n=1$, consider the coefficient of $x^{\lambda-1}$ terms:

$$9(1+\lambda)\lambda a_1 + 18(1+\lambda)a_1 = 0$$

$$\Rightarrow (9\lambda^2 + 27\lambda + 18)a_1 = 0$$

When $\lambda = 0$, $9\lambda^2 + 27\lambda + 18 \neq 0$ $\Rightarrow a_1 = 0$

When $\lambda = -1$, $9\lambda^2 + 27\lambda + 18 = 0$ $\Rightarrow a_1$ is uncertain

When $n \geq 2$, consider the coefficient of $x^{n+\lambda-2}$ terms:

$$9(n+\lambda)(n+\lambda-1)a_n + 18(n+\lambda)a_n + a_{n-2} = 0$$

$$\Rightarrow \quad -9(n+\alpha+1)(n+\alpha)a_n = a_{n-2}$$

Thus

☆1 计算最后一项

$$a_{2m} = \frac{a_0}{(-9)^m(2m+\alpha+1)(2m+\alpha)(2m-2+\alpha+1)(2m-2+\alpha)\cdots\underbrace{(2+\alpha+1)(2+\alpha)}_{2m-2(m-1)+\alpha+1}}$$

$$a_{2m+1} = \frac{a_1}{(-9)^m(2m+1+\alpha+1)(2m+1+\alpha)(2m-1+\alpha+1)(2m-1+\alpha)\cdots\underbrace{(3+\alpha+1)(3+\alpha)}_{2m+1-2(m-1)+\alpha+1}}$$

① When $\alpha=0$

$$a_{2m} = \frac{a_0}{(-9)^m(2m+1)2m(2m-1)(2m-2)\cdots 2}$$

$$= \frac{a_0}{(-9)^m(2m+1)!} \qquad m=1,2,3\cdots$$

$$a_{2m+1} = 0 \qquad m=1,2,3\cdots$$

② When $\alpha=-1$

$$a_{2m} = \frac{a_0}{(-9)^m(2m)(2m-1)(2m-2)(2m-3)\cdots 2\cdot 1}$$

$$= \frac{a_0}{(-9)^m(2m)!} \qquad m=1,2,3\cdots$$

☆2 $a_{2m+1} = \dfrac{a_1}{(-9)^m(2m+1)2m(2m-1)(2m-2)\cdots 3\cdot 2}$

别忘了 a_1 和.

$$= \frac{a_1}{(-9)^m(2m+1)!} \qquad m=1,2,3\cdots$$

Two linearly independent solutions are:

$$\sum_{n=0}^{\infty} \frac{1}{(-9)^n(2n+1)!} x^{2n} \qquad \text{取}\ a_0=1$$

☆3 a_0、a_1 的选取

$$\sum_{n=0} \left(\frac{1}{(-9)^n(2n)!} x^{2n-1} + \frac{1}{(-9)^n(2n+1)!} x^{2n} \right) \text{ 取 } a_0 = a_1 = 1$$

general solution? -1

☆4. General solution.

$$c_1 \sum_{n=0}^{\infty} \frac{1}{(-9)^n(2n+1)!} x^{2n} + c_2 \sum_{n=0}^{\infty} \left(\frac{1}{(-9)^n(2n)!} x^{2n-1} + \frac{1}{(-9)^n(2n+1)!} x^{2n} \right).$$

$c_1, c_2 = \text{const.}$

Dominant Balance.

$a > b \overset{x\to 0}{\Rightarrow} x^a < x^b$

$\boxed{x \to 0} \Rightarrow x^{-a} > x^{-b}$

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2. Consider the equation

$$y'' + \frac{2}{x}y' - \frac{1}{x^6}y = 0. \quad (*)$$

Find the asymptotic behavior of the solutions as $x \to 0^+$ (Find the first 4 leading terms in $S(x)$).

Substituting $y(x) = e^{S(x)}$

Then $y'(x) = S'(x)e^{S(x)}$

$y''(x) = S''(x)e^{S(x)} + (S')^2 e^{S(x)}$

Equation becomes

$S''e^S + (S')^2 e^S + \frac{2}{x}S'e^S - \frac{1}{x^6}e^S = 0$

$\Rightarrow S'' + (S')^2 + \frac{2}{x}S' - \frac{1}{x^6} = 0 \qquad (1)$

剩2项

① Assume $S'', (S')^2 \ll S', \frac{1}{x^6}$

Drop $S'', (S')^2$, we have

$\frac{2}{x}S' - \frac{1}{x^6} \sim 0$

$\frac{2}{x}S' \sim \frac{1}{x^6}$

$S' \sim \frac{1}{2}x^{-5}$

Then $S'' \sim -\frac{5}{2}x^{-6}$, $(S')^2 \sim \frac{1}{4}x^{-10}$

Verify: $(S')^2, S'' \gg S, S'$

It's inconsistent with the assumption.

✓ ② Assume $S'', S' \ll (S')^2, \frac{1}{x^6}$

Drop S'', S', we have

$(S')^2 - \frac{1}{x^6} \sim 0$

$(S')^2 \sim \frac{1}{x^6}$

$S' \sim \pm\frac{1}{x^3}$

Then $(s')^2 \sim \frac{1}{x^6}$, $s'' \sim \mp 3 \frac{1}{x^4}$

Verify: $(s')^2 \cdot \frac{1}{x^6} \gg s' \cdot s''$

It's consistent with the assumption.

③ Assume $s'' \cdot \frac{1}{x^6} \ll (s')^2 \cdot s'$

Drop $s'' \cdot \frac{1}{x^6}$, we have

$$(s')^2 + \frac{2}{x} s' \sim 0$$

$$(s' + \frac{2}{x}) s' \sim 0$$

$$s' \sim -\frac{2}{x}$$

Then $s'' \sim 2x^{-2}$. $(s')^2 \sim 4x^{-2}$

Verify: $\frac{1}{x^6} \gg s' . (s')^2$

It's inconsistent with the assumption.

④ Assume $(s')^2 \cdot s' \ll s'' \cdot \frac{1}{x^6}$

Drop $(s')^2 \cdot s'$, we have

$$s'' \sim x^{-6}$$

Then $s' \sim -\frac{1}{5} x^{-5}$, $(s')^2 \sim \frac{1}{25} x^{-10}$

Verify: $(s')^2 \gg s'' \cdot \frac{1}{x^6}$

It's inconsistent with the assumption.

⑤ Assume $(s')^2 \cdot \frac{1}{x^6} \ll s'' , s'$

Drop $(s')^2 \cdot \frac{1}{x^6}$, we have

$$s'' + \frac{2}{x} s' \sim 0$$

$$s' \sim x^{-2}$$

Then $(s')^2 \sim x^{-4}$ $s'' \sim 2x^{-3}$

Verify: $(s')^2 \cdot x^{-6} \gg s' \cdot s''$

It's inconsistent with the assumption.

⑥ Assume $s', \frac{1}{x^6} \ll s'', (s')^2$

Drop $s', \frac{1}{x^6}$, we have

$$s'' + (s')^2 \sim 0$$

Let $u = s'$

$$\frac{du}{dx} + u^2 = 0 \Rightarrow -\frac{1}{u^2} du = dx \Rightarrow \int -u^{-2} du = \int dx$$

$$\Rightarrow u = s' = x^{-1}$$

Then $(s')^2 = x^{-2}$. $s'' = -x^{-2}$

Verify: $\frac{1}{x^6} \gg s'', (s')^2$

It's inconsistent with the assumption.

Therefore, by method of dominant balance,

$$S(x) \sim \pm \frac{1}{2} x^{-2}$$

When $S(x) = \frac{1}{2} x^{-2}$

To get more accurate asymptotic behavior,

we assume $S(x) = \frac{1}{2} x^{-2} + C(x)$. $\underline{C(x) \ll x^{-2}}$ as $x \to 0^+$

Eq.(1) becomes

$$s'' + (s')^2 + \frac{2}{x} s' - \frac{1}{x^6} = 0 \qquad (2)$$

Since $s' = -x^{-3} + c'$

$$(s')^2 = x^{-6} - 2c' x^{-3} + (c')^2$$

$$s'' = 3x^{-4} + c''$$

Eq.(2): $3x^{-4} + c'' + x^{-6} - 2c' x^{-3} + (c')^2 + \frac{2}{x}(-x^{-3} + c') - \frac{1}{x^6} = 0$

$$\Rightarrow \quad x^{-4} + c'' - 2c' x^{-3} + (c')^2 + 2c' x^{-1} = 0$$

Since $c \ll x^{-2}$. $c' \ll -2x^{-3}$. $c'' \ll -3x^{-4}$

$(c')^2$ can be dropped compare with x^{-3}

c'' can be dropped compare with x^{-4}

$2c'x^{-1}$ can be dropped compare with $2c'x^{-3}$

Then $$x^{-4} - 2c'x^{-3} \sim 0$$

$$2c' \sim x^{-1}$$

$$c' \sim \frac{1}{2}x^{-1}$$

$$c \sim \frac{1}{2}\ln x$$

Verify: $c \sim \frac{1}{2}\ln x \ll x^{-2}$ is self-consistent

Thus $s(x) = \frac{1}{2}x^{-2} + \frac{1}{2}\ln x$

Next, we assume $S = \frac{1}{2}x^{-2} + \frac{1}{2}\ln x + d(x)$ $\quad d(x) \ll \frac{1}{2}\ln x$ as $x \to 0$

Then $$s' = -x^{-3} + \frac{1}{2}x^{-1} + d'$$

$$(s')^2 = x^{-6} - x^{-4} + \frac{1}{4}x^{-2} - 2d'x^{-3} + d'x^{-1} + (d')^2$$

$$s'' = 3x^{-4} - \frac{1}{2}x^{-2} + d''$$

Eq. (1) becomes

$$3x^{-4} - \frac{1}{2}x^{-2} + d'' + x^{-6} - x^{-4} + \frac{1}{4}x^{-2} - 2d'x^{-3} + d'x^{-1} + (d')^2$$

$$+ \frac{2}{x}(-x^{-3} + \frac{1}{2}x^{-1} + d') - x^{-6} = 0$$

$$\Rightarrow \frac{3}{4}x^{-2} - 2d'x^{-3} + 3d'x^{-1} + (d')^2 + d'' = 0$$

Since $d \ll \frac{1}{2}\ln x$, $d' \ll x^{-1}$, $d'' \ll -x^{-2}$

d', d'', $3d'x^{-1}$ can be dropped

Then $$\frac{3}{4}x^{-2} - 2d'x^{-3} \sim 0$$

$$d' \sim \frac{3}{8}x$$

$$d \sim \frac{3}{16}x^2 + C_0 \quad C_0 \text{ is constant}$$

-1

Verify: $d \sim \frac{3}{16}x^2 \overset{+C_0}{\gg} 2\ln x$ is self-consistent

Thus $$S(x) = \frac{1}{2}x^{-2} + \frac{1}{2}\ln x + \frac{3}{16}x^2 + C_0$$

$$y(x) \sim C_1 e^{\frac{1}{2}x^{-2} + \frac{1}{2}\ln x + \frac{3}{16}x^2}, \quad C_1 = e^{C_0}$$

When $S(x) = -\frac{1}{2}x^{-2}$

To get more accurate asymptotic behavior.

we assume $S(x) = -\frac{1}{2}x^{-2} + C(x)$. $C(x) \ll x^{-2}$ as $x \to 0^+$

Eq.(1) becomes

$$S'' + (S')^2 + \frac{2}{x}S' - \frac{1}{x^6} = 0 \qquad (2)$$

Since $S' = x^{-3} + C'$

$(S')^2 = x^{-6} + 2C'x^{-3} + (C')^2$

$S'' = -3x^{-4} + C''$

Eq.(2): $-3x^{-4} + C'' + x^{-6} + 2C'x^{-3} + (C')^2 + \frac{2}{x}(x^{-3} + C') - \frac{1}{x^6} = 0$

$\Rightarrow \quad -x^{-4} + C'' + 2C'x^{-3} + (C')^2 + 2C'x^{-1} = 0$

Since $C \ll x^{-2}$. $C' \ll -2x^{-3}$. $C'' \ll -3x^{-4}$

$(C')^2$ can be dropped compare with x^{-3}

C'' can be dropped compare with x^{-4}

$2C'x^{-1}$ can be dropped compare with $2C'x^{-3}$

Then $\quad -x^{-4} + 2C'x^{-3} \sim 0$

$$2C' \sim x^{-1}$$

$$C' \sim \frac{1}{2}x^{-1}$$

$$C \sim \frac{1}{2}\ln x$$

Verify: $C \sim \frac{1}{2}\ln x \ll x^{-2}$ is self-consistent

Thus $S(x) = -\frac{1}{2}x^{-2} + \frac{1}{2}\ln x$

Next, we assume $S = -\frac{1}{2}x^{-2} + \frac{1}{2}\ln x + d(x)$ $d(x) \ll \frac{1}{2}\ln x$ as $x \to 0$

Then $S' = x^{-3} + \frac{1}{2}x^{-1} + d'$

$(S')^2 = x^{-6} + x^{-4} + \frac{1}{4}x^{-2} + 2d'x^{-3} + d'x^{-1} + (d')^2$

$S'' = -3x^{-4} - \frac{1}{2}x^{-2} + d''$

Eq.(1) becomes

$$-3x^{-4} - \frac{1}{2}x^{-2} + d'' + x^{-6} + x^{-4} + \frac{1}{4}x^{-2} + 2d'x^{-3} + d'x^{-1} + (d')^2$$

$$+ \frac{2}{x}(x^{-3} + \frac{1}{2}x^{-1} + d') - x^{-6} = 0$$

$$\Rightarrow \quad \frac{3}{4}x^{-2} + 2d'x^{-3} + 3d'x^{-1} + (d')^2 + d'' = 0$$

Since $d \ll \frac{1}{2}\ln x$, $d' \ll x^{-1}$, $d'' \ll -x^{-2}$

d', d'', $3d'x^{-1}$ can be dropped

Then $\quad \frac{3}{4}x^{-2} + 2d'x^{-3} \sim 0$

$$d' \sim -\frac{3}{8}x$$

$$d \sim -\frac{3}{16}x^2 + C_0' \qquad C_0' \text{ is constant}$$

Verify: $d \sim -\frac{3}{16}x^2 \gg 2\ln x$ is self-consistent

Thus $\quad S(x) = -\frac{1}{2}x^{-2} + \frac{1}{2}\ln x - \frac{3}{16}x^2 + C_0'$

$$y(x) \sim C_2 e^{-\frac{1}{2}x^{-2} + \frac{1}{2}\ln x - \frac{3}{16}x^2}, \quad c_1 = e^{C_0'}$$

Thus, the two solutions are

$$y_1(x) \sim C_1 e^{\frac{1}{2}x^{-2} + \frac{1}{2}\ln x + \frac{3}{16}x^2}, \quad c_1 = e^{C_0}$$

$$y_2(x) \sim C_2 e^{-\frac{1}{2}x^{-2} + \frac{1}{2}\ln x - \frac{3}{16}x^2}, \quad c_2 = e^{C_0'}$$

Asymptotic Expansion of Integrals.

10/10 3. Use Taylor expansion of e^{-t^2} to find an asymptotic expansion of the integral

$$I(x) = \int_x^\infty e^{-t^2}\,dt, \quad x \to 0.$$

You are required to show that the asymptotic expansion of the integrand holds uniformly for t.

Taylor expansion of e^{-t^2} is

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} \qquad t \in (x, +\infty)$$

$\sum_{n=0}^{\infty} y_n(x)$

$$R_N(t) = \frac{e^{\xi}}{(N+1)!}(-t^2)^{N+1} \qquad \xi \in (-t^2, 0)$$

$\lim_{x\to x_0} \frac{R_N(x)}{y_N(x)}$

$$= \lim_{x\to 0} \frac{R_N(t)}{(-t^2)^N} = \lim_{x\to 0} \frac{e^{\xi}}{(N+1)!}(-t^2) \neq 0 \qquad \xi \in (-t^2, 0) \quad t \in [x, +\infty)$$

bounded

$t \in [x, +\infty)$

Thus,

$\int_x^\infty \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}\,dt = \sum_{n=0}^{\infty} \int_x^\infty \frac{(-1)^n}{n!} t^{2n}\,dt$ is not an asymptotic expansion of the integral.

$$\int_x^\infty e^{-t^2}\,dt = \underbrace{\int_0^\infty e^{-t^2}\,dt}_{I_1} - \underbrace{\int_0^x e^{-t^2}\,dt}_{I_2}$$

$$I_1 = \frac{\sqrt{\pi}}{2}$$

When $x \to 0$, $\int_0^x \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}\,dt = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n}{n!} t^{2n}\,dt$ is an asymptotic expansion of I_2

$$I_2 = \sum_{n=0}^{\infty} \int_0^x \frac{(-1)^n}{n!} t^{2n}\,dt$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n t^{2n+1}}{(2n+1)\cdot n!} \Big|_0^x \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} x^{2n+1}$$

Thus, the asymptotic expansion of the integral is

$$\frac{\sqrt{\pi}}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\, n!} x^{2n+1}$$

Also, since

$$\lim_{x\to 0} \frac{R_N(t)}{(-t^2)^N} = \lim_{x\to 0} \frac{e^{\xi}}{(N+1)!}(-t^2) = 0 \qquad \xi \in (-t^2, 0),\ t \in (0, x)$$

The asymptotic expansion holds uniformly for t.

Integration by parts.

4. Consider the integral

$$I(x) = \int_0^\infty \frac{e^{-t}}{x^2+t}dt, \quad x \to +\infty.$$

(1) Using integration by parts, show that

$$I(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} + (-1)^N N! \int_0^\infty \frac{e^{-t}}{(x^2+t)^{N+1}}dt.$$

(2) Using definition, show that

$$I(x) \sim \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{x^{2n}}, \quad x \to +\infty.$$

(1) $I(x) = \int_0^\infty \frac{e^{-t}}{x^2+t} dt$

$= -\int_0^\infty \frac{1}{x^2+t} d(e^{-t})$

$= -\left(e^{-t}\frac{1}{x^2+t}\Big|_0^\infty - (-1)\int_0^\infty e^{-t} d\left(\frac{1}{x^2+t}\right)\right)$ 是dt! 褪dx!

$= \frac{1}{x^2} - \int_0^\infty \frac{1}{(x^2+t)^2} e^{-t} dt$

Denote $\int_0^\infty \frac{1}{(x^2+t)^2} e^{-t} dt$ as $I_1(x)$

$I_1(x) = -\int_0^\infty \frac{1}{(x^2+t)^2} de^{-t}$

$= -\left(e^{-t}\frac{1}{(x^2+t)^2}\Big|_0^\infty - \int_0^\infty e^{-t} d\left(\frac{1}{(x^2+t)^2}\right)\right)$

$= -\left(-\frac{1}{x^4} - (-2)\int_0^\infty e^{-t}\frac{1}{(x^2+t)^3} dt\right)$

$= \frac{1}{x^4} - 2\int_0^\infty \frac{e^{-t}}{(x^2+t)^3} dt$

Denote $\int_0^\infty \frac{e^{-t}}{(x^2+t)^3}\,dt$ as $I_2(x)$

Similarly, $I_2(x) = \frac{1}{x^6} - 3\cdot 2\int_0^\infty \frac{e^{-t}}{(x^2+t)^4}\,dt$

Repeat this process, we have 订正后的在后面.

$$I(x) = \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} + \cancel{(-1)}^N N! \int_0^\infty \frac{e^{-t}}{(x^2+t)^{N+1}}\,dt$$

Mathematical induction. −1.5

(2) $R_N(x) = (-1)^N N! \int_0^\infty \frac{e^{-t}}{(x^2+t)^{N+1}}\,dt$

$$\frac{R_N(x)}{y_N(x)} = \frac{(-1)^N N! \int_0^\infty \frac{e^{-t}}{(x^2+t)^{N+1}}\,dt}{(-1)^{N-1}\frac{(N-1)!}{x^{2N}}}$$

$$= -N\cdot x^{2N}\int_0^\infty \frac{e^{-t}}{(x^2+t)^{N+1}}\,dt$$

$\to 0$ as $x\to\infty$ Why? −1

Thus, $I(x) \sim \sum_{n=1}^{N} \frac{(-1)^{n-1}(n-1)!}{x^{2n}}$

$$-Nx^{2N}\int_0^\infty \frac{e^{-t}}{(x^2+t)^{N+1}}\,dt$$

$$= -Nx^{2N}\left(\frac{1}{x^{2N+2}} - (N+1)\int_0^\infty \frac{e^{-t}}{(x^2+t)^{N+2}}\,dt\right)$$

$$= -\frac{N}{x^2} + N(N+1)\,x^{2N}\int_0^\infty \frac{e^{-t} \le 1}{(x^2+t)^{N+2}}\,dt.$$

$$\le -\frac{N}{x^2} + N(N+1)\int_0^\infty \frac{1\cdot x^{2N}}{x^{2N+4}}\cdot dt.$$

$$= -\frac{N}{x^2} + N(N+1)\frac{t}{x^4}$$

$\to 0$ as $x\to\infty$.

4. (1) Use Induction.

① When $N=1$.

$$I(x) = \int_0^\infty \frac{e^{-t}}{x^2+t}\,dt$$

$$= -\int_0^\infty \frac{1}{x^2+t}\,d(e^{-t})$$

$$= -\left(e^{-t}\frac{1}{x^2+t}\Big|_0^\infty - (-1)\int_0^\infty e^{-t}\,d\left(\frac{1}{x^2+t}\right)\right)$$

$$= \frac{1}{x^2} - \int_0^\infty \frac{1}{(x^2+t)^2}e^{-t}dt.$$

satisfies the equation.

② Assume that when $N=k$, The equation holds

i.e. $$I(x) = \sum_{n=1}^{k} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} + (-1)^k k! \int_0^\infty \frac{e^{-t}}{(x^2+t)^{k+1}}\,dt$$

When $N=k+1$.

Let $$J(x) = \int_0^\infty \frac{e^{-t}}{(x^2+t)^{k+1}}\,dt.$$

$$= -\int_0^\infty \frac{1}{(x^2+t)^{k+1}}\,d(e^{-t})$$

$$= -\frac{e^{-t}}{(x^2+t)^{k+1}}\Big|_0^\infty + \int_0^\infty e^{-t}\,d\frac{1}{(x^2+t)^{k+1}}$$

$$= \frac{1}{x^{2(k+1)}} - (k+1)\int_0^\infty \frac{e^{-t}}{(x^2+t)^{k+2}}\,dt$$

$$\therefore\ I(x) = \sum_{n=1}^{k} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} + (-1)^k k!\left[\frac{1}{x^{2(k+1)}} - (k+1)\int_0^\infty \frac{e^{-t}}{(x^2+t)^{k+2}}\,dt\right]$$

$$= \sum_{n=1}^{k+1} \frac{(-1)^{n-1}(n-1)!}{x^{2n}} + (-1)^{k+1}(k+1)!\int_0^\infty \frac{e^{-t}}{(x^2+t)^{k+2}}\,dt.$$

$\therefore$ Eq holds.