HW3

inner region 区为 x 邻 2.

1. Consider the equation

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$$\begin{cases} \epsilon y'' + (\frac{x}{4} - 1)y' + \frac{1}{4}y = 0, & 0 \le x \le 2. \\ y(0) = 3, \; y(2) = 2. \end{cases}$$

Assume that it is a boundary layer problem. The boundary layer is at $x = 2$, and the boundary layer thickness is ϵ.

(1) Find the outer limit, inner limit and the intermediate limit of the solution.

(2) Write down a uniform leading order approximation of the solution.

Solution:

- In the outer region, $0 \overset{<<}{\le} x \cancel{\overset{<<}{\le}} 2-\varepsilon \qquad \varepsilon \to 0$

$y_{out} = y(x) \sim \sum_{n=0}^{\infty} y_n(x)\varepsilon^n \qquad \varepsilon \to 0$

Boundary condition at $x=0$: $y(0)=3$

Thus, $y_0(0)=3$

$y_n(0)=0 \quad n \ge 1$

Substituting the expansion into the equation:

$$\sum_{n=0}^{\infty} y_n''(x)\varepsilon^{n+1} + (\frac{x}{4}-1)\sum_{n=0}^{\infty} y_n'(x)\varepsilon^n + \frac{1}{4}\sum_{n=0}^{\infty} y_n(x)\varepsilon^n \sim 0 \qquad \varepsilon \to 0$$

For $n=0$, we have

$$\varepsilon y_0'' + (\frac{x}{4}-1)y_0' + \frac{1}{4}y_0 = 0$$

$$\Rightarrow (\frac{x}{4}-1)y_0' + \frac{1}{4}y_0 = 0 \qquad \varepsilon \to 0$$

$$\Rightarrow y_0 = \frac{c}{x-4} \qquad c = \text{const}$$

$-\ln y_0 = \ln(x-4)+C$

$e^{-\ln y_0} = ce^{\ln(x-4)}$

$e^{\ln\frac{1}{y_0}} = ce^{\ln(x-4)}.$

Since $y_0(0)=3 \Rightarrow c=-12$

Therefore, the outerlimit is $y_0(x) = -\frac{12}{x-4}$

- In the inner region, $2-O(\varepsilon) \le x \le 2$

Using inner variable $X = \frac{x\boxed{-2}}{\varepsilon} \Rightarrow x = \varepsilon X + 2$

Then $Y_{in}(X) = y(x) = y(\varepsilon X + 2)$

$$\Rightarrow \frac{d}{dx} = \frac{1}{\varepsilon}\frac{d}{dX}, \quad \frac{d^2}{dx^2} = \frac{1}{\varepsilon^2}\frac{d^2}{dX^2}$$

The equation becomes 别忘记把x换成X

$$\frac{1}{\varepsilon}\frac{d^2Y_{in}}{dX^2}+\left(\frac{\varepsilon X+2}{4}-1\right)\frac{1}{\varepsilon}\frac{dY_{in}}{dX}+\frac{1}{4}y=0$$

$$\Rightarrow \frac{d^2Y_{in}}{dX^2}+\left(\frac{\varepsilon X+2}{4}-1\right)\frac{dY_{in}}{dX}+\frac{\varepsilon}{4}y=0$$

Assume that

$Y_{in}(X)\sim\sum_{n=0}^{+\infty}Y_n(X)\varepsilon^n \qquad \varepsilon\to 0$

Boundary condition at $x=2$: $y(2)=2 \Rightarrow Y_0(0)=2$

Substituting it into the equation

$$\sum_{n=0}^{\infty}Y_n''(X)\varepsilon^n+\left(\frac{\varepsilon X}{4}-\frac{1}{2}\right)\sum_{n=0}^{\infty}Y_n'(X)\varepsilon^n+\frac{1}{4}\sum_{n=0}^{\infty}Y_n(X)\varepsilon^{n+1}=0$$

When $n=0$.

$Y_0''(X)-\frac{1}{2}Y_0'(X)=0 \qquad$ as $\varepsilon\to 0$

$\Rightarrow Y_0'(X)=e^{\frac{X}{2}}+C_1$

$\Rightarrow Y_0(X)=C_2e^{\frac{X}{2}}+C_1$. $\quad C_1, C_2=$ constant.

Since $Y_0(0)=2 \Rightarrow C_1+C_2=2$

$x=2$. $X=0$

- In the overlapping region, $x\to 2$. $X\to-\infty$ as $\varepsilon\to 0$

The intermediate limit is

$$\lim_{x\to 2}y_0(x)=\lim_{X\to-\infty}Y_0(X)$$

$\underbrace{(x-2)}_{<0}/\underbrace{\varepsilon}_{\searrow 0}\to-\infty$ when $x\to 2$

i.e. $\lim_{x\to 2}-\frac{12}{x-4}=\lim_{X\to-\infty}C_2e^{\frac{X}{2}}+C_1$

$\Rightarrow \quad 6=C_1$

Thus, $C_1=6$, $C_2=-4$

Therefore, the inner limit is $Y_0(X)=-4e^{\frac{X}{2}}+6$

(2) The leading order uniform approximation in $0\le x\le 2$ is

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$$y_{unif}(x) = y^{a}_{out}(x) + y^{a}_{inner}(x) - y_{match}(x)$$

$$= \frac{12}{4-x} - 4e^{\frac{x-2}{2\varepsilon}} + 6 - 6$$

$$= \frac{12}{4-x} - 4e^{\frac{x-2}{2\varepsilon}}$$

Thickness, location, higher-order

2. Consider the equation

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$$\begin{cases} \epsilon y'' + x^2 y' + y = 0, & 1 \le x \le 2, \\ y(1) = 1, \ y(2) = 1. \end{cases}$$

(1) Determine the thickness and location of the boundary layer.

(2) Obtain a uniform approximation accurate to order ϵ as $\epsilon \to 0$.

先 thickness 后 location.

(1) Assume that the location of the boundary layer is at $x = x_0$ with thickness $\delta(\varepsilon) \ll 1$

In the inner region, let $X = \frac{x - x_0}{\delta}$

$Y_{in}(X) = y(x) = y(\delta X + x_0)$

The equation becomes 为什么不用换 写换的. $\frac{\varepsilon}{\delta^2} Y'' + \delta X^2 Y' + 2x_0 X Y' + \frac{1}{\delta} x_0^2 Y' + Y = 0$

$$\frac{\varepsilon}{\delta^2} \frac{d^2 Y_{in}}{dX^2} + \frac{x^2}{\delta} \frac{dY_{in}}{dX} + Y_{in} = 0$$

There are $O\left(\frac{\varepsilon}{\delta^2}\right)$, $O\left(\frac{1}{\delta}\right)$, $O(1)$, $O(\delta)$ terms

Since $\delta \ll 1$,

$$\frac{\varepsilon}{\delta^2}, \ \frac{1}{\delta} \gg 1 \gg \delta$$

The dominant balance is

$$\frac{\varepsilon}{\delta^2} \frac{d^2 Y_{in}}{dX^2} + \frac{1}{\delta} \cdot x^2 \frac{dY_{in}}{dX} \sim 0 \quad \text{as } \varepsilon \to 0$$

Thus, $\frac{\varepsilon}{\delta^2} = O\left(\frac{1}{\delta}\right) \Rightarrow \delta = O(\varepsilon)$

we can choose $\delta = \varepsilon$

Then, we can write $X = \frac{x - x_0}{\varepsilon} \Rightarrow x = \varepsilon X + x_0$

The equation becomes,

$$Y_{in}''(X) + (\varepsilon X + x_0)^2 \, Y_{in}'(X) + \varepsilon Y_{in}(X) = 0$$

Assume $Y_{in} \sim \sum_{n=0}^{\infty} Y_n(X) \varepsilon^n$

Substiting into the equation

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$$\sum_{n=0}^{\infty} Y_n''(X)\varepsilon^n + (\varepsilon X + x_0)^2 \sum_{n=0}^{\infty} Y_n'(X)\varepsilon^n + \sum_{n=0}^{\infty} Y_n(X)\varepsilon^{n+1} = 0$$

When $n=0$, we have

$$Y_0''(X) + x_0^2\, Y_0'(X) = 0 \quad \text{as } \varepsilon \to 0$$

$$\Rightarrow \quad Y_0'(X) = c_1 e^{-x_0^2 X}$$

$$\Rightarrow \quad Y_0(X) = c_1 e^{-x_0^2 X} + c_2 \qquad c_1, c_2 = \text{constant}$$

If $1 < x_0 \le 2$, X may go to $-\infty$ in the overlapping region.

$$\lim_{X \to -\infty} e^{-x_0^2 X} = +\infty$$

For $\lim_{X \to -\infty} Y_0(X)$ to exist, $c_1 = 0$. $Y_0(X) = c_2$

It is not a rapid varing solution. $\Rightarrow$ No boundary layer at $1 < x \le 2$

Therefore, the possible boundary layer is at $x_0 = 1$.

(2) In the outer region $(1+\varepsilon \ll x \ll 2)$, assume that

$$y_{out}(x) \sim y_0(x) + \varepsilon y_1(x) + \cdots \qquad \varepsilon \to 0^+$$

$$x^2 y_0' + y_0 = 0 \qquad y_0(2) = 1$$

$$x^2 y_n' + y_n = -y_{n-1}'' \qquad y_n(2) = 0 \qquad n = 1, 2, 3 \cdots$$

$O(1)$: $\begin{cases} x^2 y_0' + y_0 = 0 \\ y_0(2) = 1 \end{cases}$

Solution: $y_0(x) = e^{\frac{1}{x} - \frac{1}{2}}$

$O(\varepsilon)$: $\begin{cases} x^2 y_1' + y_1 = -y_0'' = -\left(\frac{1}{x^4} + \frac{2}{x^3}\right) e^{\frac{1}{x} - \frac{1}{2}} \\ y_1(2) = 0 \end{cases}$

Solution: $y_1(x) = \left(-\frac{3}{5 \cdot 2^4} + \frac{1}{5} x^{-5} + \frac{1}{2} x^{-4}\right) e^{\frac{1}{x} - \frac{1}{2}}$

In the inner region, $1 \le x \le 1 + O(\varepsilon)$

$$\xi = \frac{x-1}{\varepsilon} \Rightarrow x = \varepsilon \xi + 1$$

$$Y_{in}(\xi) = y(x) = y(\varepsilon \xi + 1)$$

The equation becomes

$$\varepsilon \cdot \frac{1}{\varepsilon^2} \cdot \frac{d^2 Y_{in}}{d\xi^2} + (\varepsilon\xi+1)^2 \cdot \frac{1}{\varepsilon} \frac{dY_{in}}{d\xi} + Y_{in} = 0$$

$$\Rightarrow \frac{1}{\varepsilon} Y_{in}'' + \left(\varepsilon\xi^2 + 2\xi + \frac{1}{\varepsilon}\right) Y_{in}' + Y_{in} = 0$$

$$\Rightarrow Y_{in}'' + (\varepsilon^2\xi^2 + 2\varepsilon\xi + 1) Y_{in}' + \varepsilon Y_{in} = 0$$

Assume that $Y_{in}(\xi) \sim Y_0(\xi) + \varepsilon Y_1(\xi) + \cdots$, $\varepsilon \to 0$

$Y_{in}(0) = y(1) = 1$

$\Rightarrow Y_0(0) = 1$

$Y_n(0) = 0 \quad n \geq 1$

$O(1)$: $\begin{cases} Y_0'' + Y_0' = 0 \\ Y_0(0) = 1 \end{cases}$

Solution: $Y_0(\xi) = 1 + c(e^{-\xi} - 1)$

$O(\varepsilon)$: $\begin{cases} Y_1'' + Y_1' = -2\xi Y_0' - Y_0 = 2c\xi e^{-\xi} - ce^{-\xi} + c - 1 \\ Y_1(0) = 0 \end{cases}$

Solution: $Y_1(\xi) = -(c\xi^2 + c\xi + c - d)e^{-\xi} + (c-1)\xi + c - d$

$= -(c\xi^2 + c\xi + d)e^{-\xi} + (c-1)\xi + d$

$O(1)$ matching:

$$\lim_{x \to 1} y_0(x) = \lim_{\xi \to +\infty} Y_0(\xi)$$

$$e^{1-\frac{1}{2}} = 1 + c(0-1)$$

$$c = 1 - e^{\frac{1}{2}}$$

$$Y_0(\xi) = 1 + (1 - e^{\frac{1}{2}})(e^{-\xi} - 1)$$

$$= e^{-\xi} - e^{-\xi + \frac{1}{2}} + e^{\frac{1}{2}}$$

$O(\varepsilon)$ matching:

$$y_{out}(x) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$$

$$= e^{\frac{1}{x} - \frac{1}{2}} + \varepsilon \cdot \left(-\frac{3}{5 \cdot 2^4} + \frac{1}{5}x^{-5} + \frac{1}{2}x^{-4}\right)e^{\frac{1}{x} - \frac{1}{2}} + O(\varepsilon^2) \quad \varepsilon \to 0$$

$$Y_{in}(\gamma) = Y_0(\gamma) + \varepsilon Y_1(\gamma) + O(\varepsilon^2)$$
$$= 1 + (1-e^{\frac{1}{2}})(e^{-\gamma}-1) + \varepsilon[((1-e^{\frac{1}{2}})\gamma^2 + (1+e^{\frac{1}{2}})\gamma + d)e^{-\gamma} - e^{\frac{1}{2}}\gamma + d] + O(\varepsilon^2)$$
$$\varepsilon \to 0$$

We should have

$[y_0(x) + \varepsilon y_1(x)] - [Y_0(\gamma) + \varepsilon Y_1(\gamma)] = o(\varepsilon)$ as $\varepsilon \to 0$ uniformly in the overlapping region.

Keep $O(1)$, $o(\varepsilon)$, $O(x)$ terms in the overlapping region.

When $1+\varepsilon \ll x \ll 2$,

$$y_{out}(x) \sim y_0(x) + \varepsilon y_1(x)$$
$$= e^{\frac{1}{x}-\frac{1}{2}} + \varepsilon \cdot \left(-\frac{3}{5\cdot 2^4} + \frac{1}{5}x^{-5} + \frac{1}{2}x^{-4}\right)e^{\frac{1}{x}-\frac{1}{2}}$$
$$= e^{\frac{1}{2}} - e^{\frac{1}{2}}(x-1) + O((x-1)^2) + \varepsilon\left(-\frac{3}{5\cdot 2^4} + x - \frac{4}{5} + O((x-1)^2) + \frac{5}{2} - 2x + O((x-1)^2)\right)\left(2e^{\frac{1}{2}} - e^{\frac{1}{2}}x + O((x-1)^2)\right)$$
$$= 2e^{\frac{1}{2}} - e^{\frac{1}{2}}x + \frac{53}{80}e^{\frac{1}{2}}\varepsilon + O(\varepsilon^2) + O(\varepsilon(x-1)) + O((x-1)^2)$$ 内由

$$Y_{in}(\gamma) \sim Y_0(\gamma) + \varepsilon Y_1(\gamma)$$
$$= 1 + (1-e^{\frac{1}{2}})(e^{-\gamma}-1) + \varepsilon[((1-e^{\frac{1}{2}})\gamma^2 + (1+e^{\frac{1}{2}})\gamma + d)e^{-\gamma} - e^{\frac{1}{2}}\gamma + d]$$
$$= 1 + (1-e^{\frac{1}{2}})(e^{-1} - 1 - e^{-1}(x-1) + O((x-1)^2)) + \varepsilon[((1-e^{\frac{1}{2}})(\gamma^2+\gamma) + d)(e^{-1} - e^{-1}(\gamma-1) + O((\gamma-1)^2) - e^{\frac{1}{2}}\gamma + d]$$

(Let $x = \varepsilon\gamma$) $= 2e^{\frac{1}{2}} - e^{\frac{1}{2}}x + d\varepsilon + O(\varepsilon^2) + O(\varepsilon(x-1)) + O((x-1)^2)$

$x = \varepsilon\gamma + 1$

Comparing the coefficients, we have $d = \frac{53}{80}e^{\frac{1}{2}}$

Therefore, $Y_1(\gamma) = ((1-e^{\frac{1}{2}})\gamma^2 + (1+e^{\frac{1}{2}})\gamma + \frac{53}{80}e^{\frac{1}{2}})e^{-\gamma} - e^{\frac{1}{2}}\gamma + \frac{53}{80}e^{\frac{1}{2}}$

$y_{match} = 2e^{\frac{1}{2}} - e^{\frac{1}{2}}x + \frac{53}{80}e^{\frac{1}{2}}\varepsilon$

(Pls see next page :))

Uniform approximation up to $O(\varepsilon)$

$$y_{unif}(x) = y_{out}^{a}(x) + y_{in}^{a}(x) - y_{match}(x)$$

$$= \underline{e^{\frac{1}{x}-\frac{1}{2}} + \varepsilon\left[\left(-\frac{1}{5\cdot 2^4} + \frac{1}{5}x^{-5} + \frac{1}{2}x^{-4}\right)e^{\frac{1}{x}-\frac{1}{2}}\right]} + \underline{1 + (1-e^{\frac{1}{2}})(e^{-\frac{x}{\varepsilon}}-1)} +$$

$$\underline{\varepsilon\left[(1-e^{\frac{1}{2}})\left(\frac{x}{\varepsilon}\right)^2 + (1-e^{\frac{1}{2}})\left(\frac{x}{\varepsilon}\right) + \frac{53}{80}e^{\frac{1}{2}}\right)e^{-\frac{x}{\varepsilon}} - e^{\frac{1}{2}}\left(\frac{x}{\varepsilon}\right) + \frac{53}{80}e^{\frac{1}{2}}\right]}$$

$$\underline{-\left(2e^{\frac{1}{2}} - e^{\frac{1}{2}}x + \frac{53}{80}e^{\frac{1}{2}}\varepsilon\right)}$$

$$= e^{\frac{1}{x}-\frac{1}{2}} + e^{-\frac{x}{\varepsilon}} - e^{\frac{1}{2}-\frac{x}{\varepsilon}} - e^{\frac{1}{2}} + e^{\frac{1}{2}}x$$

$$+ \varepsilon\left[\left(\left(-\frac{3}{80} + \frac{1}{5}x^{-5} + \frac{1}{2}x^{-4}\right)e^{\frac{1}{x}-\frac{1}{2}} + (1-e^{\frac{1}{2}})\left(\frac{x}{\varepsilon}\right)^2 + (1-e^{\frac{1}{2}})\left(\frac{x}{\varepsilon}\right) + \frac{53}{80}e^{\frac{1}{2}}\right)e^{-\frac{x}{\varepsilon}} - e^{\frac{1}{2}}\left(\frac{x}{\varepsilon}\right)\right]$$

3. Find the leading order approximation to the solution of the problem

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$$\begin{cases} \epsilon y'' + x^{\frac{1}{3}} y' - y = 0, & 0 \le x \le 1. \\ y(0) = 0, \ y(1) = e^{3/2}. \end{cases}$$

Assume that $y(x) \sim y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \cdots \quad \varepsilon \to 0$

$y_0(x)$ satisfies

$$x^{\frac{1}{3}} \frac{dy_0}{dx} - y_0 = 0$$

Solution: $y_0 = Ce^{\frac{3}{2}x^{\frac{2}{3}}}$

It cannot satisfy the two boundary conditions simultaneously.

Assume that the location of the boundary layer is at $x = x_0$ with thickness $\delta(\varepsilon) \ll 1$

In the inner region, let $\xi = \frac{x - x_0}{\delta} \Rightarrow x = \delta\xi + x_0$

$Y_{in}(\xi) = y(x) = y(\delta\xi + x_0)$

The equation becomes

$$\frac{\varepsilon}{\delta^2} \frac{d^2 Y_{in}}{d\xi^2} + \frac{(\delta\xi + x_0)^{\frac{1}{3}}}{\delta} \frac{dY_{in}}{d\xi} - Y_{in} = 0$$

There are $O(\frac{\varepsilon}{\delta^2})$. $O(\frac{(\delta\xi + x_0)^{\frac{1}{3}}}{\delta})$. $O(1)$ terms

Since $\delta \ll 1$.

$$\frac{\varepsilon}{\delta^2} \cdot \frac{1}{\delta} \gg 1$$

The dominant balance is

$$\frac{\varepsilon}{\delta^2} \frac{d^2 Y_{in}}{d\xi^2} + \frac{(\delta\xi + x_0)^{\frac{1}{3}}}{\delta} \frac{dY_{in}}{d\xi} \sim 0 \quad \text{as } \varepsilon \to 0$$

只有x可取0的时候要讨论.

Thus, $\frac{\varepsilon}{\delta^2} \sim \frac{(\delta\xi + x_0)^{\frac{1}{3}}}{\delta} \Rightarrow (\frac{\varepsilon}{\delta^1})^3 \sim \frac{(\delta\xi + x_0}{\delta^3}$

$\delta\xi + x_0 \to x_0$

① If $x_0 \neq 0$. $(\frac{\varepsilon}{\delta^1})^3 \sim \frac{1}{\delta^3}$, we can choose $\delta = \varepsilon$

② If $x_0 = 0$ $(\frac{\varepsilon}{\delta^2})^3 \sim \frac{1}{\delta^2}$, we can choose $\delta = \varepsilon^{\frac{3}{4}}$

$\frac{\delta\xi}{\delta^3} \sim \frac{1}{\delta^2}$

Then, we discuss the location of boundary layer

① If $x_0 \neq 0$, $\delta = \varepsilon$. Then we can write $\xi = \frac{x - x_0}{\varepsilon} \Rightarrow x = \varepsilon\xi + x_0$

The equation becomes

$$\varepsilon \cdot \frac{1}{\varepsilon^2} Y_{in}'' + (\varepsilon\xi + x_0)^{\frac{1}{3}} \cdot \frac{1}{\varepsilon} Y_{in}' - Y_{in} = 0$$

$$\Rightarrow Y_{in}'' + (\varepsilon\xi + x_0)^{\frac{1}{3}} Y_{in}' - \varepsilon Y_{in} = 0$$

Assume $Y_{in} \sim \sum_{n=0}^{\infty} Y_n(\xi)\varepsilon^n$

Substiting into the equation

$$\sum_{n=0}^{\infty} Y_n''(X)\varepsilon^n + (\varepsilon\xi + x_0)^{\frac{1}{3}} \sum_{n=0}^{\infty} Y_n'(X)\varepsilon^n - \sum_{n=0}^{\infty} Y_n(X)\varepsilon^{n+1} = 0$$

When $n=0$, we have

$$Y_0'' + x_0^{\frac{1}{3}} Y' = 0 \quad \text{as} \quad \varepsilon \to 0$$

$$\Rightarrow Y_0'(\xi) = ce^{-x_0^{\frac{1}{3}}\xi}$$

$$\Rightarrow Y_0(\xi) = c_1 e^{-x_0^{\frac{1}{3}}\xi} + c_2$$

If $0 < x_0 \le 1$, ξ may go to $-\infty$ in the overlapping region.

$\lim_{\xi \to -\infty} e^{-x_0^{\frac{1}{3}}\xi} = +\infty$

For $\lim_{\xi \to -\infty} Y_0(\xi)$ to exist, $c_1 = 0$. $Y_0(\xi) = c_2$

It is not a rapidly varying solution. So there is no boundary layer when $0 < x_0 \le 1$.

Therefore, the possible boundary layer is at $x_0 = 0$. and the layer thickness is $\varepsilon^{\frac{3}{4}}$

Now find the leading order approximation.

- In the outer region, $O(\varepsilon^{\frac{3}{4}}) << x << 1$

$$y(x) = y_{out} \sim \sum_{n=0}^{+\infty} y_n(x)\varepsilon^n$$

boundary condition at $x=1$: $y(1) = e^{\frac{3}{2}}$

Thus $y_0(1) = e^{\frac{3}{2}}$

$y_n(1)=0 \qquad n\ge 1$

Substituting the expansion into the equation,

$$\sum_{n=0}^{\infty} y_n''(x)\,\varepsilon^{n+1} + x^{\frac13}\sum_{n=0}^{+\infty} y_n'(x) - \sum_{n=0}^{\infty} y_n(x) \sim 0 \qquad \varepsilon\to 0$$

We have $\begin{cases} x^{\frac13} y_0' - y_0 = 0 \\ y_0(1) = e^{\frac32} \end{cases}$

Solution : $y_0(x) = e^{\frac32 x^{\frac23}}$

- In the inner region, $0 \le x \le O(\varepsilon^{\frac34})$

Let $\xi = \dfrac{x}{\varepsilon^{\frac34}}$. $Y_{in}(\xi) = y(x) = y(\varepsilon^{\frac34}\xi)$

The equation becomes

$$\varepsilon\cdot\varepsilon^{-\frac32} Y_{in}'' + (\varepsilon^{\frac34}\xi)^{\frac13}\cdot\varepsilon^{-\frac34} Y_{in}' - Y_{in} = 0$$

$$\Rightarrow \quad Y_{in}'' + \xi^{\frac13} Y_{in}' - \varepsilon^{\frac12} Y_{in} = 0$$

Assume $Y_{in}(\xi) \sim \sum_{n=0}^{\infty} Y_n(\xi)\,\varepsilon^n$ 为什么不是 $\sum_{n=0}^{\infty} Y_n(\xi)(\varepsilon^{\frac34})^n$?

Substituting the expansion into the equation

$$\Rightarrow \sum_{n=0}^{\infty} Y_n''(\xi)\,\varepsilon^{\frac34 n} + \xi^{\frac13}\sum_{n=0}^{\infty} Y_n'(\xi)\,\varepsilon^{\frac34 n} - \sum_{n=0}^{\infty} Y_n(\xi)\,\varepsilon^{\frac34 n+\frac12} = 0$$

When $n=0$. we have

$\begin{cases} Y_0'' + \xi^{\frac13} Y_0' = 0 \\ Y_0(0) = 0 \end{cases}$ $\dfrac{dY}{d\xi} = Ce^{-\frac34\xi^{\frac43}}$

why?

Solution : $Y_0(\xi) = C\int_0^{\xi} e^{-\frac34 x^{\frac43}}\,dx$

In the overlapping region . $x\to 0$. $\xi\to+\infty$ as $\varepsilon\to 0$

The intermediate limit is

$$\lim_{x\to 0} y_0(x) = \lim_{\xi\to+\infty} Y_0(\xi)$$

$$\lim_{x\to 0} e^{\frac32 x^{\frac23}} = \lim_{\xi\to+\infty} C\int_0^{\xi} e^{-\frac34 x^{\frac43}}\,dx$$

$$1 = C\int_0^{+\infty} e^{-\frac34 x^{\frac43}}\,dx$$

Thus, the inner limit is $Y_0(z) = \left(\int_0^{+\infty} e^{-\frac{3}{4}x^{\frac{4}{3}}} dx\right)^{-1} \int_0^{z} e^{-\frac{3}{4}x^{\frac{4}{3}}} dx$

The leading order approximation is

$$y_{unif}(x) = y_{out}^{a}(x) + y_{in}^{a}(x) - y_{match}(x)$$

$$= e^{\frac{3}{2}x^{\frac{2}{3}}} + \left(\int_0^{+\infty} e^{-\frac{3}{4}x^{\frac{4}{3}}} dx\right)^{-1} \int_0^{z} e^{-\frac{3}{4}x^{\frac{4}{3}}} dx - 1$$