HW4.

Liu Yonglin WKB theory

1. Find the WKB approximation correct up to $O(\epsilon^{\frac{1}{2}})$ to the general solution of the problem

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$$\epsilon y'' + (x^2 + \epsilon x)y = 0, \quad x > 0,$$

where $\epsilon > 0$.

$\varepsilon y'' + (x^2+\varepsilon x)y = 0$

$\Rightarrow \varepsilon y'' = -(x^2+\varepsilon x)y$

Let $\cancel{Q(x) = -(x^2+\varepsilon x)}$ ε不能放进去.

$y(x) = e^{S(x)}$

$S(x) \sim \frac{1}{\delta} S_0(x) + S_1(x) + \delta S_2(x) + \delta^2 S_3(x) + \cdots \quad \delta \to 0$

Then $y'(x) = S'(x) e^{S(x)}$

$y''(x) = S''(x) e^{S(x)} + (S'(x))^2 e^{S(x)}$

Substitute them into the equation

$\cancel{\varepsilon(S''e^S + (S')^2 e^S) = Q(x) e^S}$ $\Rightarrow \varepsilon[(S')^2 e^S + S'' e^S] + x^2 e^S + \varepsilon x e^S = 0$

$\Rightarrow \cancel{\varepsilon(S'' + (S')^2) = Q(x)}$ $\Rightarrow \varepsilon(S')^2 + \varepsilon S'' + x^2 + \varepsilon x = 0$

$\Rightarrow \cancel{\varepsilon\left(\frac{1}{\delta^2}(S_0')^2 + \frac{1}{\delta} S_0'' + \frac{2S_0'S_1'}{\delta} + S_1'' + (S_1')^2 + 2S_0'S_2' + \cdots\right) \sim Q(x)}$

$\varepsilon\left[\frac{1}{\delta^2} S_0'^2 + \frac{1}{\delta} S_0'' + \frac{2}{\delta} S_0' S_1' + S_1'' + (S_1')^2 + 2S_0'S_2' + \cdots\right] + x^2 + \varepsilon x = 0$

Since $O(\frac{\varepsilon}{\delta^2}) \gg O(\frac{\varepsilon}{\delta}) \gg O(\varepsilon)$ & $O(\varepsilon\delta)$

$\therefore$ Dominant balance $\frac{\varepsilon}{\delta^2} = O(1)$

$\Rightarrow \delta^2 = \varepsilon \quad \Rightarrow \delta = \varepsilon^{\frac{1}{2}}$

Thus, $S(x) \sim \varepsilon^{-\frac{1}{2}} S_0(x) + S_1(x) + \varepsilon^{\frac{1}{2}} S_2(x) + \varepsilon S_3(x) + \cdots, \ \varepsilon \to 0$

The equation becomes $x^2 + (S_0')^2 + \varepsilon^{\frac{1}{2}}(S_0'' + 2S_0'S_1') + \varepsilon[S_1'' + (S_1')^2 + 2S_0'S_2' + x] \sim 0$

$\cancel{(S_0')^2 + \varepsilon^{\frac{1}{2}}(S_0'' + 2S_0'S_1') + \varepsilon(S_1'' + (S_1')^2 + 2S_0'S_2') + \cdots \sim Q(x) \quad \varepsilon \to 0}$

- $O(1)$ equation

$(S_0')^2 = Q(x) = -(x^2 \cancel{+\varepsilon x})$ 看订正. -8

$\Rightarrow S_0' = \pm\sqrt{Q(x)} = \pm i\sqrt{x^2+\varepsilon x}$

$\Rightarrow S_0(x) = \pm \int_0^x i\sqrt{t^2+\varepsilon t}\, dt + A_0$

$$= \pm i\left(\left(\frac{x}{2}+\frac{\varepsilon}{4}\right)\sqrt{x^2+\varepsilon x} - \frac{\varepsilon^2}{8}\ln\left|\left(x+\frac{\varepsilon}{2}\right)+\sqrt{x^2+\varepsilon x}\right| + \frac{\varepsilon^2}{8}\ln\frac{\varepsilon^2}{2} + A_0\right.$$

$(*)$

· $O(\varepsilon^{\frac{1}{2}})$ equation

$$S_0' = \pm i\sqrt{x^2+\varepsilon x}$$

$$S_0'' = \pm i\,(x^2+\varepsilon x)^{\cancel{-\frac{1}{2}}}(2x+\varepsilon)$$

$$= \pm i\,\frac{2x+\varepsilon}{2\sqrt{x^2+\varepsilon x}}$$

$$2S_0'S_1' + S_0'' = 0$$

$$\Rightarrow S_1' = \frac{-S_0''}{2S_0'} = \pm\frac{2x+\varepsilon}{4(x^2+\varepsilon x)}$$

$$\Rightarrow S_1(x) = -\frac{1}{2}\,|S_0'(x)| + A_1$$

$$= -\frac{1}{4}\log(-(x^2+\varepsilon x)) + A_1$$

Special solution:

$$S(x) \sim \frac{\overbrace{i\int_0^x \sqrt{t^2+\varepsilon t}\,dt + A_0}^{=(*)}}{\sqrt{\varepsilon}} - \frac{1}{4}\log(-(x^2+\varepsilon x)) + A_1$$

or $$S(x) \sim -\frac{i\int_0^x \sqrt{t^2+\varepsilon t}\,dt + A_0}{\sqrt{\varepsilon}} - \frac{1}{4}\log(-(x^2+\varepsilon x)) + A_1$$

$$y(x) = e^{S(x)}$$

$$\sim C_1(-(x^2+\varepsilon x))^{-\frac{1}{4}} e^{\frac{i}{\sqrt{\varepsilon}}\int_0^x \sqrt{t^2+\varepsilon t}\,dt}$$

or $$y(x) \sim C_1(-(x^2+\varepsilon x))^{-\frac{1}{4}} e^{-\frac{i}{\sqrt{\varepsilon}}\int_0^x \sqrt{t^2+\varepsilon t}\,dt}$$

General solution:

$$y(x) \sim C_1(-(x^2+\varepsilon x))^{-\frac{1}{4}} e^{\frac{i}{\sqrt{\varepsilon}}\int_0^x \sqrt{t^2+\varepsilon t}\,dt}$$

$$+ C_1(-(x^2+\varepsilon x))^{-\frac{1}{4}} e^{-\frac{i}{\sqrt{\varepsilon}}\int_0^x \sqrt{t^2+\varepsilon t}\,dt}$$

订正：

- $O(1)$ equation

$$(S_0')^2 + x^2 \sim 0$$
$$(S_0')^2 = -x^2$$
$$S_0' = \pm i x$$
$$S_0 = \pm \frac{i}{2} x^2 + A_0$$

- $O(\varepsilon^{\frac{1}{2}})$ equation

$$S_0'' + 2S_0'S_1' \sim 0$$
$$S_1' = -\frac{1}{2x}$$
$$S_1 = -\frac{1}{2}\ln x + A_1$$

- $O(\varepsilon)$ equation.

$$S_1'' + (S_1')^2 + 2S_0'S_2' + x \sim 0$$
$$S_2' = \pm\left(-\frac{1}{2i} - \frac{3}{4ix^3}\right)$$
$$S_2 = \pm\left(-\frac{x}{2i} + \frac{3}{16ix^4}\right) = \pm\left(\frac{ix}{2} - \frac{3i}{16x^4}\right) + A_2$$

$$S(x) = \varepsilon^{-\frac{1}{2}} S_0(x) + S_1(x) + \varepsilon^{\frac{1}{2}} S_2(x) + \cdots$$
$$= \varepsilon^{-\frac{1}{2}}\left(\pm\frac{i}{2}x^2 + A_0\right) - \frac{1}{2}\ln x + A_1 \pm \varepsilon^{\frac{1}{2}}\left(\frac{ix}{2} - \frac{3i}{16x^4} + A_2\right)$$

$$y = e^{S(x)} = \cdots$$

Method of strained coordinates.

2. Use the method of strained coordinates to find the leading order approximation to the solution, such that there is no secular terms at $O(\epsilon)$ in the error.

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$$\begin{cases} y'' + 4y = \epsilon y(y')^2, & t \geq 0, \\ y(0) = 1, \ y'(0) = 0. \end{cases}$$

Assume that $y(\tau) \sim y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \cdots \quad \varepsilon \to 0$

To eliminate secular terms up to $O(\varepsilon)$, let

$$\tau = (1 + \omega_1 \varepsilon) t$$

Then $\frac{dy}{dt} = (1+\omega_1 \varepsilon) \frac{dy}{d\tau}$

$$\frac{d^2 y}{dt^2} = (1+\omega_1\varepsilon)^2 \frac{d^2 y}{d\tau^2}$$

Substitute them into the equation

$$(1+\omega_1\varepsilon)^2 y'' + 4y = \varepsilon (1+\omega_1\varepsilon)^2 y (y')^2$$

$O(1)$:

$$\begin{cases} y_0'' + 4y_0 = 0 \\ y_0(0) = 1 . \ y'(0) = 0 \end{cases}$$

$\Rightarrow y_0(\tau) = \cos 2\tau$

$O(\varepsilon)$:

$$\begin{cases} y_1'' + 4y_1 = y_0 (y_0')^2 - 2\omega_1 y_0'' \\ y_0(0) = 0 , \ y(0) = 0 \end{cases}$$

$$\Rightarrow y_1'' + 4y_1 = y_0 (y_0')^2 - 2\omega_1 y_0'' = \cos 2\tau \cdot 4 \sin^2 2\tau + 8\omega_1 \cos 2\tau$$

$$= \cos 2\tau (1 - \tfrac{1}{2} \cos 4\tau) + 8\omega_1 \cos 2\tau$$

$$= (1 + 8\omega_1) \cos 2\tau - \tfrac{1}{2} \cos 2\tau \cos 4\tau$$

If $1 + 8\omega_1 = 0$ i.e. $\omega_1 = -\frac{1}{8}$. there will be no secular terms

The leading order approximation is $y \sim \cos 2\tau$ with $\tau = (1 - \frac{1}{8}\varepsilon)t$ $\quad \varepsilon \to 0$

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