MATH 5411 - Advanced Probability I
Homework 3

(due: December 4, 2022)

Q1: Let $X, X_1, X_2, \ldots$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Further, let $g : \mathbb{R} \to \mathbb{R}$. Let D_g be the set of the discontinuity points of g. Assume that $\mathbb{P}(X \in D_g) = 0$. Prove the following continuous mapping theorem for convergence in distribution: If $X_n \xrightarrow{D} X$, then $g(X_n) \xrightarrow{D} g(X)$.

Q2: Suppose $g, h : \mathbb{R} \to \mathbb{R}$ are continuous with $g(x) > 0$, and $|h(x)|/g(x) \to 0$ as $|x| \to 0$. Let $F, F_1, F_2, \ldots$ be a sequence of distribution functions. Suppose $F_n \to F$ weakly and $\int g(x)\mathrm{d}F_n(x) \le C < \infty$ uniformly in n. Prove

$$\int h(x)\mathrm{d}F_n(x) \to \int h(x)\mathrm{d}F(x).$$

Q3: Let $X_1, X_2, \ldots$ be i.i.d. and have the standard normal distribution. It is known that

$$\mathbb{P}(X_i > x) \sim \frac{1}{\sqrt{2\pi}x}\exp(-\frac{x^2}{2}) \qquad \text{as } x \to \infty.$$

where $a(x) \sim b(x)$ means $a(x)/b(x) \to 1$ if $x \to \infty$.

(i): Prove that for any real number θ,

$$\mathbb{P}(X_i > x + \frac{\theta}{x})/\mathbb{P}(X_i > x) \to \exp(-\theta), \qquad \text{as } x \to \infty$$

(ii) Show that if we define b_n by $\mathbb{P}(X_i > b_n) = 1/n$,

$$\mathbb{P}(b_n(\max_{1\le i\le n} X_i - b_n) \le x) \to \exp(-e^{-x}).$$

(iii) Show that $b_n \sim (2\log n)^{\frac{1}{2}}$ and conclude $\max_{1\le i\le n} X_i/(2\log n)^{\frac{1}{2}} \to 1$ in probability.

Q4: Let $X_1, X_2, \ldots.$ be independent taking values 0 and 1 with probability 1/2 each. Let $X = 2\sum_{j\ge1} X_j/3^j$. Compute the characteristic function of X.

Q5: Let $S_n = X_1 + \cdots X_n$ in the following problems.

(a): Suppose that X_i's are independent and $\mathbb{P}(X_i = i) = \mathbb{P}(X_i = -i) = \frac{i^{-\alpha}}{4}$ and $\mathbb{P}(X_i = 0) = 1 - \frac{i^{-\alpha}}{2}$ for some nonnegative parameter α. Find $a_n(\alpha), b_n(\alpha)$ such that $(S_n - a_n(\alpha))/b_n(\alpha) \Rightarrow N(0,1)$ when $\alpha \in (0,1)$ and prove this CLT.

(b):Suppose that X_i's are independent and $\mathbb{P}(X_i = 1) = \frac{1}{i} = 1 - \mathbb{P}(X_i = 0)$. Find a_n and b_n such that $(S_n - a_n)/b_n \Rightarrow N(0,1)$ and prove this CLT.

Q6: Suppose that X_n and Y_n are independent, and $X_n \to X_\infty$ in distribution and $Y_n \to Y_\infty$ in distribution. Show that $X_n^2 + Y_n^2$ converges in distribution.

Q7: Let $X_1, X_2, \ldots$ be i.i.d. with a density that is symmetric about 0, and continuous and positive 0. Find the limiting distribution of

$$\frac{1}{n}\Big(\frac{1}{X_1} + \ldots + \frac{1}{X_n}\Big).$$

Q8: Do some self-study and explain why the *Stable distributions* and *Infinitely divisible distributions* bear such names.