Q1: Let $X, X_1, X_2, \ldots$ be a sequence of random variables defined on the same probability space. Further, let $g : \mathbb{R} \to \mathbb{R}$ be a coninuous function. Prove the following continuous mapping theorem

(i): If $X_n \xrightarrow{\mathbb{P}} X$, then $g(X_n) \xrightarrow{\mathbb{P}} g(X)$;

(ii): If $X_n \xrightarrow{\text{a.s.}} X$, then $g(X_n) \xrightarrow{\text{a.s.}} g(X)$.

Proof :

(1) $\because$ g is continuous

$\therefore \forall \varepsilon > 0.\ \exists \delta > 0$ s.t. $|g(X_n) - g(X)| < \varepsilon$ when $|X_n - X| < \delta$

$\because X_n \xrightarrow{\mathbb{P}} X$

$\therefore \forall \delta > 0,\ \lim_{n\to\infty} \mathbb{P}(\{w : |X_n(w) - X(w)| < \delta\}) = 1$

$\Rightarrow \forall \varepsilon > 0,\ \lim_{n\to\infty} \mathbb{P}(\{w : |g(X_n(w)) - g(X(w))| < \varepsilon\}) = 1$

$\Rightarrow \forall \varepsilon > 0.\ \lim_{n\to\infty} \mathbb{P}(\{w : |g(X_n(w)) - g(X(w))| > \varepsilon\}) = 0$

$\Rightarrow g(X_n) \xrightarrow{\mathbb{P}} g(X)$

(2) $\because$ g is continuous

$\therefore \forall \varepsilon > 0.\ \exists \delta > 0$ s.t. $|g(X_n) - g(X)| < \varepsilon$ when $|X_n - X| < \delta$

$\because X_n \xrightarrow{\text{a.s.}} X$

$\therefore \forall \delta > 0,\ \exists n \in \mathbb{N}$ s.t. $\mathbb{P}(\{w : |X_n(w) - X(w)| < \delta\}) = 1$

$\Rightarrow \forall \varepsilon > 0.\ \exists n \in \mathbb{N}$ s.t. $\mathbb{P}(\{w : |g(X_n(w)) - g(X(w))| < \varepsilon\}) = 1$

$\therefore g(X_n) \xrightarrow{\text{a.s.}} g(X)$

?

Q2: Prove the following statements:

(a) If $X_n \xrightarrow{a.s.} X$ and $Y_n \xrightarrow{a.s.} Y$ then $X_n + Y_n \xrightarrow{a.s.} X + Y$.

(b) If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then $X_n + Y_n \xrightarrow{P} X + Y$.

(c) It is not in general true that $X_n + Y_n \xrightarrow{D} X + Y$ if $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{D} Y$.

Proof:

(1) $\because X_n \xrightarrow{a.s.} X,\ Y_n \xrightarrow{a.s.} Y$

$\therefore \forall \varepsilon > 0.\ \exists n \in \mathbb{N}$ s.t. $\mathbb{P}(\{w: |X_n(w) - X(w)| < \frac{\varepsilon}{2}\}) = 1$

$\mathbb{P}(\{w: |Y_n(w) - Y(w)| < \frac{\varepsilon}{2}\}) = 1$

$|X_n(w) + Y_n(w) - X(w) - Y(w)|$

$\leq |X_n(w) - X(w)| + |Y_n(w) - Y(w)|$

$\therefore \forall \varepsilon > 0.\ \exists n \in \mathbb{N}$ s.t. $\mathbb{P}(\{w: |(X_n(w) + Y_n(w)) - (X(w) + Y(w))| < \varepsilon\}) = 1$

$\therefore X_n + Y_n \xrightarrow{a.s.} X + Y$

(2) $\because X_n \xrightarrow{\mathbb{P}} X,\ Y_n \xrightarrow{\mathbb{P}} Y$

$\therefore \forall \varepsilon > 0.$ st. $\lim_{n \to \infty} \mathbb{P}(\{w: |X_n(w) - X(w)| < \frac{\varepsilon}{2}\}) = 1$

$\lim_{n \to \infty} \mathbb{P}(\{w: |Y_n(w) - Y(w)| < \frac{\varepsilon}{2}\}) = 1$

$|X_n(w) + Y_n(w) - X(w) - Y(w)|$

$\leq |X_n(w) - X(w)| + |Y_n(w) - Y(w)|$

$\therefore \forall \varepsilon > 0.$ s.t. $\lim_{n \to \infty} \mathbb{P}(\{w: |(X_n(w) + Y_n(w)) - (X(w) + Y(w))| < \varepsilon\}) = 1$

$\therefore X_n + Y_n \xrightarrow{\mathbb{P}} X + Y$

(3) Assume $X_n \sim U[-1,1]$. $X \sim U[-1,1]$

$Y_n = -X_n \sim U[-1,1]$. $Y \sim U[-1,1]$

Then $X_n \xrightarrow{d} X$ $Y_n \xrightarrow{d} Y$

However, $X_n + Y_n = 0 \not\xrightarrow{d} X + Y \sim U[-1,1]$

Q3: Let $X_1, X_2, \ldots$ be uncorrelated with $\mathbb{E}X_i = \mu_i$ and $\mathrm{Var}(X_i)/i \to 0$ as $i \to \infty$. Let $S_n = \sum_{i=1}^n X_i$, and $\mu_n = \mathbb{E}S_n/n$. Show that $S_n/n - \mu_n \to 0$ in mean square and thus in probability.

(1)
$$E\left|\frac{S_n - n\mu_n}{n}\right|^2 = \frac{E|S_n - ES_n|^2}{n^2}$$
$$= \frac{1}{n^2}Var(S_n)$$
$$= \frac{1}{n^2}Var(\Sigma X_i)$$
$$= \frac{1}{n^2}\Sigma Var(X_i)$$
$$\to 0 \quad (n \to \infty)$$

$\Rightarrow \frac{S_n}{n} - \mu_n \to 0$ in square mean.

$$\mathbb{P}\left(\left|\frac{S_n}{n} - \mu_n\right| > \varepsilon\right) \le \frac{E\left|\frac{S_n}{n} - \mu_n\right|^2}{\varepsilon^2} \to 0 \quad (n \to \infty)$$

$\Rightarrow \frac{S_n}{n} - \mu_n \xrightarrow{\mathbb{P}} 0$

Q4: Let $\xi_1, \xi_2, \ldots.$ be i.i.d Cauchy r.v.s. with common density $1/[\pi(1+x^2)]$. Let $X_n = |\xi_n|$ and $S_n = \sum_{i=1}^{n} X_i$. Find b_n such that $S_n/b_n \to 1$ in probability.

The expectation of Cauchy distribution does not exist.

$\because \xi_i \sim C(0,1)$, $X_i = |\xi_i|$, X_i i.i.d.

$\therefore s\,\mathbb{P}(|X_i| > s) \to 0$ as $s \to \infty$

and $a_n = EX_i \mathbb{1}(|X_i| \le n) \to 1$ as $n \to \infty$

From the Corollary of Weak Law of Large number

$$\frac{S_n}{n} - a_n \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty$$

Let $b_n = n$, then

$$\frac{S_n}{b_n} \xrightarrow{\mathbb{P}} 1 \quad \text{as } n \to \infty$$

Q5: Let $p_k = 1/(2^k k(k+1))$, $k = 1, 2, \cdots$, and $p_0 = 1 - \sum_{k\ge1} p_k$. Notice that

$$\sum_{k=1}^{\infty} 2^k p_k = 1.$$

So, if we let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_n = -1) = p_0$ and

$$\mathbb{P}(X_n = 2^k - 1) = p_k, \qquad \forall k \ge 1,$$

then $\mathbb{E}X_n = 0$. Let $S_n = X_1 + \ldots + X_n$. Show that

$$S_n/(n/\log_2 n) \to -1, \qquad \text{in probability.}$$

Let $X_{nk} = X_k$, $b_n = 2^{m_n}$, $m_n = \min\{m : 2^{-m} m^{-\frac{3}{2}} \ge \frac{1}{n}\}$

X_{nk} satisfies

(i) $$\sum_{k=1}^{n} \mathbb{P}(|X_{nk}| > b_n) = n\,\mathbb{P}(|X_1| > 2^{m_n})$$
$$= n \sum_{k=m_n+1}^{\infty} p_k$$
$$= \sum_{k=m_n+1}^{\infty} \frac{n}{2^k \cdot k(k+1)}$$
$$\le \sum_{k=m_n+1}^{\infty} \frac{n}{2^k m_n^2}$$
$$= \frac{n}{2^{m_n} m_n^2}$$
$$\le m_n^{-\frac{1}{2}} \longrightarrow 0$$

(ii) $$b_n^{-2} \sum_{k=1}^{n} \mathbb{P}(|X_{nk}| > b_n) = 2^{-2m_n} \cdot n\, E X_1^2 \mathbb{1}(|X_1| \le 2^{m_n})$$
$$\le 2^{-2m_n} \cdot m_n^{\frac{3}{2}} E X_1^2 \mathbb{1}(|X_1| \le 2^{m_n})$$
$$(*) \quad = 2^{-2m_n} m_n^{\frac{3}{2}} \sum_{k=1}^{m_n} \left((2^k - 1)^2 p_k + p_0 \right)$$

$$\sum_{k=1}^{m_n} (2^k - 1)^2 p_k + p_0 \le 1 + \sum_{k=1}^{m_n} 2^{2k} p_k$$
$$= 1 + \sum_{k=1}^{m_n} 2^k \frac{1}{k(k+1)}$$
$$= 1 + \sum_{k < \frac{m_n}{2}} 2^k \cdot \frac{1}{k(k+1)} + \sum_{\frac{m_n}{2} \le k \le m_n} 2^k \cdot \frac{1}{k(k+1)}$$
$$\le 1 + \sum_{k < \frac{m_n}{2}} 2^k + \frac{4}{m_n(m_n+2)} \sum_{\frac{m_n}{2} \le k \le m_n} 2^k \cdot \frac{1}{k(k+1)}$$
$$= 1 + 2^{\frac{m_n}{2}+1} - 2 + \frac{4}{m_n(m_n+2)} \left(2^{m_n+1} - 2^{\frac{m_n}{2}+1} \right)$$

$$\leq \frac{2^{m_n}}{m_n^2}\cdot C \quad \text{for large } n$$

$$\therefore (*) \leq \frac{m_n^{\frac{3}{2}}}{2^{m_n}} \cdot \frac{2^{m_n}}{m_n^2} \cdot C = C\cdot m_n^{-\frac{1}{2}} \to 0$$

From the Weak Law for Triangular arrays,

$$\frac{S_n - a_n}{b_n} \xrightarrow{\mathbb{P}} 0$$

$$a_n = \sum_{k=1}^{n} E\bar{X}_{nk} = n \sum_{k=1}^{m_n} (2^k-1)p_k - p_0$$

$$= n\left(-\frac{1}{m_n+1} + \sum_{k=m_n+1}^{\infty} \frac{1}{2^k \cdot k(k+1)}\right)$$

$$\leq n\left(-\frac{1}{m_n+1} + \frac{1}{m_n^2}\sum_{k=m_n+1}^{\infty}\frac{1}{2^k}\right)$$

$$\sim -\frac{n}{m_n}$$

$\because n \leq 2^{m_n} m_n^{\frac{3}{2}} \leq C_n$, $\log_2 n \sim m_n$

$\therefore a_n \sim \frac{n}{\log_2 n}$

$$b_n = 2^{m_n} \sim n m_n^{-\frac{3}{2}} \sim n(\log_2 n)^{-\frac{3}{2}}$$

$$\therefore \frac{S_n - a_n}{b_n} \sim \frac{S_n + \frac{n}{\log_2 n}}{(n\log_2 n)^{-\frac{3}{2}}} \xrightarrow{\mathbb{P}} 0$$

$\because (\log_2 n)^{-\frac{1}{2}} \to 0$

$$\therefore \frac{S_n + \frac{n}{\log_2 n}}{\frac{n}{\log_2 n}} \xrightarrow{\mathbb{P}} 0$$

$$\therefore \frac{S_n}{\frac{n}{\log_2 n}} \xrightarrow{\mathbb{P}} 0$$

Q6: Suppose X_n are independent Poisson r.v.s with rate λ_n, i.e., $\mathbb{P}(X_n = k) = \lambda_n^k e^{-\lambda_n}/k!$ for $k = 0, 1, 2, \ldots$. Show that $S_n/\mathbb{E}S_n \to 1$ a.s. if $\sum_n \lambda_n = \infty$.

When $\lambda_n \le 1$

- Prove $\frac{S_n}{ES_n} \xrightarrow{\mathbb{P}} 1$

$\because \sum \lambda_n = \infty$

$$\therefore \mathbb{P}\left(\left|\frac{S_n - ES_n}{ES_n}\right| > \varepsilon\right) \le \frac{Var(S_n)}{\varepsilon^2 (ES_n)^2} = \frac{\sum_n \lambda_n}{\varepsilon^2 (\sum_n \lambda_n)^2} = \frac{1}{\varepsilon^2 \sum_n \lambda_n} \to 0$$

$$\therefore \frac{S_n - ES_n}{ES_n} \xrightarrow{\mathbb{P}} 0 \Rightarrow \frac{S_n}{ES_n} \xrightarrow{\mathbb{P}} 1$$

- $\because \sum \lambda_n = \infty \quad \therefore \exists n_k$ s.t. $n_k = \inf\{n : \sum \lambda_i \ge k^2\}$

Let $T_k = S_{n_k}$, then

$$\mathbb{P}\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \varepsilon\right) \le \frac{Var(T_k)}{\varepsilon^2 (ET_k)^2} = \frac{1}{\varepsilon^2 \sum_{n=1}^{n_k} \lambda_i} \le \frac{1}{\varepsilon^2 k^2}$$

$$\because \sum_{k=1}^{\infty} \mathbb{P}\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \varepsilon\right) < \infty$$

From Borel-Cantelli Lemma,

$$\mathbb{P}\left(\left|\frac{T_k - ET_k}{ET_k}\right| > \varepsilon \text{ i.o.}\right) = 0$$

$$\Rightarrow \frac{T_k}{ET_k} \xrightarrow{a.s.} 1$$

- Prove $\frac{S_n}{ES_n} \xrightarrow{a.s.} 1$

For $n \in \mathbb{N}$ s.t. $n_k < n < n_{k+1}$, $\because \lambda_n \le 1$

$$\therefore k^2 \le \sum_{i=1}^{n_k} \lambda_i = ET_k \le k^2 + 1 \le (k+1)^2 \le ET_{k+1} \le (k+1)^2 + 1$$

For $\omega \in \{\omega : \frac{T_k(\omega)}{ET_k} \to 1\}$.

$$\frac{ET_k}{ET_{k+1}} \cdot \frac{T_k(\omega)}{ET_k} = \frac{T_k(\omega)}{ET_{k+1}} \le \frac{S_n(\omega)}{ES_n} \le \frac{T_{k+1}(\omega)}{ET_k} = \frac{T_{k+1}(\omega)}{ET_{k+1}} \cdot \frac{ET_{k+1}}{ET_k}$$

Note that $\frac{E_{k+1}}{E_k} \le \frac{(k+1)^2+1}{k^2} = 1 + \frac{2(k+1)}{k^2} \to 1$

$$\therefore \frac{S_n(\omega)}{ES_n} \to 1 .$$

From Step 2, $\mathbb{P}(\Omega_0) = 1$. $\therefore \frac{S_n}{ES_n} \xrightarrow{a.s.} 1$

When $\lambda_n > 1$

$\lambda_n = \sum_{k=1}^{m} \lambda_{n_k}$ where $\lambda_{n_k} \leq 1$

We can prove in the situation that $\lambda_{n_k} \leq 1$

Q7: Let $Y_1, Y_2, \ldots$ be i.i.d. Find necessary and sufficient conditions for

(i) $Y_n/n \to 0$ almost surely;
(ii) $(\max_{m \le n} Y_m)/n \to 0$ almost surely;
(iii) $(\max_{m \le n} Y_m)/n \to 0$ in probability;
(iv) $Y_n/n \to 0$ in probability.

(i) $E|Y_1| < \infty$.

- If $E|Y_1| < \infty$, then $E|Y_1| < \infty$

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\left|\frac{Y_1}{n}\right| > \varepsilon\right) = \sum_{n=1}^{\infty} \mathbb{P}(|Y_1| > \varepsilon n) \le \int_0^\infty \mathbb{P}(|Y_1| > t)\,dt = E|Y_1| < \infty$$

From Borel-Cantelli Lemma, $\mathbb{P}(|\frac{Y_1}{n}| > \varepsilon) \to 0$ $\quad \therefore \frac{Y_1}{n} \xrightarrow{a.s.} 0$

- If $E|Y_1| = \infty$

Then $\sum_{n=1}^{\infty} \mathbb{P}(|Y_1| > \varepsilon n) \ge \int_0^\infty \mathbb{P}(|Y_1| > t)\,dt = E|Y_1| = \infty$

From Borel-Cantelli Lemma, $\mathbb{P}(|\frac{Y_1}{n}| > \varepsilon) \not\to 0$

(ii) $EY_1^+ < \infty$

- If $EY_1^+ < \infty$, then $\frac{Y_n^+}{n} \xrightarrow{a.s.} 0$

$\therefore \forall \varepsilon > 0 \ \exists N \in \mathbb{N}^+$ s.t. $\forall n > N$, $\frac{Y_n^+}{n} < \varepsilon$

$$\Rightarrow \frac{\max_{m \le n} Y_m^+}{n} \le \max\left\{\frac{Y_1^+}{n}, \ldots, \frac{Y_N^+}{n}, \varepsilon\right\} \xrightarrow{a.s.} 0$$

i.e. $\lim_{n \to \infty} \frac{\max_{m \le n} Y_m^+}{n} = 0$

$$\Rightarrow \limsup_{n \to \infty} \frac{\max_{m \le n} Y_m}{n} \le \limsup_{n \to \infty} \frac{\max_{m \le n} Y_m^+}{n} = 0$$

Below we prove $\limsup_{n \to \infty} \frac{\max_{m \le n} Y_m}{n} \ge 0$

$\mathbb{P}\left(\frac{\max_{m \le n} Y_m}{n} < -\varepsilon\right) = F(-n\varepsilon)^n$, where F is the distribution function

Let m be large enough s.t. $F(-m\varepsilon) < \delta < 1$

Then

$$\sum_{n=1}^{\infty} \mathbb{P}\left(\frac{\max_{m\le n} Y_m}{n} < -\varepsilon\right) \le \sum_{n=1}^{M} \mathbb{P}\left(\frac{\max_{m\le n} Y_m}{n} < -\varepsilon\right) + \sum_{n=M+1}^{\infty} \delta^n$$

$$\le M + \frac{\delta^{M+1}}{1-\delta}$$

$$< \infty$$

From Borel - Cantelli Lemma.

$$\liminf_{n\to\infty} \frac{\max_{m\le n} Y_m}{n} \ge -\varepsilon \quad \text{a.s.}$$

$$\therefore \limsup_{n\to\infty} \frac{\max_{m\le n} Y_m}{n} = 0 \quad \text{a.s.}$$

- Conversely, if $EY^+ = \infty$, then

$$\sum_{n=0}^{\infty} \mathbb{P}(Y_1 > n) \ge \int_0^{\infty} \mathbb{P}(Y_1 > t)\,dt = EY_1^+ = \infty$$

$$\Rightarrow \frac{Y_n}{n} > 1 \quad \text{i.o.}$$

(iii) $n\mathbb{P}(|Y_1| > n) \to 0$

- $\mathbb{P}(\max_{m\le n} Y_m \ge \delta n) \le n\mathbb{P}(Y_i \ge n\delta) \to 0$

$$\therefore \frac{\max_{m\le n} Y_m}{n} \xrightarrow{\mathbb{P}} 0$$

- if $n\mathbb{P}(|Y_1| > n) \not\to 0$

$\exists\, \delta,\ 0<\delta<1,\ n_k \to \infty,\ m_k \le n_k$ s.t. $m_k \mathbb{P}(Y_1 > n_k) \to \delta$

From second Bonferroni inq, we have

$$\mathbb{P}(\max_{m\le m_k} Y_m > n_k) \ge m_k \mathbb{P}(Y_1 > n_k) - \binom{m_k}{2} \mathbb{P}(Y_1 > n_k)^2 \to \delta - \frac{\delta^2}{2} > 0$$

(iv) $\mathbb{P}(|Y_1| < \infty) = 1$

$$\frac{Y_n}{n} \xrightarrow{\mathbb{P}} 0 \Leftrightarrow \mathbb{P}(|Y_n| > n\varepsilon) \to 0 \Leftrightarrow \mathbb{P}(|Y_1| = \infty) = 0$$

Q8: Let X_i's be i.i.d. random variables. Consider the random power series

$$\sum_{n=0}^{\infty} X_n z^n$$

Is there any deterministic (almost surely) radius of convergence of the above series in the following two cases (a): $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$, (b): $X_i \sim N(0,1)$? If so, find the radius.

From Kolmogorov 0-1 Law, the radius of convergence is constant.

$$R = \frac{1}{\limsup_{n\to\infty} |X_n(w)|^{\frac{1}{n}}} = 1$$

$\therefore \forall \varepsilon > 0$, $\mathbb{P}(\limsup_{n\to\infty} |X_n|^{\frac{1}{n}} \geq 1+\varepsilon) = 0$

$\mathbb{P}(\limsup_{n\to\infty} |X_n|^{\frac{1}{n}} \geq 1-\varepsilon) = 1$

If $\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1+\varepsilon) < \infty \overset{B\text{-}C}{\Rightarrow} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1+\varepsilon \text{ i.o.}) = 0$

$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1-\varepsilon) = \infty \overset{B\text{-}C}{\Rightarrow} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1-\varepsilon \text{ i.o.}) = 1$

$\therefore R = 1$

(a) When $\mathbb{P}(X_i=1) = \mathbb{P}(X_i=-1) = \frac{1}{2}$

$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1+\varepsilon) < \infty \overset{B\text{-}C}{\Rightarrow} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1+\varepsilon \text{ i.o.}) = 0$

$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1-\varepsilon) = \infty \overset{B\text{-}C}{\Rightarrow} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1-\varepsilon \text{ i.o.}) = 1$

$\therefore R = 1$

(b) When $X_i \sim N(0,1)$

$$\mathbb{P}(|X_n| > (1+\varepsilon)^n) = \frac{2}{\sqrt{2\pi}} \int_{(1+\varepsilon)^n}^{+\infty} e^{-\frac{x^2}{2}} dx$$

$\therefore \sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1+\varepsilon) < \infty$

$$\mathbb{P}(|X_n| > (1-\varepsilon)^n) = \frac{2}{\sqrt{2\pi}} \int_{(1-\varepsilon)^n}^{+\infty} e^{-\frac{x^2}{2}} dx$$

$\therefore \sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1-\varepsilon) = \infty$

$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1+\varepsilon) < \infty \overset{B\text{-}C}{\Rightarrow} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1+\varepsilon \text{ i.o.}) = 0$

$\sum_{n=1}^{\infty} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1-\varepsilon) = \infty \overset{B\text{-}C}{\Rightarrow} \mathbb{P}(|X_n|^{\frac{1}{n}} \geq 1-\varepsilon \text{ i.o.}) = 1$

$\therefore R = 1$