Q1: Let $A_1, A_2, \cdots$ be a sequence of events. Define

$$B_n = \cup_{m=n}^{\infty} A_m, \qquad C_n = \cap_{m=n}^{\infty} A_m.$$

Clearly $C_n \subset A_n \subset B_n$. The sequences $\{B_n\}$ and $\{C_n\}$ are decreasing and increasing respectively with limits

$$\lim B_n = B = \cap_n B_n = \cap_n \cup_{m \geq n} A_m, \qquad \lim C_n = C = \cup_n C_n = \cup_n \cap_{m \geq n} A_m.$$

The events B and C are denoted $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$, respectively. Show that
(a) $B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$,
(b) $C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$,
We say that the sequence $\{A_n\}$ converges to a limit $A = \lim A_n$ if B and C are the same set A. Suppose that $A_n \to A$ and show that
(c) $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Proof:

(a) $\omega \in B \Leftrightarrow \omega \in \limsup\limits_{n\to\infty} A_n$

$\Leftrightarrow \omega \in \bigcap\limits_{n=1}^{\infty} \bigcup\limits_{m=n}^{\infty} A_m$

$\Leftrightarrow \forall n \in \mathbb{N}, \omega \in \bigcup\limits_{m=n}^{\infty} A_m$

$\Leftrightarrow \forall n \in \mathbb{N}, \exists m \geq n$ s.t. $\omega \in A_m$

$\Leftrightarrow \omega \in A_n$ for infinitely many values of n

$\therefore B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$

(b) $\omega \in C \Leftrightarrow \omega \in \liminf\limits_{n\to\infty} A_n$

$\Leftrightarrow \omega \in \bigcup\limits_{n=1}^{\infty} \bigcap\limits_{m=n}^{\infty} A_m$

$\Leftrightarrow \exists n \in \mathbb{N}, x \in \bigcap\limits_{m=n}^{\infty} A_m$

$\Leftrightarrow \exists n \in \mathbb{N}, \forall m \geq n.\ x \in A_m$

$\Leftrightarrow \omega \in A_n$ for all but finitely many values of n

$\therefore C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$

(c) First, we need to prove $\limsup\limits_{n\to\infty} \mathbb{P}(A_n) \leq \mathbb{P}(\limsup\limits_{n\to\infty} A_n)$

$\because B_n = \bigcup\limits_{m \geq n} A_m$

$\therefore B_{n+1} \subseteq B_n$. $A_n \subseteq B_n$ for all $n \geq 1$

$\therefore \mathbb{P}(A_n) \leq \mathbb{P}(B_n)$

From

$\lim_{n\to\infty} \mathbb{P}(B_n) = \lim_{n\to\infty} \mathbb{P}(\sup_n A_n) = \mathbb{P}(\limsup_{n\to\infty} A_n)$

$\therefore \limsup_{n\to\infty} \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_{n\to\infty} A_n)$

Similarly, we have $\liminf_{n\to\infty} \mathbb{P}(A_n) \geq \mathbb{P}(\liminf_{n\to\infty} A_n)$

$\because \liminf_{n\to\infty} \mathbb{P}(A_n) \leq \limsup_{n\to\infty} \mathbb{P}(A_n)$

$\therefore \mathbb{P}(\liminf_{n\to\infty} A_n) \leq \liminf_{n\to\infty} \mathbb{P}(A_n) \leq \limsup_{n\to\infty} \mathbb{P}(A_n) \leq \mathbb{P}(\limsup_{n\to\infty} A_n)$

$\because A_n \to A$

$\therefore A = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$

$\therefore \mathbb{P}(A) \leq \liminf_{n\to\infty} \mathbb{P}(A_n) = \limsup_{n\to\infty} \mathbb{P}(A_n) \leq \mathbb{P}(A)$

$\therefore \lim_{n\to\infty} \mathbb{P}(A_n) = \liminf_{n\to\infty} \mathbb{P}(A_n) = \limsup_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(A)$

Q2: Let $\mathcal{F}$ be a σ-field, and let $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ be two sub σ-fields.

(i) Give one example which shows that $\mathcal{G} \cup \mathcal{H}$ is not a σ-field.

(ii) Prove that $\mathcal{G} \cap \mathcal{H}$ is a σ-field.

(iii) $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ is a sequence of sub σ-fields, prove that $\cup_{i=1}^{\infty} \mathcal{F}_i$ is a field. Give an example to show that $\cup_{i=1}^{\infty} \mathcal{F}_i$ is not necessarily a σ-field.

Solution:

(i) $\Omega = \{a, b, c\}$

Let $\mathcal{G} = \{\{a\}, \{b, c\}, \phi, \Omega\}$

$\mathcal{H} = \{\{b\}, \{a, c\}, \phi, \Omega\}$

Obviously, $\mathcal{G}$ and $\mathcal{H}$ are σ-fields since they satisfy the definition.

$\mathcal{G} \cup \mathcal{H} = \{\{a\}, \{b\}, \{a, c\}, \{b, c\}, \phi, \Omega\}$

$\{a\} \in \mathcal{G} \cup \mathcal{H}$. $\{b\} \in \mathcal{G} \cup \mathcal{H}$

However, $\{a\} \cup \{b\} = \{a, b\} \notin \mathcal{G} \cup \mathcal{H}$

$\therefore \mathcal{G} \cup \mathcal{H}$ is not a σ-field

(ii) Since $\mathcal{G}$ and $\mathcal{H}$ are both σ-fields.

From the definition, we have

$$\begin{cases} \Omega \in \mathcal{G} \\ A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G} \\ A_1, A_2, \cdots \in \mathcal{G} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{G} \end{cases}$$

$$\begin{cases} \Omega \in \mathcal{H} \\ A \in \mathcal{H} \Rightarrow A^c \in \mathcal{H} \\ A_1, A_2, \cdots \in \mathcal{H} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{H} \end{cases}$$

Then

- $\Omega \in \mathcal{G}.\ \Omega \in \mathcal{H} \Rightarrow \Omega \in \mathcal{G} \cap \mathcal{H}$
- $A \in \mathcal{G} \cap \mathcal{H} \Rightarrow A \in \mathcal{G}.\ A \in \mathcal{H}$
  $\Rightarrow A^c \in \mathcal{G},\ A^c \in \mathcal{H}$
  $\Rightarrow A^c \in \mathcal{G} \cap \mathcal{H}$
- $A_1, A_2, \cdots \in \mathcal{G} \cap \mathcal{H} \Rightarrow A_1, A_2 \cdots \in \mathcal{G},\quad A_1, A_2 \cdots \in \mathcal{H}$
  $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{G},\ \bigcup_{n=1}^{\infty} A_n \in \mathcal{H}$
  $\Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{G} \cap \mathcal{H}$

Therefore, $\mathcal{G} \cap \mathcal{H}$ is a σ-field.

(iii) $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ are sub σ-fields.

- $\Omega \in \mathcal{F}_i$ for any $i \Rightarrow \Omega \in \bigcup_{i=1}^{\infty} \mathcal{F}_i$
- $A \in \bigcup_{i=1}^{\infty} \mathcal{F}_i \Rightarrow \exists j\quad A \in \mathcal{F}_j$
  $\Rightarrow A^c \in \mathcal{F}_j^c$
  $\Rightarrow A^c \in \bigcup_{i=1}^{\infty} \mathcal{F}_i^c$
- $A_1, \cdots, A_n \subseteq \bigcup_{i=1}^{\infty} \mathcal{F}_i \Rightarrow \exists j,\ A_1, \cdots, A_n \subseteq \mathcal{F}_j$
  $\Rightarrow \bigcup_{i=1}^{n} A_i \in \mathcal{F}_j$
  $\Rightarrow \bigcup_{i=1}^{n} A_i \in \bigcup_{i=1}^{\infty} \mathcal{F}_i$

$\therefore \bigcup_{i=1}^{\infty} \mathcal{F}_i$ is a field

Define $\mathcal{F}_1 = \{1\},\ \mathcal{F}_2 = \{1, 2\},\ \cdots,\ \mathcal{F}_i = \{1, 2, \cdots, i\}$.

Let $A_i = \{2i\}$.

Each A_i is in $\bigcup_{i=1}^{\infty} \mathcal{F}_i$, however, $\bigcup_{i=1}^{\infty} A_i$ is not in $\bigcup_{i=1}^{\infty} \mathcal{F}_i$

Thus, $\bigcup_{i=1}^{\infty} \mathcal{F}_i$ is not a σ-field.

Q3: Suppose that X and Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $A \in \mathcal{F}$. We set $Z(\omega) = X(\omega)$ for all $\omega \in A$ and $Z(\omega) = Y(\omega)$ for all $\omega \in A^c$. Prove that Z is a random variable.

Proof:

$\because X, Y$ are random variables,

$\therefore \forall B \in \mathcal{B}(\mathbb{R})$, $X^{-1}(B) \in \mathcal{F}$

$Y^{-1}(B) \in \mathcal{F}$

$\because Z(\omega) = \begin{cases} X(\omega) & \omega \in A \\ Y(\omega) & \omega \in A^c \end{cases}$

$Z^{-1}(B) = Z^{-1}((A \cap B) \cup (A^c \cap B))$

$= Z^{-1}(A \cap B) \cup Z^{-1}(A^c \cap B)$

$= X^{-1}(A \cap B) \cup Y^{-1}(A^c \cap B)$

$\in \mathcal{F}$

$\therefore Z$ is a random variable

Q4: Prove the following two definitions of random vector are equivalent.

Def.1: $X = (X_1, \ldots, X_d) : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a random vector if it is $\mathcal{F}$-measurable.

Def.2: $X = (X_1, \ldots, X_d)$ is a random vector if $X_i : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\mathcal{F}$-measurable for all $i = 1, \ldots, d$.

Proof:

$X : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$

$X_i : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$\because \mathbb{R}^d = \overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{d}$

Given $B \in \mathcal{B}(\mathbb{R}^d)$. B can be decomposed as $B_1 \times \cdots \times B_d$

$\because X^{-1}(B) = X^{-1}(B_1 \times \cdots \times B_d)$

$= X_1^{-1}(B_1) \cap X_2^{-1}(B_2) \cap \cdots \cap X_d^{-1}(B_d)$

$= \bigcap_{i=1}^{d} X_i^{-1}(B_i)$

$\therefore X$ is $\mathcal{F}$-measurable $\Rightarrow X^{-1}(B) \in \mathcal{F} \Rightarrow X_i^{-1}(B_i) \in \mathcal{F} \Rightarrow X_i$ is measurable

X_i is measurable $\Rightarrow X_i^{-1}(B_i) \in \mathcal{F} \Rightarrow X^{-1}(B) \in \mathcal{F} \Rightarrow X$ is $\mathcal{F}$-measurable

Q5: Prove the following reverse Fatou's lemma: Let $f_1, f_2, \ldots$ be a sequence of Lebesgue integrable functions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that there exists a non-negative integrable function g on Ω such that $f_n \leq g$ for all n. Prove

$$\limsup_{n\to\infty} \int f_n \mathrm{d}\mu \leq \int \limsup_{n\to\infty} f_n \mathrm{d}\mu.$$

Proof:

$\because g - f_n \geq 0$

$\therefore$ We can apply Fatou's lemma on the sequence $\{g - f_n\}$

From Fatou's lemma,

$$\int \liminf_{n\to\infty} (g - f_n)\, d\mu \leq \liminf_{n\to\infty} \int (g - f_n)\, d\mu$$

$$\int g\, d\mu + \int \liminf_{n\to\infty} (-f_n)\, d\mu \leq \int g\, d\mu + \liminf_{n\to\infty} \int (-f_n)\, d\mu$$

$\because \int g\, d\mu \neq \pm\infty$

$\therefore \limsup_{n\to\infty} \int f_n\, d\mu \leq \int \limsup_{n\to\infty} f_n\, d\mu.$