MATH 5411 - Advanced Probability I
Homework 2

(due: November 10, 2022)

Q1: Let $X, X_1, X_2, \ldots$ be a sequence of random variables defined on the same probability space. Further, let $g : \mathbb{R} \to \mathbb{R}$ be a coninuous function. Prove the following continuous mapping theorem

(i): If $X_n \xrightarrow{\mathbb{P}} X$, then $g(X_n) \xrightarrow{\mathbb{P}} g(X)$;
(ii): If $X_n \xrightarrow{\text{a.s.}} X$, then $g(X_n) \xrightarrow{\text{a.s.}} g(X)$.

Q2: Prove the following statements:

(a) If $X_n \stackrel{\text{a.s.}}{\to} X$ and $Y_n \stackrel{\text{a.s.}}{\to} Y$ then $X_n + Y_n \stackrel{\text{a.s.}}{\to} X + Y$.
(b) If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$ then $X_n + Y_n \stackrel{P}{\to} X + Y$.
(c) It is not in general true that $X_n + Y_n \stackrel{D}{\to} X + Y$ if $X_n \stackrel{D}{\to} X$ and $Y_n \stackrel{D}{\to} Y$.

Q3: Let $X_1, X_2, ...$ be uncorrelated with $\mathbb{E}X_i = \mu_i$ and $\mathrm{Var}(X_i)/i \to 0$ as $i \to \infty$. Let $S_n = \sum_{i=1}^n X_i$, and $\mu_n = \mathbb{E}S_n/n$. Show that $S_n/n - \mu_n \to 0$ in mean square and thus in probability.

Q4: Let $\xi_1, \xi_2, ...$ be i.i.d Cauchy r.v.s. with common density $1/[\pi(1+x^2)]$. Let $X_n = |\xi_n|$ and $S_n = \sum_{i=1}^n X_i$. Find b_n such that $S_n/b_n \to 1$ in probability.

Q5: Let $p_k = 1/(2^k k(k+1))$, $k = 1, 2, \cdots$, and $p_0 = 1 - \sum_{k\geq 1} p_k$. Notice that

$$\sum_{k=1}^{\infty} 2^k p_k = 1.$$

So, if we let $X_1, X_2, \ldots$ be i.i.d. with $\mathbb{P}(X_n = -1) = p_0$ and

$$\mathbb{P}(X_n = 2^k - 1) = p_k, \qquad \forall k \geq 1,$$

then $\mathbb{E}X_n = 0$. Let $S_n = X_1 + \ldots + X_n$. Show that

$$S_n/(n/\log_2 n) \to -1, \qquad \text{in probability.}$$

Q6: Suppose X_n are independent Poisson r.v.s with rate λ_n, i.e., $\mathbb{P}(X_n = k) = \lambda_n^k \mathrm{e}^{-\lambda_n}/k!$ for $k = 0, 1, 2, \ldots$. Show that $S_n/\mathbb{E}S_n \to 1$ a.s. if $\sum_n \lambda_n = \infty$.

Q7: Let $Y_1, Y_2, \ldots$ be i.i.d. Find necessary and sufficient conditions for

(i) $Y_n/n \to 0$ almost surely;
(ii) $(\max_{m\leq n} Y_m)/n \to 0$ almost surely;
(iii) $(\max_{m\leq n} Y_m)/n \to 0$ in probability;
(iv) $Y_n/n \to 0$ in probability.

Q8: Let X_i's be i.i.d. random variables. Consider the random power series

$$\sum_{n=0}^{\infty} X_n z^n$$

Is there any deterministic (almost surely) radius of convergence of the above series in the following two cases (a): $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$, (b): $X_i \sim N(0, 1)$? If so, find the radius.