MATH 5411 - Advanced Probability I
Homework 1

(due: October 10, 2022)

Q1: Let $A_1, A_2, \cdots$ be a sequence of events. Define

$$B_n = \cup_{m=n}^{\infty} A_m, \qquad C_n = \cap_{m=n}^{\infty} A_m.$$

Clearly $C_n \subset A_n \subset B_n$. The sequences $\{B_n\}$ and $\{C_n\}$ are decreasing and increasing respectively with limits

$$\lim B_n = B = \cap_n B_n = \cap_n \cup_{m \geq n} A_m, \qquad \lim C_n = C = \cup_n C_n = \cup_n \cap_{m \geq n} A_m.$$

The events B and C are denoted $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$, respectively. Show that
(a) $B = \{\omega \in \Omega : \omega \in A_n$ for infinitely many values of $n\}$,
(b) $C = \{\omega \in \Omega : \omega \in A_n$ for all but finitely many values of $n\}$,
We say that the sequence $\{A_n\}$ converges to a limit $A = \lim A_n$ if B and C are the same set A. Suppose that $A_n \to A$ and show that
(c) $\mathbb{P}(A_n) \to \mathbb{P}(A)$.

Q2: Let $\mathcal{F}$ be a σ-field, and let $\mathcal{G}, \mathcal{H} \subseteq \mathcal{F}$ be two sub σ-fields.

(i) Give one example which shows that $\mathcal{G} \cup \mathcal{H}$ is not a σ-field.

(ii) Prove that $\mathcal{G} \cap \mathcal{H}$ is a σ-field.

(iii) $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ is a sequence of sub σ-fields, prove that $\cup_{i=1}^{\infty} \mathcal{F}_i$ is a field. Give an example to show that $\cup_{i=1}^{\infty} \mathcal{F}_i$ is not necessarily a σ-field.

Q3: Suppose that X and Y are random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and let $A \in \mathcal{F}$. We set $Z(\omega) = X(\omega)$ for all $\omega \in A$ and $Z(\omega) = Y(\omega)$ for all $\omega \in A^c$. Prove that Z is a random variable.

Q4: Prove the following two definitions of random vector are equivalent.

Def.1: $X = (X_1, \ldots, X_d) : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is a random vector if it is $\mathcal{F}$-measurable.

Def.2: $X = (X_1, \ldots, X_d)$ is a random vector if $X_i : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\mathcal{F}$-measurable for all $i = 1, \ldots, d$.

Q5: Prove the following reverse Fatou's lemma: Let $f_1, f_2, \ldots$ be a sequence of Lebesgue integrable functions on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that there exists a non-negative integrable function g on Ω such that $f_n \leq g$ for all n. Prove

$$\limsup_{n\to\infty} \int f_n \mathrm{d}\mu \leq \int \limsup_{n\to\infty} f_n \mathrm{d}\mu.$$