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Q1: Let $X, X_1, X_2, \dots$ be a sequence of random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Further, let $g : \mathbb{R} \to \mathbb{R}$. Let D_g be the set of the discontinuity points of g. Assume that $\mathbb{P}(X \in D_g) = 0$. Prove the following continuous mapping theorem for convergence in distribution: If $X_n \xrightarrow{D} X$, then $g(X_n) \xrightarrow{D} g(X)$.

$\because$ $X_n \xrightarrow{D} X$

$\therefore$ $\exists$ Y_n, Y has the same distribution with X_n, X s.t. $Y_n \xrightarrow{a.s.} Y$

For any bounded, continuous $f: \mathbb{R} \to \mathbb{R}$, $D_{f\circ g} \subset D_g$

$0 \le \mathbb{P}(Y \in D_{f\circ g}) = \mathbb{P}(X \in D_{f\circ g}) \le \mathbb{P}(X \in D_g) = 0$

$\Rightarrow$ $\mathbb{P}(Y \in D_{f\circ g}) = 0$

$\Rightarrow$ $f(g(Y_n)) \xrightarrow{a.s.} f(g(Y))$

Also, $\because$ f is bounded. by the bounded convergence thm we have

$E[f(g(Y_n))] \to Ef(g(Y))$ $\quad \forall f$ bounded

$\Rightarrow$ $g(Y_n) \xrightarrow{D} g(Y)$

$\Rightarrow$ $g(X_n) \xrightarrow{D} g(X)$

Q2: Suppose $g, h : \mathbb{R} \to \mathbb{R}$ are continuous with $g(x) > 0$, and $|h(x)|/g(x) \to \cancel{0}\ \infty$ as $|x| \to 0$. Let $F, F_1, F_2, \ldots$ be a sequence of distribution functions. Suppose $F_n \to F$ weakly and $\int g(x) \mathrm{d}F_n(x) \le C < \infty$ uniformly in n. Prove

$$\int h(x) \mathrm{d}F_n(x) \to \int h(x) \mathrm{d}F(x).$$

$\because F_n \to F$

$\therefore \exists\, \xi_n, \xi$ s.t. $\xi_n \overset{D}{=} F_n$, $\xi = F$ s.t. $\xi_n \xrightarrow{a.s.} \xi$

We can assume $h(0)=0$, since $\int h(x)+c \, d F_n(x) = \int h(x) \, d F_n(x) + c$

we can always substract a constant $h(0)$ from h and the conclusion will not be affected.

Select a large enough M s.t. $\mathbb{P}(|\xi| = M) = 0$

Let $\overline{\xi} = \xi \mathbb{1}(|\xi| \le M)$. $\overline{\xi}_n = \xi_n \mathbb{1}(|\xi_n| \le M)$

Then, $\overline{\xi}_n \xrightarrow{a.s.} \overline{\xi}$ and $h(\overline{\xi}_n)$ is bounded

By bounded convergence thm, we have $E h(\overline{\xi}_n) \to E h(\overline{\xi})$

$\because h(0)=0$. Let $\varepsilon_n = \sup \left\{ \frac{h(x)}{g(x)} : |x| \ge M \right\}$

$$\begin{aligned}
|E h(\overline{\xi}_n) - E h(\xi_n)| &\le E|h(\overline{\xi}_n) - h(\xi_n)| \\
&\le E[\, h(\xi_n) \mathbb{1}(|\xi_n| > M)] \\
&\le \varepsilon_m E g(\xi_n) \\
&\le C \varepsilon_m
\end{aligned}$$

By Fatou's Lemma,

$$E g(x) = E \lim_{n\to\infty} g(x_n) \le \liminf_{n\to\infty} E g(x_n) \le C$$

Similarly,

$$\begin{aligned}
|E h(\overline{\xi}) - E h(\xi)| &\le E|h(\overline{\xi}) - h(\xi)| \\
&\le E|h(\xi) \mathbb{1}(|\xi| > M)| \\
&\le \varepsilon_m E g(x) \\
&\le C \varepsilon_m
\end{aligned}$$

As a result,

$$\begin{aligned}
|E h(\xi_n) - E h(\xi)| &\le |E h(\xi_n) - E h(\overline{\xi}_n)| + |E h(\overline{\xi}_n) - E h(\overline{\xi})| + |E h(\overline{\xi}) - E h(\xi)| \\
&\le 3 c h_m \to 0
\end{aligned}$$

Q3: Let $X_1, X_2, \ldots$ be i.i.d. and have the standard normal distribution. It is known that

$$\mathbb{P}(X_i > x) \sim \frac{1}{\sqrt{2\pi}x}\exp(-\frac{x^2}{2}) \qquad \text{as } x \to \infty.$$

where $a(x) \sim b(x)$ means $a(x)/b(x) \to 1$ if $x \to \infty$.

(i): Prove that for any real number θ,

$$\mathbb{P}(X_i > x + \frac{\theta}{x})/\mathbb{P}(X_i > x) \to \exp(-\theta), \qquad \text{as } x \to \infty$$

(ii) Show that if we define b_n by $\mathbb{P}(X_i > b_n) = 1/n$,

$$\mathbb{P}(b_n(\max_{1\le i\le n} X_i - b_n) \le x) \to \exp(-e^{-x}).$$

(iii) Show that $b_n \sim (2\log n)^{\frac{1}{2}}$ and conclude $\max_{1\le i\le n} X_i/(2\log n)^{\frac{1}{2}} \to 1$ in probability.

(i) $$\lim_{x\to\infty} \frac{\mathbb{P}(X_i > x+\frac{\theta}{x})}{\mathbb{P}(X_i > x)} = \lim_{x\to\infty} \frac{x}{x+\frac{\theta}{x}} e^{-\frac{(x+\frac{\theta}{x})^2}{2}+\frac{x^2}{2}}$$

$$= \lim_{x\to\infty} \frac{x}{x+\frac{\theta}{x}} e^{-\theta-\frac{\theta^2}{2x^2}}$$

$$= e^{-\theta}$$

(ii) $$\mathbb{P}(b_n(\max X_i - b_n) \le x) = \mathbb{P}(b_n(X_i - b_n) \le x)^n$$

$$= (1-\mathbb{P}(b_n(X_i-b_n) > x))^n$$

$$= (1-\mathbb{P}(X_i > b_n + \frac{x}{b_n}))^n$$

From (i), $\frac{\mathbb{P}(X_i > b_n + \frac{x}{b_n})}{\mathbb{P}(X_i > b_n)} = n\,\mathbb{P}(X_i > b_n + \frac{x}{b_n}) \to e^{-x}$ as $b_n \to \infty$

Then $\mathbb{P}(b_n(\max X_i - b_n) \le x) \to \lim_{n\to\infty}(1-\frac{e^{-x}}{n})^n = e^{-e^{-x}}$

(iii) $$\mathbb{P}(X_i > (2\log n)^{\frac{1}{2}}) \sim \frac{1}{\sqrt{2\pi}(2\log n)^{\frac{1}{2}}} e^{-\frac{2\log n}{2}}$$

$$= \frac{1}{2\sqrt{\pi}(\log n)^{\frac{1}{2}}}\cdot\frac{1}{n}$$

$$= \frac{1}{2\sqrt{\pi}(\log n)^{\frac{1}{2}}}\,\mathbb{P}(X_i > b_n)$$

$b_n \le (2\log n)^{\frac{1}{2}}$ when n is sufficiently large

Moreover,

$$\mathbb{P}(X_i > (2\log n - 2\log\log n)^{\frac{1}{2}}) \sim \frac{1}{\sqrt{2\pi}(2\log n - 2\log\log n)^{\frac{1}{2}}} e^{-\frac{2\log n - 2\log\log n}{2}}$$

$$\sim \frac{1}{\sqrt{2\pi}(\log n)^{\frac{1}{2}}} \frac{\log n}{n}$$

$$= \frac{(\log n)^{\frac{1}{2}}}{2\sqrt{\pi}} \mathbb{P}(Z_i > b_n)$$

For sufficiently large n. $(2\log n - 2\log\log n)^{\frac{1}{2}} \le b_n$

$\therefore b_n \sim (2\log n)^{\frac{1}{2}}$

From (ii), we have

$$\mathbb{P}(b_n(\max Z_i - b_n) \le x) \to e^{-e^{-x}} \quad ①$$

$$\therefore \mathbb{P}(b_n(\max Z_i - b_n) \ge x) \to 1-e^{-e^{-x}} \quad ②$$

Let $x = Z_n$ in ①

$$\mathbb{P}(\max Z_i - b_n \le \frac{Z_n}{b_n}) \to e^{-e^{-Z_n}}$$

Let $Z_n = o(b_n) \to \infty$, we have

$$\mathbb{P}(\max Z_i - b_n \le 0) \to 1$$

Moreover, let $x = Y_n$ in ②. we have

$$\mathbb{P}(\max Z_i - b_n \ge \frac{Y_n}{b_n}) \to 1-e^{-e^{-Y_n}}$$

Let $Y_n = o(b_n) \to -\infty$, we have

$$\mathbb{P}(\max Z_i - b_n \ge 0) \to 1$$

$$\therefore \mathbb{P}\left(\frac{\max Z_i}{b_n} = 1\right) \to 1$$

$$\therefore \mathbb{P}\left(\frac{\max Z_i}{(2\log n)^{\frac{1}{2}}} = 1\right) \to 1$$

Q4: Let $X_1, X_2, ...$ be independent taking values 0 and 1 with probability 1/2 each. Let $X = 2\sum_{j\geq 1} X_j/3^j$. Compute the characteristic function of X.

$\mathbb{P}(\mathfrak{Z}_j = 0) = \frac{1}{2}$, $\mathbb{P}(\mathfrak{Z}_j = 1) = \frac{1}{2}$

$\varphi_j(t) = E e^{it\mathfrak{Z}_j} = \frac{1}{2} + \frac{1}{2} e^{it}$

Then $E e^{it\frac{2\mathfrak{Z}_j}{3^j}} = \varphi_j\left(\frac{2}{3^j} t\right) = \frac{1}{2} + \frac{1}{2} e^{\frac{2it}{3^j}}$

Denote $S_n = \sum_{j=1}^{n} \frac{2\mathfrak{Z}_j}{3^j}$

Then $E e^{itS_n} = E e^{it\sum\frac{2\mathfrak{Z}_j}{3^j}} = \prod_{j=1}^{n} E e^{\frac{2it\mathfrak{Z}_j}{3^j}}$

$\because S_n \xrightarrow{a.s.} \mathfrak{Z}$

$\therefore S_n \xrightarrow{D} \mathfrak{Z}$

Then $E^{itS_n} \longrightarrow E^{it\mathfrak{Z}}$

$\therefore E^{it\mathfrak{Z}} = \prod_{j=1}^{\infty} E e^{\frac{2it\mathfrak{Z}_j}{3^j}} = \prod_{j=1}^{\infty} \frac{1+e^{\frac{2it}{3^j}}}{2}$

Q5: Let $S_n = X_1 + \cdots X_n$ in the following problems.

(a): Suppose that X_i's are independent and $\mathbb{P}(X_i = i) = \mathbb{P}(X_i = -i) = \frac{i^{-\alpha}}{4}$ and $\mathbb{P}(X_i = 0) = 1 - \frac{i^{-\alpha}}{2}$ for some nonnegative parameter α. Find $a_n(\alpha), b_n(\alpha)$ such that $(S_n - a_n(\alpha))/b_n(\alpha) \Rightarrow N(0,1)$ when $\alpha \in (0,1)$ and prove this CLT.

(b): Suppose that X_i's are independent and $\mathbb{P}(X_i = 1) = \frac{1}{i} = 1 - \mathbb{P}(X_i = 0)$. Find a_n and b_n such that $(S_n - a_n)/b_n \Rightarrow N(0,1)$ and prove this CLT.

(a) $E(X_i) = 0 \qquad Var(X_i) = E(X_i^2) = \dfrac{i^{2-\alpha}}{2}$

Then $E(S_n) = E(\Sigma X_i) = 0$

$Var(S_n) = Var(\Sigma X_i) = \Sigma \dfrac{i^{2-\alpha}}{2} \sim \dfrac{1}{2(3-\alpha)} n^{3-\alpha}$

Let $X_{n,m} = \dfrac{(2(3-\alpha))^{\frac{1}{2}} X_m}{n^{\frac{3-\alpha}{2}}}$

Then $E X_{n,m} = 0$.

$\Sigma E X_{n,m}^2 \longrightarrow 1$

Since $0<\alpha<1$.

$\forall \varepsilon > 0$. $\exists n_\varepsilon$ s.t. when $n \geq n_\varepsilon$. $|X_{n,m}| = \left| \dfrac{\sqrt{2(3-\alpha)}\, X_m}{n^{\frac{3-\alpha}{2}}} \right| \leq \dfrac{\sqrt{2(3-\alpha)}\, n}{n^{\frac{3-\alpha}{2}}} < \varepsilon$

$\therefore \Sigma E[X_{n,m}^2 \mathbb{1}(|X_{n,m}| > \varepsilon)] = 0$ when $n \geq n_\varepsilon$

Then $\dfrac{S_n}{\sqrt{Var(S_n)}} = \dfrac{\sqrt{2(3-\alpha)}\, S_n}{n^{\frac{3-\alpha}{2}}} \sim N(0,1)$

Thus $a_n(\alpha) = 0$

$b_n(\alpha) = \dfrac{1}{\sqrt{2(3-\alpha)}} n^{\frac{3-\alpha}{2}}$

(b) $E(X_i) = \frac{1}{i}$

$Var(X_i) = E(X_i^2) - (EX_i)^2 = \frac{1}{i} - \frac{1}{i^2}$

$E(S_n) = E(\Sigma X_i) = \Sigma \frac{1}{i} \sim \log n$

$Var(S_n) = Var(\Sigma X_i) = \Sigma \frac{1}{i} - \frac{1}{i^2} \sim \log n$

Let $X_{n,m} = \dfrac{X_m - \frac{1}{m}}{(\log n)^{\frac{1}{2}}}$

Then $E X_{n,m} = 0$, $\Sigma E X_{n,m}^2 \to 1$

$\forall \varepsilon > 0$. $\exists n_\varepsilon$ s.t. when $n \geq n_\varepsilon$.

$|X_{n,m}| \leq \dfrac{1}{(\log n)^{\frac{1}{2}}} < \varepsilon$

$$\sum_{m=1}^{n} E[\gamma_{n,m}^2 \mathbb{1}(|\gamma_{n,m}|>\varepsilon)] = 0$$

Then

$$\frac{S_n - \sum \frac{1}{i}}{(\log n)^{\frac{1}{2}}} \sim N(0,1) \quad \text{or} \quad \frac{S_n - \log n}{(\log n)^{\frac{1}{2}}} \sim N(0,1)$$

$\therefore$ $a_n = \log n$, $b_n = (\log n)^{\frac{1}{2}}$

Q6: Suppose that X_n and Y_n are independent, and $X_n \to X_\infty$ in distribution and $Y_n \to Y_\infty$ in distribution. Show that $X_n^2 + Y_n^2$ converges in distribution.

$\because$ $X_n \xrightarrow{D} X_\infty$, $Y_n \xrightarrow{D} Y_\infty$

Also, $g(x) = x^2$ is continuous

By Continuous Mapping Thm. $X_n^2 \xrightarrow{D} X_\infty^2$

$Y_n^2 \xrightarrow{D} Y_\infty^2$

and $X_n^2 \perp\!\!\!\perp Y_n^2$, $X_\infty^2 \perp\!\!\!\perp Y_\infty^2$

By Lévy's continuity Thm. $\forall t$

$$\varphi_{X_n^2}(t) \longrightarrow \varphi_{X_\infty^2}(t)$$

$$\varphi_{Y_n^2}(t) \longrightarrow \varphi_{Y_\infty^2}(t)$$

$\because$ $X_n^2 \perp\!\!\!\perp Y_n^2$, $X_\infty^2 \perp\!\!\!\perp Y_\infty^2$

$\therefore$ $\varphi_{X_n^2+Y_n^2}(t) = \varphi_{X_n^2}(t)\,\varphi_{Y_n^2}(t) \rightarrow \varphi_{X_\infty^2}(t)\,\varphi_{Y_\infty^2}(t)$

and $\varphi_{\infty}^2(t)$, $\varphi_{Y_\infty^2}(t)$ are continuous at 0

Then, $\varphi_{X_\infty^2}(t)\,\varphi_{Y_\infty^2}(t)$ is the characteristic function of $X_\infty^2 + Y_\infty^2$

$\therefore$ $X_n^2 + Y_n^2 \xrightarrow{D} X_\infty^2 + Y_\infty^2$

Q7: Let $X_1, X_2, \ldots$ be i.i.d. with a density that is symmetric about 0, and continuous and positive 0. Find the limiting distribution of

$$\frac{1}{n}\Big(\frac{1}{X_1} + \ldots + \frac{1}{X_n}\Big).$$

$\because \mathbb{P}\left(\frac{1}{X_i} > x\right) = \mathbb{P}\left(0 < X_i < \frac{1}{x}\right)$

$= \int_0^{\frac{1}{x}} f(y)\,dy \sim \frac{f(0)}{x}$ as $x \to \infty$

$\mathbb{P}\left(\frac{1}{X_i} < -x\right) = \mathbb{P}\left(-\frac{1}{x} < X_i < 0\right)$

$= \int_{-\frac{1}{x}}^{0} f(y)\,dy \sim \frac{f(0)}{x}$ as $x \to \infty$

$\therefore \lim_{x\to\infty} \frac{\mathbb{P}(X_1 > x)}{\mathbb{P}(|X_1| > x)} = \frac{1}{2}$

$\mathbb{P}(|X_1| > x) \sim 2f(0) \cdot x^{-1}$

Let $a_n = \inf\{x : \mathbb{P}(|X_1| > x) \le \frac{1}{n}\}$.

Then $a_n \sim 2f(0)n$

$\because$ Density of X_i is symmetric about 0

$\therefore b_n = nE(X_1 \mathbb{1}(|X_1| \le a_n)) = 0$

Then, $\frac{S_n - b_n}{a_n} \sim \frac{S_n}{2f(0)n} \Rightarrow Y$

where the characteristic function of Y is $e^{itc - b|t|^{-1}}$.

Q8: Do some self-study and explain why the *Stable distributions* and *Infinitely divisible distributions* bear such names.

Stable distribution :

A distribution is said to be stable if summing independent random variables from it results in a random variable in the same distribution.

It is called stable because its shape remains unchanged (thus stable) when being summed.

Infinitely divisible distribution:

A sufficient condition for Z be a limit of sums of the form

$$S_n = X_{n,1} + \cdots + X_{n,n}$$

is that Z has an infinitely divisible distribution, i.e. for each n $\exists$ i.i.d. sequence $Y_{n,1} \cdots Y_{n,n}$ s.t.

$$Z \stackrel{d}{=} Y_{n,1} + \cdots + Y_{n,n}$$