Ph.D. Written Qualifying Exam, Spring 2015
Advanced Calculus

函数极限 1. Find $\lim_{x\to 0}\left(\dfrac{3-e^x}{2+x}\right)^{\frac{1}{\sin x}}$. $= e^{-1}$

二元函数微分 2. Let

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0).\end{cases}$$

(1) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.

(2) Is $f(x,y)$ differentiable at $(0,0)$? If yes, write down the total differential. If not, justify your conclusion.

多重积分次序 3. Evaluate $\int_0^1 dx\int_1^x \sin(t^2)dt$.

多重积分 球坐标变换 4. Evaluate

$$\iint_S x^2dydz+y^2dzdx+z^2dxdy,$$

where S is the surface $(x-a)^2+(y-b)^2+(z-c)^2=R^2$ with outward-pointing normal direction.

积分? 5. Suppose that

$$\int_{-1}^{1}\frac{dx}{x^2+\varepsilon^2+5x^4+11x^6}=\frac{A}{\varepsilon}+O(1),\ \text{as } \varepsilon\to 0^+.$$

Find the number A.

级数 6. Show that when $x>0$, the following inequalities hold

$$\sum_{k=0}^{2n+1}(-1)^k\frac{x^{2k+1}}{(2k+1)!}<\sin x<\sum_{k=0}^{2n}(-1)^k\frac{x^{2k+1}}{(2k+1)!},$$

$$\sum_{k=0}^{2n+1}(-1)^k\frac{x^{2k}}{(2k)!}<\cos x<\sum_{k=0}^{2n+2}(-1)^k\frac{x^{2k}}{(2k)!},$$

for any non-negative integer n.

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《 Advanced Calculus 2014-2015 Spring 》

1. $$\lim_{x\to 0}\left(\frac{3-e^x}{2+x}\right)^{\frac{1}{\sin x}} = \lim_{x\to 0} e^{\frac{1}{\sin x}\ln\left(\frac{3-e^x}{2+x}\right)}$$

洛必达 $$= \lim_{x\to 0} e^{\frac{\left(\ln\frac{3-e^x}{2+x}\right)'}{\cos x}}$$

$$= e^{-1}$$

sin y bounded.

2. (1) $$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} x\sin\frac{1}{\sqrt{x^2}} \le \lim_{x\to 0} x = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0} y\sin\frac{1}{\sqrt{y^2}} \le \lim_{y\to 0} y = 0$$

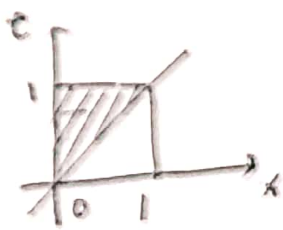
(2) $$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)x-\frac{\partial f}{\partial y}(0,0)y}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)}\frac{(x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

$$= \lim_{(x,y)\to(0,0)}\sqrt{x^2+y^2}\,\sin\frac{1}{\sqrt{x^2+y^2}}$$

$$= 0$$

3. $$\int_0^1 dx\int_1^x \sin(t^2)\,dt = \int_0^1 dt\int_0^t \sin t^2\,dx$$



$$= \int_0^1 t\sin t^2\,dt$$

$$= \frac{1}{2}\int_0^1 \sin t^2\,d(t^2)$$

$$= \frac{1}{2}\int_0^1 \sin s\,ds$$

$$= \frac{1}{2}\,-\cos s\Big|_0^1$$

$$= \frac{1}{2}(1-\cos 1)$$

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3. ① Fresnel Integral

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(t^2) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!}$$

$$\int_1^x \sin t^2 dt = \int_1^x \sum_{n=0}^{\infty} \frac{(-1)^n t^{4n+2}}{(2n+1)!} dt = \sum \frac{(-1)^n}{(2n+1)!} \int_1^x t^{4n+2} dt$$

$$= \sum \frac{(-1)^n}{(2n+1)!} \frac{t^{4n+3}}{4n+3} \Big|_1^x = \sum_{n=0}^{\infty} \frac{(-1)^n (x^{4n+3}-1)}{(2n+1)!(4n+3)}$$

$$\int_0^1 dx \int_1^x \sin t^2 dt = \int_0^1 \sum \frac{(-1)^n (x^{4n+3}-1)}{(2n+1)!(4n+3)} dx = \sum \frac{(-1)^n}{(2n+1)!(4n+3)} \int_0^1 (x^{4n+3}-1) dx$$

$$= \sum \frac{(-1)^n}{(2n+1)!(4n+3)} \left(\frac{x^{4n+4}}{4n+4} - x \right) \Big|_0^1 = \sum \frac{(-1)^{n+1}}{(2n+1)!(4n+4)}$$

②
$$\int_0^1 dx \int_1^x \sin(t^2) dt = \int_0^1 dt \int_t^1 \sin(t^2) dx = \int_0^1 \sin(t^2) dt - \underbrace{\int_0^1 t \sin(t^2) dt}_{= \frac{1}{2}\sin 1}$$

$$\int_0^1 \sin(t^2) dt = \sum \frac{(-1)^n}{(2n+1)!} \frac{t^{4n+3}}{4n+3} \Big|_0^1 = \sum \frac{(-1)^n}{(2n+1)!(4n+3)}$$

$$\text{原式} = \sum \frac{(-1)^n}{(2n+1)!(4n+3)} - \frac{1}{2} \sin 1$$

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4. 见 14-15 Fall #4

5. $\because \int_{-1}^{1} \frac{dx}{x^2+\varepsilon^2+5x^4+11x^6} = \frac{A}{\varepsilon} + O(1)$

$\therefore \int_{-1}^{1} \frac{\varepsilon\, dx}{x^2+\varepsilon^2+5x^4+11x^6} = A + O(\varepsilon).$

Let $x = \varepsilon y$.

$$LHS = \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\varepsilon^2 dy}{\varepsilon^2(1+y^2)+5\varepsilon^4y^4+11\varepsilon^6y^6}.$$

$$\approx \int_{-\frac{1}{\varepsilon}}^{\frac{1}{\varepsilon}} \frac{1}{1+y^2} dy + O(\varepsilon)$$

$$\varepsilon \to 0^+ \quad = \int_{\infty}^{\infty} \frac{1}{1+y^2} dy + O(\varepsilon)$$

$$= \arctan y \Big|_{-\infty}^{+\infty} + O(\varepsilon)$$

$$= \pi + O(\varepsilon)$$

$\therefore A = \pi$

扫描全能王 创建

Ph.D. Written Qualifying Exam, Spring 2022
Advanced Calculus

函数、数列极限 1. (1) Find $\lim_{x\to+\infty}\left(\cos\sqrt{x+1}-\cos\sqrt{x}\right)$.

(2) If $\lim_{n\to+\infty} a_n = a$, find $\lim_{n\to+\infty}\frac{a_1+2a_2+\cdots+na_n}{n^2}$.

二元函数的微分连续性 2. Let

$$f(x,y)=\begin{cases}\dfrac{x^{\frac{3}{2}}y^{\beta}}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0).\end{cases}$$

Discuss when this function (i) is continuous, (ii) is differentiable, and (iii) has continuous partial derivatives.

多重积分 & 变量代换 3. Compute the volume of the region Ω bounded by the surface $x^2+y^2=8-z$ and the plane $z=2y$.

Green 公式 4. Evaluate

$$\int_L \cos(x^2+y^2)(xdx+ydy),$$

where L is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Lagrange 乘数法 5. (1) Suppose that $\sum_{i=1}^{n} x_i^2 = A^2$. Find the maximum value of $\sum_{i=1}^{n} x_i$.

Hölder Inq

$\sum x_i \cdot 1 \le (\sum x_i^2)^{\frac{1}{2}}(\sum 1^2)^{\frac{1}{2}} = \sum x_i^2\sqrt{n}$

$= A\sqrt{n}$

(2) Use Lagrange multiplier method to prove that

$$\sum_{i=1}^{n} a_i b_i \le \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

6. Using the generalized Riemann-Lebesgue lemma: $\int_a^b f(t)e^{ix\psi(t)}dt \to 0$, as $x\to+\infty$, for real functions $f(t)$ and $\psi(t)$, provided that $|f(t)|$ is integrable, $\psi(t)$ is continuously differentiable, and $\psi'(t)\neq 0$ for $t\in[a,b]$, show that

$$\int_0^{\frac{\pi}{4}} e^{ix\sin^2 t}dt \sim \frac{1}{2}e^{i\frac{\pi}{4}}\sqrt{\frac{\pi}{x}},\quad \text{as } x\to+\infty.$$

Hints and Remarks:

(i) $f(x)\sim g(x)$ as $x\to x_0 \iff \lim_{x\to x_0}\frac{f(x)}{g(x)}=1$.

(ii) You can first prove that for some appropriate ε, $\int_0^{\frac{\pi}{4}} e^{ix\sin^2 t}dt \sim \int_0^{\varepsilon} e^{ixt^2}dt$, $x\to+\infty$.

(iii) $\int_0^{+\infty} e^{iz^2}dz = \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}}$.

ε: 大 ?

扫描全能王 创建

1. (1) $\lim\limits_{x\to\infty} \cos\sqrt{x+1} - \cos\sqrt{x} = \lim\limits_{x\to\infty} -2\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)$

$= \lim\limits_{x\to\infty} -2\sin\left[\frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{2(\sqrt{x+1}+\sqrt{x})}\right]\sin\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)$

$= \lim\limits_{x\to\infty} -2\sin\left[\frac{x+1-x}{2(\sqrt{x+1}+\sqrt{x})}\right]\sin\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)$

$= \lim\limits_{x\to\infty} -2\sin\underbrace{\frac{1}{2(\sqrt{x+1}+\sqrt{x})}}_{0}\ \underbrace{\sin\frac{\sqrt{x+1}+\sqrt{x}}{2}}_{\in[-1,1]}$

$= 0$

(2) ① 积分 $\frac{1}{n}\sum\limits_{i=1}^{n}\frac{i a_i}{n} = \int a x\,dx = \frac{a}{2}$

② Stolz公式. $A_n = a_1 + 2a_2 + \cdots + n a_n$

$B_n = n^2$. (满足 $\uparrow +\infty$).

$\lim\limits_{n\to\infty}\frac{A_{n+1}-A_n}{B_{n+1}-B_n} = \lim\limits_{n\to\infty}\frac{A_n}{B_n}$

↳ $= \lim\limits_{n\to\infty}\frac{(n+1)a_{n+1}}{2n+1} = \frac{a}{2}$.

扫描全能王 创建

Qualify Exam 2022 Advanced Calculus

1. (1) $\lim_{x\to\infty}(\cos\sqrt{x+1}-\cos\sqrt{x}) = \lim_{x\to\infty} -2\sin\left(\frac{\sqrt{x+1}-\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)$

$= \lim_{x\to\infty} -2\sin\left(\frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{2(\sqrt{x+1}+\sqrt{x})}\right)\sin\left(\frac{\sqrt{x+1}+\sqrt{x}}{2}\right)$

$= \lim_{x\to\infty} -2\sin\left(\frac{x+1-x}{2(\sqrt{x+1}+\sqrt{x})}\right)\sin\frac{\sqrt{x+1}+\sqrt{x}}{2}$

$= \lim_{x\to\infty} -2\sin\frac{1}{2(\sqrt{x+1}+\sqrt{x})}\sin\frac{\sqrt{x+1}+\sqrt{x}}{2}$

$\because \lim_{x\to\infty}\sin\frac{1}{2(\sqrt{x+1}+\sqrt{x})}\to 0.$

$\sin\frac{\sqrt{x+1}+\sqrt{x}}{2}\in[-1,1]$ 有界

$\therefore$ 原式 $=0$.

(2) ① Stolz公式

令 $A_n=a_1+2a_2+\cdots+na_n$, $B_n=n^2$.

B_n 单调递增, 趋于 $+\infty$

$\therefore$ 由Stolz公式. $\lim_{n\to\infty}\frac{A_n}{B_n}=\lim_{n\to\infty}\frac{A_{n+1}-A_n}{B_{n+1}-B_n}=\lim_{n\to\infty}\frac{(n+1)a_{n+1}}{2n+1}=\frac{a}{2}$

扫描全能王 创建

2. (1) 令 $x=r\sin\theta$. $y=r\cos\theta$.

$$f(x,y)= f(r,\theta) = \begin{cases} \dfrac{(r\sin\theta)^{\frac{3}{2}}(r\cos\theta)^{\beta}}{r^2} & r\neq 0 \\ 0 & r=0 \end{cases}$$

$$\lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} r^{\beta-\frac{1}{2}}(\sin\theta)^{\frac{3}{2}}(\cos\theta)^{\beta} \le \lim_{r\to 0} r^{\beta-\frac{1}{2}}.$$

① $\beta-\frac{1}{2}>0$ i.e. $\beta>\frac{1}{2}$ $\lim_{r\to 0} f(r,\theta)=0$

② $\beta-\frac{1}{2}=0$ i.e. $\beta=\frac{1}{2}$ $\lim_{r\to 0} f(r,\theta)=1$

③ $\beta-\frac{1}{2}<0$ i.e. $\beta<\frac{1}{2}$ $\lim_{r\to 0} f(r,\theta)=\infty$.

综上. 当 $\beta>\frac{1}{2}$ 时. $f(x,y)$ 连续

(2) 取单位向量 $u=(u_1,u_2)$.

$$\frac{\partial f}{\partial u}(0,0) = \lim_{t\to 0} \frac{(tu_1)^{\frac{3}{2}}(tu_2)^{\beta}}{t^2u_1^2+t^2u_2^2} = t^{\beta-\frac{1}{2}}u_1^{\frac{3}{2}}u_2^{\beta}$$

① $\beta-\frac{1}{2}>0$ i.e. $\beta>\frac{1}{2}$ $\lim_{t\to 0}\frac{\partial f}{\partial u}(0,0)=0$

② $\beta-\frac{1}{2}=0$ i.e. $\beta=\frac{1}{2}$ $=u_1^{\frac{3}{2}}u_2^{\frac{1}{2}}$

③ $\beta-\frac{1}{2}<0$ i.e. $\beta<\frac{1}{2}$ $=\infty$

综上. 当 $\beta\ge\frac{1}{2}$ 时. $f(x,y)$ 的方向导数存在.

(2) $\frac{\partial f}{\partial x}(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0$

$\frac{\partial f}{\partial y}(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=0$

$$\lim_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)x-\frac{\partial f}{\partial y}(0,0)y}{\sqrt{x^2+y^2}}$$

$$=\lim_{r\to 0}\frac{(r\sin\theta)^{\frac{3}{2}}(r\cos\theta)^{\beta}}{r}$$

$$=\lim_{r\to 0} r^{\beta+\frac{1}{2}}(\sin\theta)^{\frac{3}{2}}(\cos\theta)^{\beta}$$

$$\le\lim_{r\to 0} r^{\beta+\frac{1}{2}}$$

when $\beta+\frac{1}{2}>0$ i.e. $\beta>-\frac{1}{2}$ $\lim=0$

$=-\frac{1}{2}$ $=1$

$<-\frac{1}{2}$ $=\infty$

$\therefore \beta\ge-\frac{1}{2}$

(3) 变量代换同(1).

$$f'_x=\frac{-\frac{1}{2}x^{\frac{5}{2}}y^{\beta}+\frac{3}{2}x^{\frac{1}{2}}y^{2+\beta}}{(x^2+y^2)^2} \le r^{\beta-\frac{3}{2}}$$

① $\beta>\frac{3}{2}$ $\lim_{r\to 0} r^{\beta-\frac{3}{2}}=0$

② $\beta=\frac{3}{2}$ $=1$

③ $\beta<\frac{3}{2}$ $=\infty$.

$$f'_y=\frac{\beta x^{\frac{7}{2}}y^{\beta-1}+(\beta-2)x^{\frac{3}{2}}y^{\beta+1}}{(x^2+y^2)^2} \le (2\beta-2)(r^{\beta-\frac{3}{2}})$$

① $\beta=1$ $\lim_{r\to 0}(2\beta-2)(r^{\beta-\frac{3}{2}})=0$

② $\beta>\frac{3}{2}$ $=0$

③ $\beta=\frac{3}{2}$ $=1$

④ $\beta<\frac{3}{2}$. $\beta\neq 1$ $=\infty$

综上. $\beta>\frac{3}{2}$

扫描全能王 创建

3. 联立 $\begin{cases} x^2+y^2=8-z \\ z=2y \end{cases}$ 得 $x^2+(y+1)^2=9$

$\therefore \text{Volume} = \int_D \left(\int_{2y}^{8-(x^2+y^2)} dz \right) dx\,dy$

$= \int_{x^2+(y+1)^2=9} -(x^2+(y+1)^2)+9 \quad dx\,dy$

令 $x=r\cos\theta$. $y=-1+r\sin\theta$. 则 $\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|=r$.

原式 $= \int_0^{2\pi}\int_0^3 (9-r^2)\,r\,dr\,d\theta$

$= \int_0^{2\pi} \frac{81}{4}\,d\theta$

$= \frac{81}{2}\pi$.

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① 坐标原点在$\partial\Omega$之外

4. 由格林公式. $\int_{\Omega}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\int_{\partial\Omega}Pdx+Qdy$

$P=\cos(x^2+y^2)x$

$Q=\cos(x^2+y^2)y$

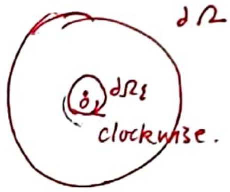
$\frac{\partial Q}{\partial x}=-2xy\sin(x^2+y^2)$

$\frac{\partial P}{\partial y}=-2xy\sin(x^2+y^2)$

$$\int_L \cos(x^2+y^2)(xdx+ydy)=\int_{\Omega}\left[-2xy\sin(x^2+y^2)+2xy\sin(x^2+y^2)\right]dxdy=0$$

② 坐标原点在$\partial\Omega$之内.

取以$(0,0)$为圆心，ε为半径的圆 $x^2+y^2=\varepsilon^2$.



$$\int_{\partial\Omega}+\int_{\partial\Omega_\varepsilon}Pdx+Qdy=\int_{\Omega-\Omega_\varepsilon}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=0.$$

$$\therefore \int_{\partial\Omega}Pdx+Qdy=\int_{\partial\Omega_\varepsilon}Pdx+Qdy$$

$$=\int_{x^2+y^2=\varepsilon^2}\frac{xdx+ydy}{\cos(x^2+y^2)}$$

Let $x=\varepsilon\cos\theta$

$y=\varepsilon\sin\theta$.

$$=0$$

扫描全能王 创建

5. (1) 设目标函数 $f(x)=\sum_{i=1}^{n} x_i$. 约束条件 $\Phi(x)=\sum_{i=1}^{n} x_i^2-A^2$

$F(x,\lambda)=\sum_{i=1}^{n} x_i-\lambda\left(\sum_{i=1}^{n} x_i^2-A\right)$

$$\begin{cases} F'_{x_i}=1-2\lambda x_i=0 \\ F'_\lambda=\sum_{i=1}^{n} x_i^2-A=0 \end{cases} \Rightarrow x_i=\sqrt{\frac{A^2}{n}}=A\sqrt{n}$$

$x_i=\frac{1}{2\lambda}$

$nx_i^2-A=0$

唯一驻点必为最大值点？

$\therefore \sum_{i=1}^{n} x_i \le \sum_{i=1}^{n}\sqrt{\frac{A^2}{n}}=A\sqrt{n}$

(2) 即证在 $\sum_{i=1}^{n} a_ib_i=A$ 的条件下，求 $f=\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}$ 的最小值.

~~数学归纳法：~~

- ~~当 $n=1$ 时. $(a_1^2)^{\frac{1}{2}}(b_1^2)^{\frac{1}{2}}=a_1b_1=A$~~
- ~~假设 $n=m$ 时. $\left(\sum_{i=1}^{m} a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{m} b_i^2\right)^{\frac{1}{2}}\ge A$.~~

~~当 $n=m+1$ 时. 令 $f(b_1,\cdots,b_{m+1})=\left(\sum_{i=1}^{m+1} a_i^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{m+1} b_i^2\right)^{\frac{1}{2}}=UV$ $U=(\sum a_i^2)^{\frac{1}{2}}, V=(\sum b_i^2)^{\frac{1}{2}}$~~

~~考虑~~

记 $\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}=U$. $\left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}=V$.

令 $F(a_1,\cdots,a_n,b_1,\cdots,b_n)=f-\lambda\left(\sum_{i=1}^{n} a_ib_i-A\right)=UV-\lambda\left(\sum_{i=1}^{n} a_ib_i-A\right)$

$$\begin{cases} \frac{\partial F}{\partial a_i}=a_i\frac{V}{U}-\lambda b_i=0 \\ \frac{\partial F}{\partial b_i}=b_i\frac{U}{V}-\lambda a_i=0 \\ \frac{\partial F}{\partial \lambda}=\sum_{i=1}^{n} a_ib_i-A=0 \end{cases} \Rightarrow \begin{matrix} a_i=\lambda\cdot\frac{U}{V} \\ b_i=\lambda\cdot\frac{V}{U} \end{matrix} \Rightarrow \begin{matrix} a_1=\cdots=a_n \\ b_1=\cdots=b_n \end{matrix}$$

$n a_ib_i=A$

$f=\left(\sum a_i^2\right)^{\frac{1}{2}}\left(\sum b_i^2\right)^{\frac{1}{2}}=(na_i^2)^{\frac{1}{2}}(nb_i^2)^{\frac{1}{2}}=na_ib_i=A$

扫描全能王 创建

Ph.D. Written Qualifying Exam, Fall 2014
Advanced Calculus

函数极限

1. Find $\lim_{x\to 0}\frac{e^{\frac{1}{x}}}{(1+x)^{\frac{1}{x^2}}}$. $= \lim_{x\to0}\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x^2}\ln(1+x)}} = \lim_{x\to0} e^{\frac{1}{x}-\frac{1}{x^2}\ln(1+x)} = \lim_{y\to\infty} e^{y-y^2\ln(1+\frac{1}{y})} = \lim_{y\to\infty} e^{y-y^2(\frac{1}{y}-\frac{1}{2y^2}+o(\frac{1}{y^2}))}$

$= e^{\frac{1}{2}}$

二元函数微分

2. Let

$$f(x,y)=\begin{cases}\dfrac{2x^2+3xy+2y^2}{x^2+y^2}, & (x,y)\neq(0,0)\\ 2, & (x,y)=(0,0).\end{cases}$$

(1) $\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=$

(1) Find $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$. 在x,y轴上为2，因此偏导恒为0

(2) Is $f(x,y)$ differentiable at $(0,0)$? If yes, write down the total differential. If not, justify your conclusion. No $\lim_{t\to0}\frac{f(tu)-f(0)}{t} = \frac{1}{t}\frac{2t^2u_1^2+3t^2u_1u_2+2t^2u_2^2}{t^2u_1^2+t^2u_2^2}-2 = \frac{3u_1u_2}{t}\to\infty$

多重积分次序

3. Evaluate $\int_0^1\int_1^x e^{-t^2}dtdx$.

多重积分

球坐标变换

4. Evaluate

$$\iint_S x^2dydz+y^2dzdx+z^2dxdy,$$

where S is the surface $(x-a)^2+(y-b)^2+(z-c)^2=R^2$ with outward-pointing normal direction.

级数

5. Show that when $x>0$, the following inequalities hold

$\sin x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$

$$\sum_{k=0}^{2n+1}(-1)^k\frac{x^{2k+1}}{(2k+1)!}<\sin x<\sum_{k=0}^{2n}(-1)^k\frac{x^{2k+1}}{(2k+1)!},$$

$$\sum_{k=0}^{2n+1}(-1)^k\frac{x^{2k}}{(2k)!}<\cos x<\sum_{k=0}^{2n+2}(-1)^k\frac{x^{2k}}{(2k)!},$$

$\cos x = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$

for any non-negative integer n.

6. (1) Show that for any $x>0$, $\int_0^{\varepsilon}e^{-x\sin^2 t}dt=\left(1+O(x\varepsilon^4)\right)\int_0^{\varepsilon}e^{-xt^2}dt$, as $\varepsilon\to0^+$.

(2) Show that for any $\varepsilon\in(0,\frac{\pi}{4})$, $\int_{\varepsilon}^{\frac{\pi}{4}}e^{-x\sin^2 t}dt=\left(1+O(\frac{1}{x\varepsilon^2})\right)\frac{e^{-x\sin^2\varepsilon}-e^{-\frac{x}{2}}\sin2\varepsilon}{x\sin2\varepsilon}$, as $x\to+\infty$.

(3) Show that $\int_0^{\frac{\pi}{4}}e^{-x\sin^2 t}dt\sim\frac{1}{2}\sqrt{\frac{\pi}{x}}$, as $x\to+\infty$.

Remarks:

(i) $f(x)=O(g(x))$ as $x\to x_0 \iff$ There exist constants $\delta, M>0$, such that $|f(x)|\le M|g(x)|$ when $|x-x_0|<\delta$.

(ii) $f(x)\sim g(x)$ as $x\to x_0 \iff \lim_{x\to x_0}\frac{f(x)}{g(x)}=1$.

(iii) $\int_0^{+\infty}e^{-s^2}ds=\frac{\sqrt{\pi}}{2}$.

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3 $\int_0^1 dx\int_1^x e^{-t^2}dt = \int_0^1 dt\int_t^1 e^{-t^2}dx = \int_0^1 (e^{-t^2}x|_t^1)dt = \int_0^1 (e^{-t^2}-te^{-t^2})dt$

$= \underbrace{\int_0^1 e^{-t^2}dt}_{I} - (-\frac{1}{2}e^{-t^2}|_0^1)$

$I^2 = \frac{1}{4}\int_{-1}^1\int_{-1}^1 e^{-(x^2+y^2)}dxdy = \frac{\pi}{4}(1-\frac{1}{e})$

原式 $= \frac{\pi}{4} - \frac{\pi}{4e} - 1 + \frac{e}{2}$

4. 球面坐标 $x = a + R\sin\varphi\cos\theta$ $y = b + R\sin\varphi\sin\theta$ $z = c + R\cos\varphi$ $\varphi \in [0,\pi]$ $\theta \in [0,2\pi]$

$\frac{\partial(y,z)}{\partial(\varphi,\theta)} = R^2\sin^2\varphi\cos\theta$ $\frac{\partial(z,x)}{\partial(\varphi,\theta)} = R^2\sin\varphi\sin\theta$ $\frac{\partial(x,y)}{\partial(\varphi,\theta)} = R^2\cos\varphi\sin\varphi$

原式 $= \int_0^\pi\int_0^{2\pi}(a+R\sin\varphi\cos\theta)^2R^2\sin^2\varphi\cos\theta\, d\varphi d\theta + \int_0^\pi\int_0^{2\pi}(c+R\cos\varphi)^2R^2\cos\varphi\sin\varphi\, d\varphi d\theta$

$+ \int_0^\pi\int_0^{2\pi}(b+R\sin\varphi\sin\theta)^2R^2\sin^2\varphi\sin\theta\, d\varphi d\theta$

$= \frac{8\pi}{3}(a+b+c)R^3$

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« Advanced Calculus 2014-2015 Fall »

1. $\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{x}}}{(1+x)^{\frac{1}{x^2}}} = \lim\limits_{x\to 0}\dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x^2}\ln(1+x)}} = \lim\limits_{x\to 0} e^{\frac{1}{x}-\frac{1}{x^2}\ln(1+x)}$

$= \lim\limits_{y\to\infty} e^{y-y^2\ln(1+\frac{1}{y})} \overset{展开}{=} \lim\limits_{y\to\infty} e^{y-y^2(\frac{1}{y}-\frac{1}{2y^2}+o(\frac{1}{y^2}))} = e^{\frac{1}{2}}$

2. (1) $f\equiv 0$ on both x-axis and y-axis, so the partial derivative of $f\equiv 0$.

$\therefore \frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)=0$

(2) No.

Let u be a unit vector. $u=(u_1,u_2)$.

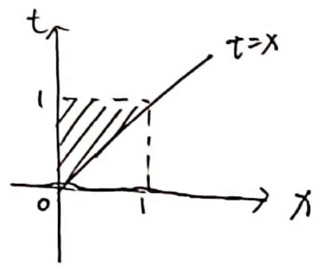
$\lim\limits_{t\to 0}\dfrac{f(tu)-f(0)}{t} = \dfrac{1}{t}\,\dfrac{2t^2u_1^2+3t^2u_1u_2+2t^2u_2^2-2}{t^2u_1^2+t^2u_2^2} = \dfrac{3u_1u_2}{t}\to\infty$

$\therefore f(x,y)$ is not differential at $(0,0)$

3. $\int_0^1 dx\int_1^x e^{-t^2}dt = \int_0^1 dt\int_0^t e^{-t^2}dx = \int_0^1 e^{-t^2}dt\int_0^t dx = \int_0^1 te^{-t^2}dt$

$=\frac{1}{2}\int_0^1 e^{-t^2}d(t^2) = \frac{1}{2}\int_0^1 e^{-s}ds$

$=\frac{1}{2}(-e^{-s}|_0^1) = \frac{1}{2}(1-\frac{1}{e})$



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4. In spherical coordinate system.

$x = a + R\sin\varphi\cos\theta \qquad y = b + R\sin\varphi\sin\theta \qquad z = c + R\cos\varphi. \qquad \varphi \in [0,\pi],\ \theta \in [0, 2\pi]$

$$\frac{\partial(y,z)}{\partial(\varphi,\theta)} = R^2\sin\varphi\cos\theta. \qquad \frac{\partial(z,x)}{\partial(\varphi,\theta)} = R^2\sin\varphi\sin\theta. \qquad \frac{\partial(x,y)}{\partial(\varphi,\theta)} = R^2\cos\varphi\sin\varphi.$$

$$\iint_S x^2 dydz + y^2 dzdx + z^2 dxdy$$

$$= \int_0^{\pi}\int_0^{2\pi} (a+R\sin\varphi\cos\theta)^2 R^2 \sin\varphi\cos\theta\, d\varphi d\theta + \int_0^{\pi}\int_0^{2\pi} (c+R\cos\varphi)^2 R^2\cos\varphi\sin\varphi\, d\varphi d\theta$$

$$+ \int_0^{\pi}\int_0^{2\pi} (b+R\sin\varphi\sin\theta)^2 R^2 \sin^2\varphi\sin\theta\, d\varphi d\theta$$ 分成两个曲面 

$$= \frac{8\pi}{3}(a+b+c)R^3$$

$$\iint x^2 dydz = \iint_D \left(a+\sqrt{R^2-(y-b)^2-(z-c)^2}\right)^2 dydz - \iint_D \left(a-\sqrt{\quad}\right)^2 dydz$$

$$= 4a\iint \sqrt{R^2 - \cdots}\, dydz$$ 令 $y = b + r\cos\theta.\ z = c + r\sin\theta$

$$\iint x^2 dydz = 4a\int_0^{2\pi} d\theta \int_0^R \sqrt{R^2-r^2}\, r\, dr = \frac{8\pi a}{3}R^3.$$

5. The Taylor expansion of $\sin x$ is $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} := S(x)$

$\cos x$ is $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!}$

Series of functional terms $S(x)$ is convergent.

$\Rightarrow$ Partial sum $S_n^{(1)} = \sum_{k=0}^{2n+2}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$, $S_n^{(2)} = \sum_{k=0}^{2n+1}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$ has limitation

and $\lim_{n\to\infty} S_n^{(1)} = \lim_{n\to\infty} S_n^{(2)} = S(x) = \sin x$

• $\sin x - \sum_{k=0}^{2k+1}(-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sum_{k=2n+2}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$$= \frac{x^{4n+5}}{(4n+5)!} + \sum_{k=2n+3}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= \frac{x^{4n+5}}{(4n+5)!} + S_n - S_n^{(1)}$$

$\forall \varepsilon > 0 \quad \exists N_1$ s.t. $n > N_1 \quad |S_n - S_n^{(1)}| < \varepsilon.$

Let $0 < \varepsilon < \frac{x^{4n+5}}{(4n+5)!}$

$$\sin x - \sum_{k=0}^{2k+1}(-1)^k\frac{x^{2k+1}}{(2k+1)!} > \frac{x^{4n+5}}{(4n+5)!} - \varepsilon > 0$$

$$\therefore \sin x > \sum_{k=0}^{2k+1}(-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

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- $\sum_{k=0}^{2n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} - \sin x = -\sum_{k=2n+1}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$

$-(-1)^{2n+1}$

$$= \frac{x^{4n+3}}{(4n+3)!} - \sum_{k=2n+2}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= \frac{x^{4n+3}}{(4n+3)!} - \left(S_n - S_n^{(2)}\right)$$

$$> \frac{x^{4n+3}}{(4n+3)!} - \varepsilon'$$

$$> 0$$

Let $\varepsilon' < \min\left\{ \frac{x^{4n+3}}{(4n+3)!}, \frac{x^{4n+5}}{(4n+5)!} \right\}$

$\therefore \sum_{k=0}^{2n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} > \sin x$

$\cos x$ 同理.

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6. (1) $\sin t = t + o(t^3)$

$\sin^2 t = t^2 + O(t^4)$

$e^{-x\sin^2 t} = e^{-x(t^2+O(t^4))} = e^{-xt^2} e^{-xO(t^4)}$

$\because e^x = 1 + x + \frac{x^2}{2!} + \cdots$

$\therefore e^{-xO(t^4)} = 1 + xO(t^4)$.

$$\therefore \int_0^{\varepsilon} e^{-x\sin^2 t} dt = \int_0^{\varepsilon} e^{-xt^2}(1 + xO(t^4))\,dt$$

$$= \int_0^{\varepsilon} e^{-xt^2} dt + \int_0^{\varepsilon} e^{-xt^2} O(xt^4)\,dt$$

$$= \int_0^{\varepsilon} e^{-xt^2} dt + O(x\varepsilon^4)\int_0^{\varepsilon} e^{-xt^2} dt$$

$$\varepsilon \to 0 \quad = (1 + O(x\varepsilon^4))\int_0^{\varepsilon} e^{-xt^2} dt.$$

(2) Let $\Phi(x) = \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} dt$

$$= \int_{\varepsilon}^{\frac{\pi}{4}} -\frac{1}{x\sin 2t}\, d e^{-x\sin^2 t}.$$

$$= -\left.\frac{e^{-x\sin^2 t}}{x\sin 2t}\right|_{\varepsilon}^{\frac{\pi}{4}} + \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} d\left(\frac{1}{x\sin 2t}\right)$$

$$= \frac{e^{-x\sin^2\varepsilon} - e^{-\frac{x}{2}}\sin 2\varepsilon}{x\sin 2\varepsilon} + \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} d\left(\frac{1}{x\sin 2t}\right)$$

$$\because \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} d\left(\frac{1}{x\sin 2t}\right) = \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t}\left(-\frac{2\cos 2t}{x\sin 2t}\right)dt = \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t}\frac{-2}{x\sin 2t\tan 2t}\,dt$$

$$\because \frac{-2}{x\sin 2t\tan 2t} = \frac{-2}{x[2t+O(t^3)][2t+O(t^3)]} = O\left(\frac{1}{xt^2}\right)$$

$$\therefore \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} d\left(\frac{1}{x\sin 2t}\right) = \int_{\varepsilon}^{\frac{\pi}{4}} O\left(\frac{1}{xt^2}\right) e^{-x\sin^2 t} dt$$

$$x \to \infty \quad = O\left(\frac{1}{x\varepsilon^2}\right)\int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} dt$$

$$\therefore \Phi(x) = \frac{e^{-x\sin^2\varepsilon} - e^{-\frac{x}{2}}\sin 2\varepsilon}{x\sin 2\varepsilon} + O\left(\frac{1}{x\varepsilon^2}\right)\int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} dt$$

$$= \frac{e^{-x\sin^2\varepsilon} - e^{-\frac{x}{2}}\sin 2\varepsilon}{x\sin 2\varepsilon} + O\left(\frac{1}{x\varepsilon^2}\right)\frac{e^{-x\sin^2\varepsilon} - e^{-\frac{x}{2}}\sin 2\varepsilon}{2\sin 2\varepsilon}$$

$$+ O\left(\frac{1}{x\varepsilon^2}\right) O\left(\frac{1}{x\varepsilon^2}\right)\int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} dt + \cdots\cdots$$

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$$= \left(1+O\left(\frac{1}{x\varepsilon^2}\right)\right) \frac{e^{-x\sin^2\varepsilon} - e^{-\frac{\pi}{2}\sin 2\varepsilon}}{x\sin 2\varepsilon}$$

(3) $$\int_0^{\frac{\pi}{4}} e^{-x\sin^2 t} dt = \int_0^{\varepsilon} e^{-x\sin^2 t} dt + \int_{\varepsilon}^{\frac{\pi}{4}} e^{-x\sin^2 t} dt \sim \int_0^{\varepsilon} e^{-xt^2} dt.$$

$$\int_0^{\varepsilon} e^{-xt^2} dt = \int_0^{\varepsilon} e^{-xt^2} d(\sqrt{x}t)$$

$$= \frac{1}{\sqrt{x}} \int_0^{\sqrt{x}\varepsilon} e^{-s^2} ds$$

$x \to \infty$ $$= \frac{1}{2}\sqrt{\frac{\pi}{x}}$$

$$\therefore \int_0^{\frac{\pi}{4}} e^{-x\sin^2 t} dt = \frac{1}{2}\sqrt{\frac{\pi}{x}}.$$

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改：

2. (1) $\frac{\partial f}{\partial x}(0,0)=\lim\limits_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim\limits_{x\to 0}\frac{\frac{2x^2}{x^2}-2}{x}=0$

$\frac{\partial f}{\partial y}(0,0)=0$.

(2) $\lim\limits_{(x,y)\to(0,0)}\frac{f(x,y)-f(0,0)-\frac{\partial f}{\partial x}(0,0)\cdot x-\frac{\partial f}{\partial y}(0,0)\cdot y}{\sqrt{x^2+y^2}}$

$=\lim\limits_{(x,y)\to(0,0)}\frac{\frac{2x^2+3xy+2y^2}{x^2+y^2}-2}{\sqrt{x^2+y^2}}$.

令 $x=r\cos\theta$; $y=r\sin\theta$

上式 $=\lim\limits_{r\to 0}\frac{\frac{2r^2\cos^2\theta+3r^2\cos\theta\sin\theta+2r^2\sin\theta}{r^2}-2}{r}$

$=\lim\limits_{r\to 0}\frac{\frac{2r^2+3r^2\cos\theta\sin\theta}{r^2}-2}{r}$

$=\lim\limits_{r\to 0}\frac{2+3\cos\theta\sin\theta-2}{r}$

$=\lim\limits_{r\to 0}\frac{3\sin 2\theta}{2r}\to\infty$.

∴ Not differentiable

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