Qualifying Exam on Linear Algebra

December 2017

Answer ALL questions.

1. Let $\lambda \in \mathbb{C}$ be given. Find all $a \in \mathbb{C}$ such that the following matrix A is diagonalizable.

$$A = \begin{bmatrix} \lambda & 0 & 0 \\ a & \lambda & 0 \\ 0 & a & \lambda \end{bmatrix}$$

2. Let $A_n \in \mathbb{R}^{n \times n}$ be defined as in the following with $a, b > 0$. Prove $\lim_{n \to \infty} \frac{1}{n} \log \det(A_n)$ exist and compute its value.

$$A_n = \begin{bmatrix} a & b & 0 & \cdots & 0 \\ -b & a & b & \ddots & \vdots \\ 0 & -b & a & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & b \\ 0 & \cdots & 0 & -b & a \end{bmatrix}$$

3. Let $v_1 = (0, 1, x)$, $v_2 = (1, x, 1)$, and $v_3 = (x, 1, 0)$. Find all $x \in \mathbb{R}$ for which v_1, v_2, v_3 are linearly independent over $\mathbb{R}$.

4. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times m}$ be two matrices. Prove that

$$\Lambda(AB) = \Lambda(BA),$$

   where $\Lambda(C) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } C, \text{ and } \lambda \neq 0\}$.

5. Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. Let $x, y \in \mathbb{C}^n$ be two eigenvectors of A corresponding to two distinct eigenvalues respectively. Prove from scratch that $x \perp y$.

6. Let A be a 2×2 real matrix such that

$$a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \leq \frac{1}{10}.$$

   Prove that $I + A$ is invertible.

7. Let $\mathcal{S}$ be a subset of $\mathbb{C}^{3 \times 3}$. The set $\mathcal{S}$ is called dense if every matrix in $\mathbb{C}^{3 \times 3}$ is a limit of a sequence of matrices in $\mathcal{S}$.

   (a) Prove that the set of matrices with distinct eigenvalues is dense in $\mathbb{C}^{3 \times 3}$.

   (b) Prove that the set of matrices with one Jordan block is not dense in $\mathbb{C}^{3 \times 3}$.