Dec 2011

Qualify Exam on Linear Algebra, Fall 2011
Department of Mathematics, HKUST

Choose five problems. Justify your answers.

1. Consider the following linear system of equations

$$\begin{aligned} x_1 + \quad x_2 \qquad\quad &= 1 \\ x_1 + \; 2x_2 + \quad x_3 &= 2 \\ x_2 + \; 2x_3 &= 3 \end{aligned}$$

(a). Does this system have a unique solution? If so, find it.

(b). Add the fourth equation

$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 = b_4$$

to the system above. Explain that under what circumstances the new system still admits a unique solution? When there is no solution, define the least-squares solution to the new system, and describe the QR-decomposition method for the least-squares solution.

2. Let A and B be two $n \times n$ matrices over C. Prove that $I - AB$ and $I - BA$ have the same determinant.

3. Show that if A is an $n \times n$ matrix, then A^n can be written as a linear combination of the matrices $I, A, A^2, \ldots, A^{n-1}$ (that is, $A^n = \alpha_0 I + \alpha_1 A + \alpha_2 A^2 + \ldots + \alpha_{n-1}A^{n-1}$ for some scalars $\alpha_0, \ldots, \alpha_{n-1}$).

4. Let A be an $n \times n$ matrix with only real eigenvalues, and γ be a real number such that $A - \gamma I$ is invertible. Let $x^{(0)}$ be an arbitrary vector with unit Euclidean norm. Consider the sequence of vectors defined by

$$(A - \gamma I)y^{(n)} = x^{(n-1)}, \quad x^{(n)} = \frac{y^{(n)}}{\|y^{(n)}\|}, \quad n > 1.$$

(a). Discuss the convergence of the sequence $\{x^{(n)}\}$. Does it always converge?

(b). Characterize the limit, if it exists, of the sequence.

5. Answer the following questions with justifications.

   (a) If $\mathbf{x}^T A\mathbf{x} > 0, \forall \mathbf{x} \neq 0$, then A must be a symmetric positive definite matrix. Is it true or or false?

   (b) For an $n \times n$ symmetric matrix A with eigenvalues $\lambda_1 > \lambda_2 > \ldots > \lambda_n$. Prove that

$$\min_{R_k} \max_{\mathbf{x} \in R_k, \|\mathbf{x}\|=1} \mathbf{x}^T A\mathbf{x} = \lambda_{n-k+1},$$

   where R_k is a k-dimensional subspace of R^n, and $\|\cdot\|$ stands for l_2 norm.

   (c) Let $\{\lambda_i\}$ and $\{\sigma_i\}$ be the eigenvalues and singular values of a general matrix A listed in descending order, separately. Prove that $|\lambda_1| \leq \sigma_1$ yet $|\lambda_n| \geq \sigma_n$.

==================== GOOD LUCK! ====================

PhD Written Qualifying Exam Dec 2012
Linear Algebra
Attempt all five problems. Justifying your answers.

1. Describe, in terms of the parameters α and β, when the system of equations

$$\left(\begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} + \alpha \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \right) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

has a unique solution, infinitely many solutions, and no solution at all.

2. Let V be a vector space over a field $\mathbf{F}$, and let $T : V \to V$ be a linear transformation satisfying $T^2 = T$. Define $W_1 = \{v \in V | T(v) = v\}$ and $W_2 = \{v \in V | T(v) = \vec{0}\}$.

   (a) Prove that W_1 is a vector subspace of V.

   (b) Prove that $W_1 \bigcap W_2 = \{\vec{0}\}$.

   (c) Prove that V is the direct sum: $V = W_1 \oplus W_2$.

3. Define $\rho(A) = \max_{1 \le i \le n} |\lambda_i|$, where $\{\lambda_i\}$ are eigenvalues of an $n \times n$ matrix A. Prove that $\lim\limits_{k\to\infty} A^k = 0$ iff $\rho(A) < 1$.

4. Prove that the nonzero eigenvalues of AB and BA are the same, where A and B are rectangular matrices of sizes $n \times m$ and $m \times n$ respectively. Express the corresponding eigenvectors of AB in terms of those of BA.

5. Let $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ be the eigenvalues of a Hermitian matrix H. For $i \ge 2$, let $u^1, \cdots, u^{i-1}$ be mutually orthogonal unit eigenvectors belonging to $\lambda_1, \cdots, \lambda_{i-1}$. Prove that

$$\lambda_i = \max_{||u||=1,(u,u^1)=\cdots=(u,u^{i-1})=0} (Hu, u)$$

and the maximum is achieved when u is an eigenvector belonging to λ_i, where $||\cdot||$ stands for the l_2 norm and $(\cdot, \cdot)$ stands for the l_2 inner product.

PhD Written Qualifying Exam
Linear Algebra

1. Let V be a vector space over the real field $\mathbb{R}$ and $(\alpha_1, \alpha_2, \alpha_3)$ is a basis of V. Let A be a linear transformation on V and the matrix representation of A under the basis $(\alpha_1, \alpha_2, \alpha_3)$ is
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}.$$
   (a) Find the minimum polynomial $m(\lambda)$ of A. Justify your answer.

   (b) Decompose V into the direct sum of nontrivial A-invariant subspaces. Justify your answer.

2. Find the inverse matrix of the following matrix $B \in \mathbb{R}^n$:
$$B = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ n & 1 & 2 & 3 & \cdots & n-1 \\ n-1 & n & 1 & 2 & \cdots & n-2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ 3 & 4 & 5 & 6 & \cdots & 2 \\ 2 & 3 & 4 & 5 & \cdots & 1 \end{pmatrix}.$$

3. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, $a_i \neq a_j$ and $b_i \neq b_j$ for $\forall i \neq j$. The (i, j) element of the matrix $A \in \mathbb{R}^{n \times n}$ is $A(i, j) = a_i - b_j$, for all $1 \leq i, j \leq n$.

   (a) Calculate $|A|$.

   (b) What is the dimension of the solution space of the equation $Ax = 0$? Justify your answer.

   (c) Find a basis of the solution space of the equation $Ax = 0$.

4. Let $A \in \mathbb{R}^{n \times n}$ be a positive definite matrix, $n > 1$. Let $\alpha \in \mathbb{R}^n$ and $\alpha \neq 0$. Define $B = A\alpha\alpha'$. Find the largest eigenvalue of B and a basis of the eigenspace corresponding to the largest eigenvalue. Justify your answer.

5. Let the (i, j) element of $H \in \mathbb{R}^{n \times n}$ be $H(i, j) = \frac{1}{i+j-1}$, $n > 1$. Define
$$\begin{aligned} f : \mathbb{R}^n \times \mathbb{R}^n &\to \mathbb{R} \\ (\alpha, \beta) &\mapsto f(\alpha, \beta) = \alpha' H \beta \end{aligned}$$
   Is the functional $f(\cdot, \cdot)$ an inner product on $\mathbb{R}^n$? Justify your answer.

6. Suppose $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n}$ and $\text{rank}(A) + \text{rank}(B) \leq n$. Show that there exists an invertible matrix C such that $ACB = 0$.

Prof. L X. Wu

Qualify Exam on Linear Algebra
Department of Mathematics, HKUST
December 3, 2013

Problems:

1. Find the Jordan canonical form of

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

2. Let V be a vector space and let U and W be finite dimensional subspaces of V. Show that $\dim(U+W) = \dim U + \dim W - \dim U \cap W$.

3. Find 3×3 matrices A and B corresponding to the indicated linear operator in each case:
   - A represents the orthogonal projection onto the plane $x + y + z = 0$.
   - B represents the rotation through an angle $\pi/2$ about $x = y = z$.

4. Let A be a matrix of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ c_1 & c_2 & c_3 & \dots & \dots & c_n \end{pmatrix}.$$

   Show that the minimal polynomial and characteristic polynomial of A are equal.

5. Let C be an $n \times n$ matrix. Show that trace$(C) = 0$ is both necessary and sufficient for C to be equal to $AB - BA$ for some $n \times n$ matrices A and B.

====================== GOOD LUCK! ======================

Qualify Exam on Linear Algebra
Department of Mathematics, HKUST
May 13, 2014

Problems:

1. Let $P_n(\mathbb{R})$ denote the vector space of polynomials of degree less than or equal to n, with coefficients in the real numbers. Define the linear transformation $T : P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$ by $T(f(x)) = xf(x) + f'(x)$. Using the standard ordered bases $B_2 = \{1, x, x^2\}$ for $P_2(\mathbb{R})$ and $B_3 = \{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$, compute the matrix $[T]_{B_2}^{B_3}$.

2. Let A and B be nonsingular $n \times n$ matrices over $\mathbb{C}$.

   (a) Show that if $A^{-1}B^{-1}AB = cI, c \in \mathbb{C}$, then $c^n = 1$.

   (b) Show that if $AB - BA = cI, c \in \mathbb{C}$, then $c = 0$.

3. (a) Let
$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$
   What are the eigenvalues of A?

   (b) Let B be an $n \times n$ real symmetric matrix with all zeros on the diagonal, and let $I + B$ be positive definite. Prove that the largest eigenvalue of B is less than $n - 1$.

4. Suppose A is an $n \times n$ skew-symmetric matrix with entries from $\mathbb{R}$ (i.e., $A^T = -A$), and that A is not the zero matrix.

   (a) Prove that if n is odd, then 0 is an eigenvalue of A.

   (b) Prove that A is diagonalizable over $\mathbb{C}$.

   (c) Prove that A is not diagonalizable over $\mathbb{R}$.

5. Let (x, y) be a positive definite inner product on the finite dimensional real vector space V. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of vectors satisfying $(v_i, v_j) < 0$ for all $i \neq j$. Prove that $\dim(\text{span } S) \geq k - 1$.

====================== GOOD LUCK! ======================

Written Qualifying Exam on Linear Algebra **2 December 2014**
Answer all five questions.

1. (15 pts) True or false.

   (a) The vector space of all 4×4 matrices that are both symmetric and anti-symmetric (also called "skew-symmetric") has dimension one.

   (b) If T is a linear transformation between the linear spaces V and W, then the set $\{v \in V | T(v) = 0\}$ is a linear subspace of V.

   (c) The vectors $v_1, v_2, \cdots, v_n$ in R^n are linearly independent if, and only if, span $\{v_1, v_2, \cdots, v_n\}$ is all of R^n.

   (d) If A is an $n \times n$ matrix such that nullity $(A) = 0$, then A is the identity matrix.

   (e) If A is an $k \times n$ matrix with rank k, then the columns of A are linearly independent.

2. (15 pts) Let L be an $n \times n$ matrix with real entries and let λ be an eigenvalue of L. In the following list, identify all the assertions that are correct.

   (a) $a\lambda$ is an eigenvalue of aL for any scalar a.

   (b) λ^2 is an eigenvalue of L^2.

   (c) $\lambda^2 + a\lambda + b$ is an eigenvalue of $L^2 + aL + bI_n$ for all real scalars a and b.

   (d) If $\lambda = a + ib$, with $a, b \neq 0$ some real numbers, is an eigenvalue of L, then $\lambda = a - ib$ is also an eigenvalue of L.

3. (20 pts) Let L be the space of polynomials of degree at most k and define the linear map $L : \mathcal{P}_k \longrightarrow \mathcal{P}_{k+1}$ by $Lp := p''(x) + xp(x)$.

   (a) Show that the polynomial $q(x) = 1$ is not in the image of L.

   (b) Let $V = \{q(x) \in \mathcal{P}_{k+1} | q(0) = 0\}$. Show that the map $L : \mathcal{P}_k \rightarrow V$ is invertible.

4. (20 pts) An $n \times n$ matrix is called *nilpotent* if A^k equals the zero matrix for some positive integer k. (For instance, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent.)

   (a) If λ is an eigenvalue of a matrix A, show that $\lambda = 0$. (Hint:start with the equation $A\overrightarrow{x} = \lambda\overrightarrow{x}$.)

   (b) Show that if A is both nilpotent and diagonalizable, then A is the zero matrix.

   (c) Let A be the matrix that represents $T : \mathcal{P}_5 \longrightarrow \mathcal{P}_5$ (polynomials of degree at most 5) given by differentiation: $Tp = dp/dx$. Without doing any computations, explain why A must be nilpotent.

5. (30 pts) Let A and B be $n \times n$ complex matrices that commute: $AB = BA$. If λ is an eigenvalue of A, let $\mathcal{V}_\lambda$ be the subspace of all eigenvectors having this eigenvalue.

   (a) Show there is an vector $v \in \mathcal{V}_\lambda$ that is also an eigenvector of B, possibly with a different eigenvalue.

   (b) Give an example showing that some vectors in $\mathcal{V}_\lambda$ may not be an eigenvectors of B.

   (c) If all the eigenvalue of A are distinct (so each has algebraic multiplicity one), show that there is a basis in which both A and B are diagonal. Also, give an example showing this may be false if some eigenvalue of A has multiplicity greater than one.

Qualify Exam on Linear Algebra
Department of Mathematics, HKUST
May 14, 2015

Problems:

1. Let

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

   (a) Find the characteristic polynomial of A.

   (b) Find the eigenvalues of A.

   (c) Find the dimensions of all eigenspaces of A.

   (d) Find the Jordan canonical form of A.

2. Let V be a vector space and let U and W be finite dimensional subspaces of V. Show that $\dim(U+W) = \dim U + \dim W - \dim U \cap W$.

3. Let A be a matrix of the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ c_1 & c_2 & c_3 & \dots & c_{n-1} & c_n \end{pmatrix}.$$

   Show that the minimal polynomial and characteristic polynomial of A are equal.

4. Find 3×3 matrices A, B and C corresponding to the indicated linear operator in each case:

   (a) A represents the orthogonal projection onto the plane $x+y+z=0$.

   (b) B represents the reflection about the plane $x+y+z=0$.

   (c) C represents the rotation through an angle $\pi/2$ about $x=y=z$.

5. Let C be an $n \times n$ matrix. Show that $\text{trace}(C) = 0$ is both necessary and sufficient for C to be equal to $AB - BA$ for some $n \times n$ matrices A and B.

Qualifying Exam on Linear Algebra

December 2015

Answer all questions.

1. Describe, in terms of the parameters α and β, when the system of equations
$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2\alpha+1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\beta \\ 2-\beta \end{pmatrix}$$
has a unique solution, infinitely many solutions, and no solution at all.

2. Let $f : V \times V \mapsto \mathbb{R}$ be a bilinear form on a finite-dimensional real vector space V. Thus, f is linear in both variables:
$$f(ax+by, z) = af(x,z) + bf(y,z)$$
$$f(x, cy+dz) = cf(x,y) + df(x,z)$$
Suppose that $v \in V$ is such that $f(v,v) \neq 0$. Let $\operatorname{span}(v)$ be the subspace of V generated by v and let $v^\perp$ be the following subset of V:
$$v^\perp = \{x \in V : f(x,v) = 0\}.$$
   (a) Prove that $v^\perp$ is a subspace of V.
   (b) Prove that $V = \operatorname{span}(v) \oplus v^\perp$.
   (c) The function $R : V \mapsto V$ is given by
   $$R(x) = x - 2\frac{f(x,v)}{f(v,v)}v.$$
   What is the determinate of (a matrix representation for) R?

3. For a 2×2 real matrix A, define
$$\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots .$$
Assuming that the above series converges and that for all 2×2 matrices B and C
$$B\sin(A)C = BAC - \frac{BA^3C}{3!} + \frac{BA^5C}{5!} - \frac{BA^7C}{7!} + \cdots ,$$
compute $\sin(A)$ where $A = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$.

4. Let A and B be two rank-r real matrices of size $n \times n$. Let P_A and P_B be the orthogonal projection operator onto the column spaces of A and B respectively. Let $\|\cdot\|$ be the spectral norm.

   (a) Prove that
   $$\|P_A - P_B\| = \|(I - P_B)P_A\| = \|P_B(I - P_A)\|,$$
   where I is the identity operator.

   (b) Prove that
   $$\|P_A - P_B\| \leq \frac{\|A - B\|}{\sigma_{\min}(A)},$$
   where $\sigma_{\min}(A)$ is the minimum nonzero singular value of A, and $\|\cdot\|$ is the spectral norm.

5. Consider the vector space of all real $m \times n$ matrices endowed with the inner product $\langle X, Y\rangle = \mathrm{trace}(X^T Y)$. Prove that the norm dual to the spectral norm is the nuclear norm, i.e.,
   $$\|X\|_* = \sup_{\|Y\| \leq 1} \langle X, Y\rangle,$$
   where $\|\cdot\|$ is the spectral norm (the maximum singular value) and $\|\cdot\|_*$ is the nuclear norm (the summation of all singular values).

Qualifying Exam on Linear Algebra

May 2016

Answer 5 of the following 7 questions.

1. Let V be the space of all 3×3 real matrices that are skew-symmetric, i.e., $A^T = -A$, where A^T denotes the transpose of A. Prove that the expression
$$\langle A, B\rangle = \frac{1}{2}\text{Trace}(AB^T)$$
defines an inner product on V. Find an orthonormal basis of V with respect to this inner product.

2. Let $P \in \mathbb{R}^{n\times m}$ and $Q \in \mathbb{R}^{m\times n}$ and $A \in \mathbb{R}^{n\times n}$. Assume that A and $(I + QA^{-1}P)$ are invertible. Prove that
$$(A + PQ)^{-1} = A^{-1} - A^{-1}P(I + QA^{-1}P)^{-1}QA^{-1}.$$

3. Let A be a nonsingular $n \times n$ matrix with entries in $\mathbb{C}$, and let A^* be its complex conjugate transpose (or adjoint). Let $\|\mathbf{x}\| = \left(\sum_{i=1}^n |x_i|^2\right)^{1/2}$ be the standard norm on $\mathbb{C}^n$.
   (a) Show that all of the eigenvalues of A^*A are positive real numbers.
   (b) Show that for all $\mathbf{x} \in \mathbb{C}^n$,
   $$\sqrt{\lambda_{\min}}\|\mathbf{x}\| \le \|A\mathbf{x}\| \le \sqrt{\lambda_{\max}}\|\mathbf{x}\|$$
   where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest eigenvalue and the largest eigenvalue of A^*A respectively.

4. Let $A \in \mathbb{C}^{n\times n}$. Prove that, for any positive integer k,
$$\left|\text{Trace}(A^k)\right|^{1/k} \le |\lambda_{\max}(A)|\,,$$
where Trace is the trace operator and $\lambda_{\max}(A)$ is the maximum eigenvalue in magnitude of A .

5. Let V be a vector space over real numbers. A linear transformation $T : V \to V$ is said to be idempotent if $T^2 = T$. Prove that if T is idempotent then $V = V_0 \oplus V_1$, where $T(v_0) = 0$ for all $v_0 \in V_0$ and $T(v_1) = v_1$ for all $v_1 \in V_1$.

6. For a 2×2 real matrix A, define
$$\sin(A) = A - \frac{A^3}{3!} + \frac{A^5}{5!} - \frac{A^7}{7!} + \cdots .$$

Assuming that the above series converges and that for all 2×2 matrices B and C

$$B \sin(A) C = BAC - \frac{BA^3C}{3!} + \frac{BA^5C}{5!} - \frac{BA^7C}{7!} + \cdots ,$$

compute $\sin(A)$ where $A = \frac{1}{2}\begin{pmatrix} 5 & -1 \\ 1 & 3 \end{pmatrix}$.

7. Suppose A and M are $n \times n$ matrices over $\mathbb{C}$, A is invertible and $AMA^{-1} = M^2$. Prove that all nonzero eigenvalues of M are roots of unity.

Qualifying Exam on Linear Algebra

December 2016

Answer ALL questions.

1. Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^4$ be given by $T(v) = Av$, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 4 & 2 \end{bmatrix}$$

   (a) Find the dimension of the null space of T.

   (b) Find a basis for the range space of T.

2. Let

$$A = \begin{bmatrix} 4 & -4 \\ 1 & 0 \end{bmatrix}$$

   (a) Find the Jordan decomposition of A.

   (b) Based on the Jordan decomposition, find an explicit formula for a_n, where a_n is defined recursively by $a_{k+1} = 4a_k - 4a_{k-1}$ with $a_0 = a$ and $a_1 = b$.

3. Let V be an n-dimensional $(n \geq 2)$ vector space over $\mathbb{C}$ with a set of basis vectors $e_1,\ e_2,\ \ldots,\ e_n$. Let T be a linear transformation satisfying $Te_1 = e_2,\ \ldots,\ Te_{n-1} = e_n,\ Te_n = e_1$.

   (a) Show that T has 1 as an eigenvalue and find the associated eigenvector. Show that up to scaling the eigenvector is unique.

   (b) Is T diagonalizable? Justify your answer.

4. Let V be the space of all $n \times n$ real matrices.

   (a) Prove that the expression

$$\langle A, B \rangle = \frac{1}{2}\mathrm{Trace}(AB^T)$$

   defines an inner product on V.

   (b) Find (with reasoning) the orthogonal complement of the subspace of all symmetric matrices in V.

5. Let A be an $n \times n$ real unitary matrix, i.e., $AA^T = A^T A = I$.

   (a) Prove that 1 or -1 must be an eigenvalue of A, if n is odd.

   (b) Is 1 or -1 an eigenvalue of A, if n is even? If yes, prove it; otherwise, give a counter example.

Qualifying Exam on Linear Algebra

May 2017

Answer ALL questions.

1. Prove that if $a, \lambda \in \mathbb{C}$ with $a \neq 0$, then the following matrix A is not diagonalizable.

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & \lambda \end{bmatrix}$$

2. Prove that the following matrix is positive definite.

$$A = \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 3 & 1 & \cdots & 1 \\ 1 & 1 & 4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}$$

3. Let T be a linear transformation of a vector space V to itself. If Tv and v are linearly dependent for any $v \in V$. Show that T must be a multiple of identity.

4. Let A be an $n \times n$ real matrix such that $A^T = -A$. Prove that $\det(A) \geq 0$.

5. Let $A, B \in \mathbb{C}^{n\times n}$ be satisfying $AB - BA = A$. Prove that A is not invertible.

6. Let V be a complex inner product space, and $u, v \in V$. Prove the Cauchy-Schwartz inequality

$$\langle u, v\rangle \leq \|u\|\|v\|.$$

7. Let V be the space of all $n \times n$ real matrices.

   (a) Prove that the expression

$$\langle A, B\rangle = \frac{1}{2}\text{Trace}(AB^T)$$

   defines an inner product on V.

   (b) Given $C \in V$, define a linear transformation Φ by $\Phi_C(A) = CA - AC$ for any $A \in V$. Find the adjoint of Φ_C.

PhD Qualifying Exam on Linear Algebra

May 2020

Answer ALL questions.

1. Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix and let $e_1 = [1, 0, \dots, 0]^T \in \mathbb{R}^n$. Show that for each $\lambda \in \mathbb{R}$, the linear transformation A_λ defined by

$$A_\lambda x = Ax + \lambda \langle e_1, x\rangle e_1$$

is invertible if and only if $1 + \lambda\langle e_1, A^{-1}e_1\rangle \neq 0$.

2. Prove that if $a, \lambda \in \mathbb{C}$ with $a \neq 0$, then the following matrix A is not diagonalizable.

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & a \\ 0 & 0 & \lambda \end{bmatrix}$$

3. 4. Let V be a vector space and let n be an integer satisfying $1 \leq n < \dim(V)$. Let $\{V_i\}$ be a collection of n-dimensional subspaces of V with the property that $\dim(V_i \cap V_j) = n - 1$ for every $i \neq j$. Show that at least one of the following holds:

   (i) All V_i share a common $(n-1)$-dimensional subspace.

   (ii) There is an $(n+1)$-dimensional subspace of V containing all V_i.

4. Given $B \in \mathbb{R}^{n\times n}$, we define a linear transformation $L_B : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by $L_B(A) = B^T AB$, where T denotes transpose.

   (a) Prove that L_B is invertible if and only if B is invertible.

   (b) Express $\operatorname{rank}(L_B)$ in terms of n and $\operatorname{rank}(B)$ for all choices of B.

5. Given two real matrices A and B whose sizes are $n \times m$ and $m \times n$ respectively. Prove that AB and BA have the same nonzero eigenvalues.

6. Consider the linear space $\mathbb{R}^{n\times n}$ endowed with inner product $\langle A, B\rangle = \operatorname{trace}(A^T B)$, where trace is the trace of a matrix.

   (a) Prove that $\operatorname{trace}(A^T B)$ indeed defines an inner product on $\mathbb{R}^{n\times n}$.

   (b) Consider the function $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$, where $\det(\cdot)$ is the determinant. Prove

   $$\nabla \det(I) = I,$$

   where I is the identity matrix. (Hint: It suffices to prove, for any matrix $X \in \mathbb{R}^{n\times n}$, $\lim_{\epsilon\to 0} \frac{\det(I+\epsilon X) - \det(I)}{\epsilon} = \langle I, X\rangle$.)

   (c) Calculate $\nabla \det(X)$ for any invertible $X \in \mathbb{R}^{n\times n}$.

PhD Qualifying Exam on Linear Algebra

December 2020

Answer ALL questions.

1. Describe, in terms of the parameters α and β, when the system of equations

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2\alpha+1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1+\beta \\ 2-\beta \end{pmatrix}$$

has a unique solution, infinitely many solutions, and no solution at all.

2. Let

$$A = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

be an $n \times n$ matrix. Find an invertible matrix T such that $B = T^{-1}AT$ is diagonal, and find this matrix B.

3. Let V be a finite dimensional linear product space over $\mathbb{C}$ with inner product $\langle \cdot, \cdot \rangle$. Prove that there is no invertible linear transformation $T : V \to V$ such that $\langle v, Tv \rangle = 0$ for all $v \in V$.

4. The result in Question 3 does not hold if $\mathbb{C}$ is replaced by $\mathbb{R}$. Show that in $\mathbb{R}^2$ with inner product $\langle u, v \rangle := \sum_{i=1}^{2} u_i v_i$ there exists an invertible linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\langle v, Tv \rangle = 0$ for all $v \in \mathbb{R}^2$.

5. Prove or disprove: There exists two $n \times n$ matrices A and B with complex entries such that $AB - BA = I$, where I is the identity matrix.

6. Suppose A and M are $n \times n$ matrices over $\mathbb{C}$, A is invertible and $AMA^{-1} = M^2$. Prove that all nonzero eigenvalues of M are roots of unity.

7. A sequence of matrices $\{A_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^{n\times n}$ is said converge to a matrix B if entries of $\{A_k\}_{k\in\mathbb{N}}$ converge to entries of B. Given a matrix $T \in \mathbb{R}^{n\times n}$, show that $\{T^k\}_{k\in\mathbb{N}}$, where T^k is the k-th power of T, converges to the zero matrix if and only if all eigenvalues λ of T are such that $|\lambda| < 1$.

PhD Qualifying Exam on Linear Algebra

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Answer ALL questions with reasoning.

1. Let $a, b, c, d \in \mathbb{R}$ and
$$A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{bmatrix}$$
Determine conditions on a, b, c, d so that there is only one Jordan block for each eigenvalue of A in the Jordan form of A.

2. For
$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & -5 & 3 \\ 1 & -1 & 2 \end{bmatrix}$$
write A^{-1} as a polynomial in A with real coefficients.

3. Let V denote the vector space of all real $n \times n$ matrices.
   (a) Prove that the form $\langle A, B \rangle := \text{trace}(A^T B)$, where A^T is the transpose of A, defines an inner product on V.
   (b) Find the orthogonal complement the subspace of all skew-symmetric matrices in V. (Recall that an $n \times n$ real matrix A is skew-symmetric if and only if $A^T = -A$.)

4. Let $T : V \to V$ be a linear operator such that $T^6 = 0$ but $T^5 \neq 0$.
   (a) Suppose $V = \mathbb{R}^6$. Prove that there is no linear operator $S : V \to V$ such that $S^2 = T$.
   (b) Does the answer change if $V = \mathbb{R}^{12}$?

5. Let A be an $n \times n$ real matrix with transpose A^T. Prove that A and AA^T have the same range.

6. Let A be an $n \times n$ complex diagonalizable matrix and I the $n \times n$ identity matrix. Show that
$$\begin{bmatrix} A & I \\ 0 & A \end{bmatrix}$$
is not diagonalizable.

7. Let A be an $n \times n$ complex matrix. Prove that $\text{rank}(A) = \text{rank}(A^2)$ if and only if $\lim_{\lambda \to 0}(A + \lambda I)^{-1} A$ exists, where I is the identity matrix.