PhD Qualifying Exam on Linear Algebra

May 2018

Answer ALL questions.

1. Let $a, b, c, d \in \mathbb{R}$ and
$$A = \begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & 0 \\ 0 & c & 3 & -2 \\ 0 & d & 2 & -1 \end{bmatrix}$$
Determine conditions on a, b, c, d so that there is only one Jordan block for each eigenvalue of A in the Jordan form of A.

2. Let A be an $n \times n$ real matrix with transpose A^T. Prove that A and AA^T have the same range.

3. Prove that eigenvalues of a Hermitian matrix are all real.

4. Let A be an $n \times n$ real matrix. Suppose A is orthogonal and symmetric.
   (a) Prove that if A is positive definite then A is the identity.
   (b) Does the answer change if A is only positive semidefinite?

5. Let V be a complex vector space and $T : V \to V$ be a linear transformation. Let $v_1, v_2, \ldots, v_n$ be non-zero vectors in T, each an eigenvector of T of a different eigenvalue. Prove that $\{v_1, v_2, \ldots, v_n\}$ are linearly independent.

6. Recall that the exponential of an $n \times n$ matrix is defined by
$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$
   (a) Prove that, if A and B are commutable, i.e., $AB = BA$, then
$$e^{A+B} = e^A e^B.$$
   (b) Give an example of non-commutable matrices when this equality fails.