Algolab 2019 – Week 5

Overview

Today's tutorial: 'Advanced' Techniques

- Greedy Algorithms
  - Example 1: Knapsack variations
  - Proof Technique: Exchange Argument
  - Example 2: Interval Scheduling
  - Proof Technique: Staying Ahead
- Split & List

Greedy Algorithms

"Greed is good.

Greed is right.

Greed works."

— Wall Street 1987

by Gordon Gekko

- Sometimes **locally optimal** choices result in a **globally optimal** solution.
- This is when we can apply **greedy algorithms**.
- **However** often choices that seem best in a particular moment turn out not to be optimal in the long run (e.g. in chess, life, etc.).

A greedy approach typically has the following steps:

1. **Modelling**: realise that your task requires you to construct a set that is in some sense **globally optimal**.
2. **Greedy choice**: given already chosen elements $c_1, \ldots, c_{k-1}$, decide how to choose c_k, based on some **local optimality criterion**.
3. **Prove** that elements obtained in this way result in a **globally optimal** set.
4. **Implement** the greedy choice to be as efficient as possible.

Equal Weights Knapsack

Given a maximum weight W and a set of n items with **values** $v_1, \ldots, v_n$ and **weights** $w_1 = \ldots = w_n = 1$, find an **assignment** $x_1, \ldots, x_n$ that maximizes $\sum v_i x_i$ subject to $\sum w_i x_i \leq W$ and $x_i \in \{0, 1\}$.

Fractional Knapsack

Given a maximum weight W and a set of n items with **values** $v_1, \ldots, v_n$ and **weights** $w_1, \ldots, w_n$, find an **assignment** $x_1, \ldots, x_n$ that maximizes $\sum v_i x_i$ subject to $\sum w_i x_i \leq W$ and $0 \leq x_i \leq 1$.

Example: Two Knapsack Problems

Greedy choice

Idea:

- sort items **decreasingly** according to $\frac{v_i}{w_i}$ ratio
- choose items in that order until knapsack is **full**

Example: Two Knapsack Problems

Prove that this yields an optimal solution.

General method: Exchange Argument

- Let A be the choices made by the greedy algorithm.
- Let O be an optimal solution.
- **Goal:** Assuming A and O are 'not equal', modify O to create O' such that
  1. O' is at least as good as O, and
  2. O' is 'more like' A.

Tip: One good way to do the last bit is to assume O is an optimal solution which **'follows A the longest'**, that is has the **longest common prefix with A**.

Look at the first point at which O differs from A and exchange some (further) element to get O' which agrees with A at that point as well.

Example: Two Knapsack Problems

Prove that this yields an optimal solution.

Proof Sketch

- Let $A = \{x_1, \ldots, x_n\}$ be the choices made by the greedy algorithm.
- Let $O = \{x'_1, \ldots, x'_n\}$ be an optimal solution (which agrees with A the longest, i.e. shares the longest prefix).
- If $A = O$ we are done.
- Let $i \in [n-1]$ be the **smallest index** such that:

$$x_j = x'_j \qquad \text{for all } j < i \qquad \text{and} \qquad x_i \neq x'_i$$

- $A = \{x_1, \ldots, x_{i-1}, x_i, \ldots, x_n\}$
- $O = \{x_1, \ldots, x_{i-1}, x'_i, \ldots, x'_n\}$

Example: Two Knapsack Problems

Proof Sketch (cont.)

- Because of the greedy choice:
  - $x_i > x'_i$
  - $\frac{v_i}{w_i} \geq \frac{v_j}{w_j}$ for all $j > i$
- How to make O 'closer' to A:
  - We want to **replace** x'_i by x_i in O
  - $(x_i - x'_i)w_i$ weight has to be available:
    - **Equal Weights**: All objects have the same weight
    - **Fractional**: Fractional assignments allow removing the exact weight
- O' has equal or better value
- O' is closer to A
- **Contradiction!**

Example: Two Knapsack Problems

Implement the algorithm efficiently.

1. **Sort** the items according to value to weight ratio.
2. Iterate over the items in this order.
3. While there is space, add the items to the knapsack.

This takes time $O(N \log N)$.

General Knapsack Problem

- Knapsack is NP-Complete, so greedy doesn't work! **Right?**
- If a greedy algorithm doesn't skip any item $\rightarrow$ Greedy is optimal!

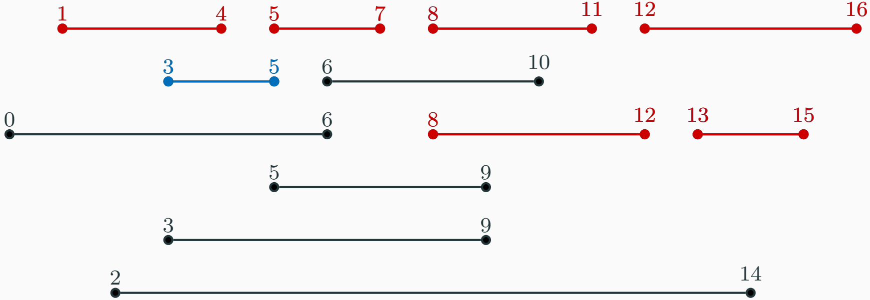
Warning

- Greedy **usually works** for many instances of most problems
- Easy to convince yourself greedy works
- Greedy **rarely** works

Example: Interval scheduling

- Your CPU needs to execute N jobs described by time intervals $[s_i, f_i]$.
- Job i starts at time s_i and ends at time f_i.
- Two jobs are **compatible** if their intervals are disjoint.
- **Goal:** find the maximum number of mutually compatible jobs.

Example: Interval Scheduling



$$A = \{[3,5],[8,11],[13,15]\}$$

Optimal:

$B = \{[1,4],[5,7],[8,11],[12,16]\}$ also $C = \{[1,4],[5,7],[8,12],[13,15]\}$

Example: Interval scheduling

Modelling done for us in the problem description—find the maximum set of compatible jobs.

Example: Interval scheduling

Greedy choice: decide how to choose the job i_k given already chosen jobs $i_1, \ldots, i_{k-1}$.

Natural candidates:

- **Earliest start time** – among compatible jobs, take the one with smallest s_k.
- **Earliest finish time** – among compatible jobs, take the one with smallest f_k.
- **Shortest length** – among compatible jobs, take the one with smallest $f_k - s_k$.
- **Fewest conflicts** – among compatible jobs, take the one which conflicts with the least amount of other compatible jobs.

Example: Interval scheduling

Earliest start time

Earliest finish time

Shortest length

Fewest conflicts

Which one do you think will work?

Example: Interval scheduling

Earliest start time

WRONG

Example: Interval scheduling

Shortest length

WRONG

Example: Interval scheduling

Fewest conflicts

WRONG

Example: Interval scheduling

Earliest finish time

Maybe???

Example: Interval scheduling

Prove that earliest finish time is correct.

General method: Staying Ahead

- Let $A = \{i_1, \ldots, i_n\}$ be the jobs chosen according to earliest finish time.
- Let $O = \{j_1, \ldots, j_m\}$ be an optimal solution (sorted by finish time).
- If $|A| = |O|$ we are done.
- **Goal:** Show that for all $k \leq n$ we have $f_{i_k} \leq f_{j_k}$ (that is, 'stays ahead').

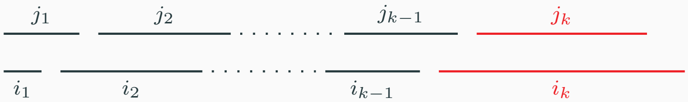
Example: Interval scheduling

Prove that earliest finish time is correct.

Proof Sketch

- **Goal:** Show that for all $k \leq n$ we have $f_{i_k} \leq f_{j_k}$ (that is, 'stays ahead').
- Proof by induction on k.
- **Base case,** $k = 1$: **Clearly holds**!
- Let $k > 1$ and assume it holds for $k - 1$ (i.e. $f_{i_{k-1}} \leq f_{j_{k-1}}$).
- Could it happen that $f_{i_k} > f_{j_k}$? **NO!**

  WHY?! $f_{i_{k-1}} \leq f_{j_{k-1}}$ and j_k is **compatible** with j_{k-1}, thus with i_{k-1} as well. The greedy algorithm would select j_k instead of i_k.



Example: Interval scheduling

Prove that earliest finish time is correct.

Proof Sketch (cont.)

- **Goal:** Show that for all $k \leq n$ we have $f_{i_k} \leq f_{j_k}$ (that is, 'stays ahead').
- For all $k \leq n$, we have $f_{i_k} \leq f_{j_k}$.
- Since $m > n$, there is j_{n+1} in O with:

$$s_{j_{n+1}} > f_{j_n} \qquad \text{and thus} \qquad s_{j_{n+1}} > f_{i_n}.$$

- Therefore, j_{n+1} is **compatible** with $i_1, \ldots, i_n$, but **does not** belong to A.

Contradiction! $\square$

Example: Interval scheduling

Implement the algorithm efficiently.

1. **Sort** the jobs according to increasing finish time.
2. Iterate over the jobs in this order.
3. For each job with interval $[s_i, f_i]$, add the job if s_i is greater than the finish time of the last job that was added.

This takes time $O(N \log N)$.

Example: Checking Change

ATM has bills with values $1, 10$, and 25 and is supposed to give you 42. What is the minimum number of bills used?

Greedy choice

$$1 \times 25 + 1 \times 10 + 7 \times 1 = 42$$

Bills used: **9**.

Optimal

$$4 \times 10 + 2 \times 1 = 42$$

Bills used: **6**.

Conclusion:

- Some (but not all!) problems can be solved with a greedy approach.
- Deciding how to make the greedy choice can be non-obvious.
- We can check whether the greedy solution works using an **exchange argument** or a **staying ahead** argument.
- Disproving greedy solutions tends to be easy
- Convincing yourself greedy works is easy (**even when it doesn't!**)

Split & List [1]

[1] This is an advanced technique intended for students aiming for top grades. If it appears in the exam, it will only be necessary for the last 20 or 40 points of an exercise.

Brute Force

Brute force: some problems are **hard** and we only know how to solve them by **trying everything**.

However, one can often do it a **little bit smarter**:

1. Heuristics (important in practice, not in AlgoLab)
2. **Improve worst case complexity :)**

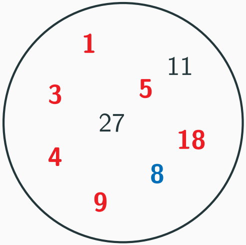
We will see a technique called **Split & List**.

This technique is why there is 'DES' and 'triple-DES' but no 'double-DES'...

Example: Subset Sum

Given a set $S \subseteq \mathbb{N}$, is there a subset $S' \subseteq S$ such that $\sum_{s \in S'} s = k$?

- $S = \{1, 3, 4, 5, 8, 9, 11, 18, 27\}$
- $k = 8$? **YES!** $S' = \{1, 3, 4\}$ or $S' = \{8\}$
- $k = 1000$? **NO!**
- $k = 37$? **YES!** $S' = \{1, 4, 5, 9, 18\}$



NP-Complete :(

n is small: brute force n **is small: brute force**

Check **all subsets**!

Recursive/Iterative algorithm

k is small: DP k **is small: DP**

EXERCISE!

Subset Sum — Recursive

Example: Subset Sum

Given a set $S = \{s_1, \ldots, s_n\} \subseteq \mathbb{N}$, is there a subset $S' \subseteq S$ such that $\sum_{s \in S'} s = k$?

We want a recursive definition of $f(i, j) :=$ 'is there $S' \subseteq \{s_1, \ldots, s_i\}$ s.t. $\sum_{s \in S'} s = j$'.

- Base cases:

  $f(i, 0) =$ `true`, for all i, and

  $f(0, j) =$ `false`, for all $j > 0$.
- $f(i, j) = f(i-1, j-s_i) \vee f(i-1, j)$

Recursive algorithm:

```
bool f(int i, int j) {
    if (j == 0) return true;
    if ((i == 0 && j > 0) || j < 0) return false;
    return f(i - 1, j - elements[i]) || f(i - 1, j);
}
```

Time complexity: $O(2^n)$, ok for $n \approx 25$.

Subset Sum — Iterative

How can we iterate over all subsets of an n element set?

Trick: encode the set in an integer.

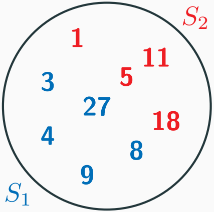
```
bool subsetsum(int k) {
  for (int s = 0; s < 1<<n; ++s) { // Iterate through all subsets
      int sum = 0;
      for (int i = 0; i < n; ++i) {
          if (s & 1<<i) sum += elements[i]; // If i-th element in subset
      }
      if (sum == k) return true;
  }
  return false;
}
```

Time complexity: $O(n \cdot 2^n)$, ok for $n \approx 25$.

Subset Sum — Faster? Split & List

Split S into $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$ of size $\approx \frac{n}{2}$.

List all subset sums of S_1 and S_2 into L_1 and L_2



Lemma: The following statements are equivalent:

- There is a $S' \subseteq S$ with $\sum_{s \in S'} s = k$
- There are $S'_1 \subseteq S_1$ and $S'_2 \subseteq S_2$ such that $\sum_{s \in S'_1} s + \sum_{s \in S'_2} s = k$

Idea: use second statement to check the first.

Algorithm sketch:

- Sort L_2
- For each k_1 in L_1 check if there is k_2 in L_2 (**binary search**!) such that $k_1 + k_2 = k$.

Time complexity: $O(n \cdot 2^{n/2})$, ok for $n \approx 50$. :)