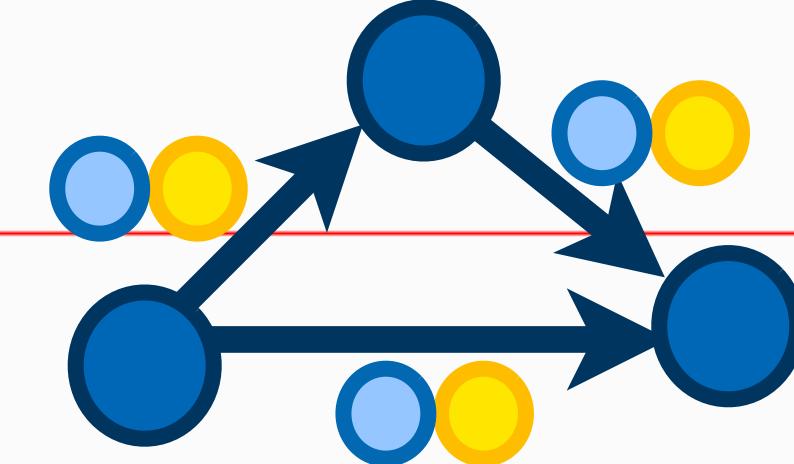


Minimum Cut, Bipartite Matching and Minimum Cost Maximum Flow with BGL

Min Cost Max Flow = MCMF



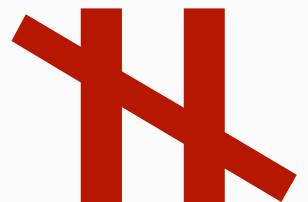
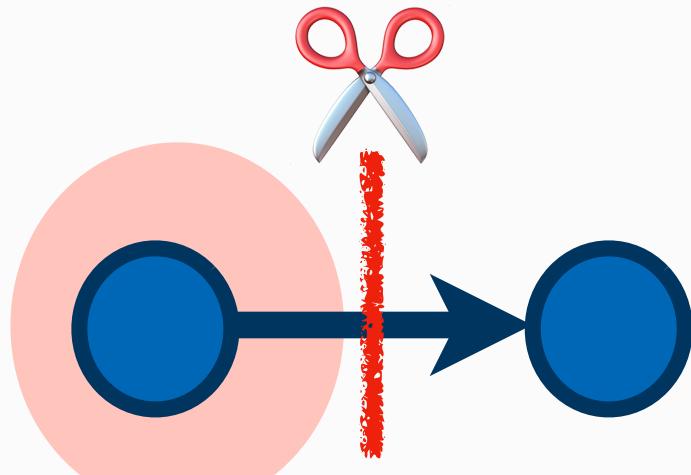
Pascal Su¹

November 13, 2019

ETH Zürich,

¹ based on material from Daniel Graf and Andreas Bärtschi

Max Flow Min Cut = MFMC



Recap: Basic Network Flows – What did we see last time?

One problem to rule them all...

Network Flow: Problem Statement

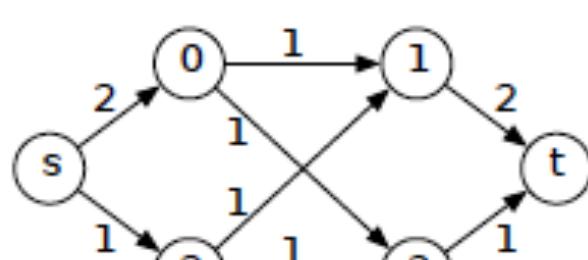
Input: A flow network consisting of

- directed graph $G = (V, E)$
- source $s \in V$ with only out edges
- sink $t \in V$ with only in edges
- edge capacity $c : E \rightarrow \mathbb{N}$.

Output: Flow function $f : E \rightarrow \mathbb{R}$ subject to

- Capacity constraints for every edge e .
 $0 \leq f(e) \leq c(e)$
- Flow conservation at every vertex u .
 $\sum_{(v,u) \in E} f((v,u)) \text{ in flow} = \sum_{(u,w) \in E} f((u,w)) \text{ out flow}$
- Maximization of total flow.
 $|f| := \sum_{(s,v) \in E} f((s,v))$

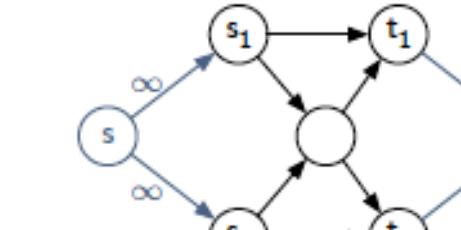
Theorem
On a graph with integral capacities, there always exists an integral maximum flow.



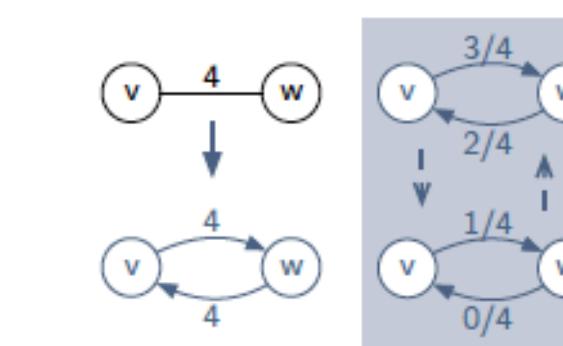
4 / 23

Common techniques

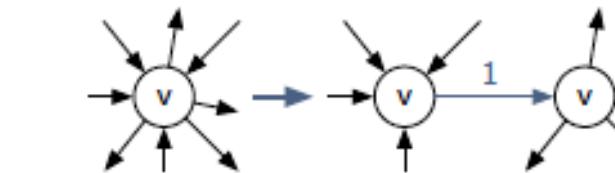
Multiple sources/sinks
with e.g. $\infty \approx \sum_{e \in E} c(e)$



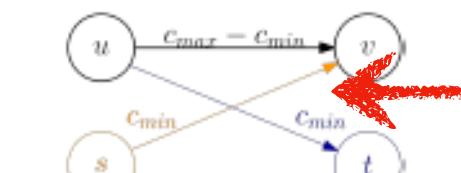
Undirected graphs
antiparallel edges with flow reducible to one direction



Vertex capacities
split into in-vertex and out-vertex



Minimum flow per edge
how to enforce $c_{\min}(e) \leq f(e) \leq c_{\max}(e)$?

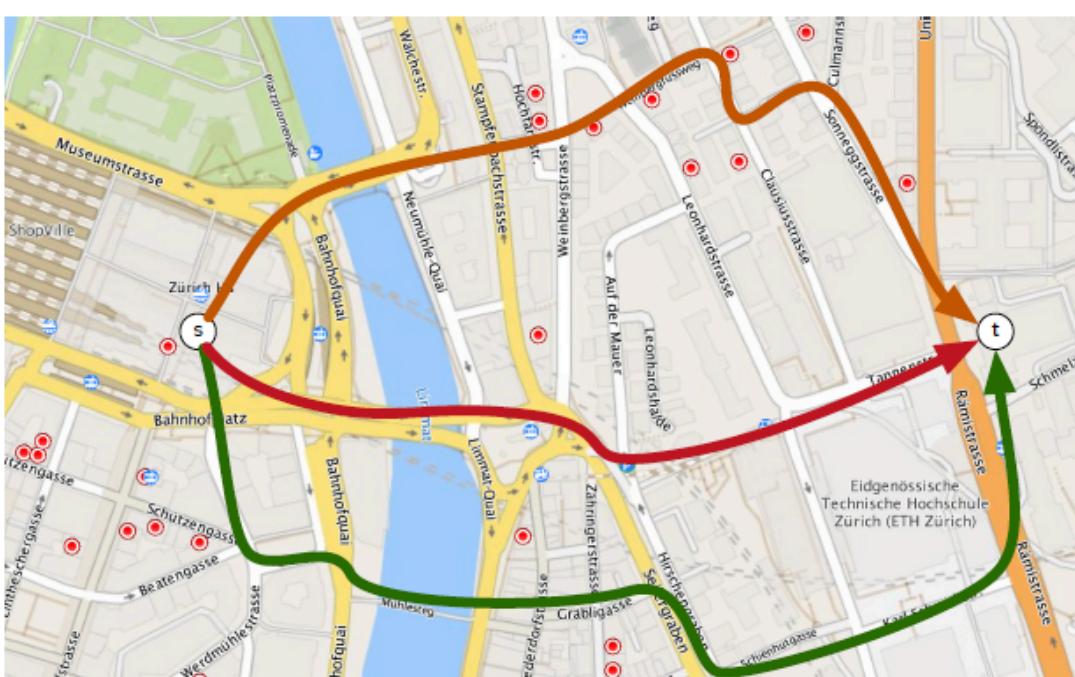


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see addition to slides
week 5

Flow Application: Edge Disjoint Paths

How many ways are there to get from HB to CAB without using the same street twice?



Map: search.ch, TomTom, swisstopo, OSM

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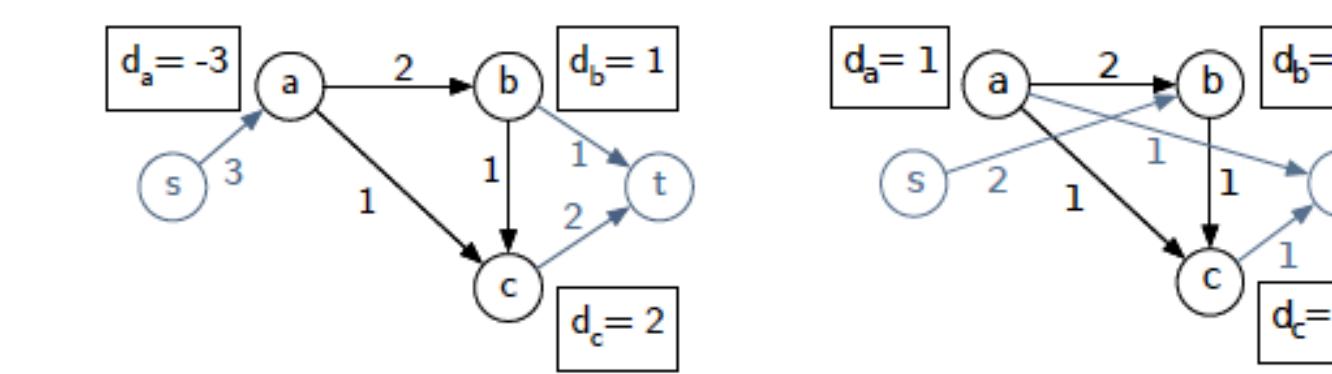
Flow Application: Circulation Problem

► Multiple sources with a certain amount of flow to give (supply).

► Multiple sinks that want a certain amount of flow (demand).

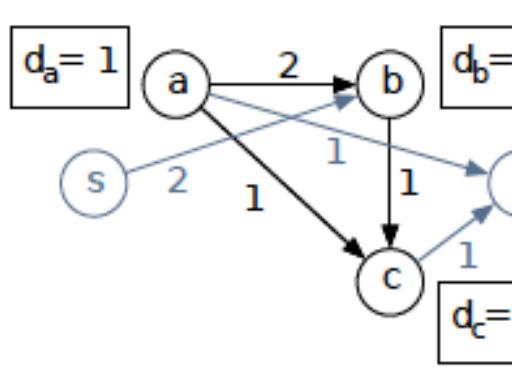
► Model these as negative or positive demand per vertex d_v .

► Question: Is there a feasible flow? Surely not if $\sum_{v \in V} d_v \neq 0$. Otherwise? Add super-source and super-sink to get a maximum flow problem.



$d_a = -3$, $d_b = 1$, $d_c = 2$, $d_s = 3$, $d_t = -2$

feasible flow exists



$d_a = 1$, $d_b = -2$, $d_c = 1$, $d_s = 2$, $d_t = 1$

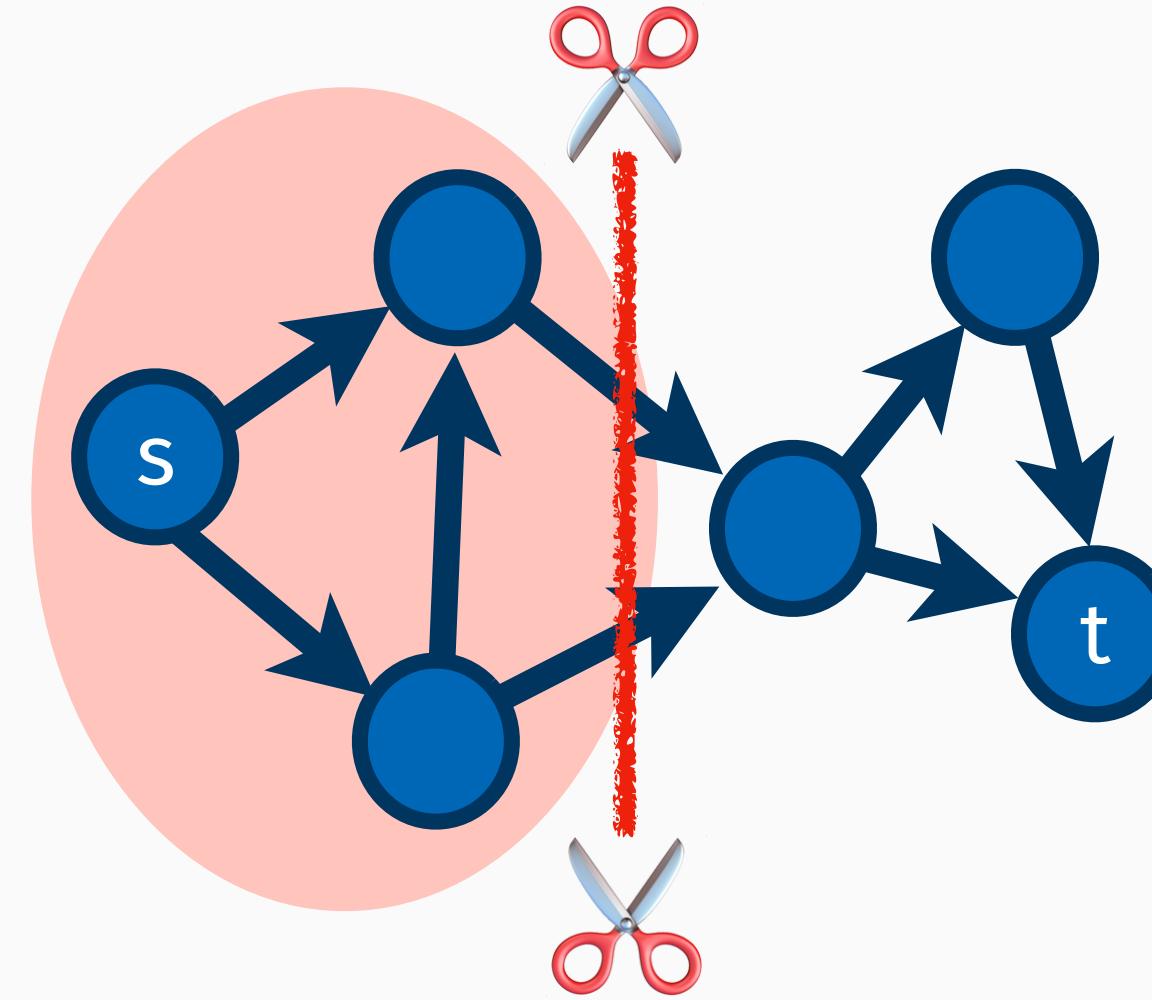
no feasible flow exists

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Today: Advanced Network Flow – What else are flows useful for?

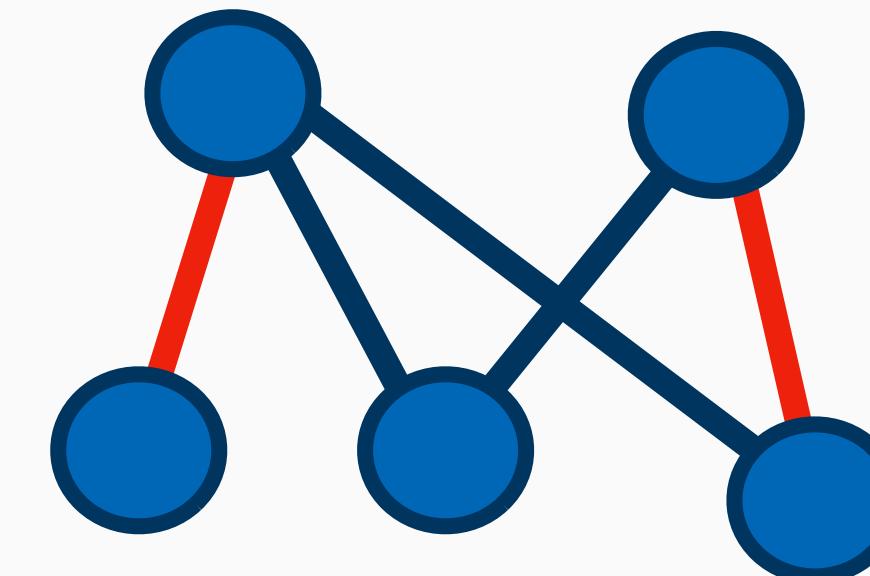
Minimum Cuts

- ▶ How to disconnect t from s cheaply?



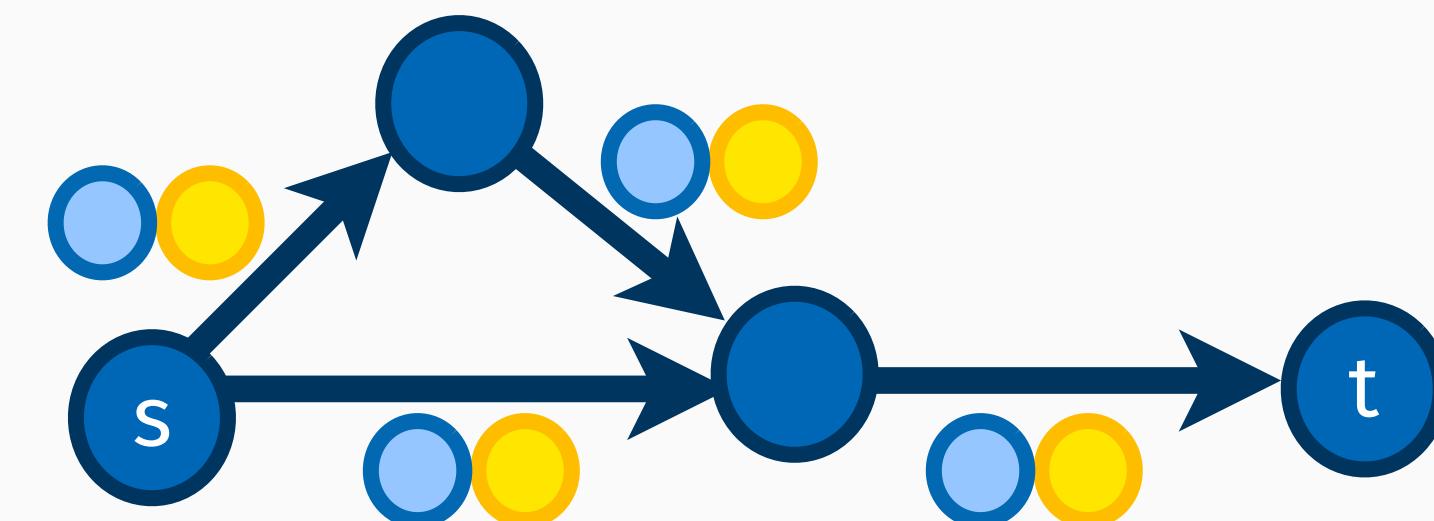
Bipartite Matching

- ▶ How to assign A's to B's effectively?



Flows with Costs

- ▶ What if sending flow comes with a price?

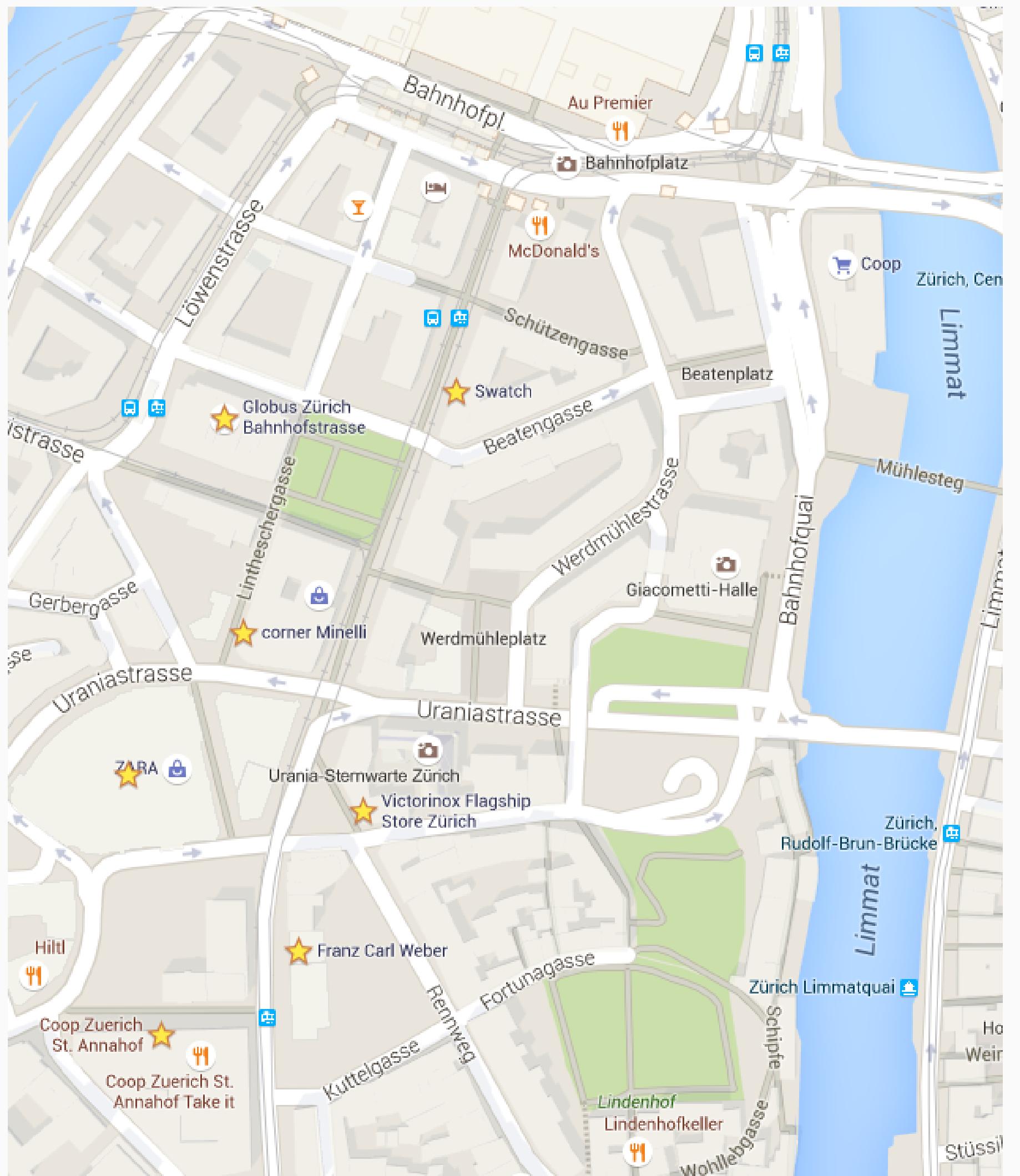


Minimum Cut

Minimum Cut: Shopping Trip



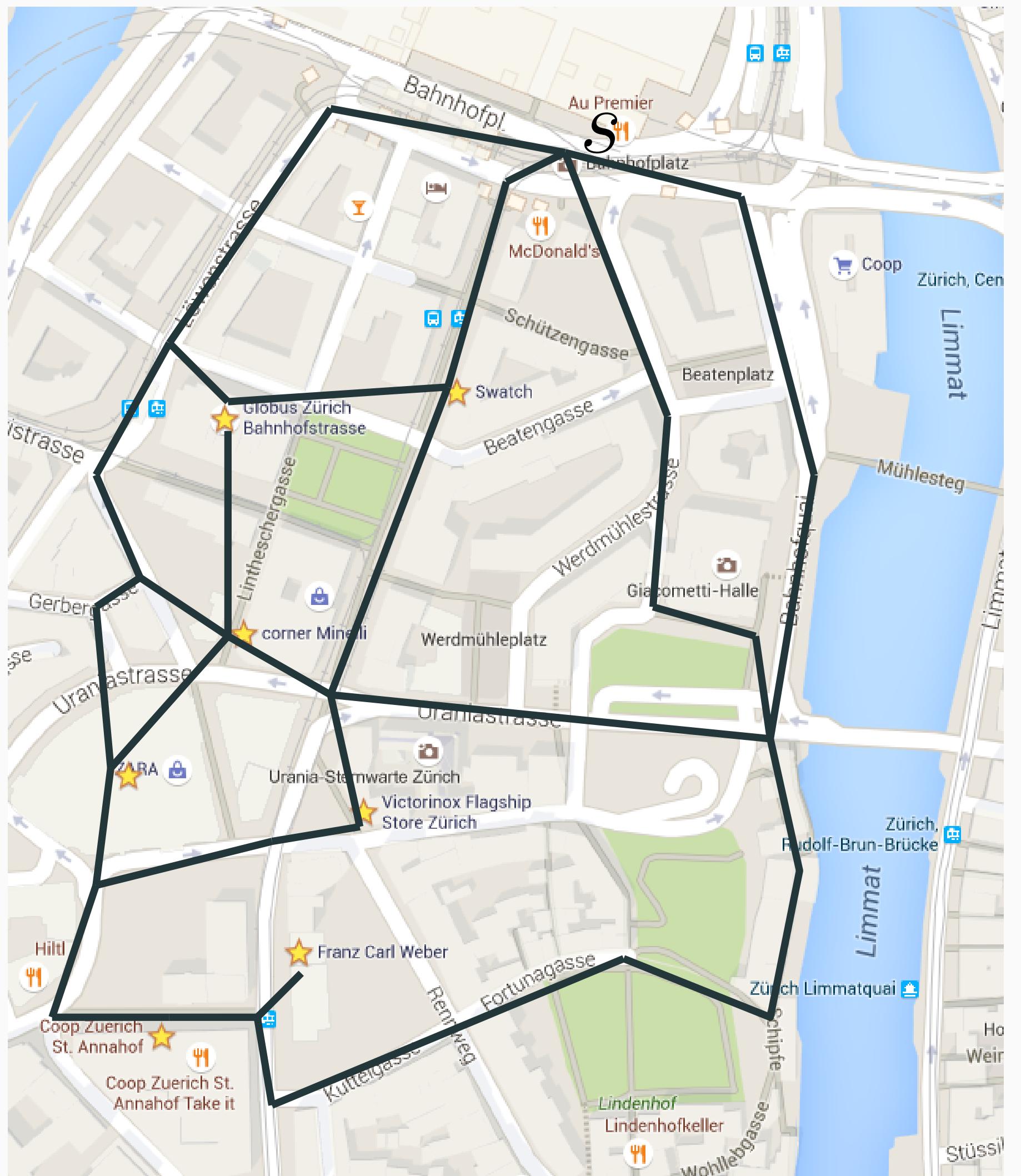
Minimum Cut: Shopping Trip



Start from HB:

- ▶ Visit as many shops as possible.
- ▶ Return to HB after each shop.

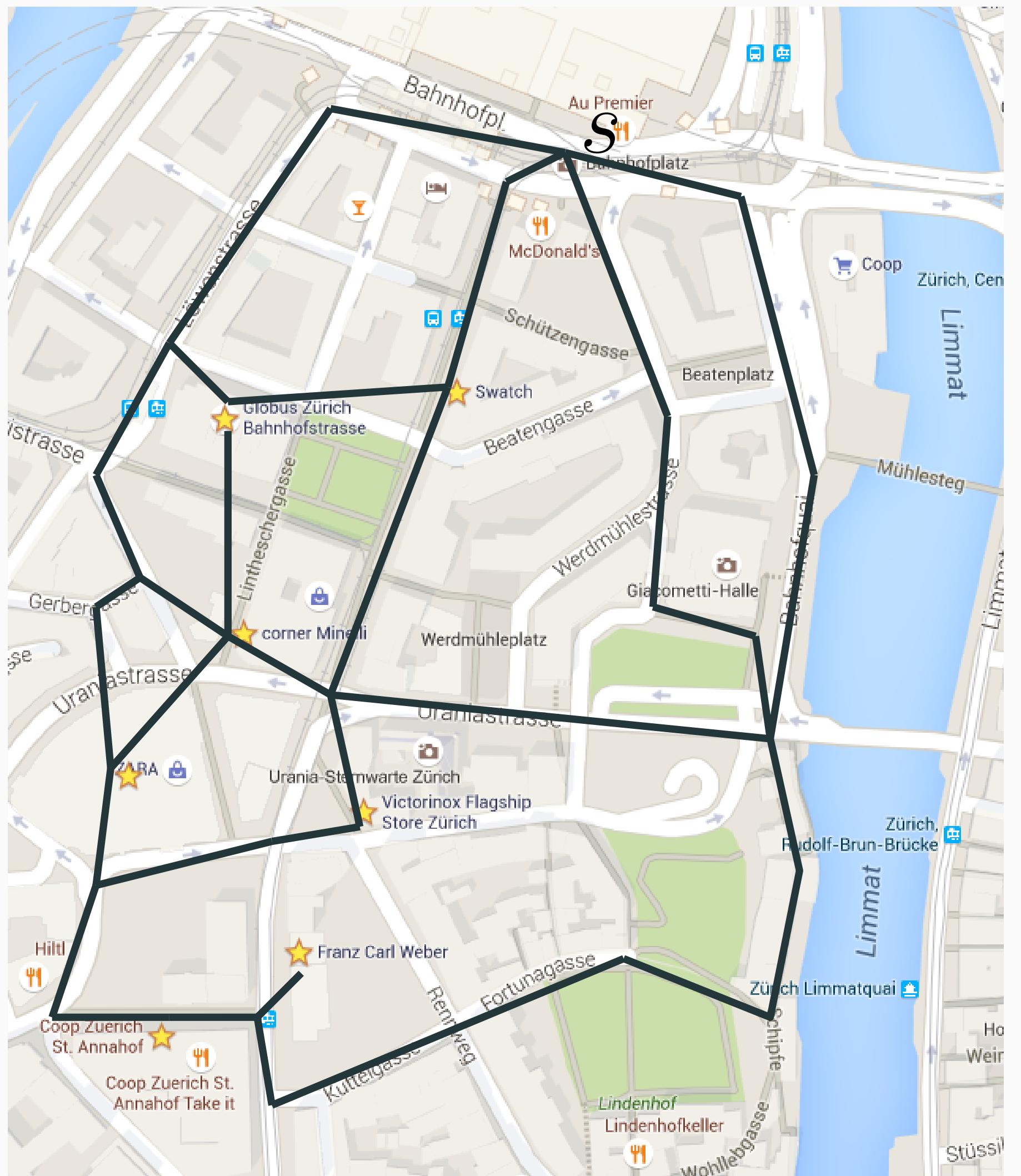
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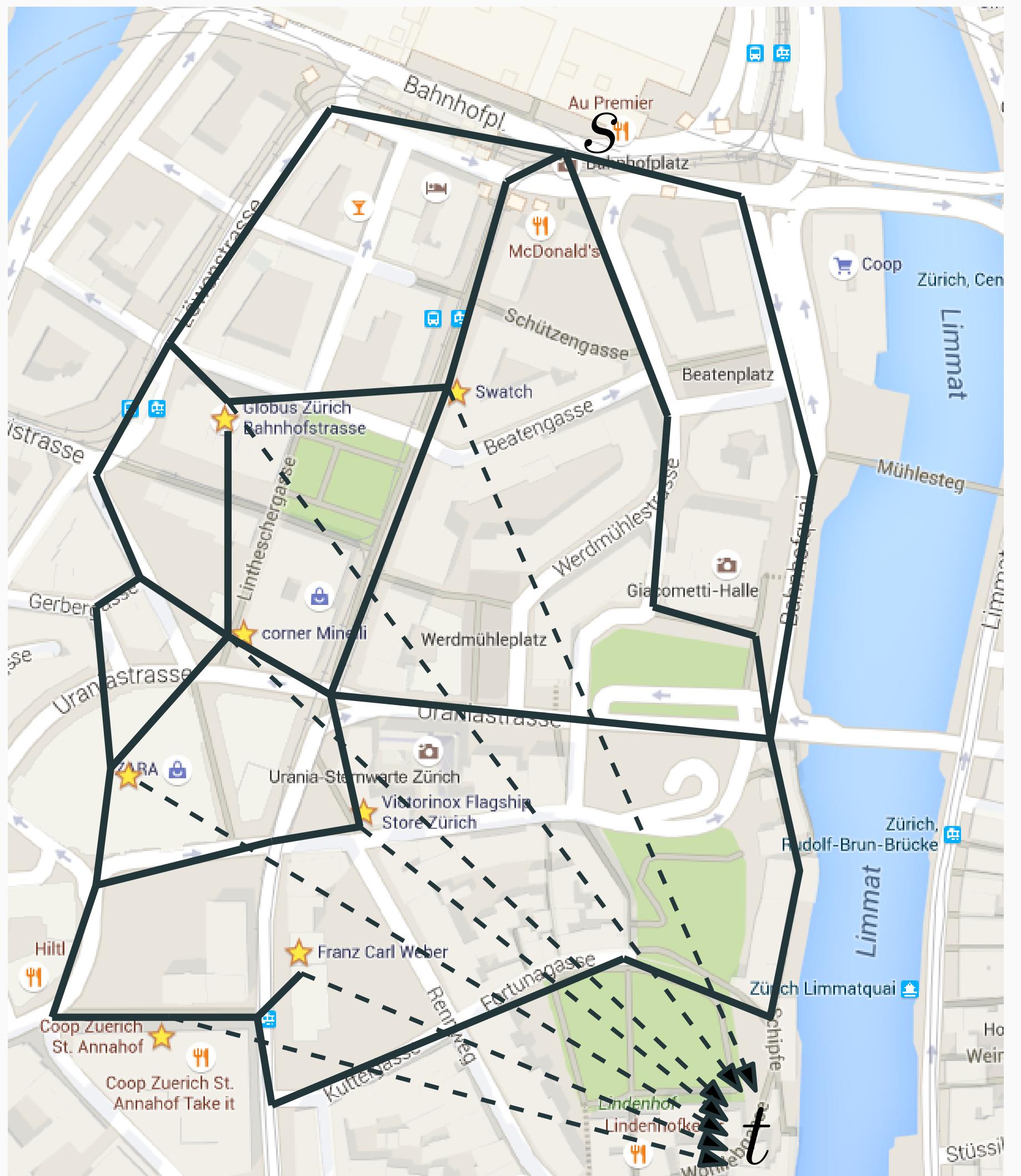


Start from HB:

- ▶ Visit as many shops as possible.
- ▶ Return to HB after each shop.

Condition: Use each road on at most one trip.

Minimum Cut: Shopping Trip

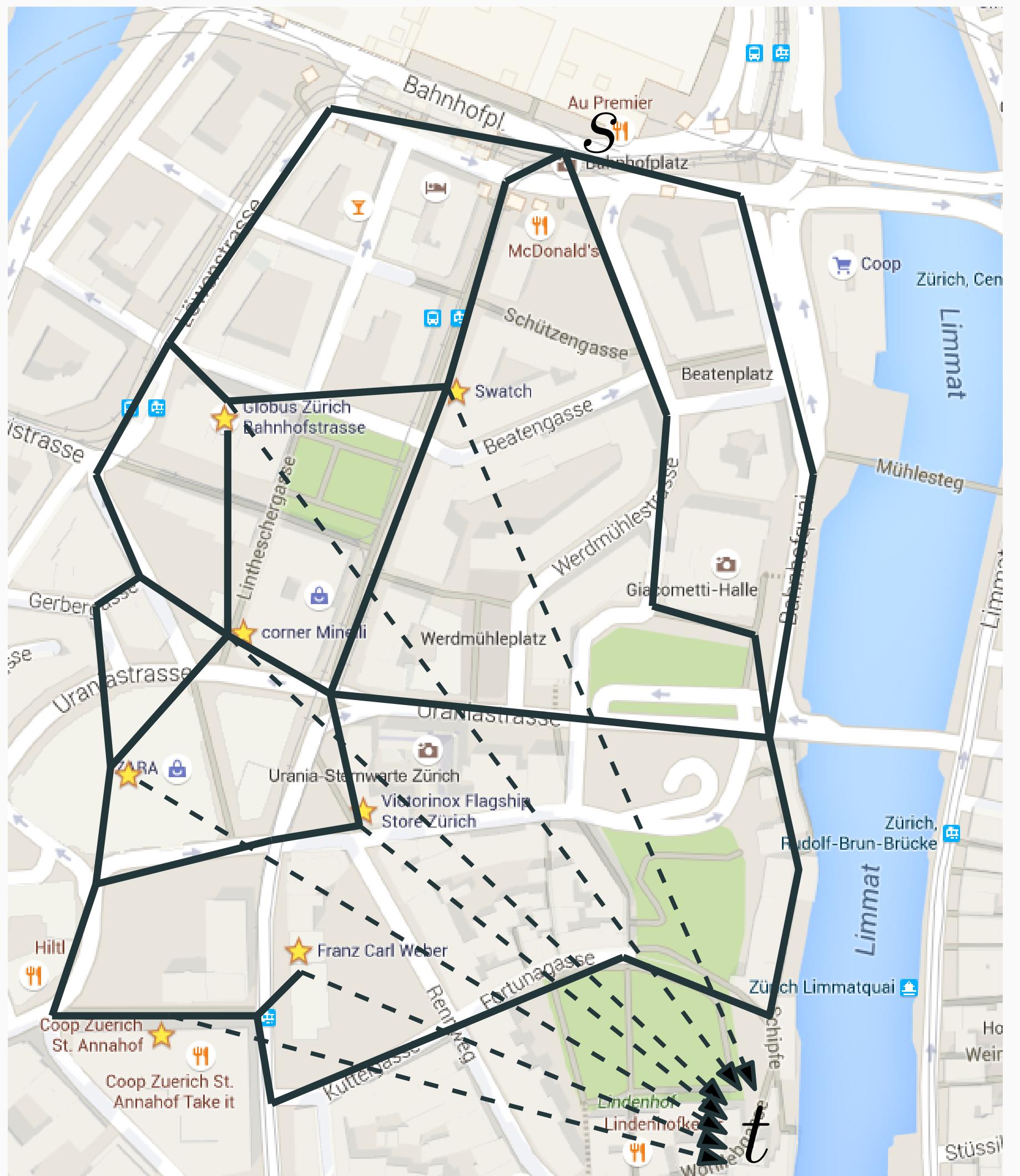


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Minimum Cut: Shopping Trip



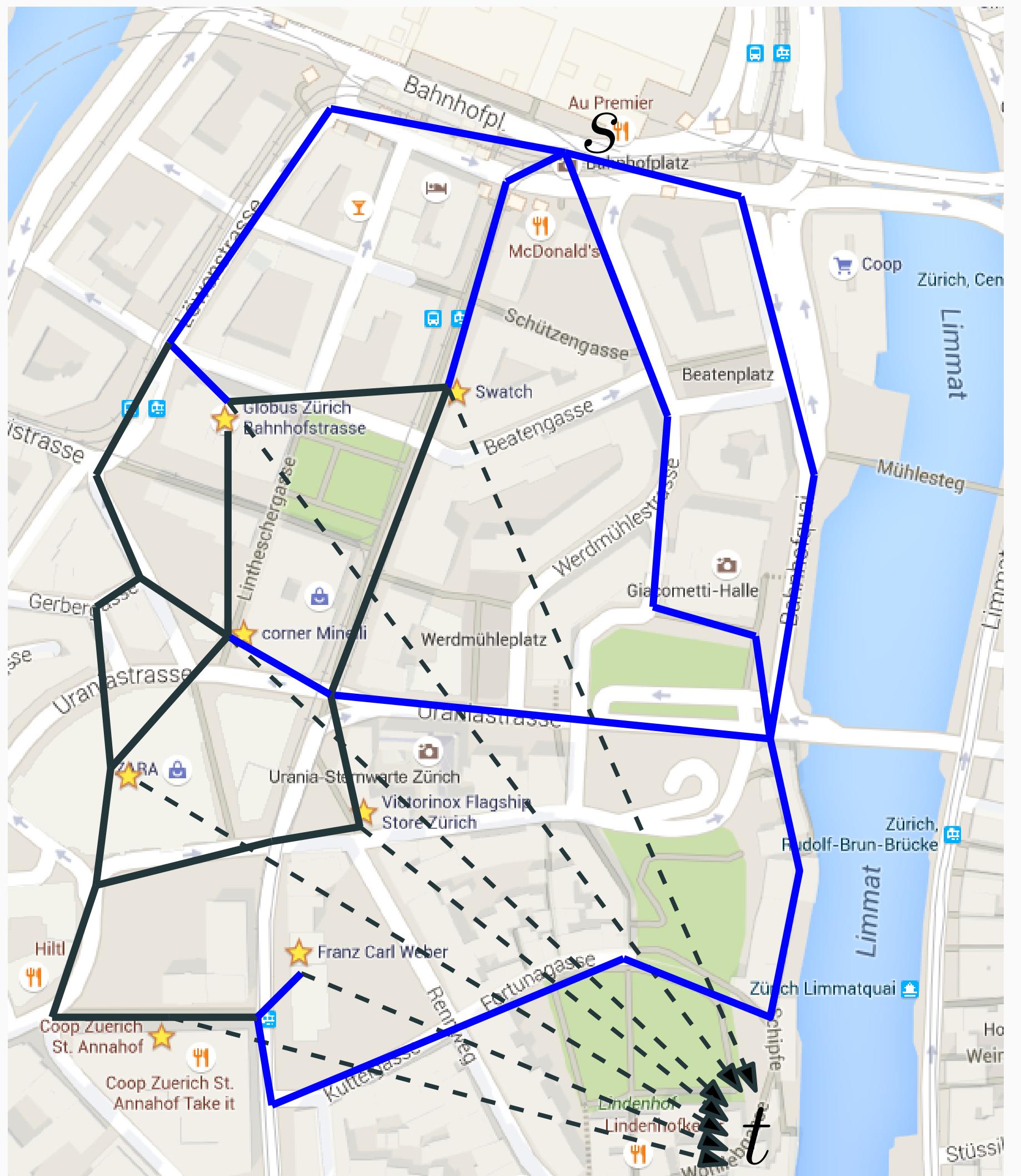
Start from HB:

- ▶ Visit as many shops as possible.
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Condition: Use each road on at most one trip.

Compute the bottleneck, i.e. the number of edge-disjoint paths.

Minimum Cut: Shopping Trip



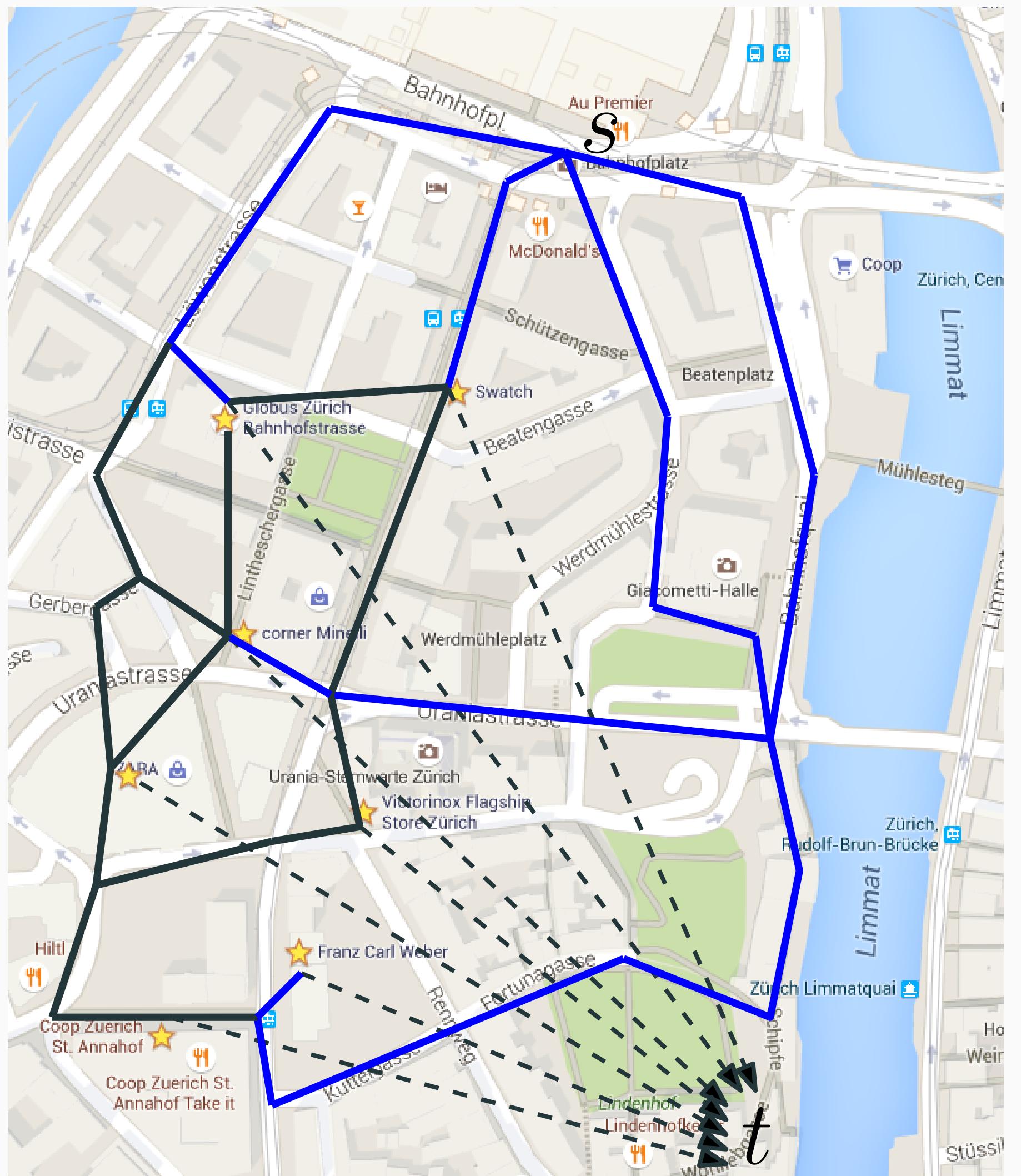
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Condition: Use each road on at most one trip.

Compute the bottleneck, i.e. the number of edge-disjoint paths. \Rightarrow Four shops.

Minimum Cut: Shopping Trip



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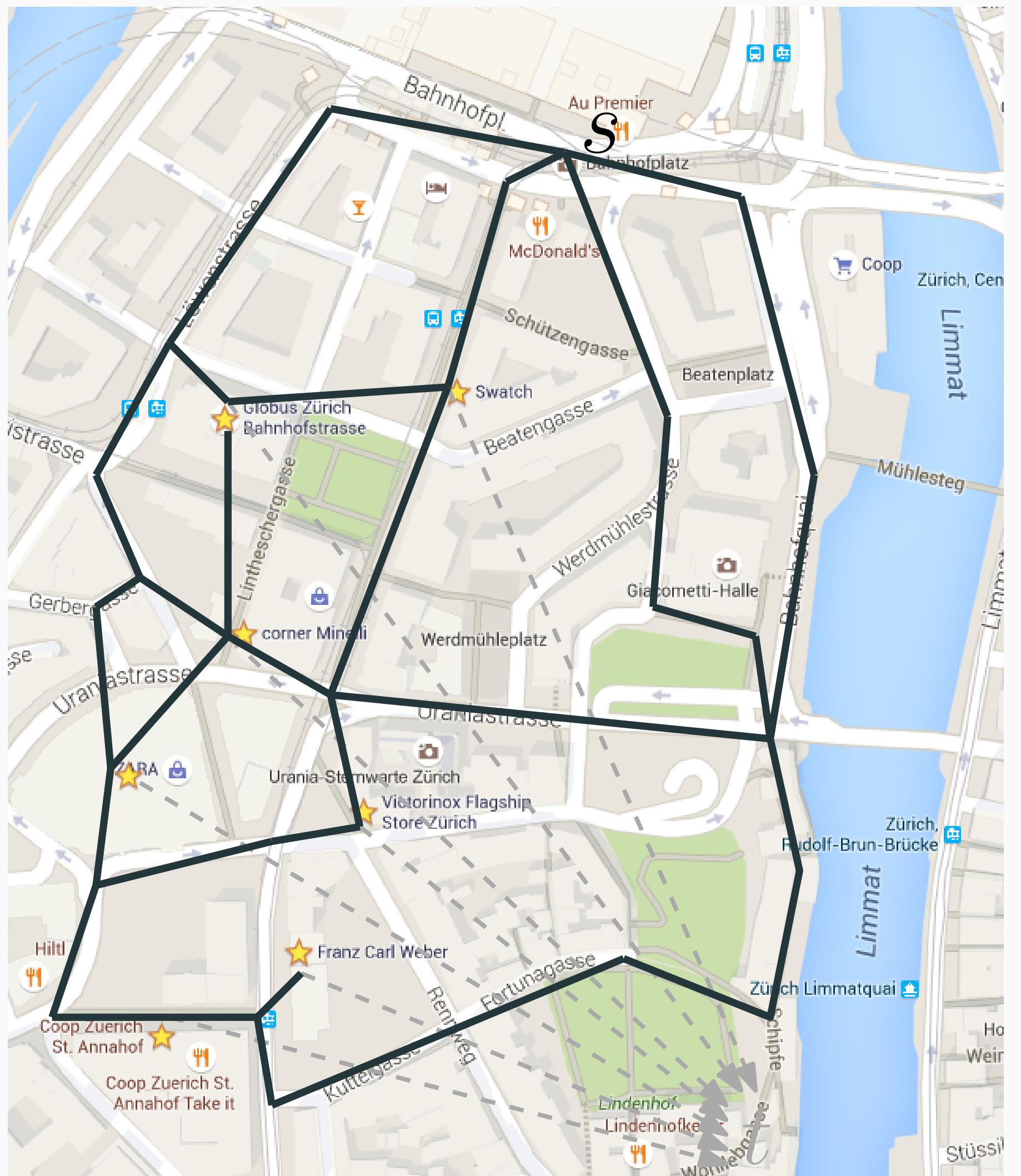
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Unrealistic condition!

(There are interesting streets in Zürich.)

Minimum Cut: Shopping Trip



Start from HB:

- ▶ Visit as many shops as possible.
- ▶ Return to HB after each shop.

Condition: Use beautiful roads more often.

Minimum Cut: Shopping Trip



Start from HB:

- ▶ Visit as many shops as possible.
- ▶ Return to HB after each shop.

Condition: Use beautiful roads more often.



Use route via Lindenhof at most twice.

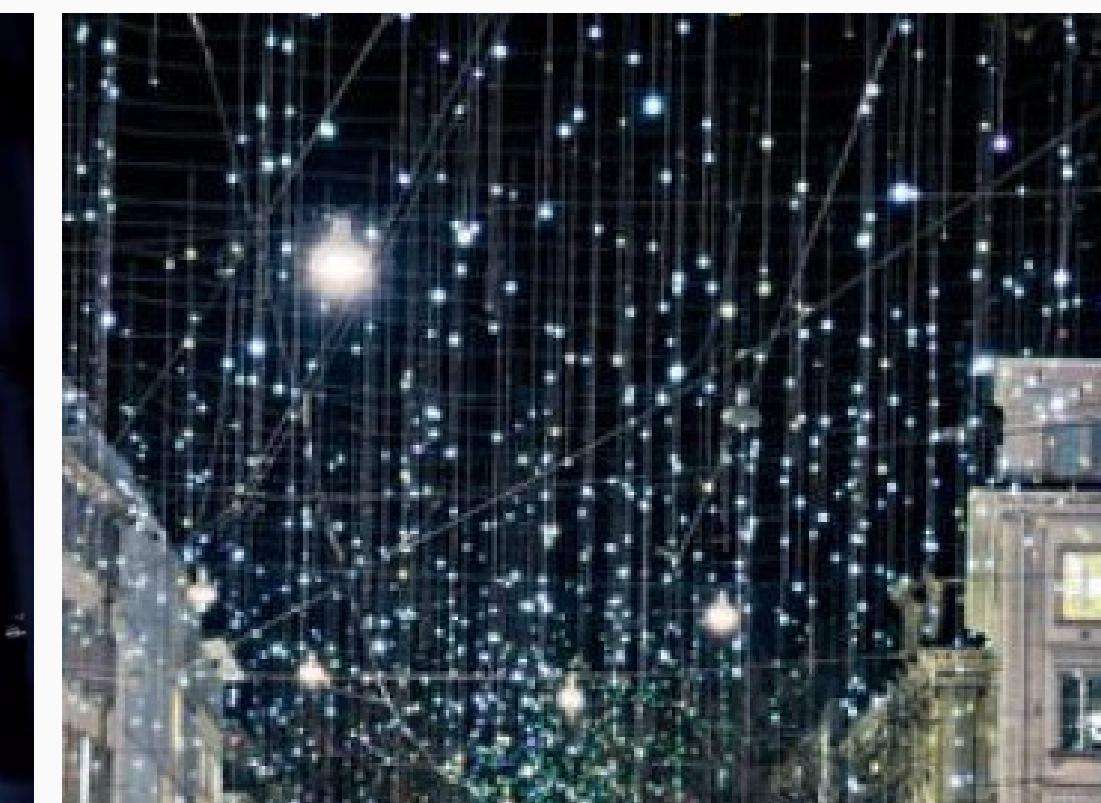
Minimum Cut: Shopping Trip



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Condition: Use beautiful roads more often.



Use Bahnhofstrasse up to three times

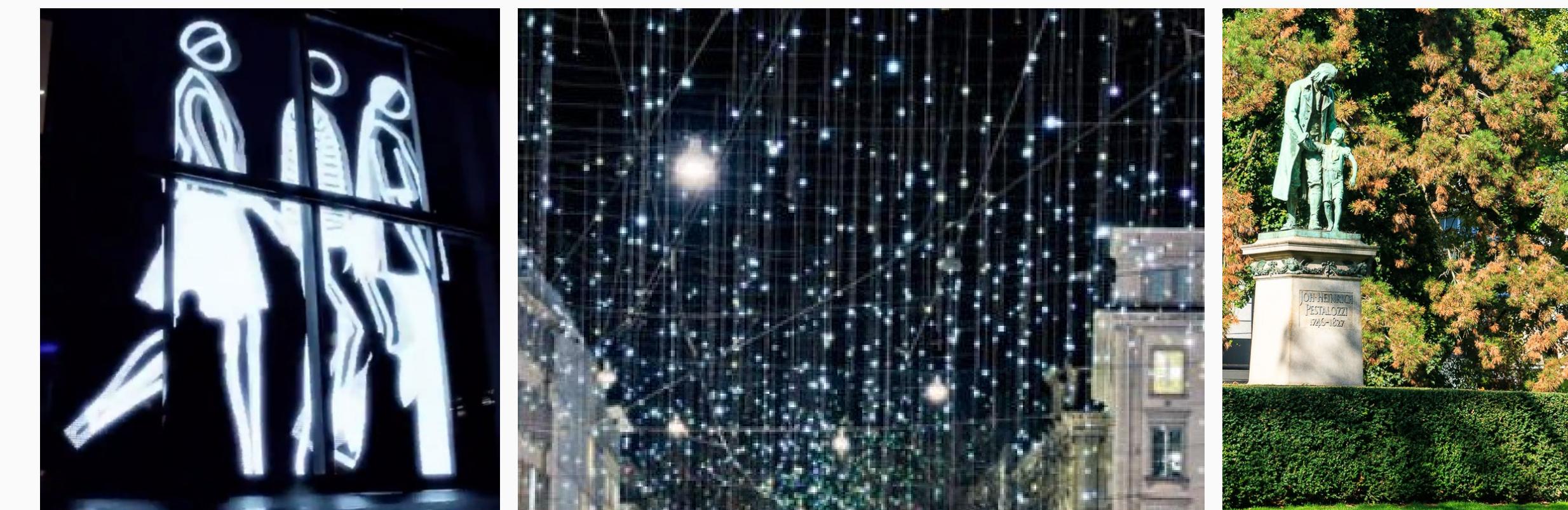
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Compute the weighted bottleneck, i.e. the minimum cut between s and t .

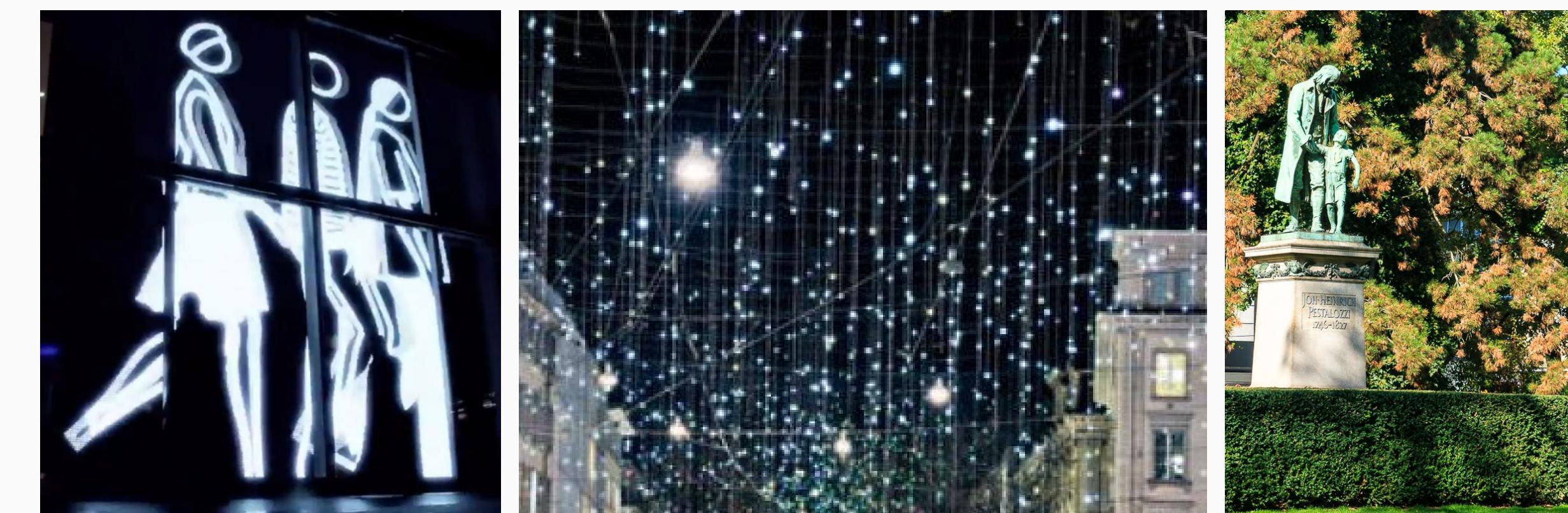
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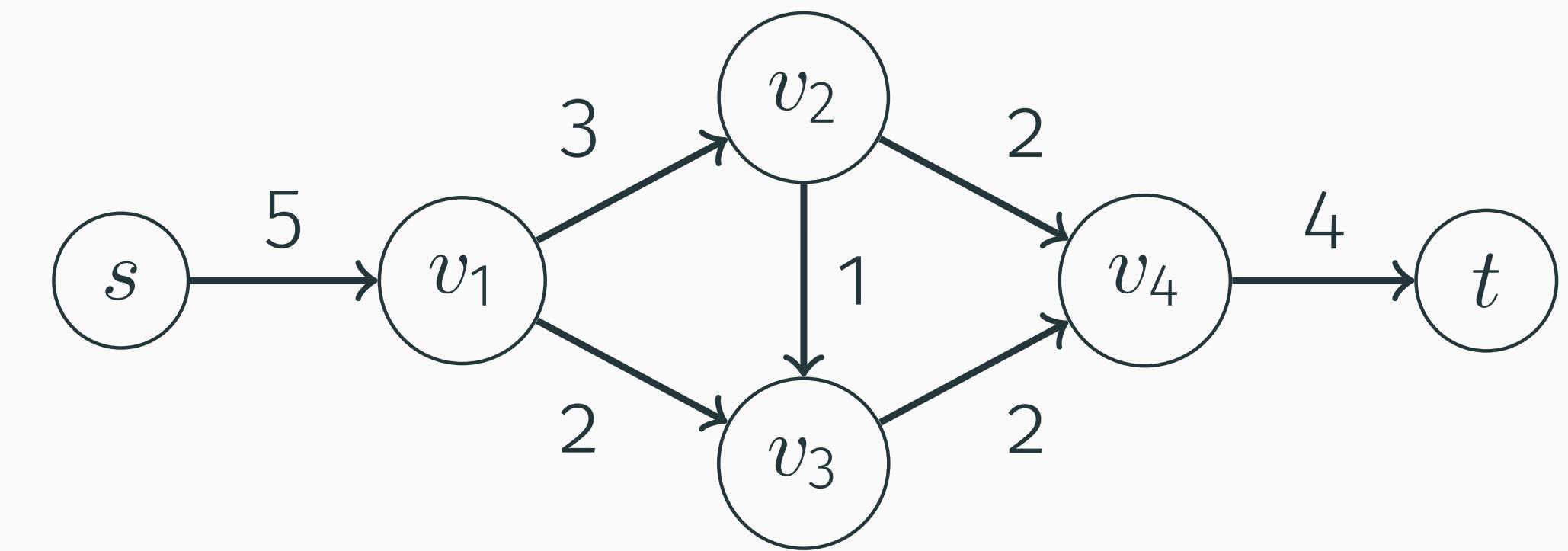


Use Bahnhofstrasse up to three times.

Compute the weighted bottleneck, i.e. the minimum cut between s and t . $\Rightarrow 6$ shops.

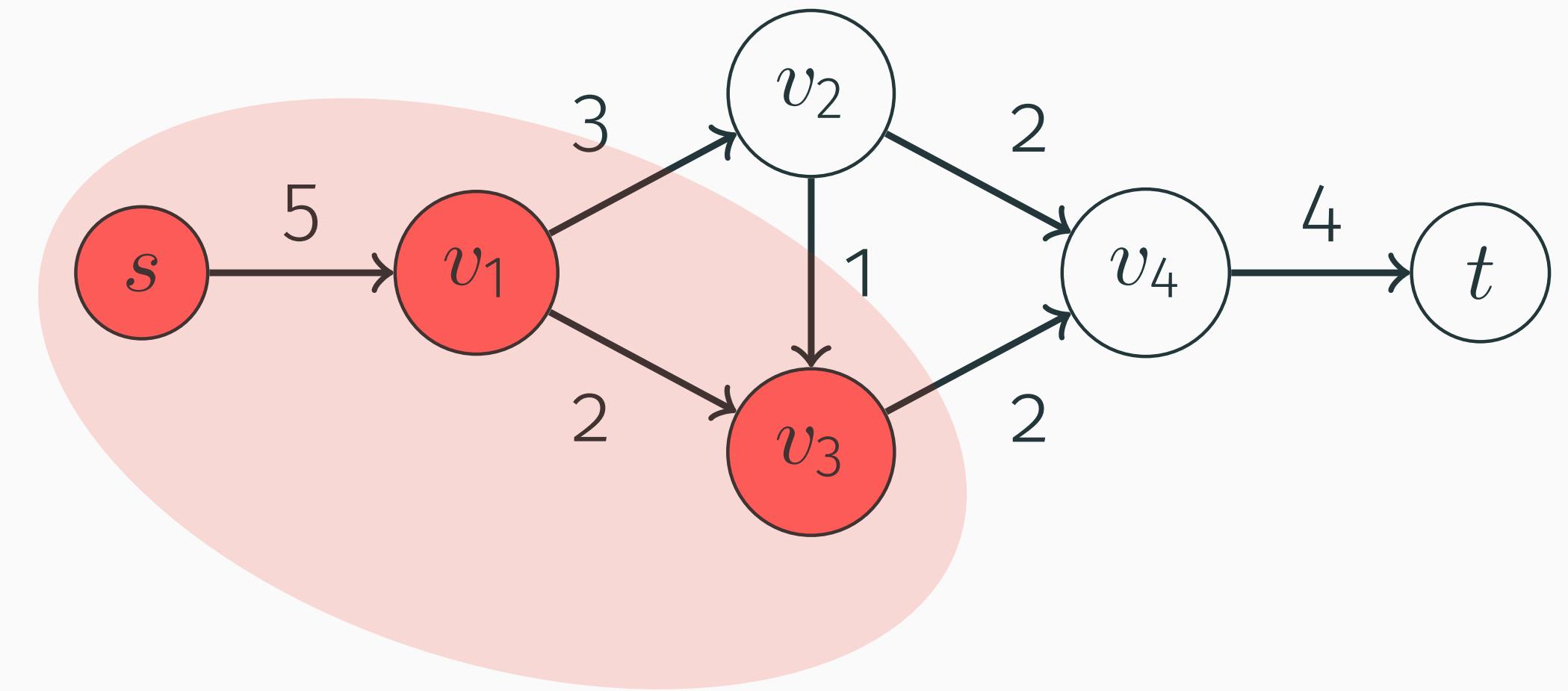
Minimum Cut: Cuts and Flows

$G = (V, E, s, t)$ a flow network. $S \subset V$ s.t. $s \in S, t \in V \setminus S$,



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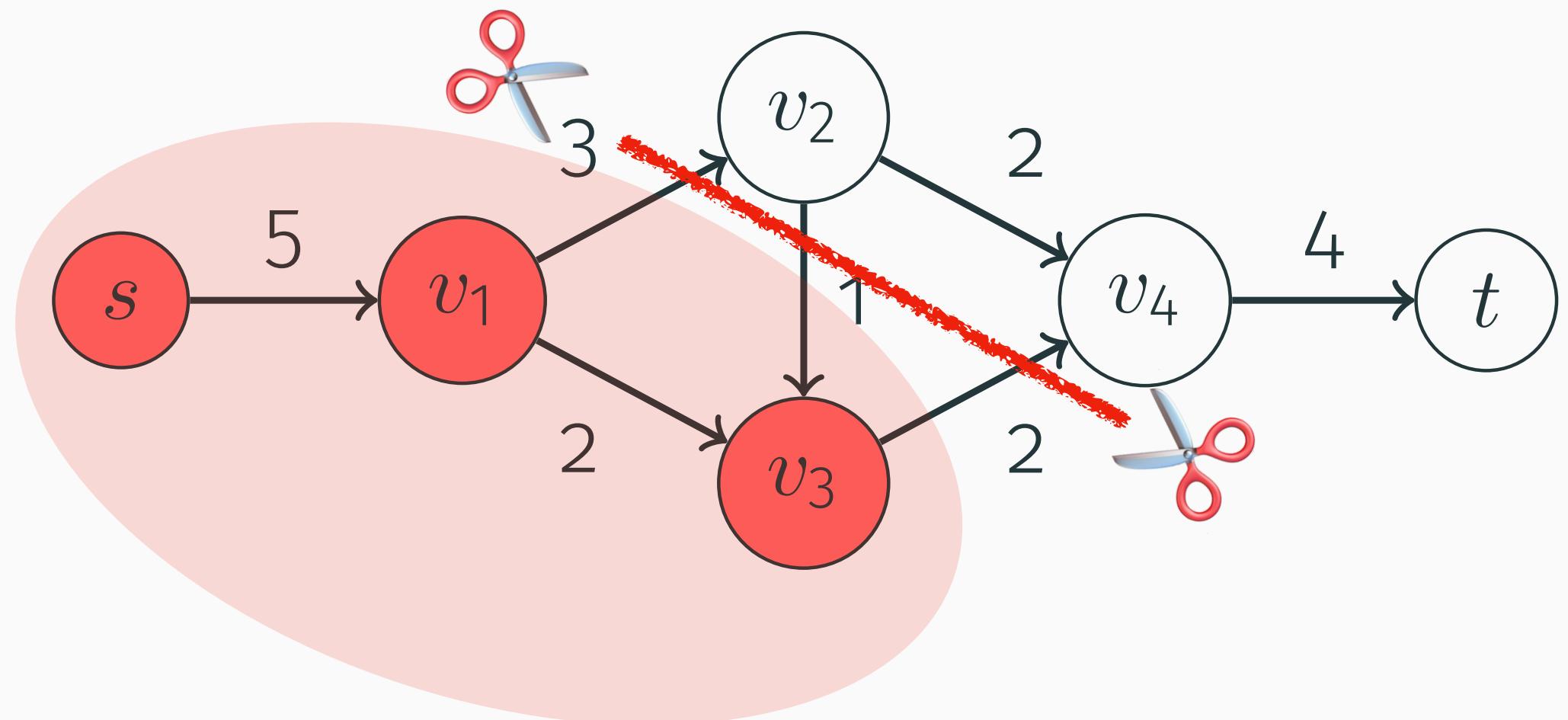


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The value of the $(S, V \setminus S)$ -cut is

$$\begin{aligned}\text{cap}(S, V \setminus S) &:= \text{outgoing capacity} \\ &= \sum_{\substack{e=(u,v) \\ u \in S, v \in V \setminus S}} \text{cap}(e)\end{aligned}$$

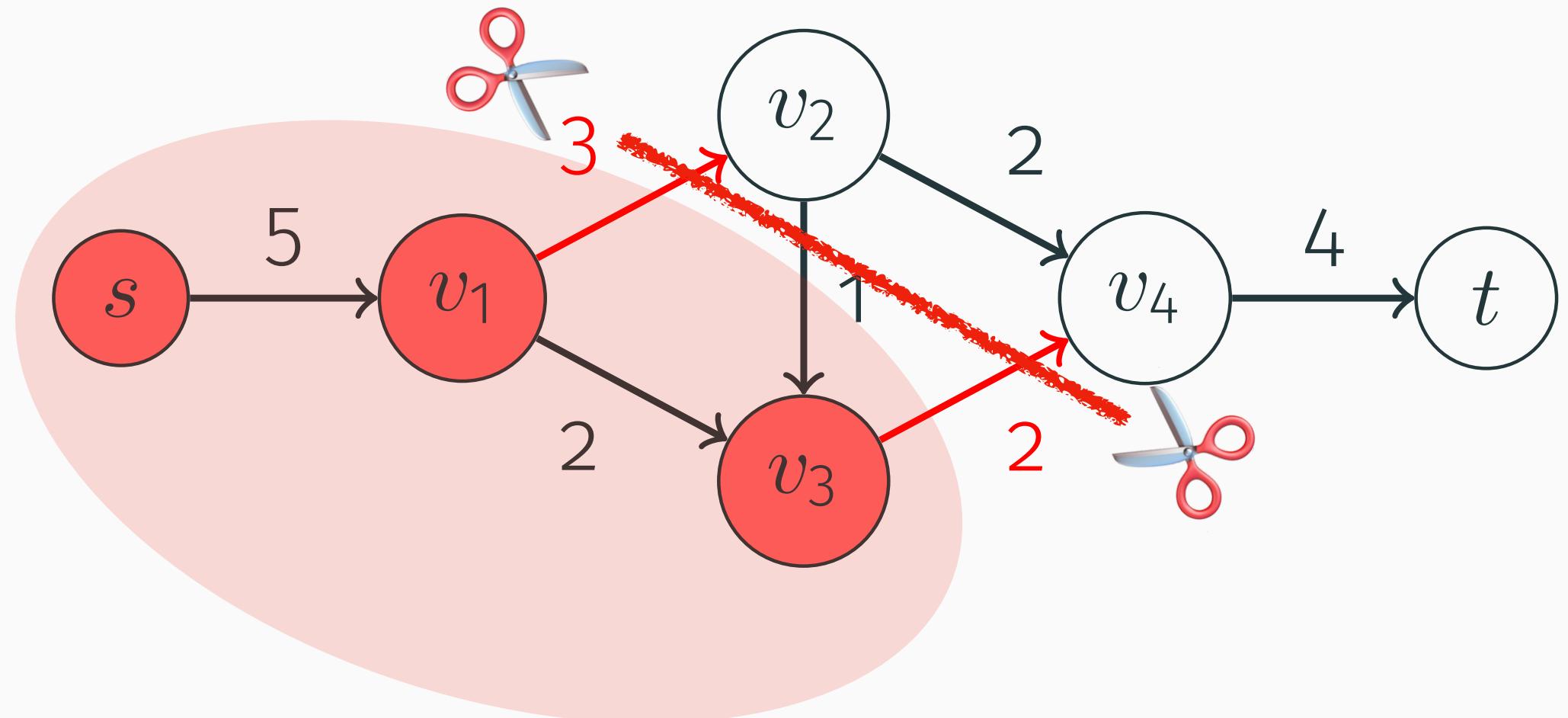


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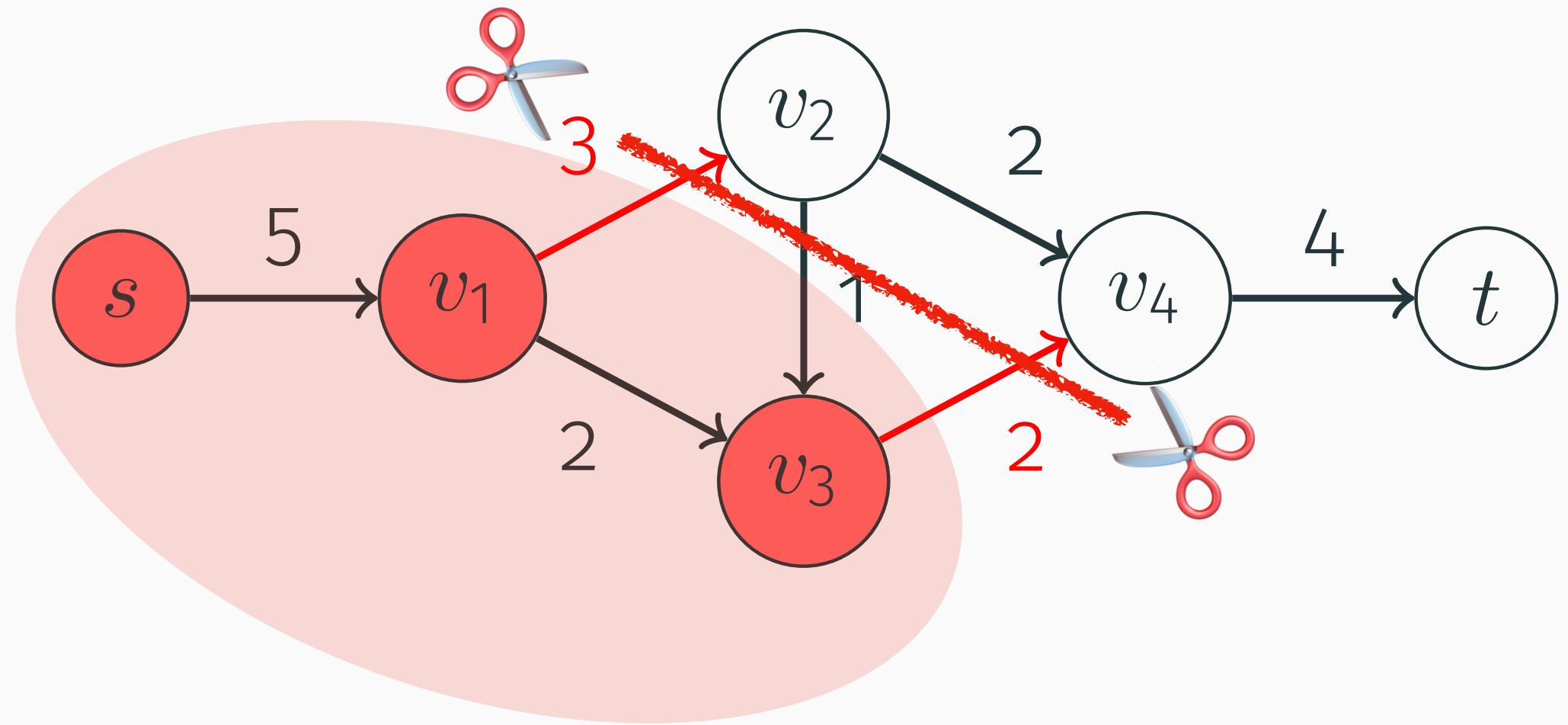


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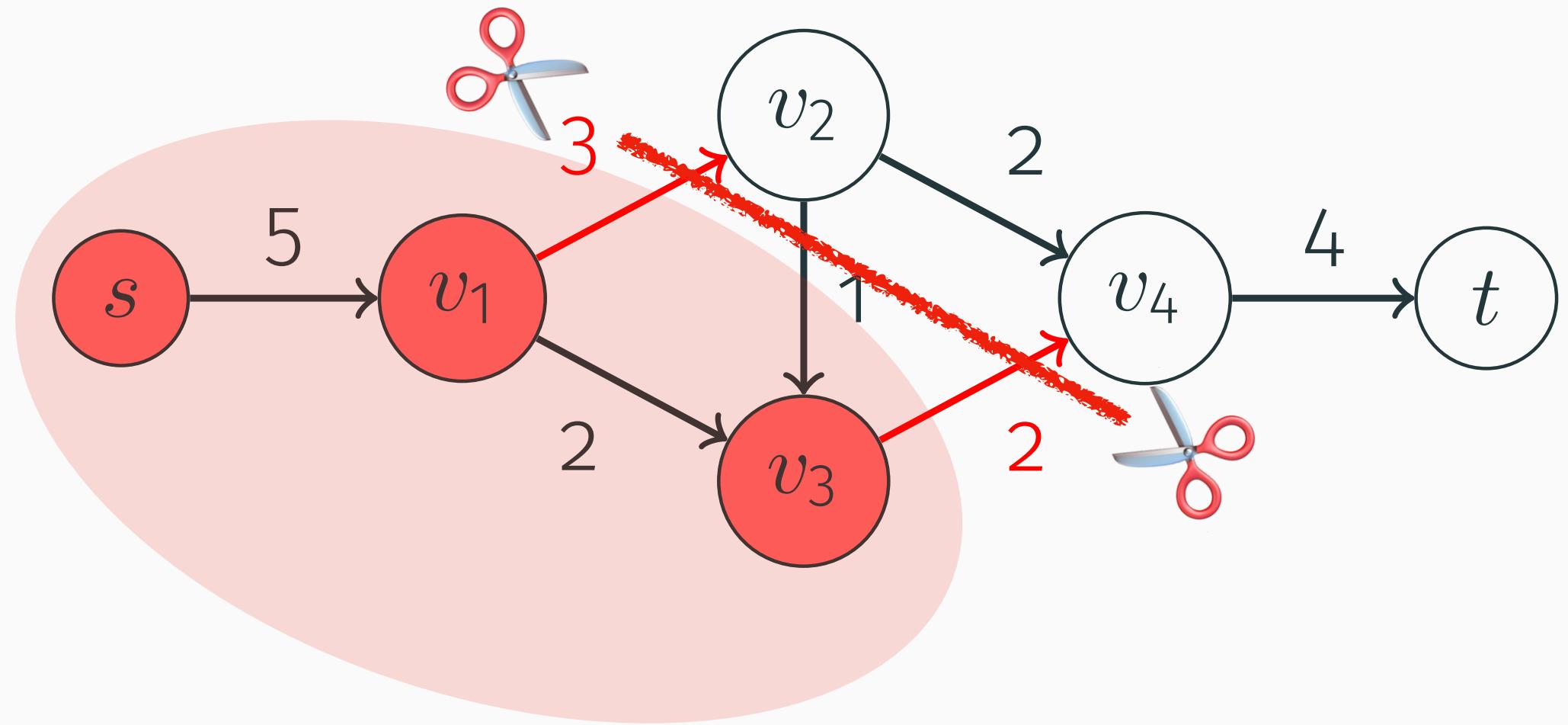


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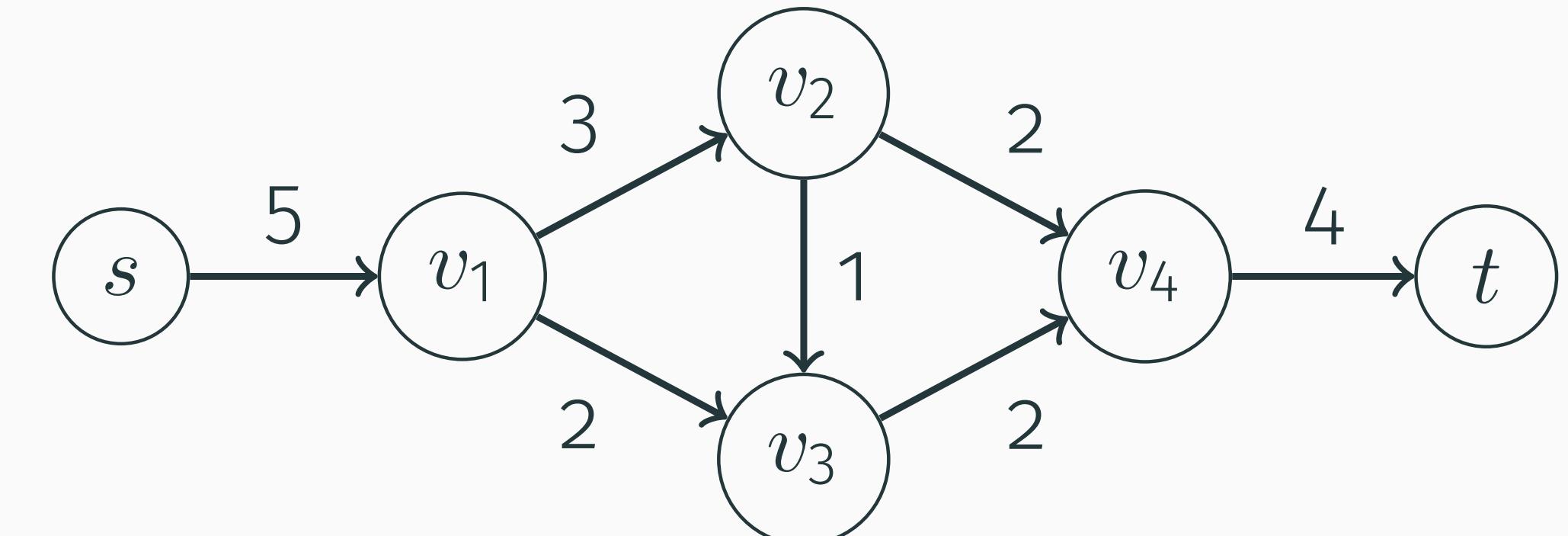
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The value of the flow f from S to $V \setminus S$ is

$$f(S, V \setminus S) := \text{outgoing flow} - \text{incoming flow}$$

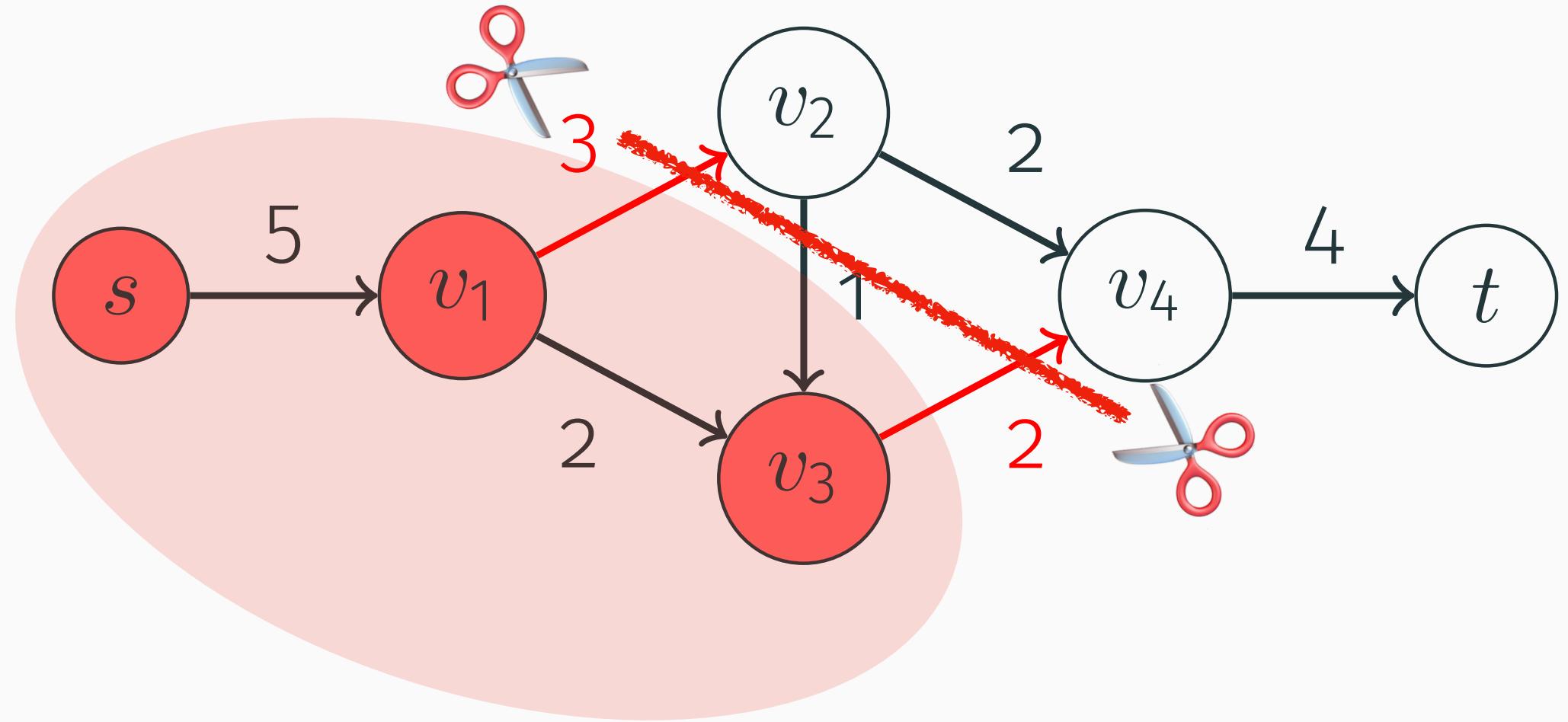


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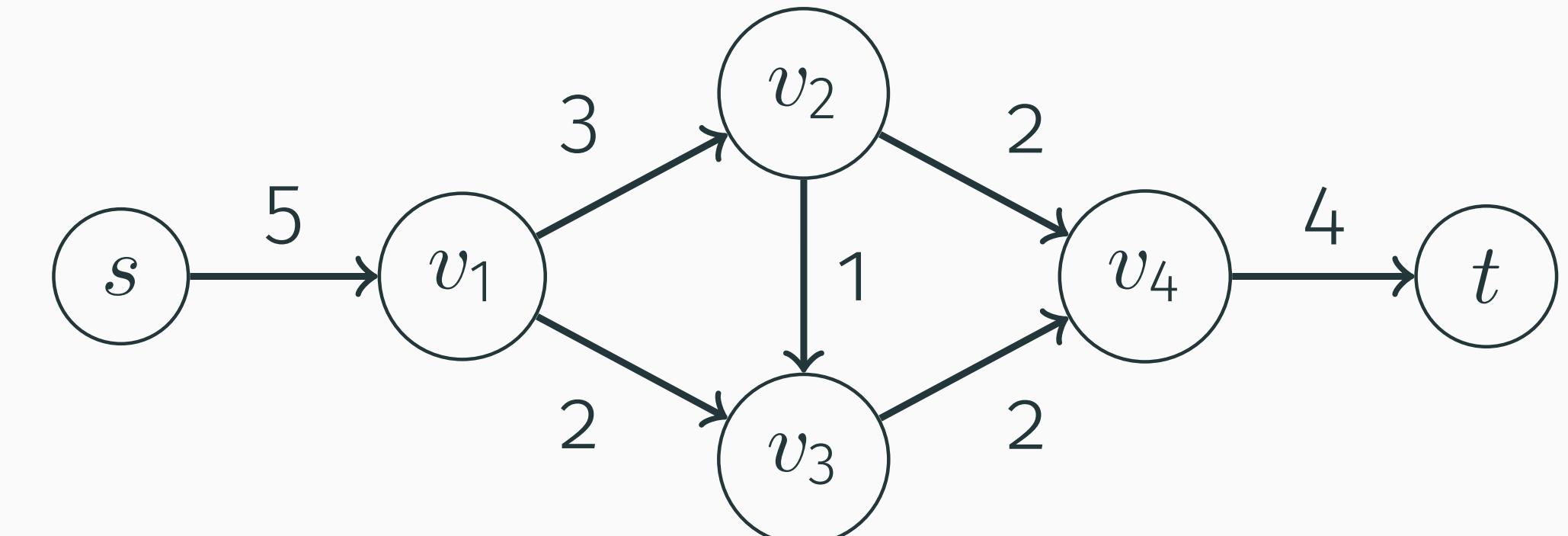
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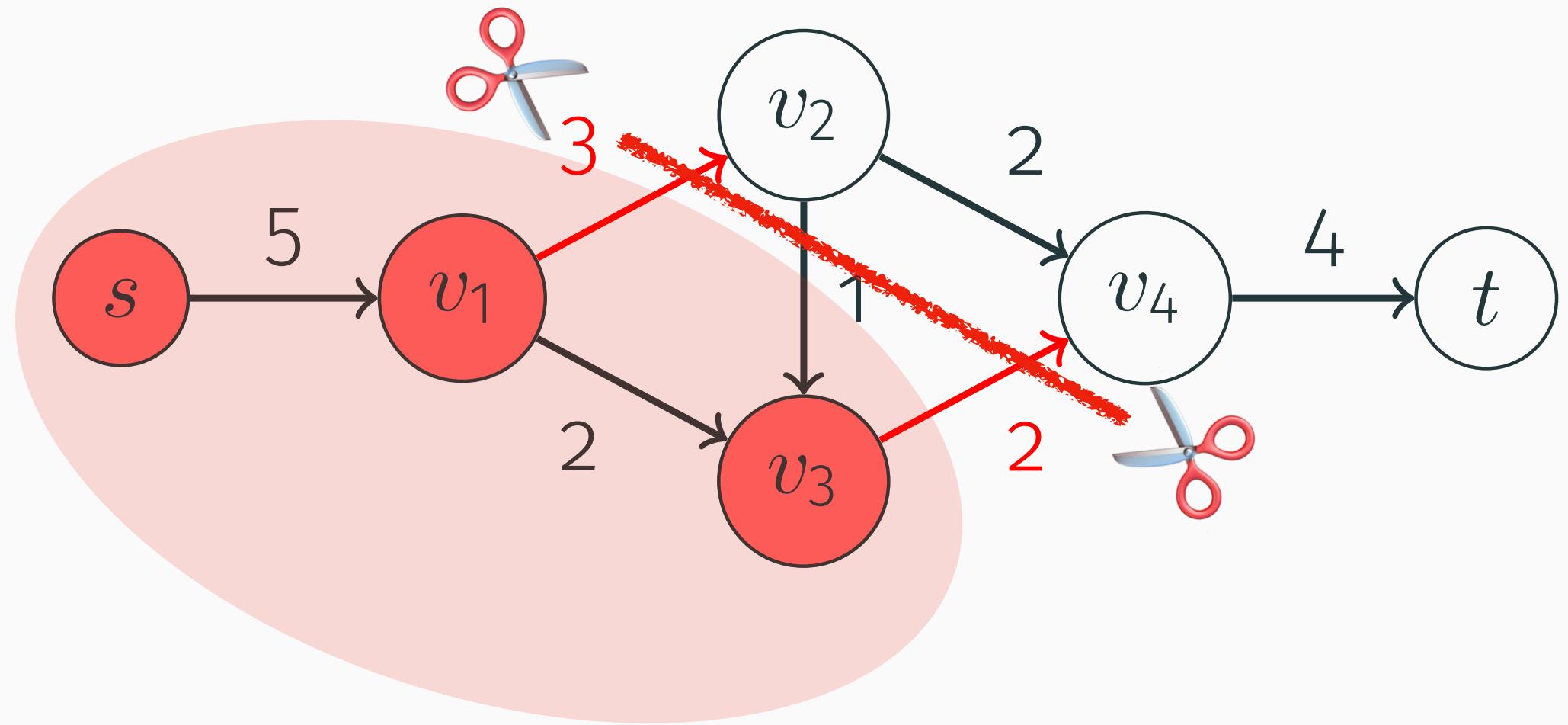


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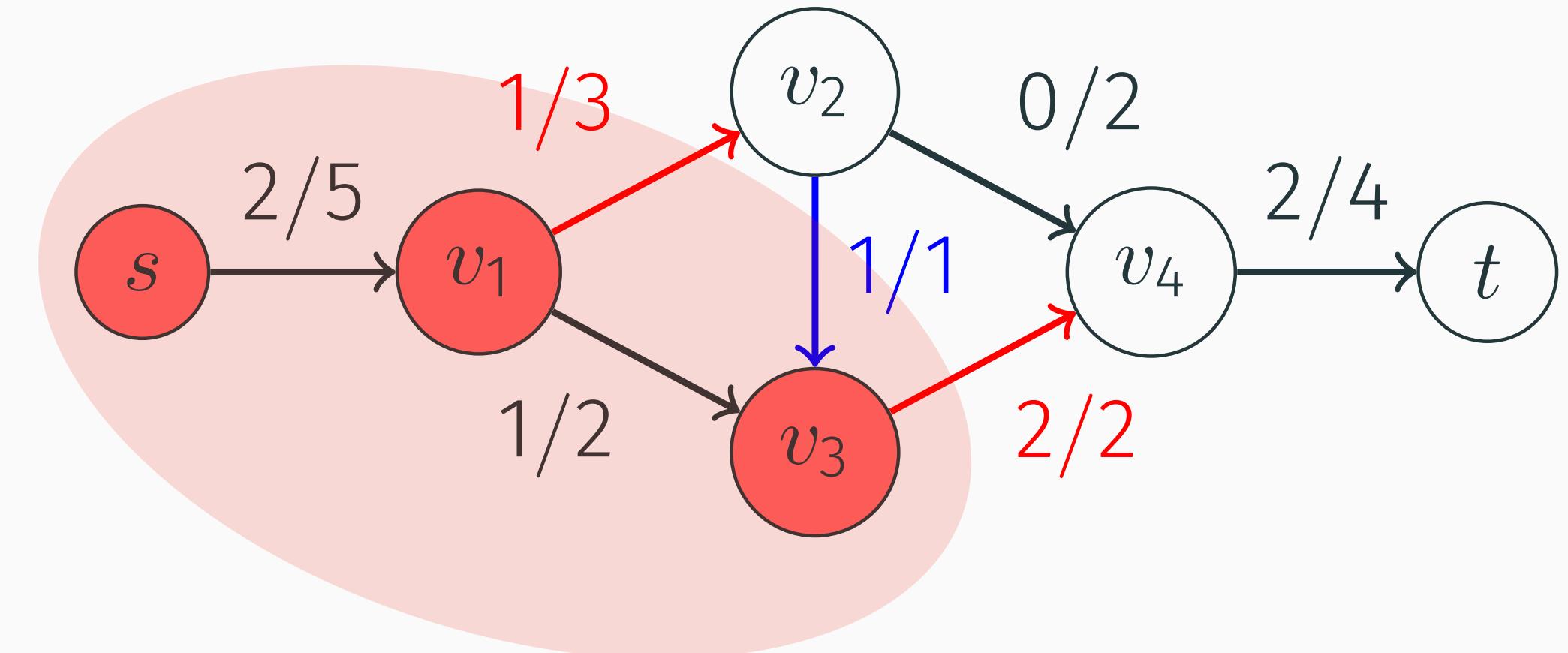
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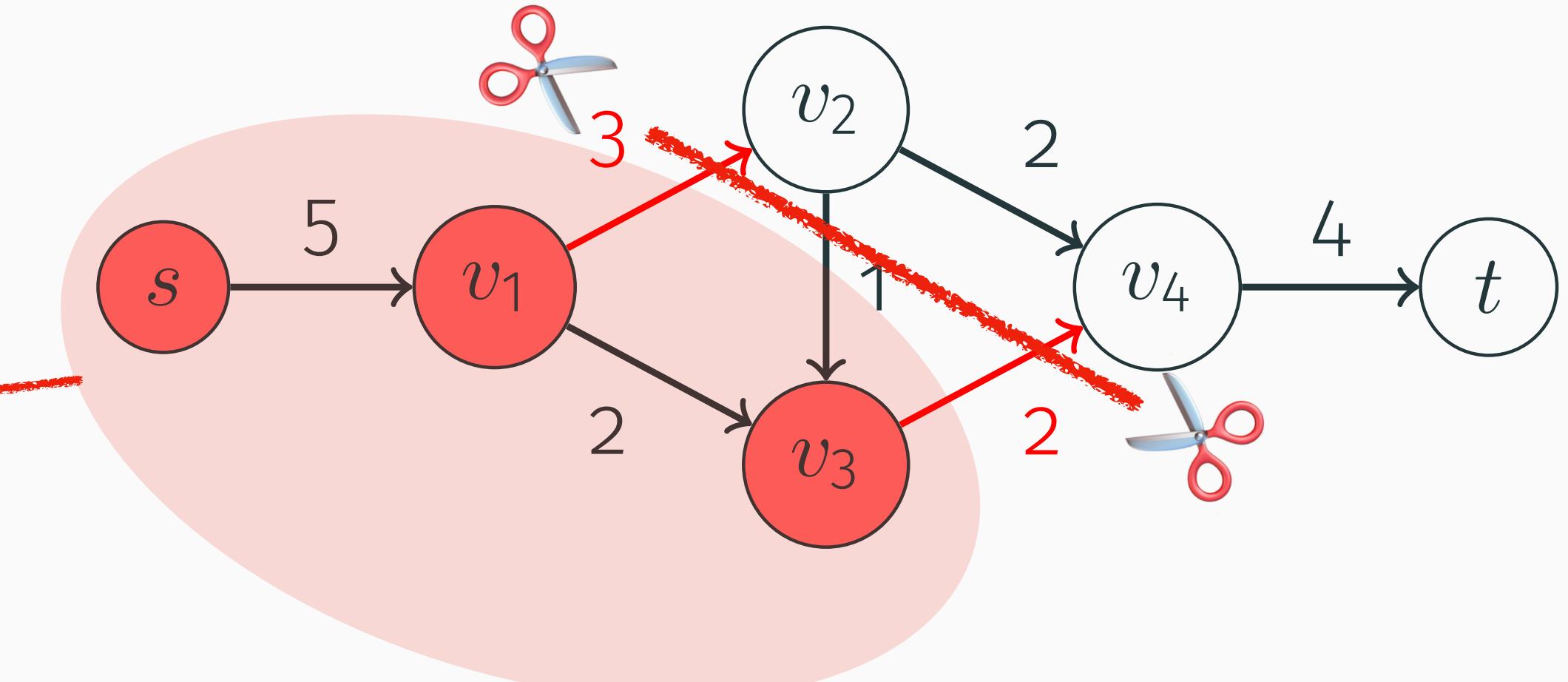
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$\text{cap}(S, V \setminus S) :=$ outgoing capacity

$$= \sum_{\substack{e=(u,v) \\ u \in S, v \in V \setminus S}} \text{cap}(e)$$

$$= 3 + 2 = 5.$$

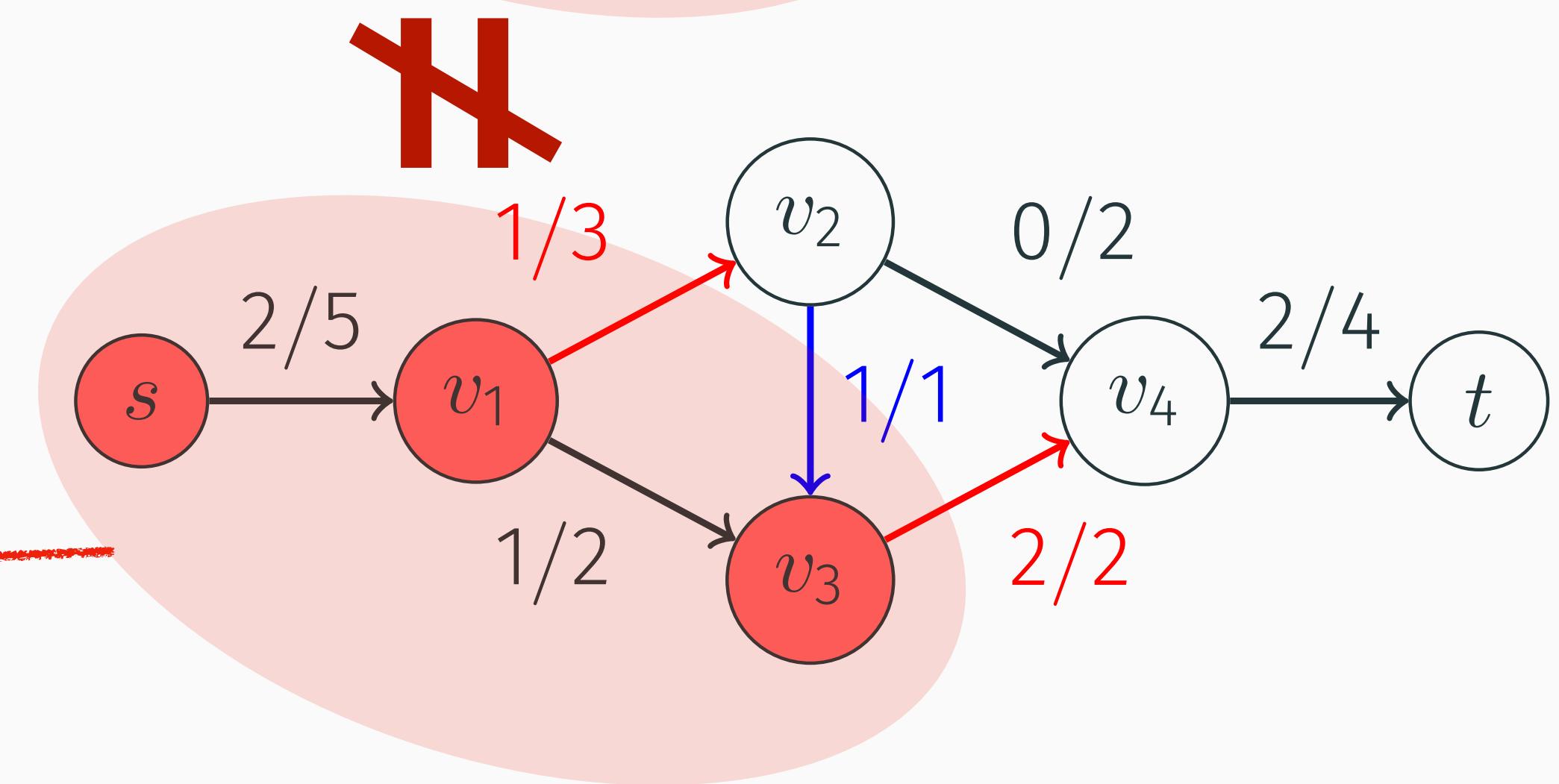


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$$= 1 + 2 - 1 = 2.$$



Minimum Cut: Maxflow-Mincut-Theorem

Theorem (Maxflow-Mincut-Theorem)

Let f be an s - t -flow in a graph G . Then f is a maximum flow if and only if

$$|f| = \min_{S: s \in S, t \notin S} \text{cap}(S, V \setminus S).$$

This allows us to easily find a minimum s - t -cut:

- ▶ Construct the residual graph $G_f := (V, E_f)$. For each edge $(u, v) \in G$ we have:
 - An edge $(u, v) \in G_f$ with capacity $\text{cap}(e) - f(e)$, if $\text{cap}(e) - f(e) > 0$.
 - An edge $(v, u) \in G_f$ with capacity $f(e)$, if $f(e) > 0$.
- ▶ Since f is a maximum flow, there is no s - t path in the residual graph G_f .
- ▶ Take S to be all vertices in G_f reachable from s .
 $\Rightarrow (S, V \setminus S)$ is a minimum s - t -cut.

Minimum Cut: Maxflow-Mincut-Theorem

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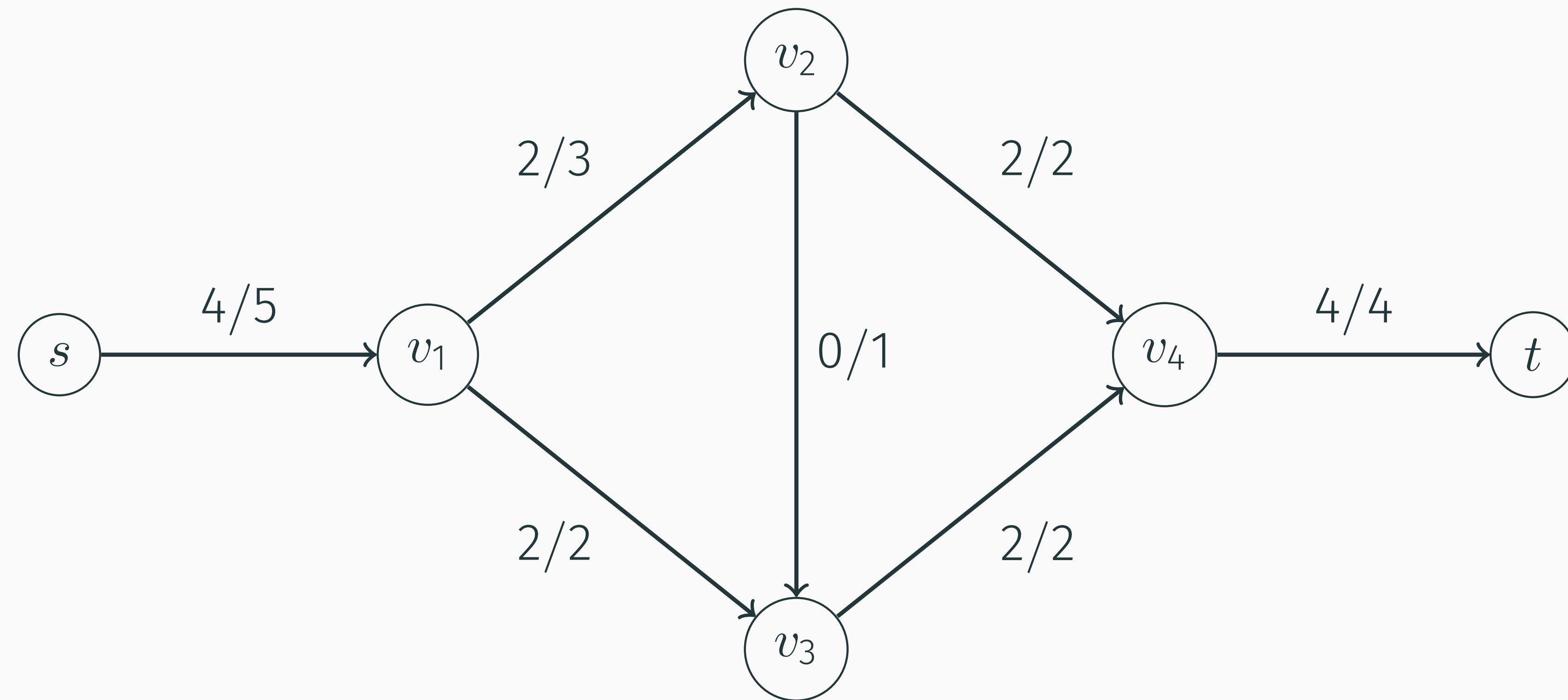
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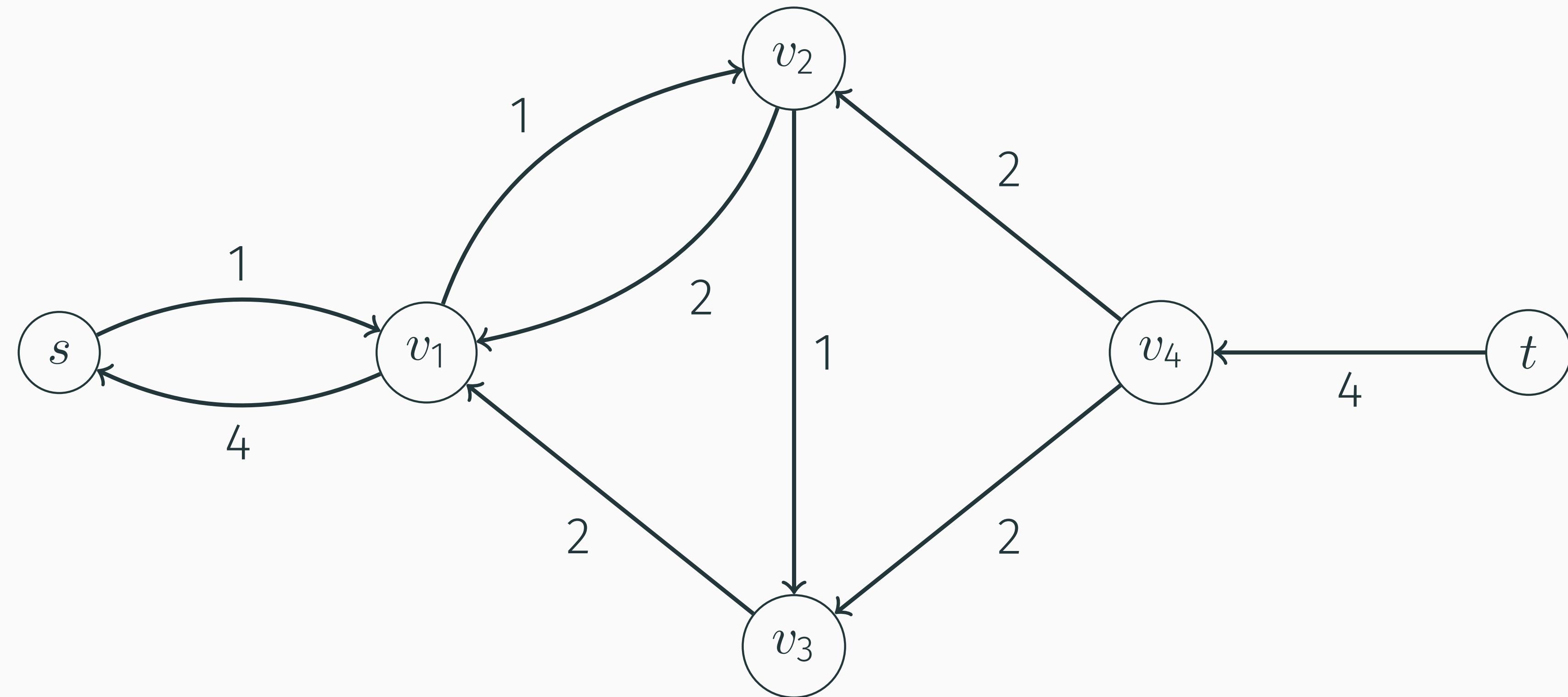
Minimum Cut: Example

Graph G and a maximum flow f .



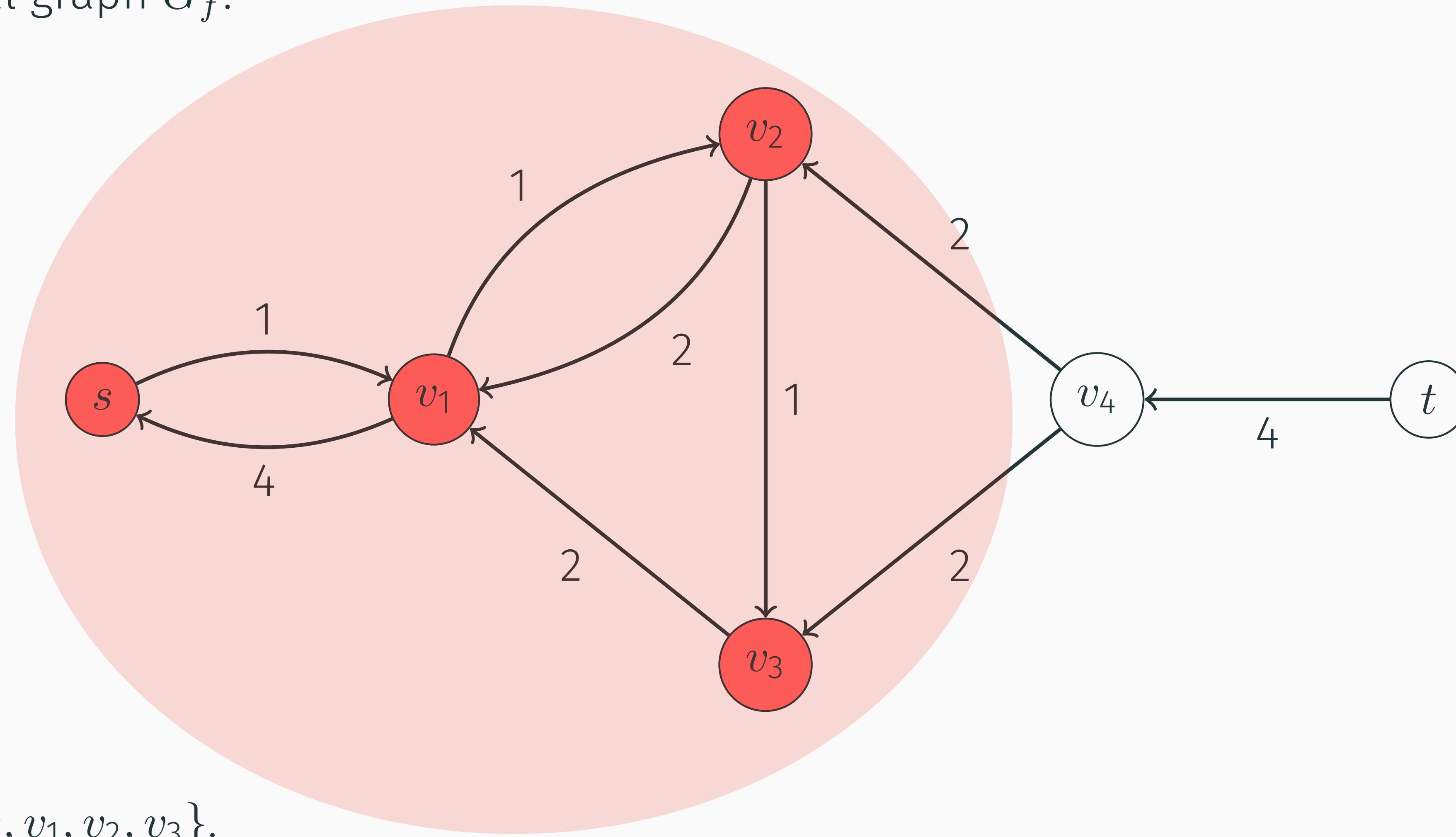
Minimum Cut: Example

Residual graph G_f .



Minimum Cut: Example

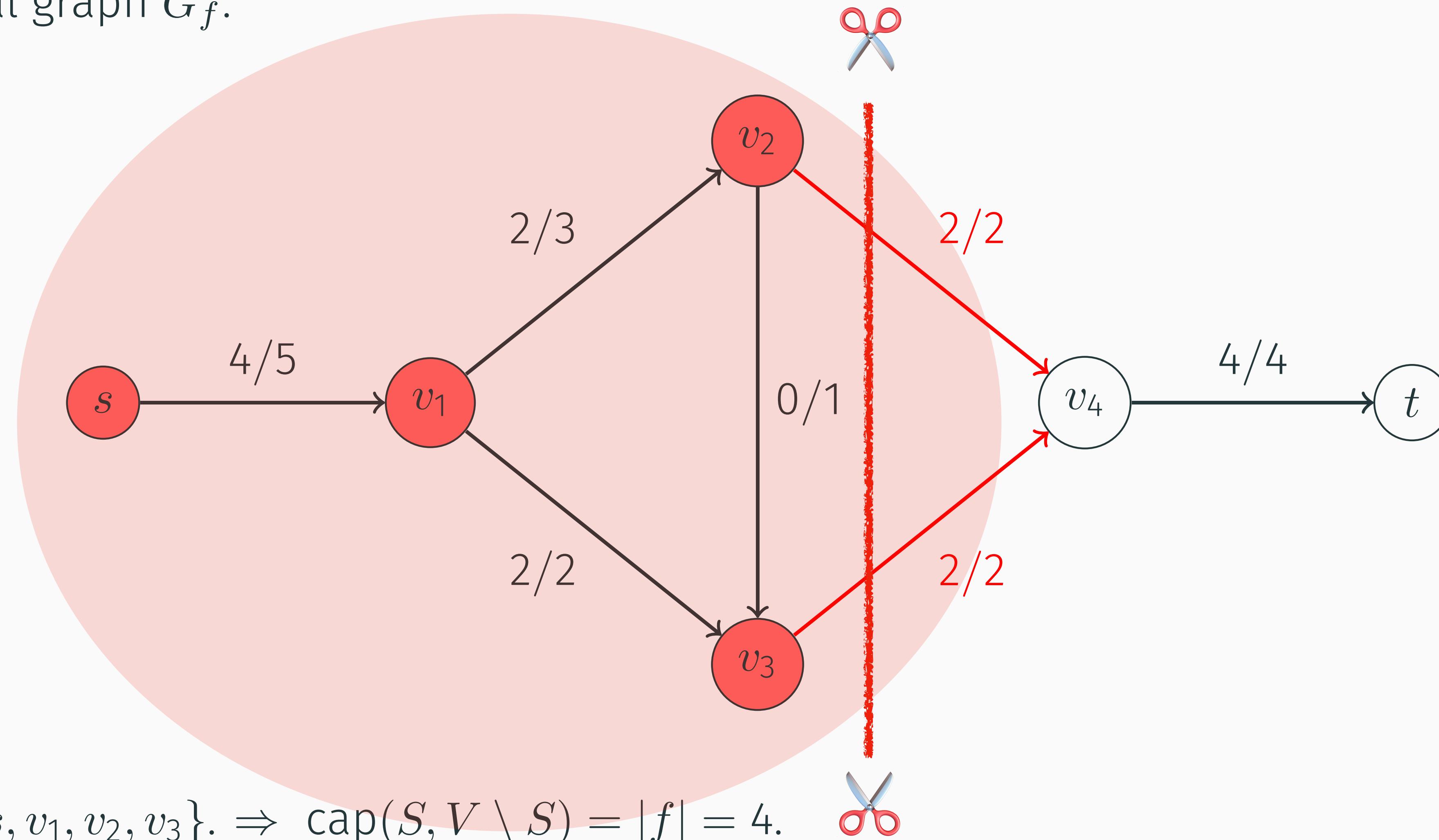
Residual graph G_f .



$$|S| = \{s, v_1, v_2, v_3\}.$$

Minimum Cut: Example

Residual graph G_f .



Minimum Cut: Code

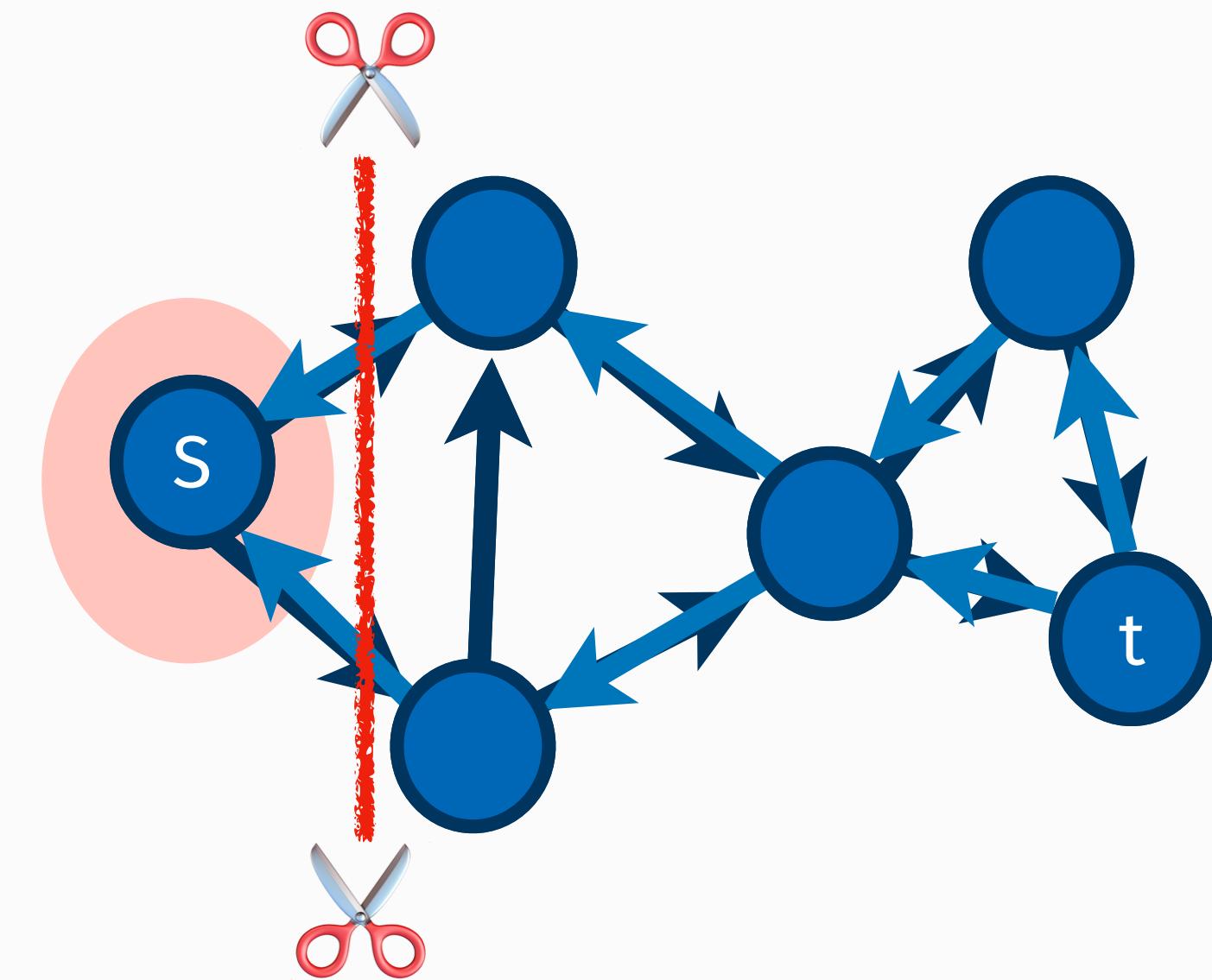
Example code: BFS on the residual graph G_f . → [bgl_residual_bfs.cpp](#)

```
90 // BFS to find vertex set S
91 std::vector<int> vis(N, false); // visited flags
92 std::queue<int> Q; // BFS queue (from std:: not boost::)
93 vis[src] = true; // Mark the source as visited
94 Q.push(src);
95 while (!Q.empty()) {
96     const int u = Q.front();
97     Q.pop();
98     out_edge_it ebeg, eend;
99     for (boost::tie(ebeg, eend) = boost::out_edges(u, G); ebeg != eend; ++ebeg) {
100         const int v = boost::target(*ebeg, G);
101         // Only follow edges with spare capacity
102         if (rc_map[*ebeg] == 0 || vis[v]) continue;
103         vis[v] = true;
104         Q.push(v);
105     }
106 }
```

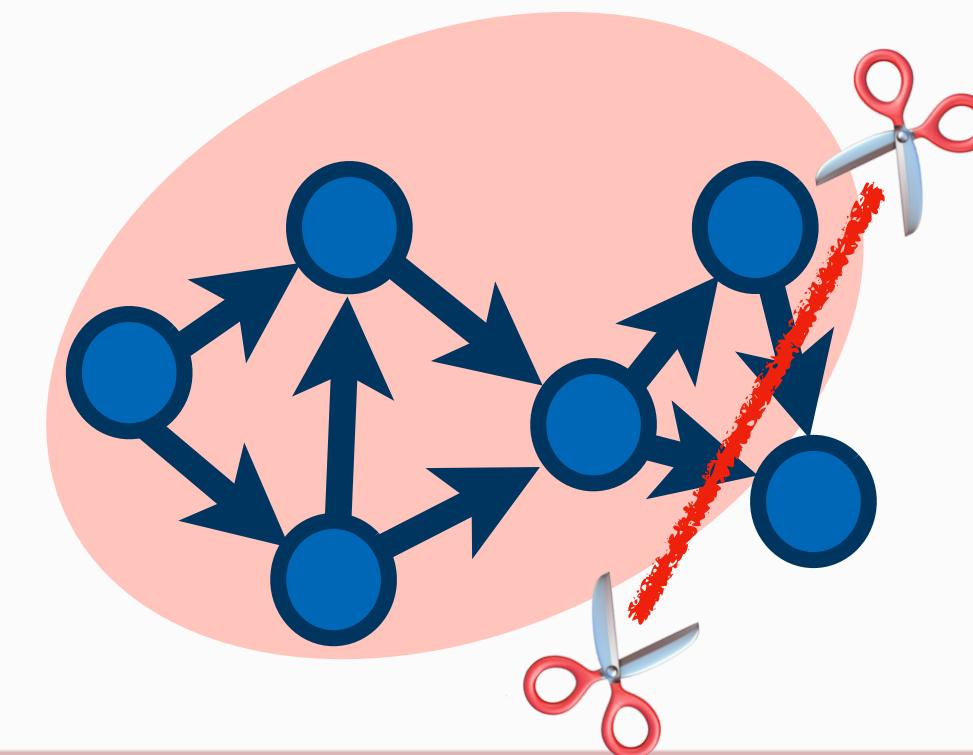
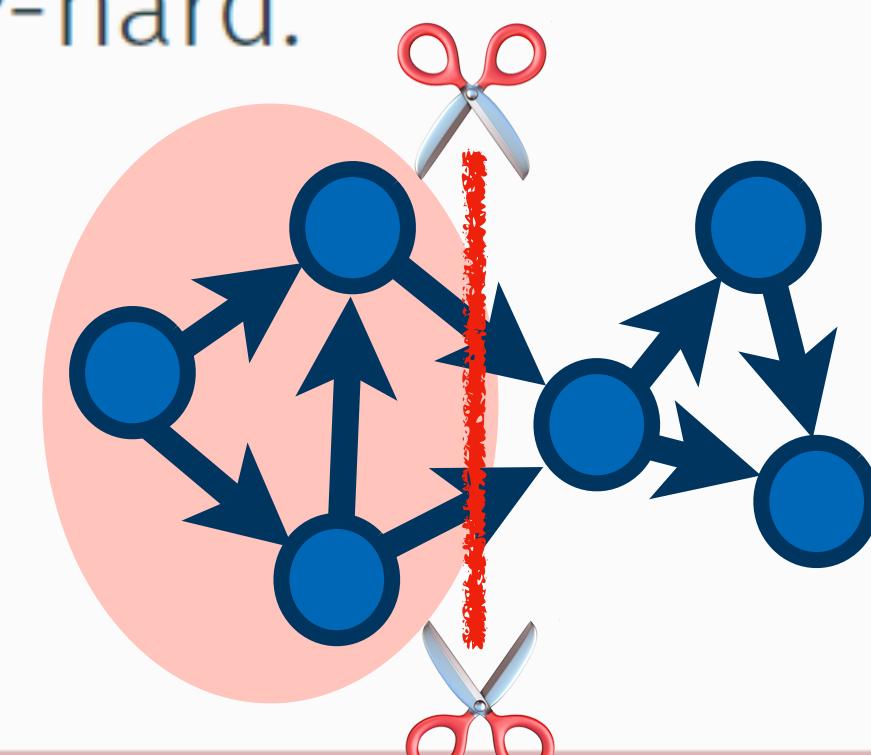
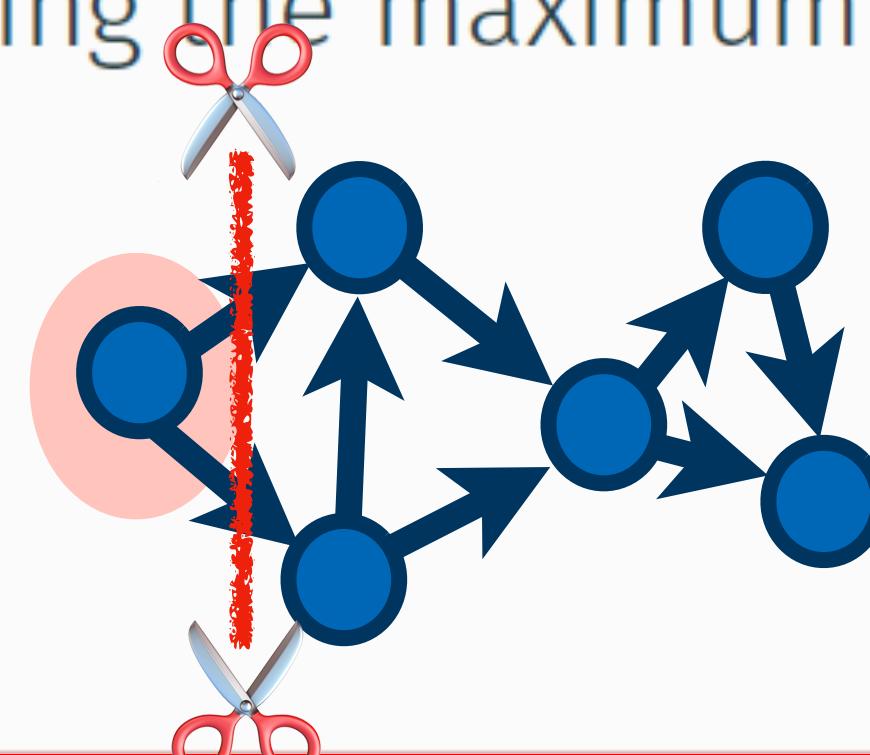
Minimum Cut: Algorithm

Summary of what you need to do to find a minimum cut:

1. Compute maximum flow f and the residual graph G_f .
2. Compute the set of vertices S :
 - S is reachable from the source s in G_f .
 - BFS on edges with residual capacity > 0 .
3. Output (depending on the task):
 - All vertices in S .
 - All edges going from S to $V \setminus S$.



Note: Minimum cuts are not necessarily unique. But earliest and latest min-cuts are. Also computing the maximum cut is NP-hard.

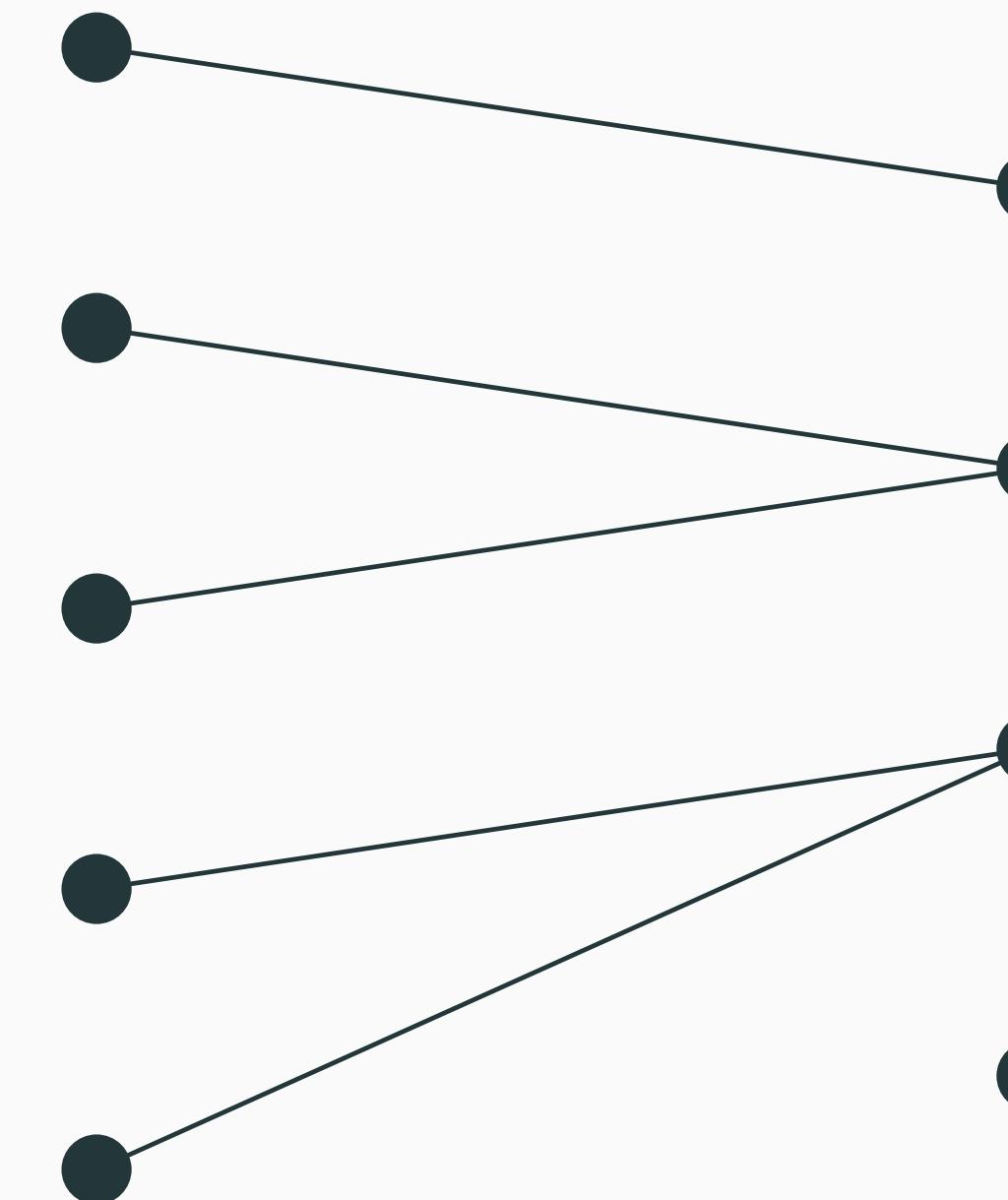


Bipartite Matchings

Maximum Matchings: Bipartite Graphs

Maximum Matching = pick as many non-adjacent edges as possible

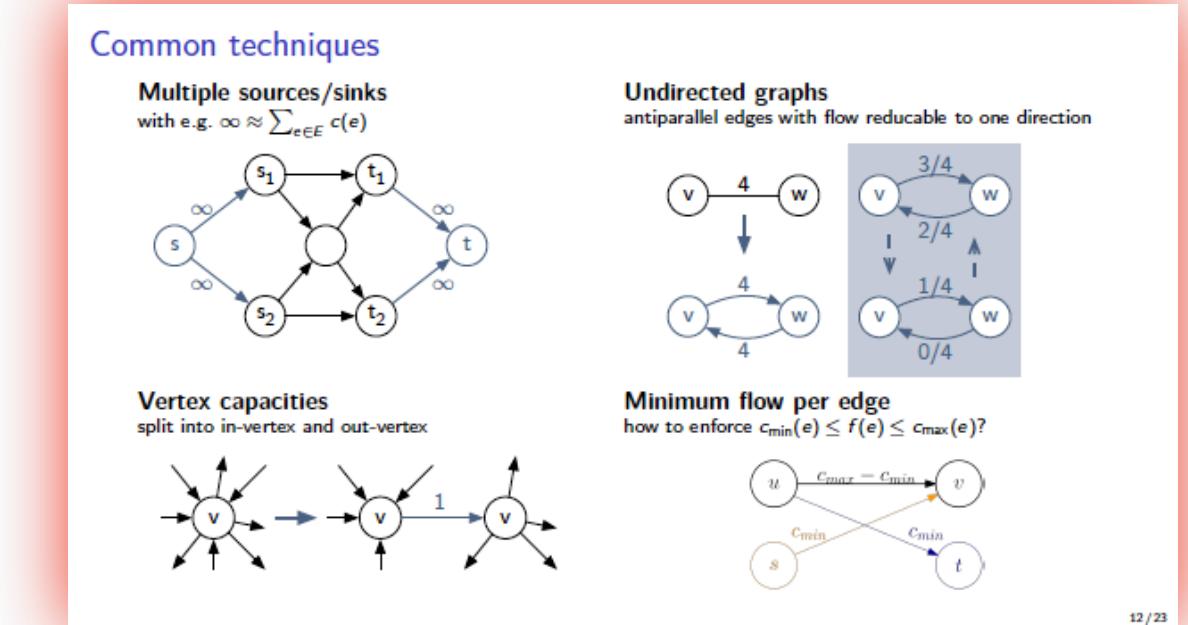
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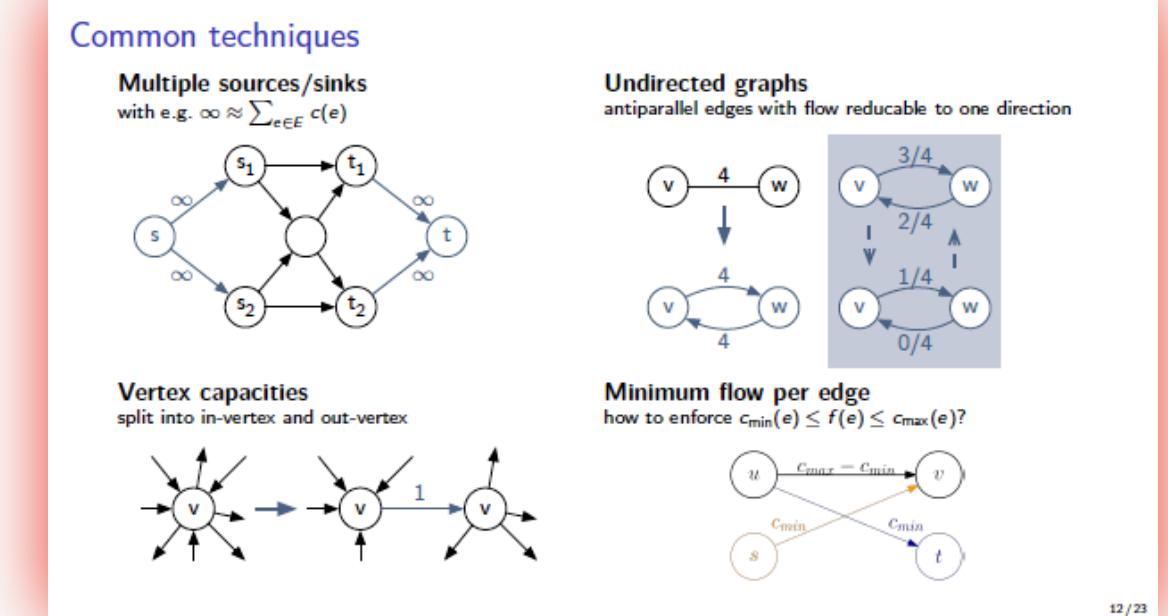
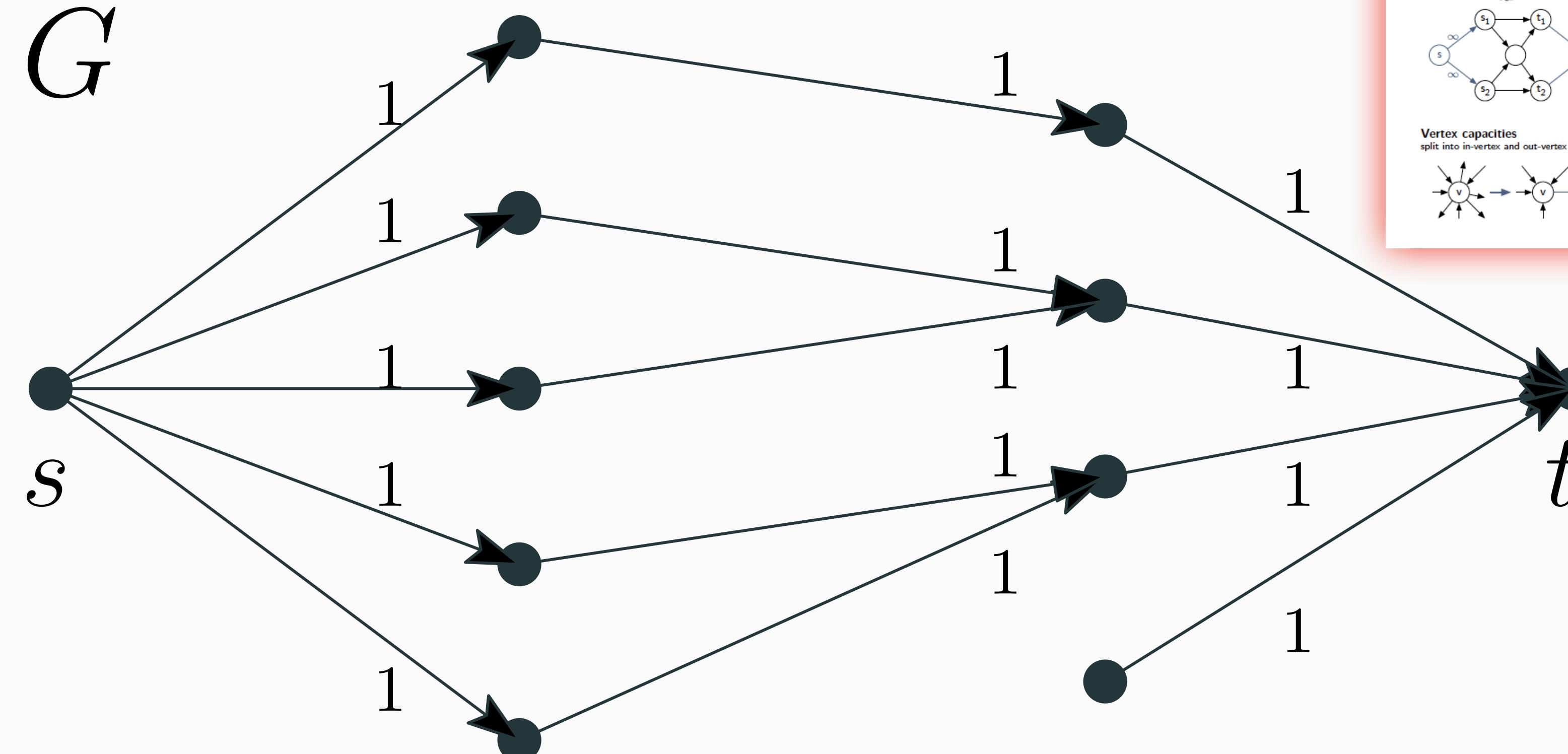
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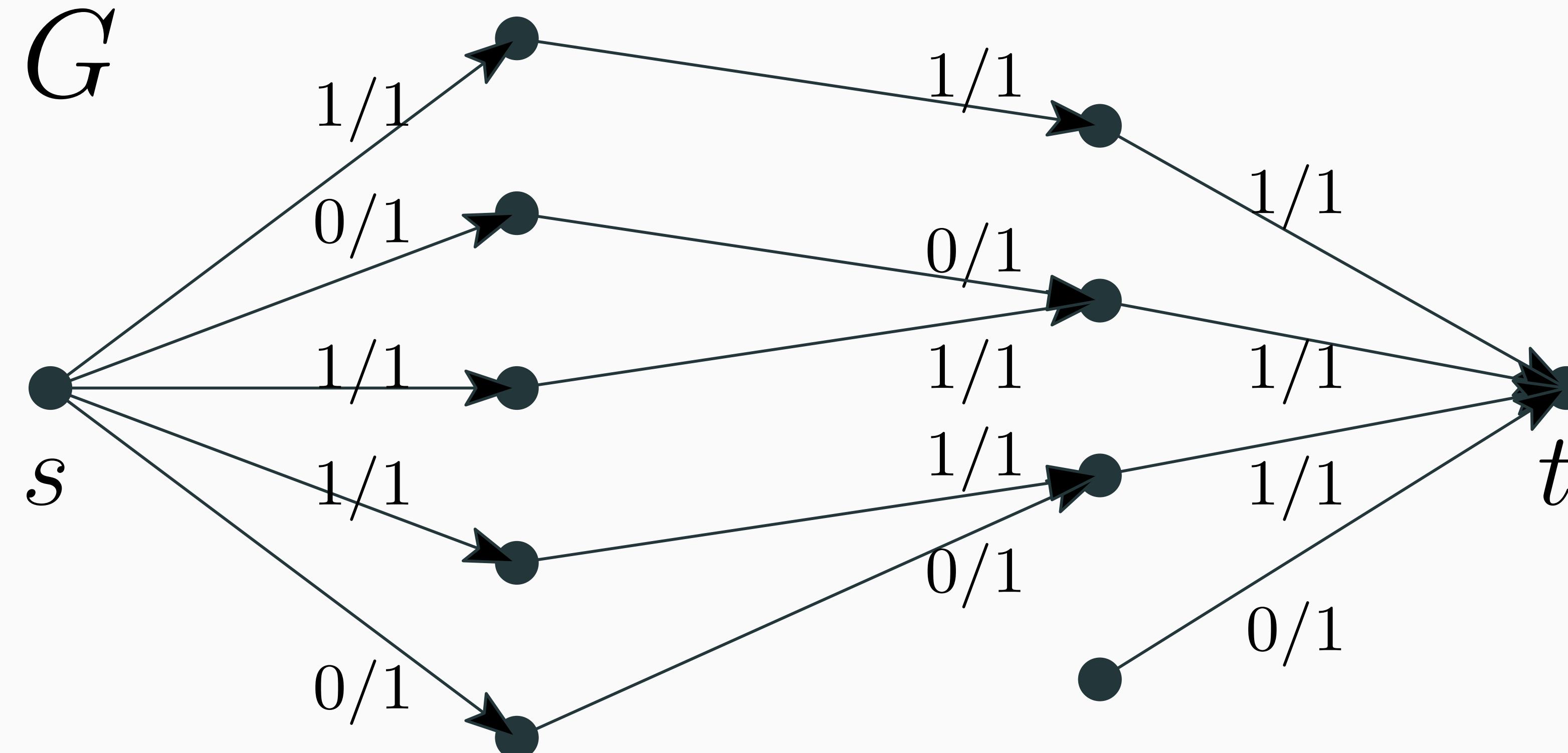
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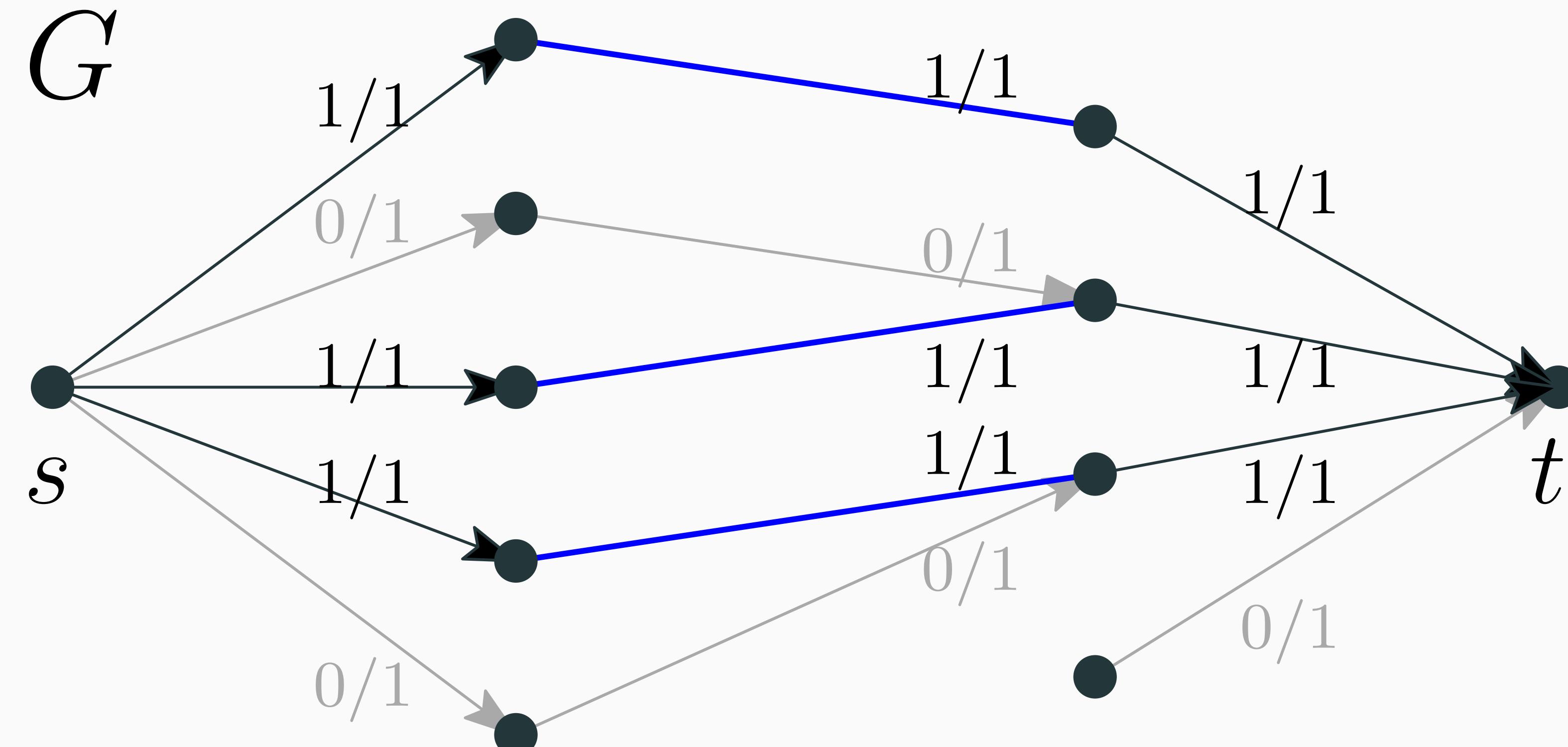
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Vertex Cover and Independent Set: General Graphs

- ▶ **Maximum independent set (MaxIS)**

Largest $T \subseteq V$, such that

$$\nexists u, v \in T : (u, v) \in E.$$

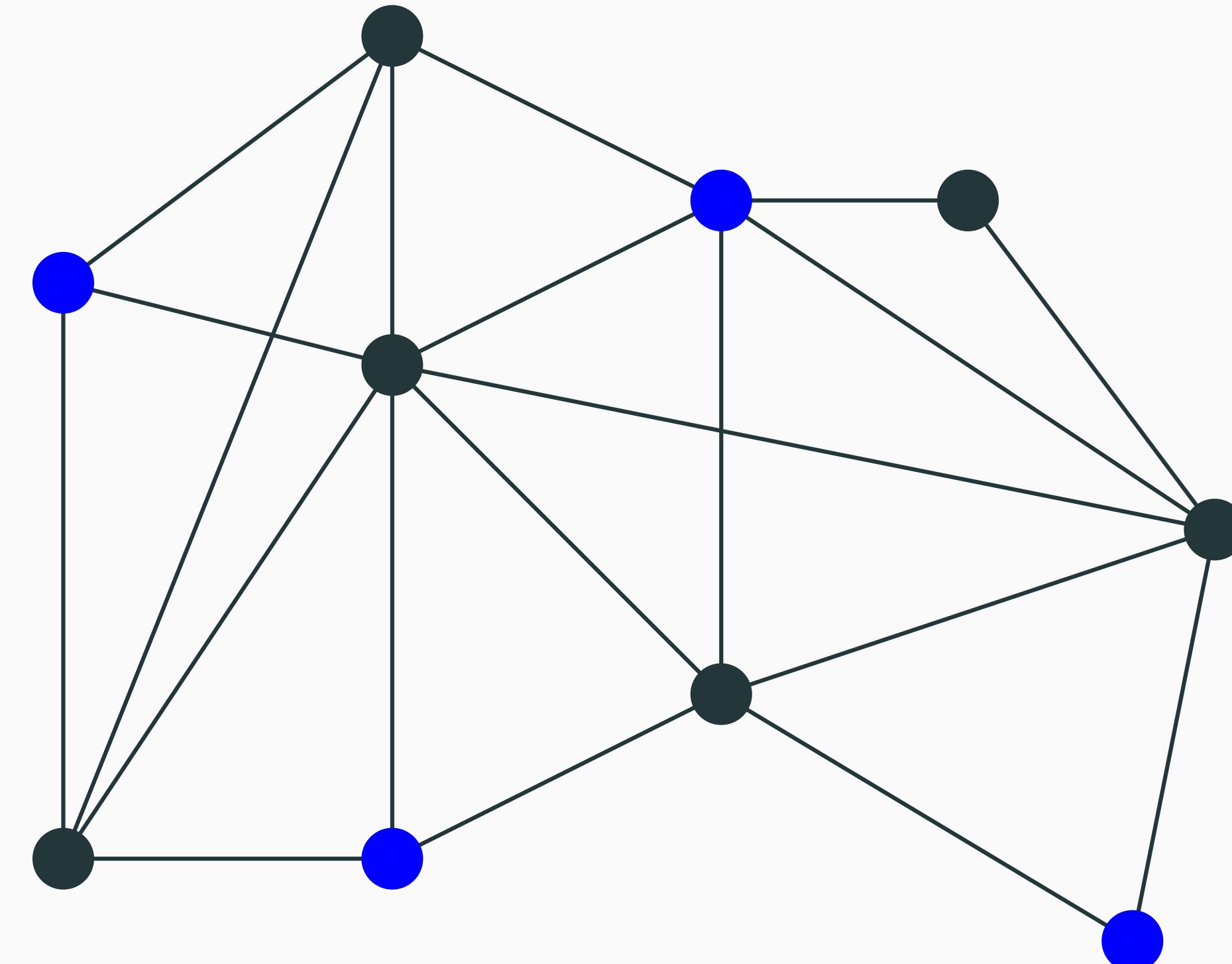
- ▶ **Minimum vertex cover (MinVC)**

Smallest $S \subseteq V$, such that

$$\forall (u, v) \in E : u \in S \vee v \in S.$$

- ▶ These problems are complementary!

$$\text{MaxIS} = V \setminus \text{MinVC}$$



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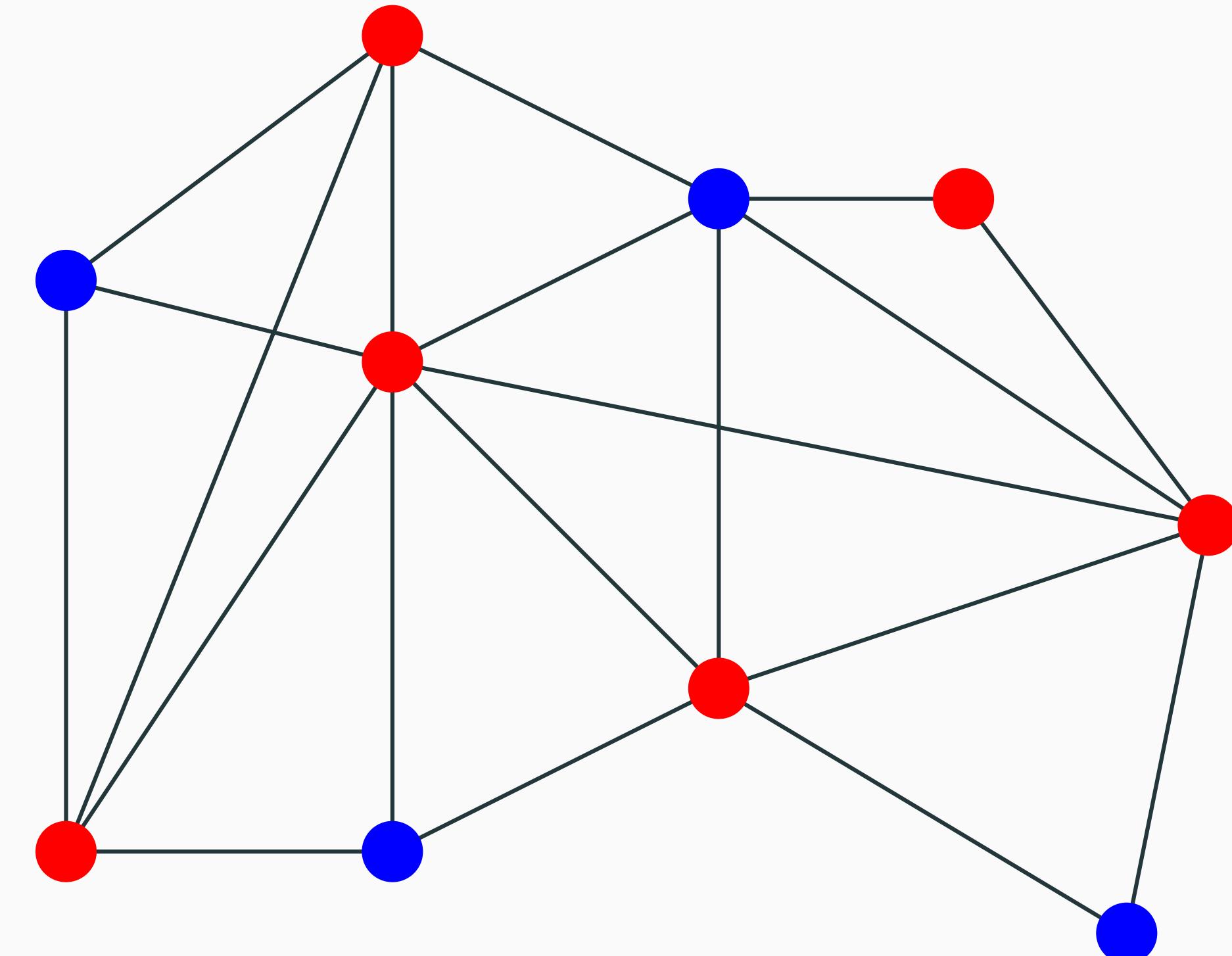
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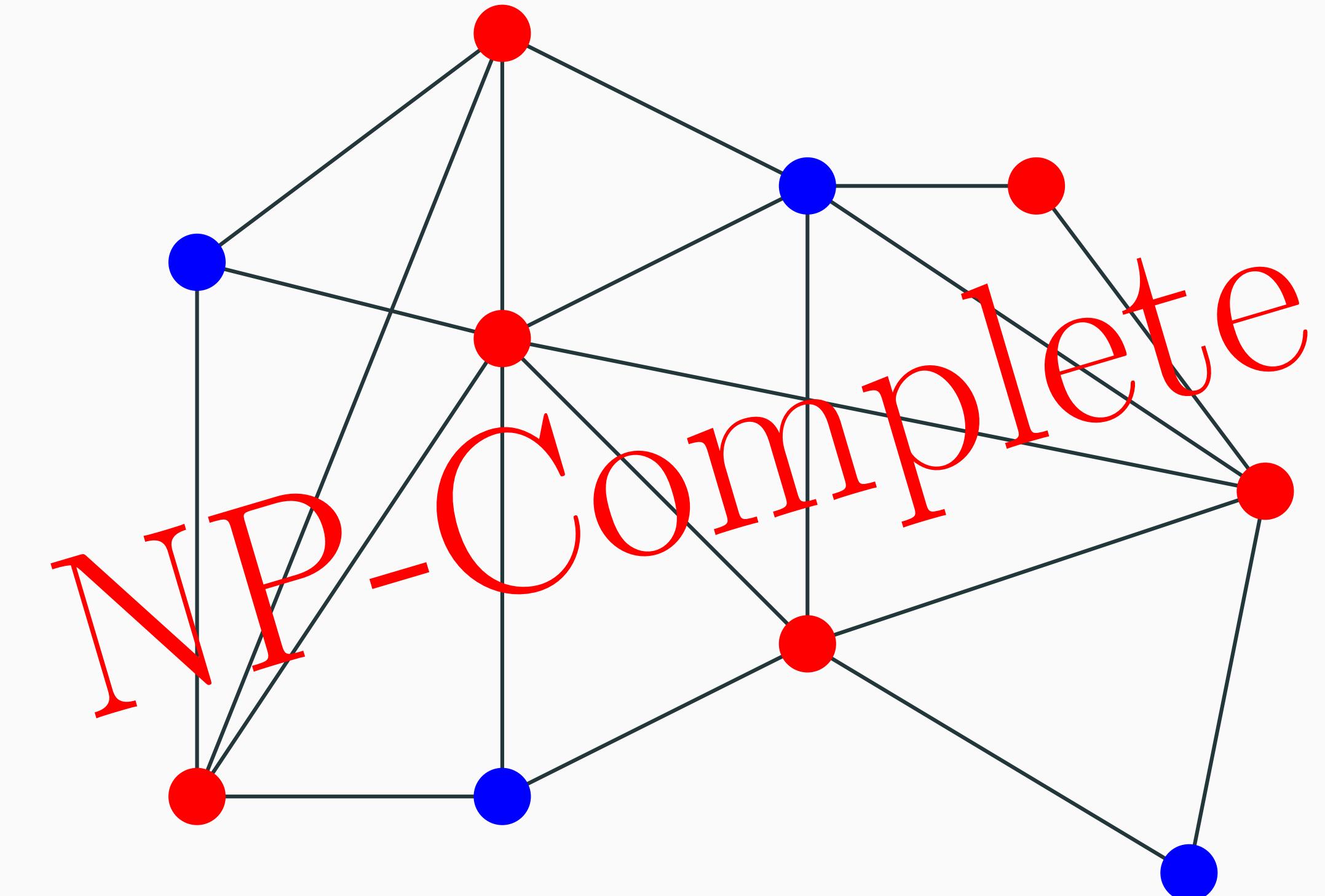
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Brief Excursion: Options for Runtime Analysis

By now, we know many ways of deciding whether an algorithm is fast or slow:

- ▶ look at the input size (the classical way)
- ▶ look at the output size (e.g. fast as the answer is guaranteed to be small)
If you know that MaxIS is very small, then it might be tractable while computing a MinVC directly is too slow
- ▶ look at some special input restrictions (e.g. Attack of the clones)
- ▶ look at detailed structure of the input (e.g. all graphs are trees)

Vertex Cover and Independent Set: Bipartite Graphs

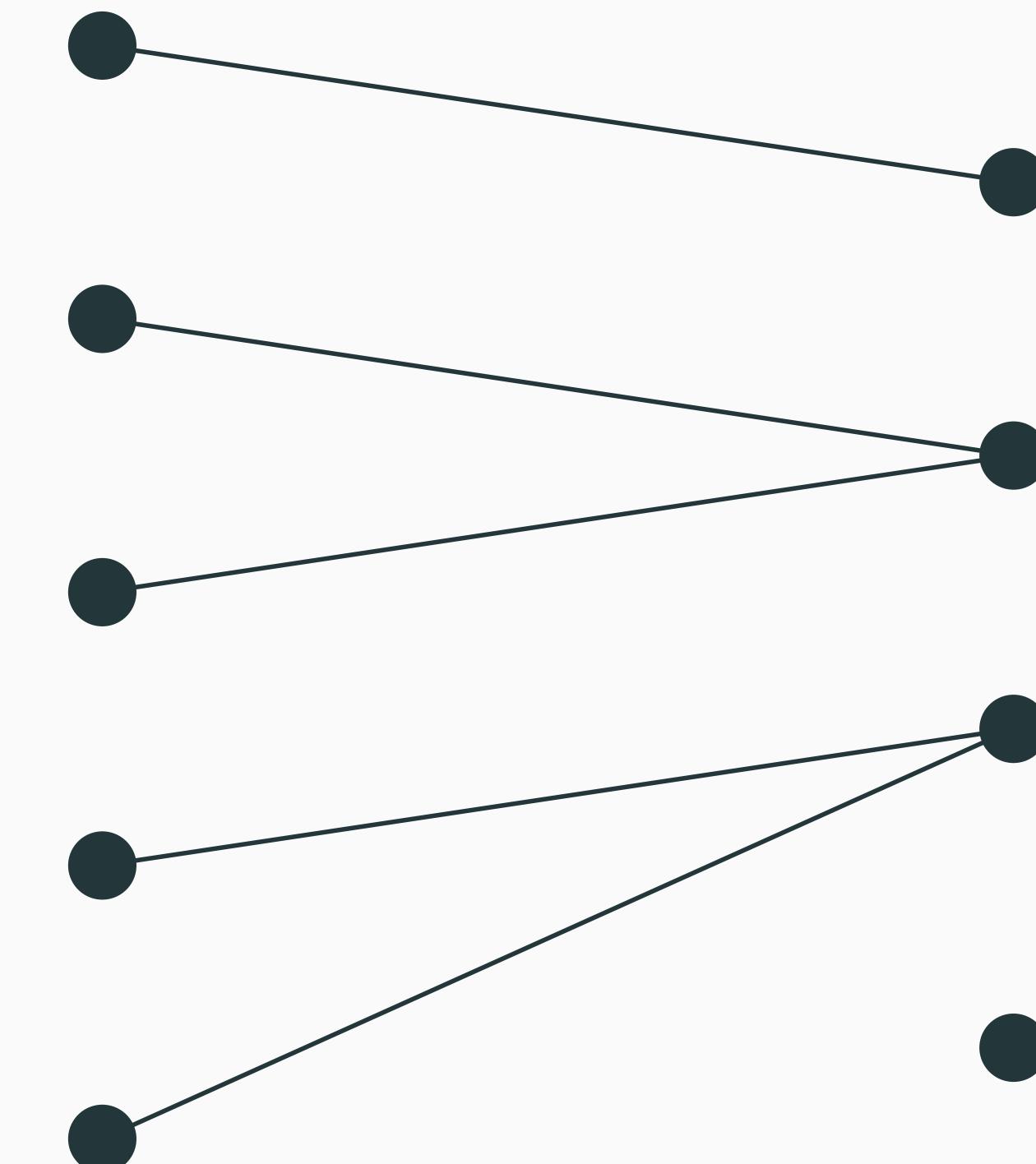
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In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.

Proof: See [Wikipedia](#) for a nice and short proof.

Algorithm:

1. Maximum matching M , $V = L \cup R$. Find all unmatched vertices in L , label them as visited.
2. Starting at visited vertices search (BFS) left to right along edges from $E \setminus M$ until all vertices in L are visited.
3. MinVC – all unvisited in L and all visited in R .
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Careful! Step 2 can take several rounds.

Easy Implementation?

Vertex Cover and Independent Set: Bipartite Graphs

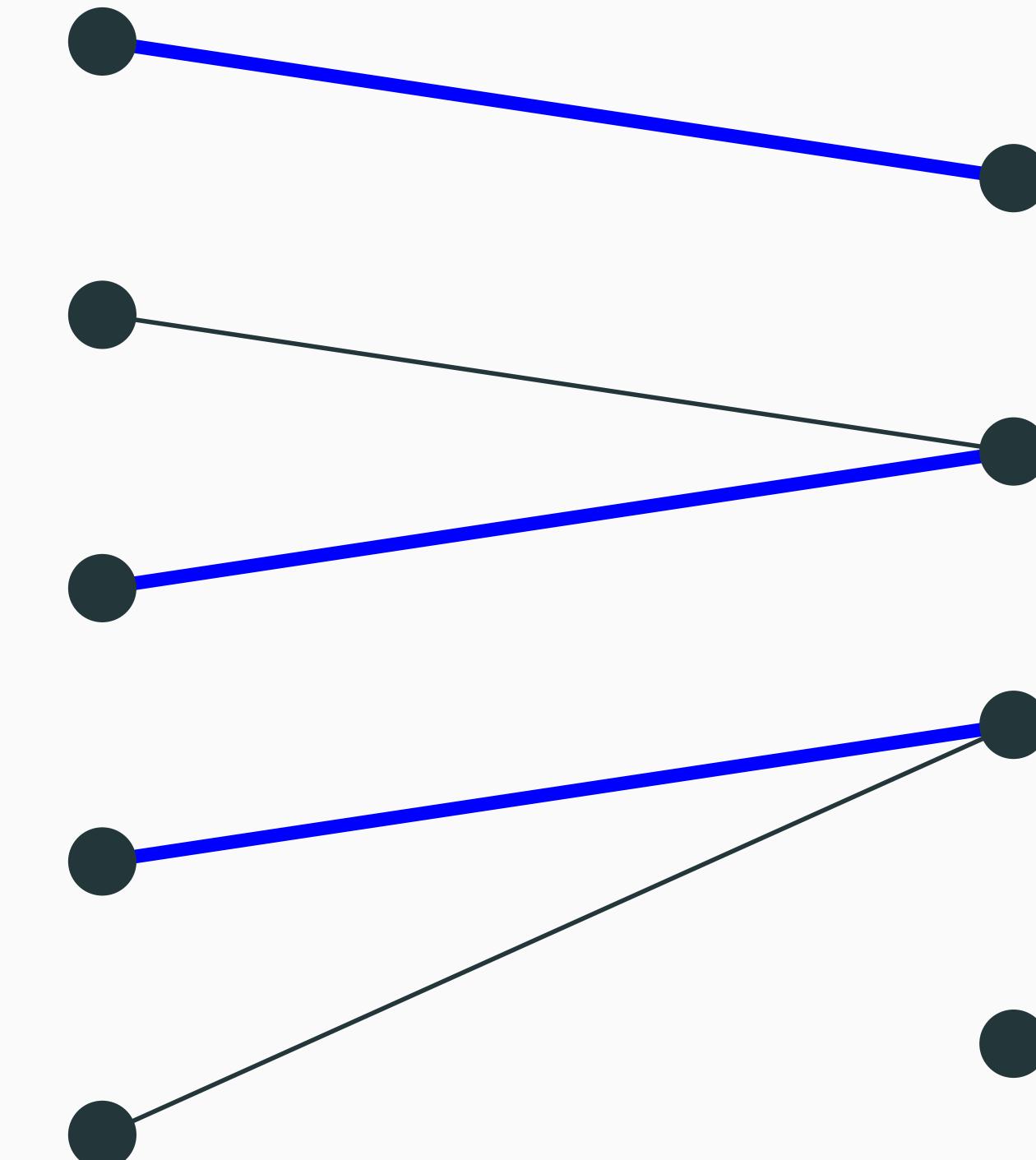
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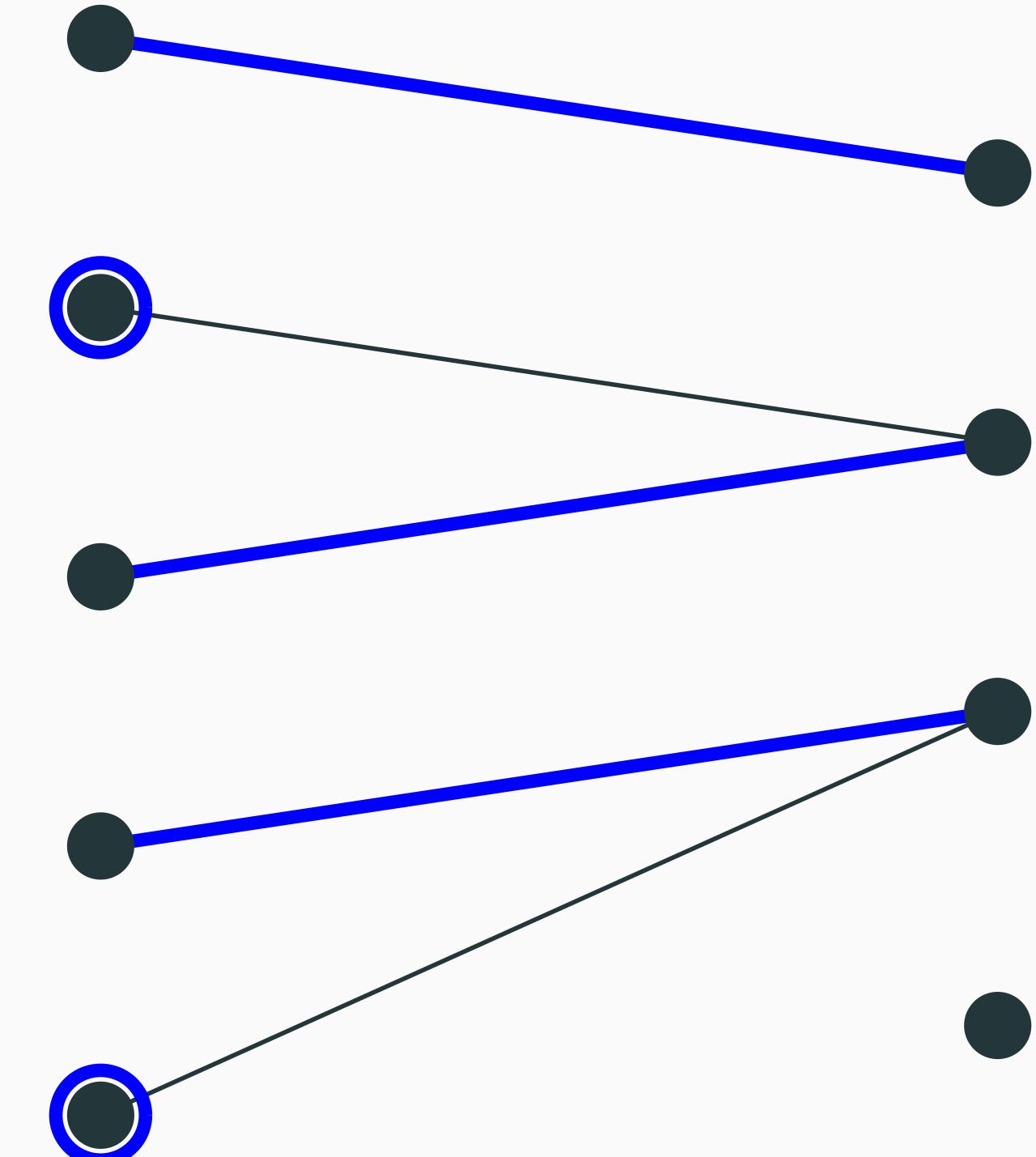
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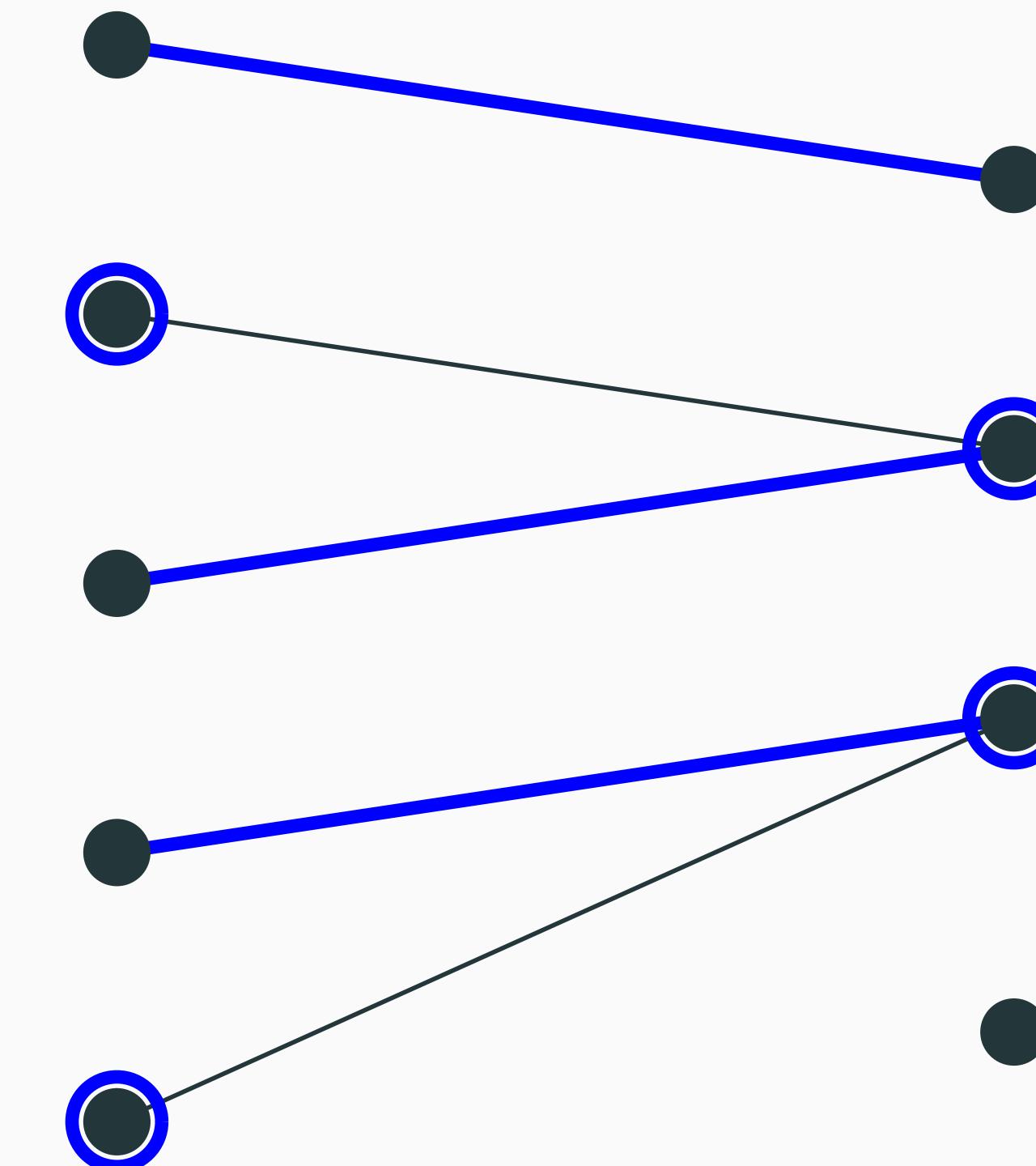
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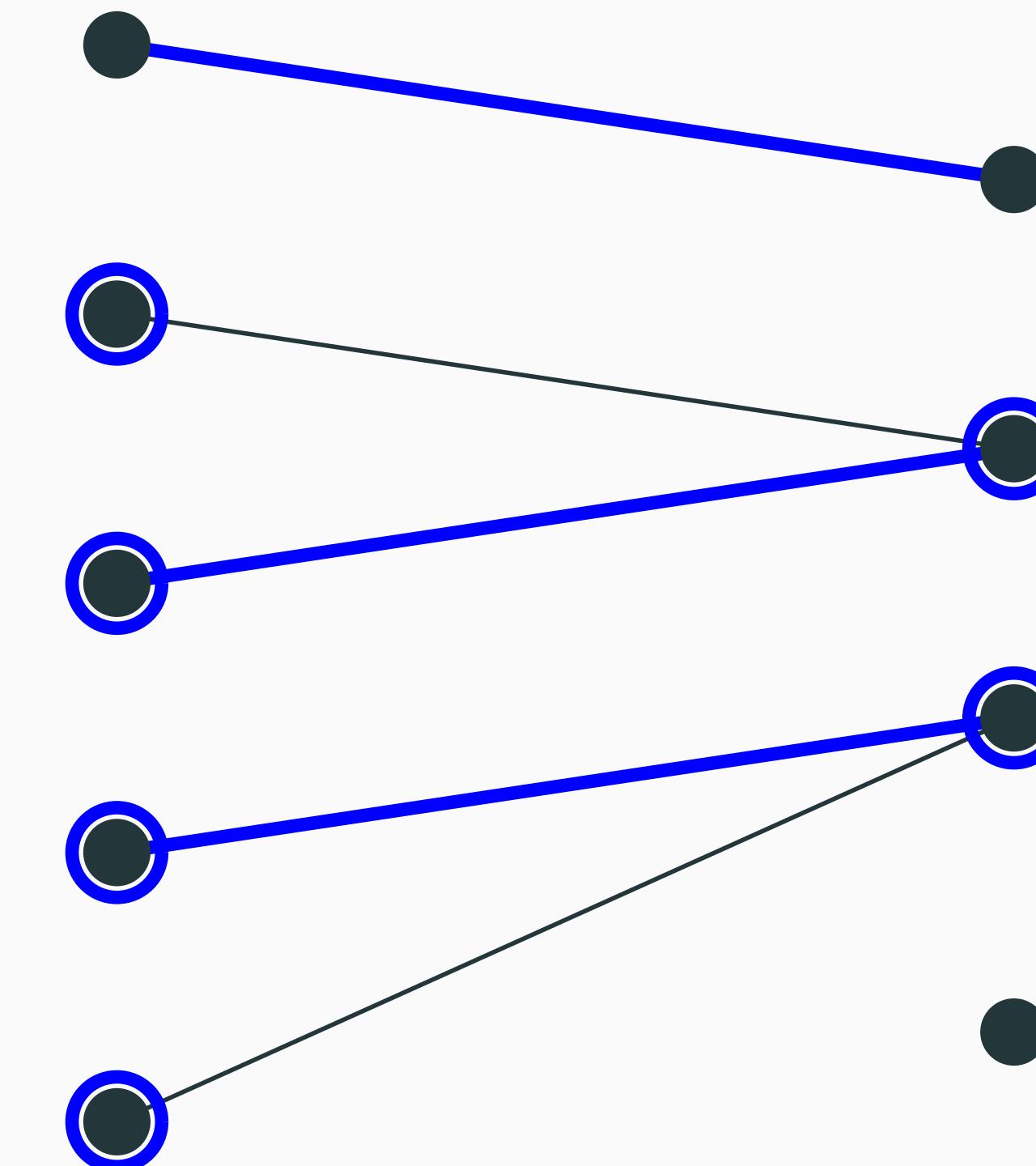
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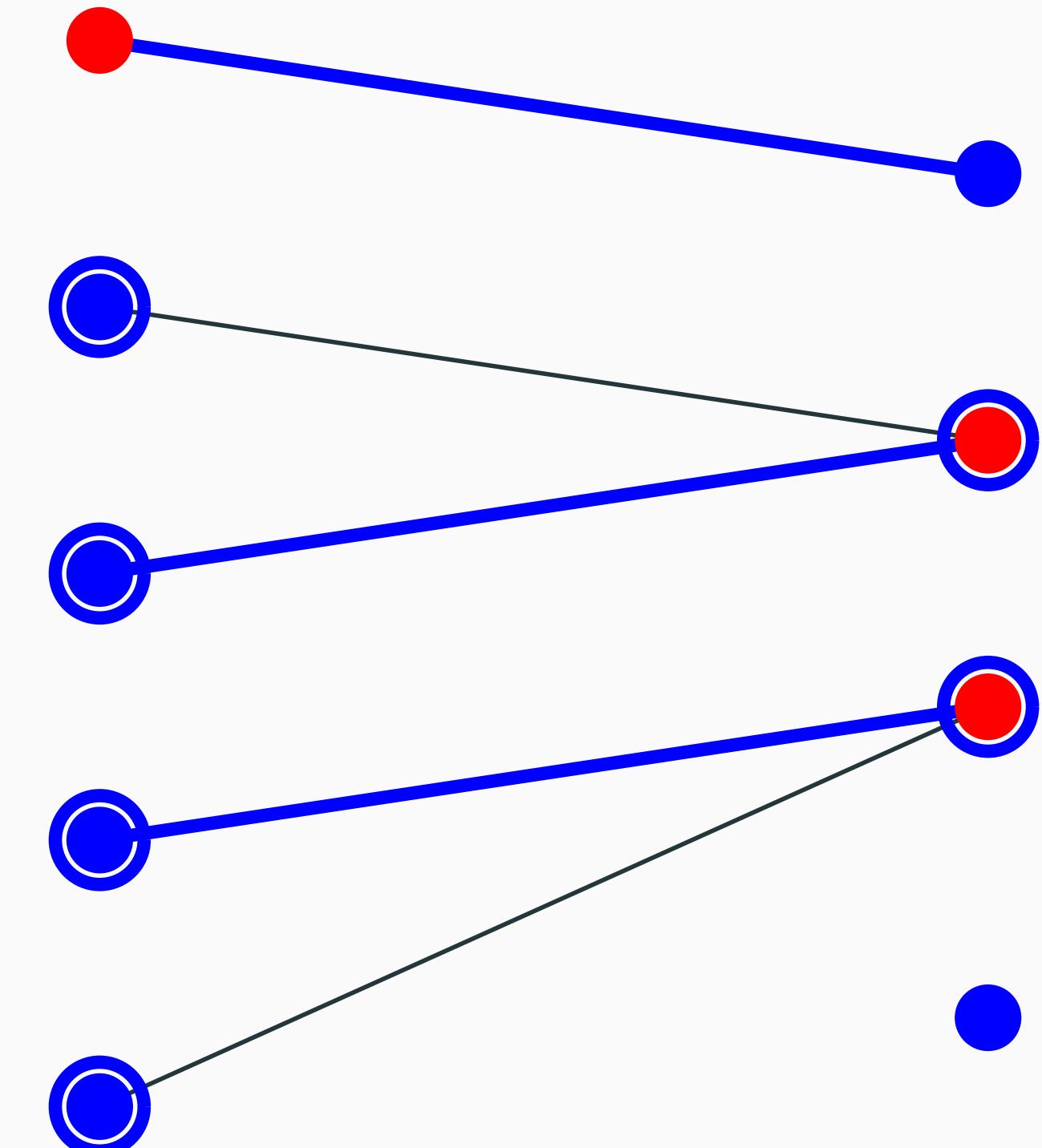
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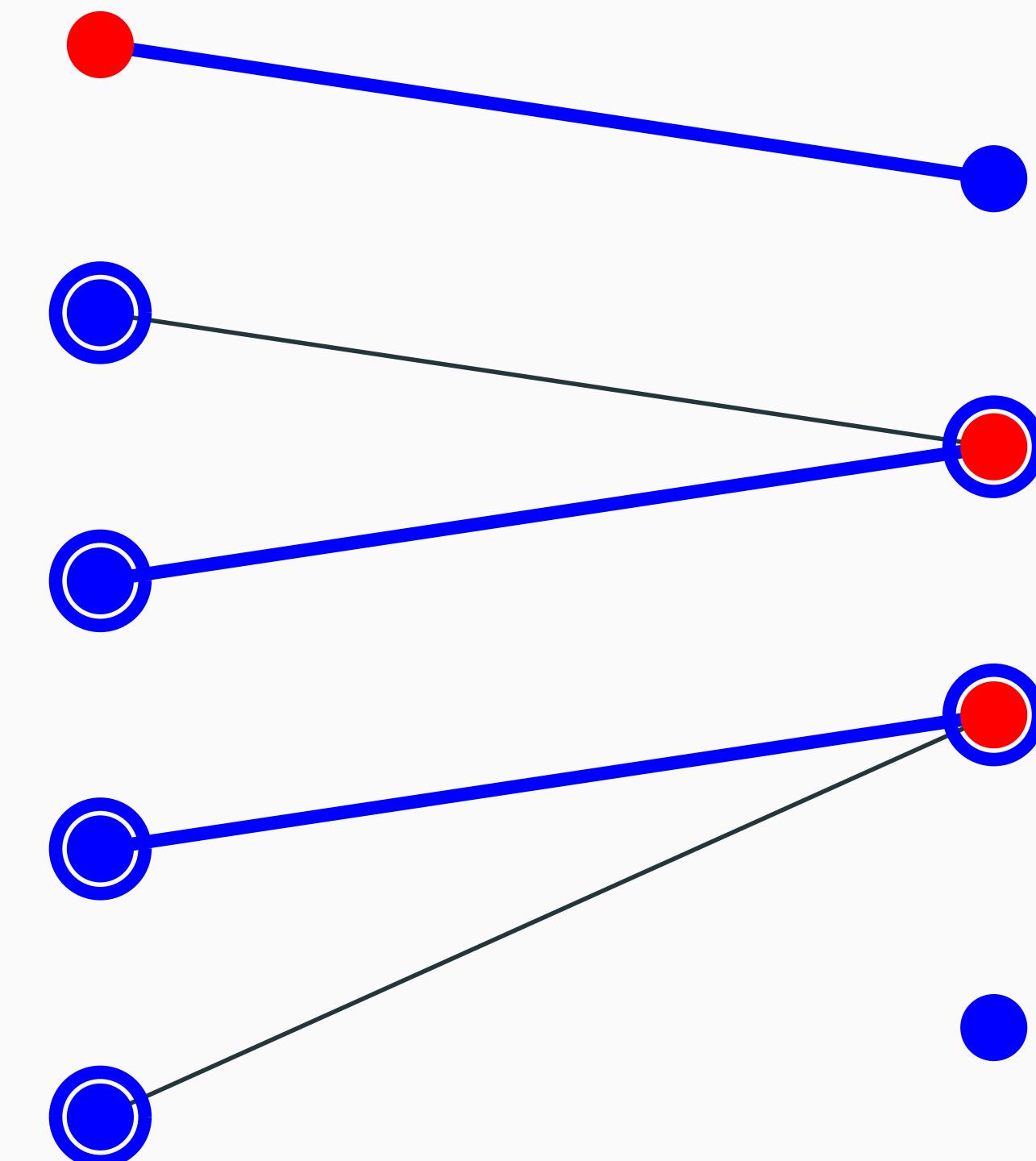
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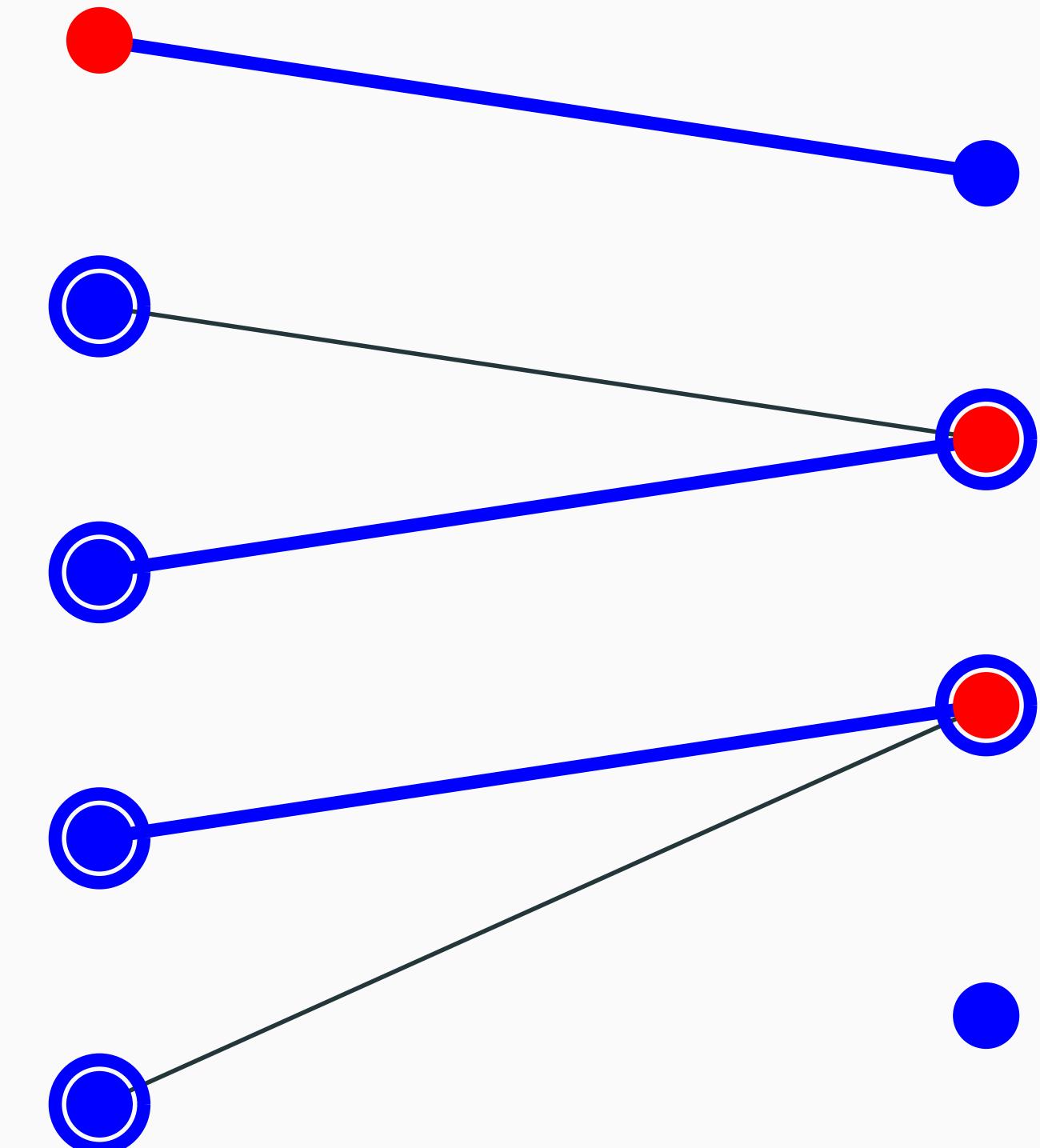
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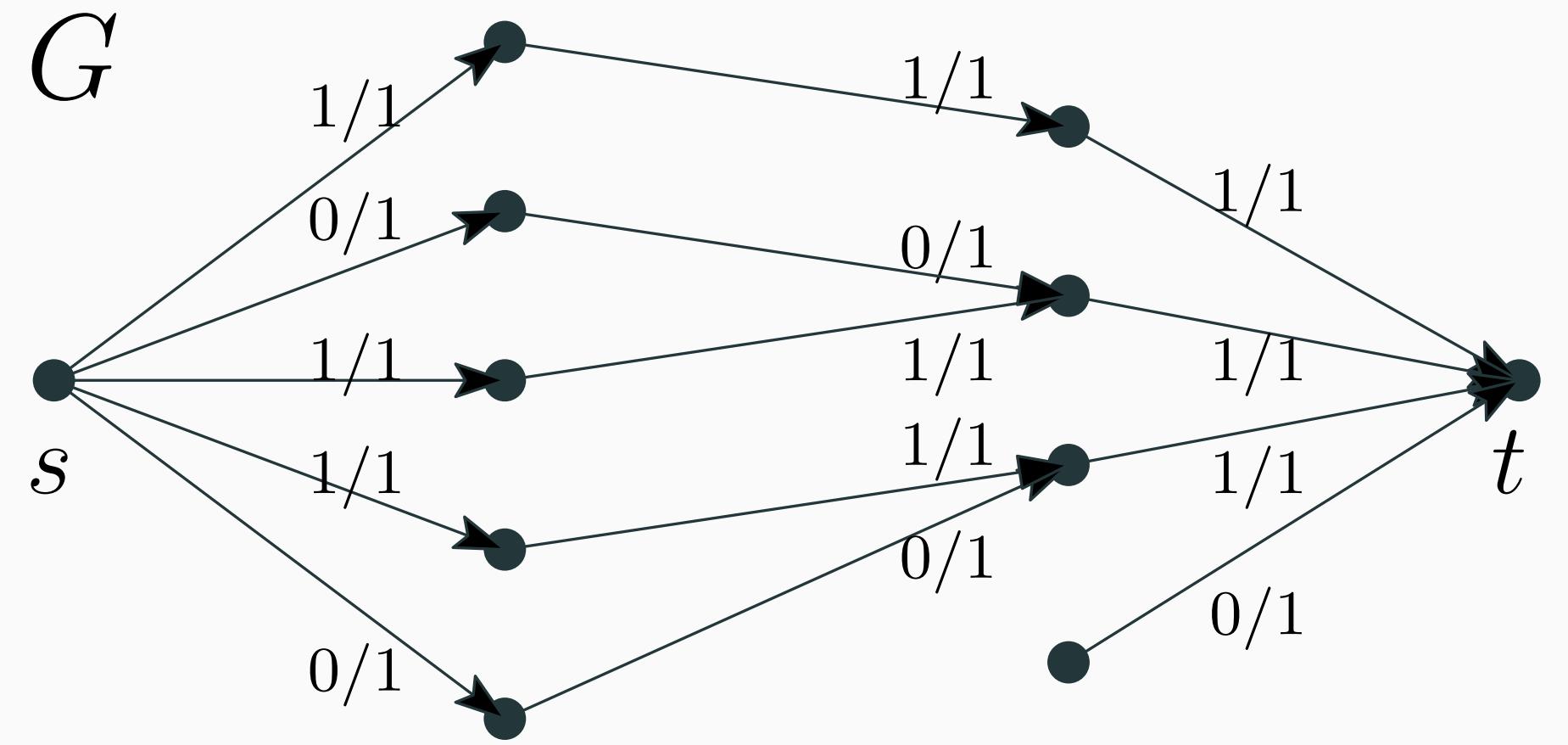
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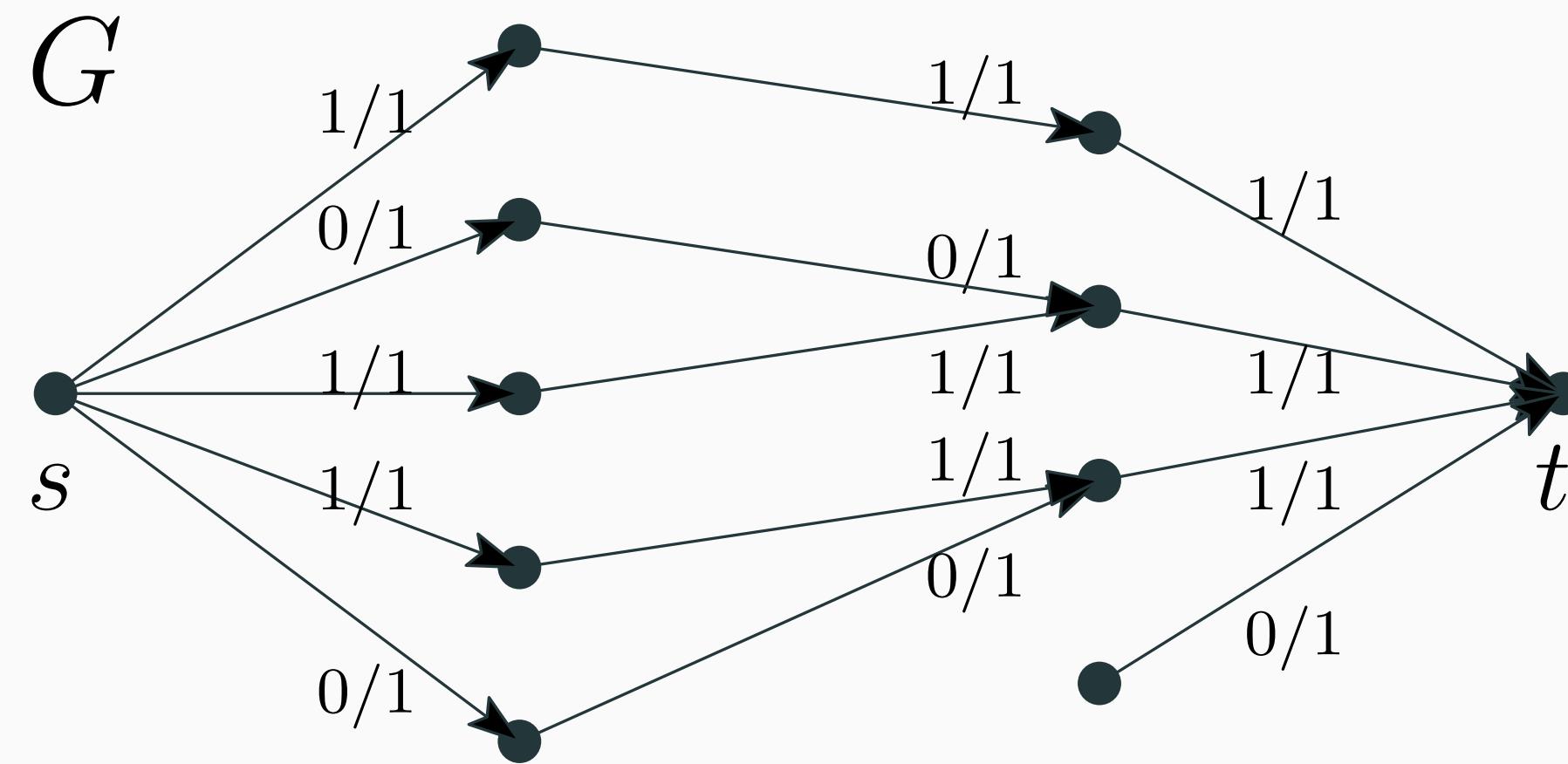
Finding a MinVC or MaxIS in bipartite graphs: step by step

1) Formulate and compute the flow:

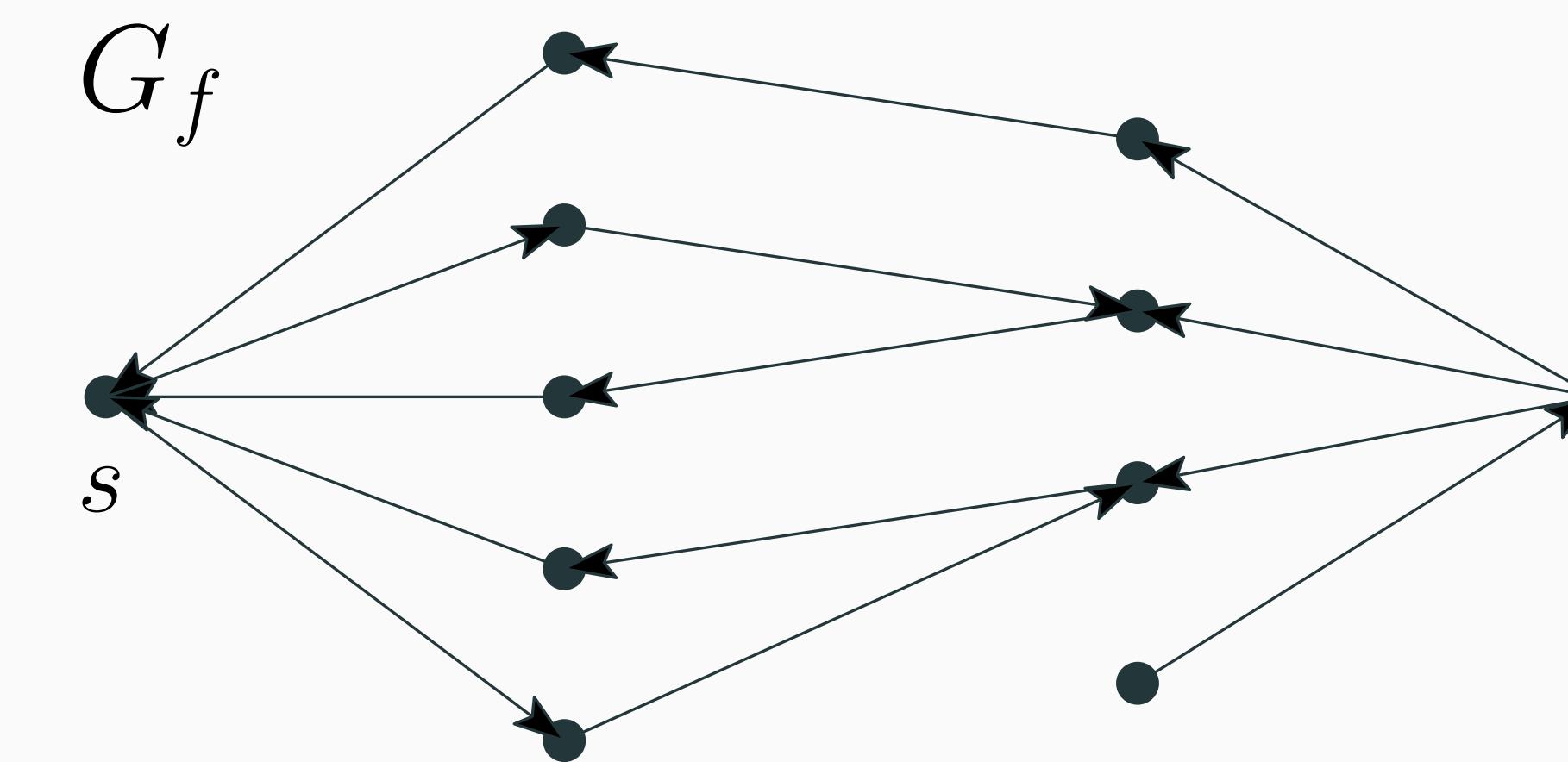


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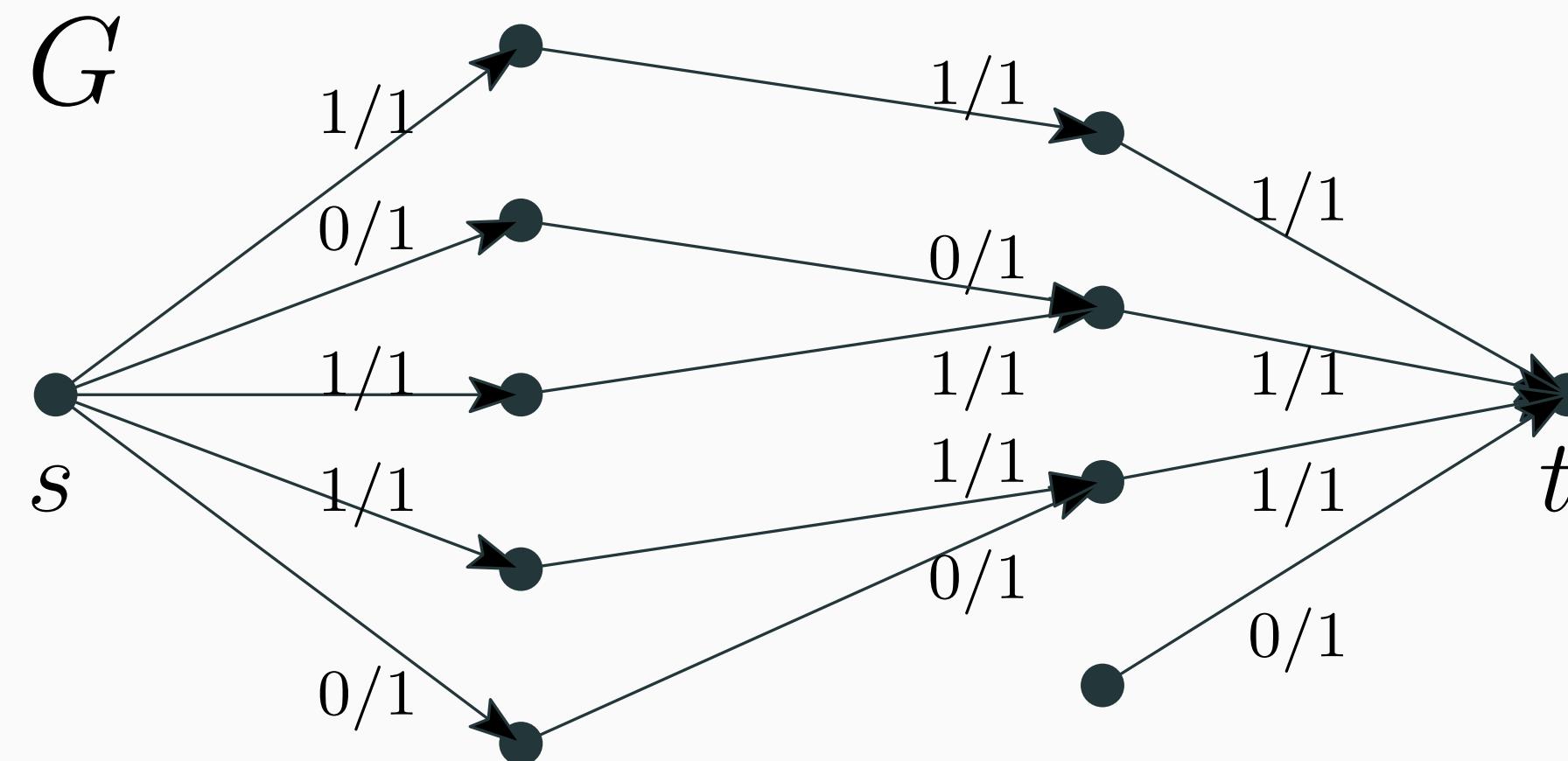


2) Compute the residual graph G_f :

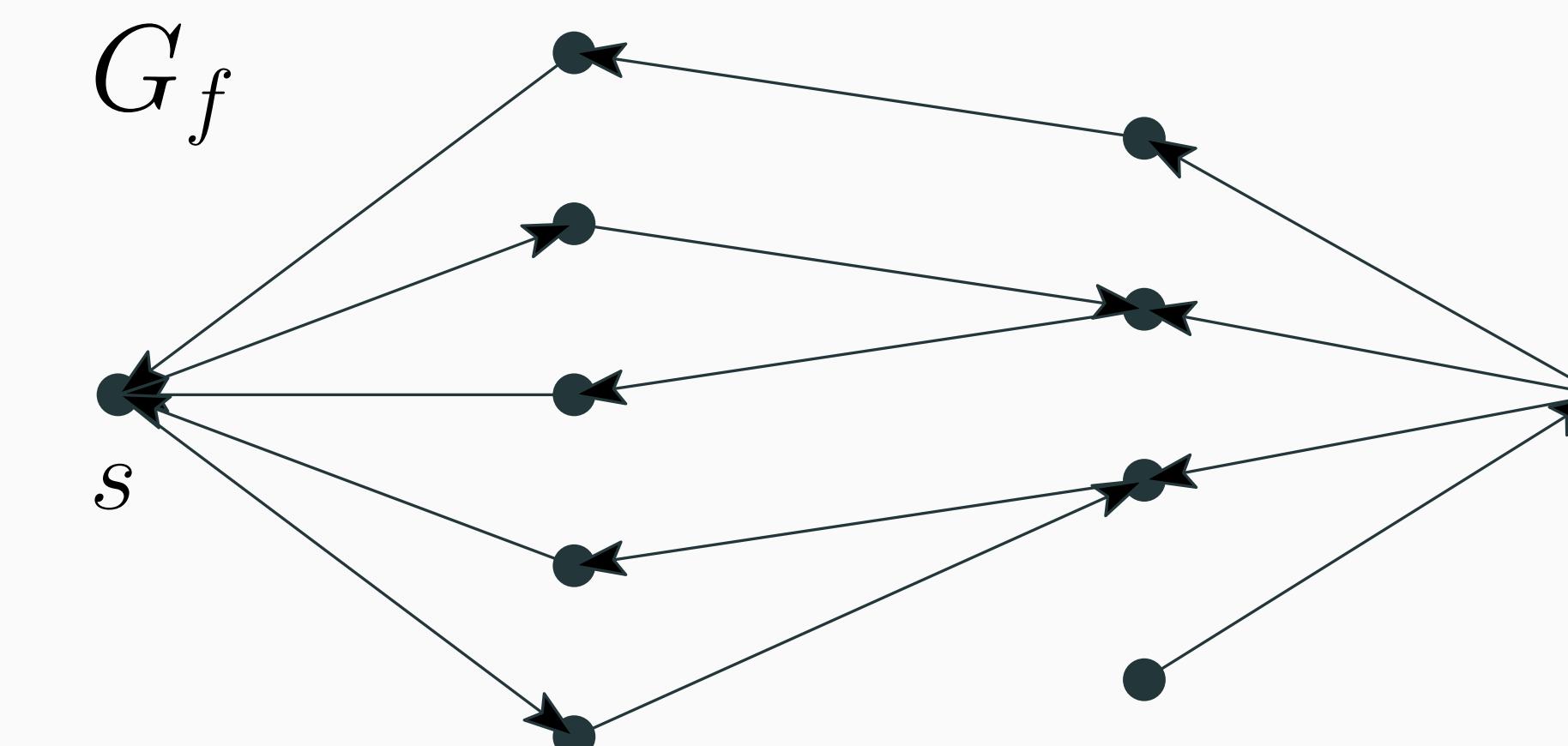


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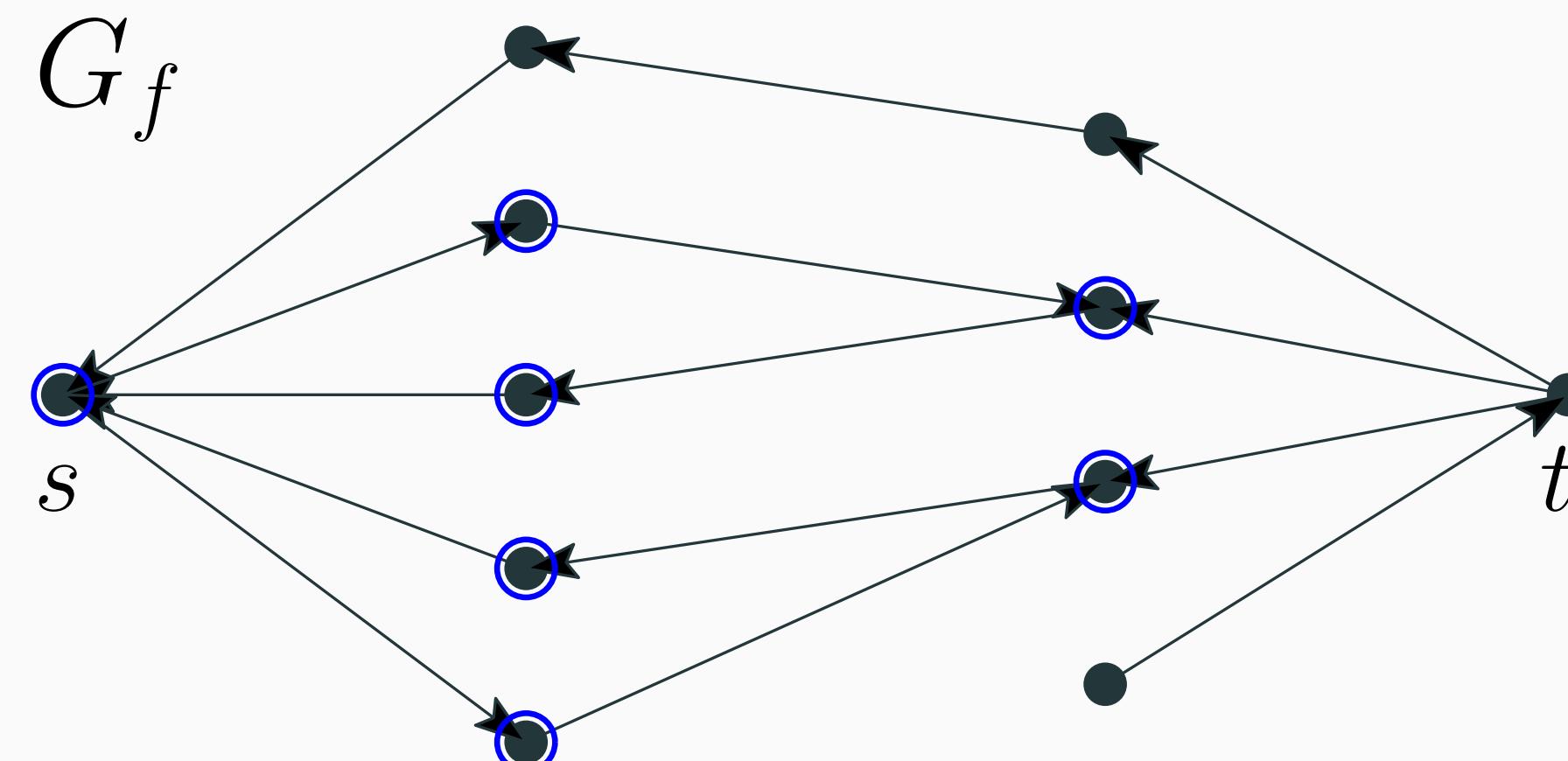
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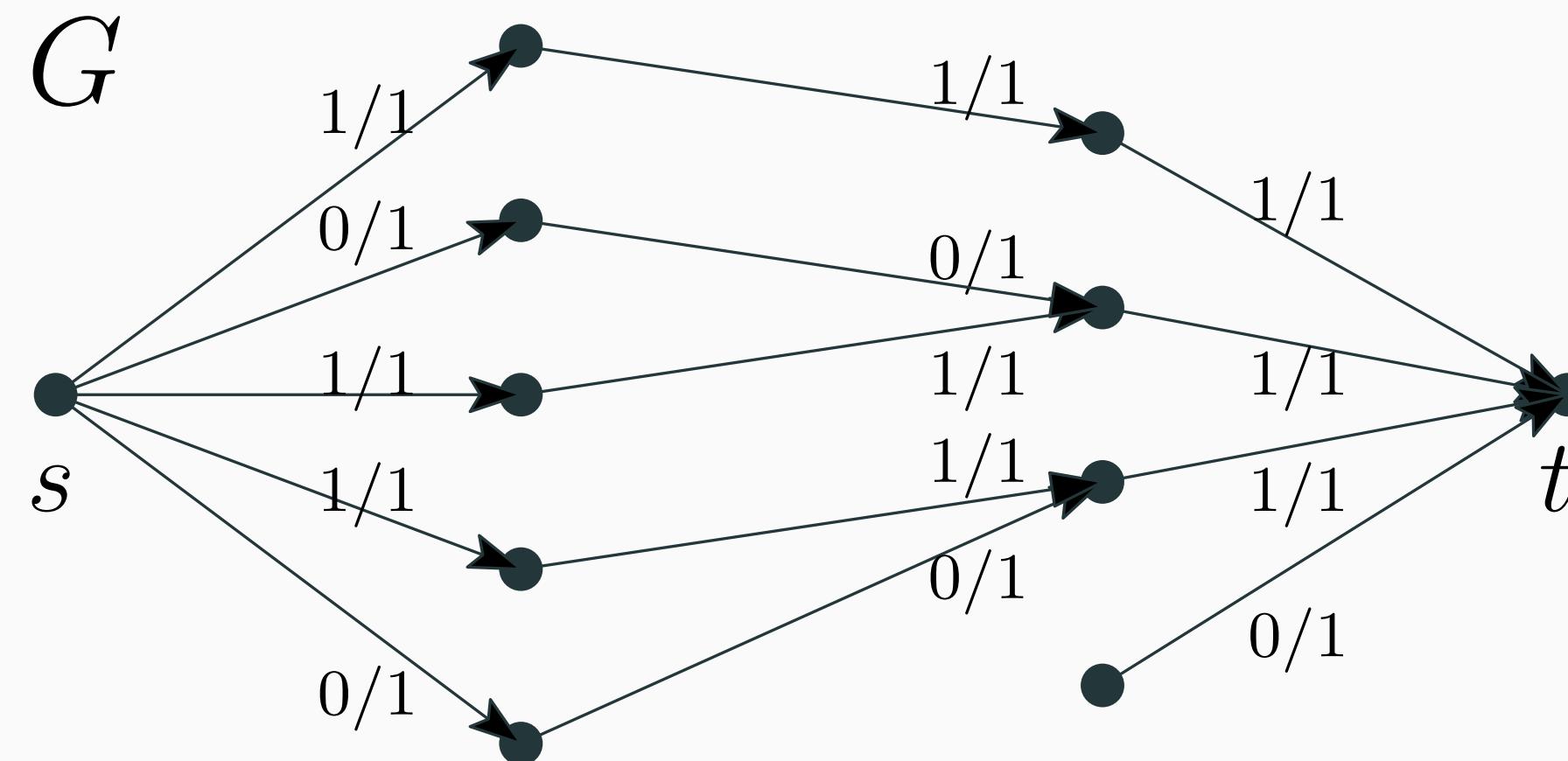


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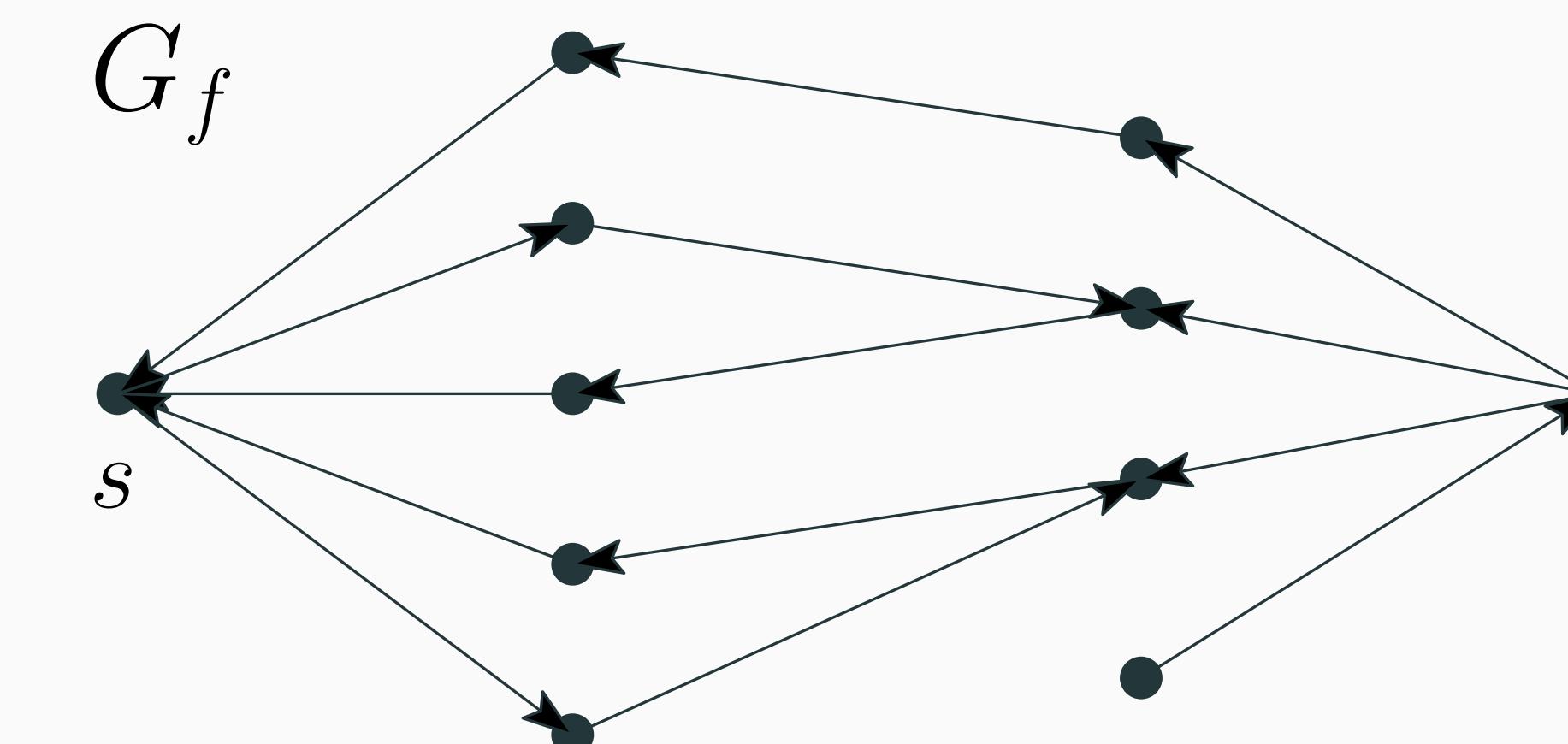


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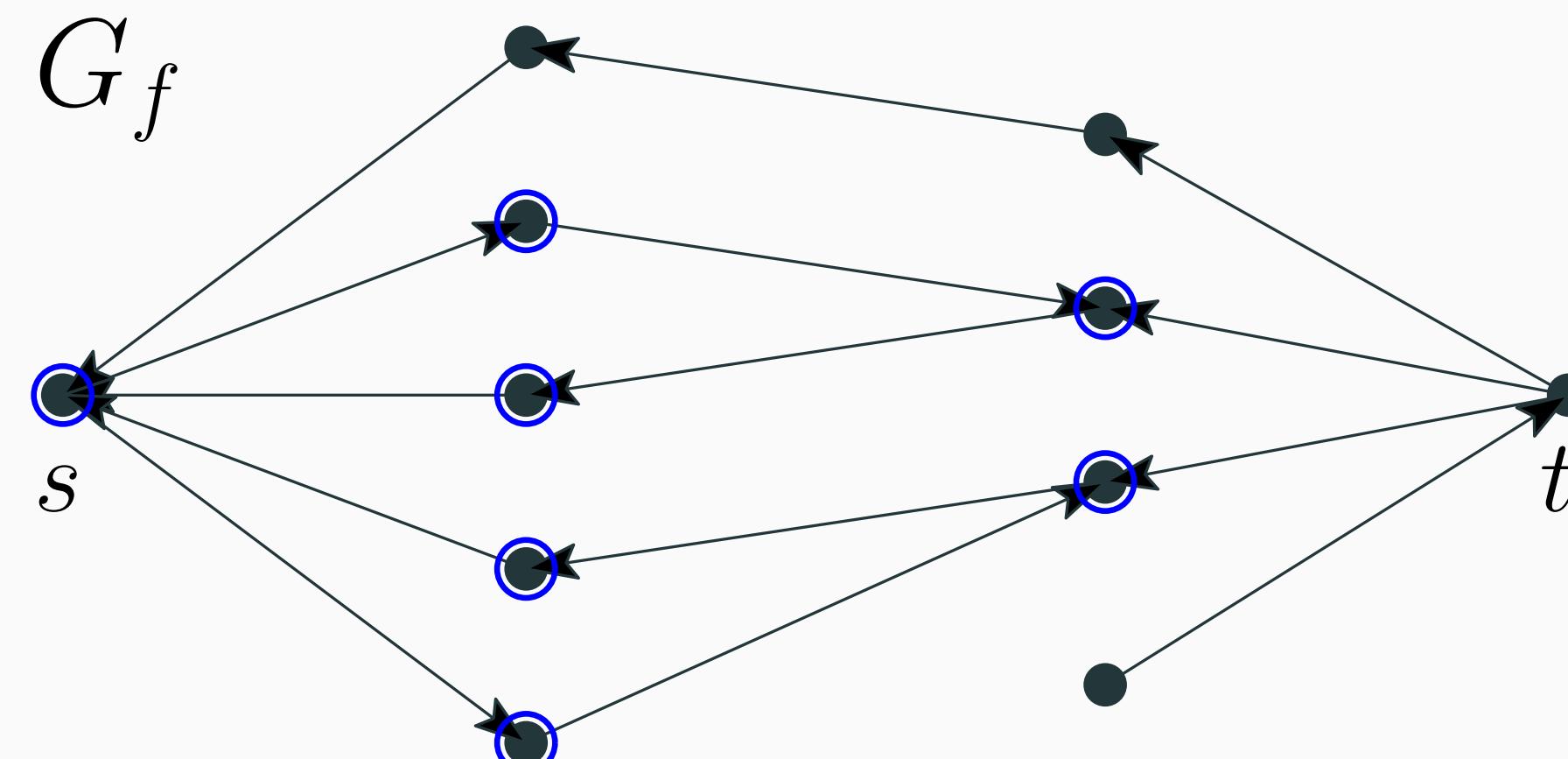
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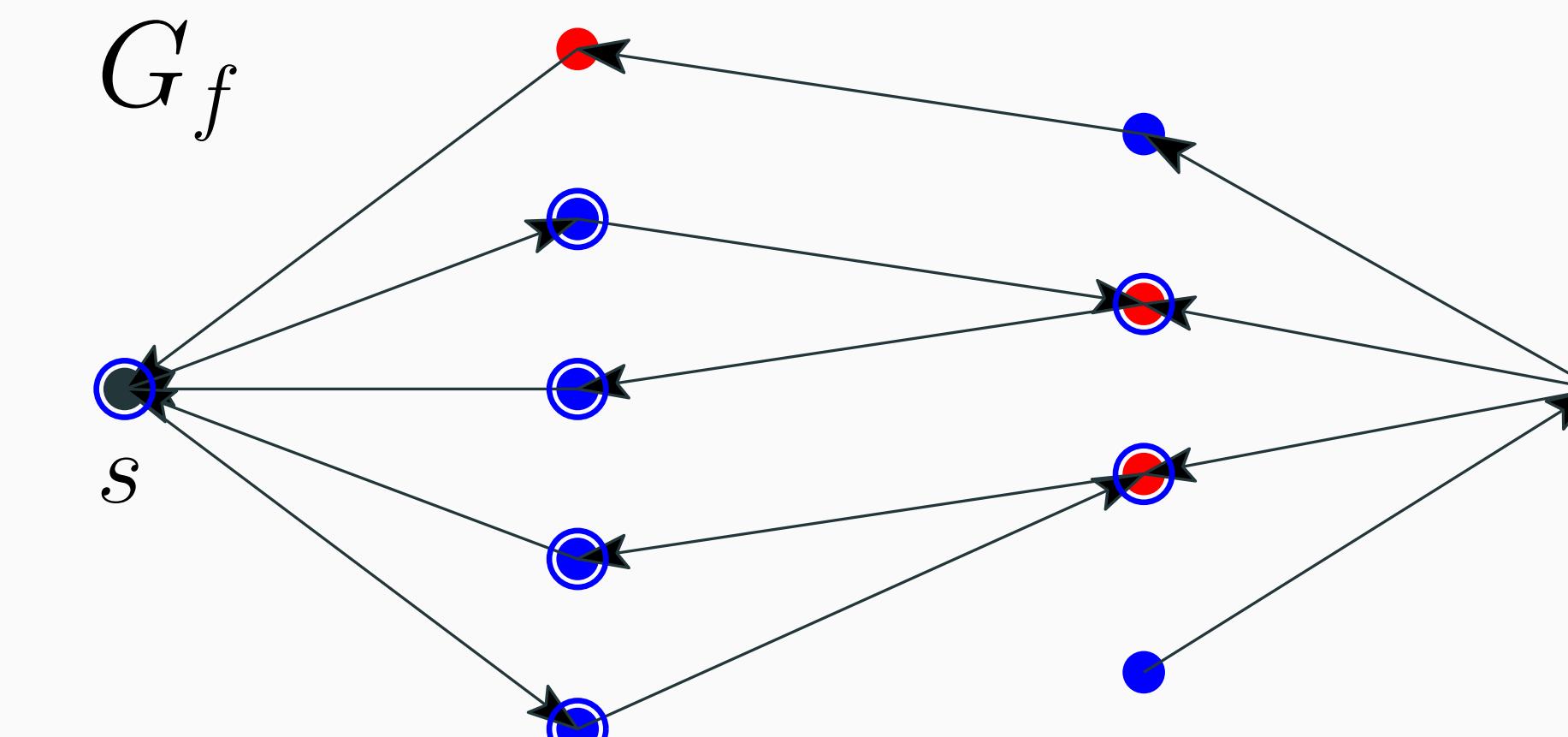
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4) Read the MinVC or MaxIS from the marks:



Summary: MaxFlowMinCut and Bipartite Matching

What you should remember:

Minimum Cut

- ▶ Theorem: maximum amount of any s - t -flow = minimum capacity of any s - t -cut
- ▶ Finding the cut: BFS/DFS on residual graph starting from s .

Vertex Cover

- ▶ Minimum vertex cover and maximum independent set are hard problems.
- ▶ Bipartite graphs allow fast MinVC and MaxIS (both on top of maximum matching).
- ▶ Finding the minimum vertex cover: BFS/DFS on residual graph from s .

Min Cost Max Flow

Minimum Cost for a Bipartite Matching

How to pick the *best* maximum matching? How to break ties among equally large ones?

E.g.: Which set of marriages is the most stable?

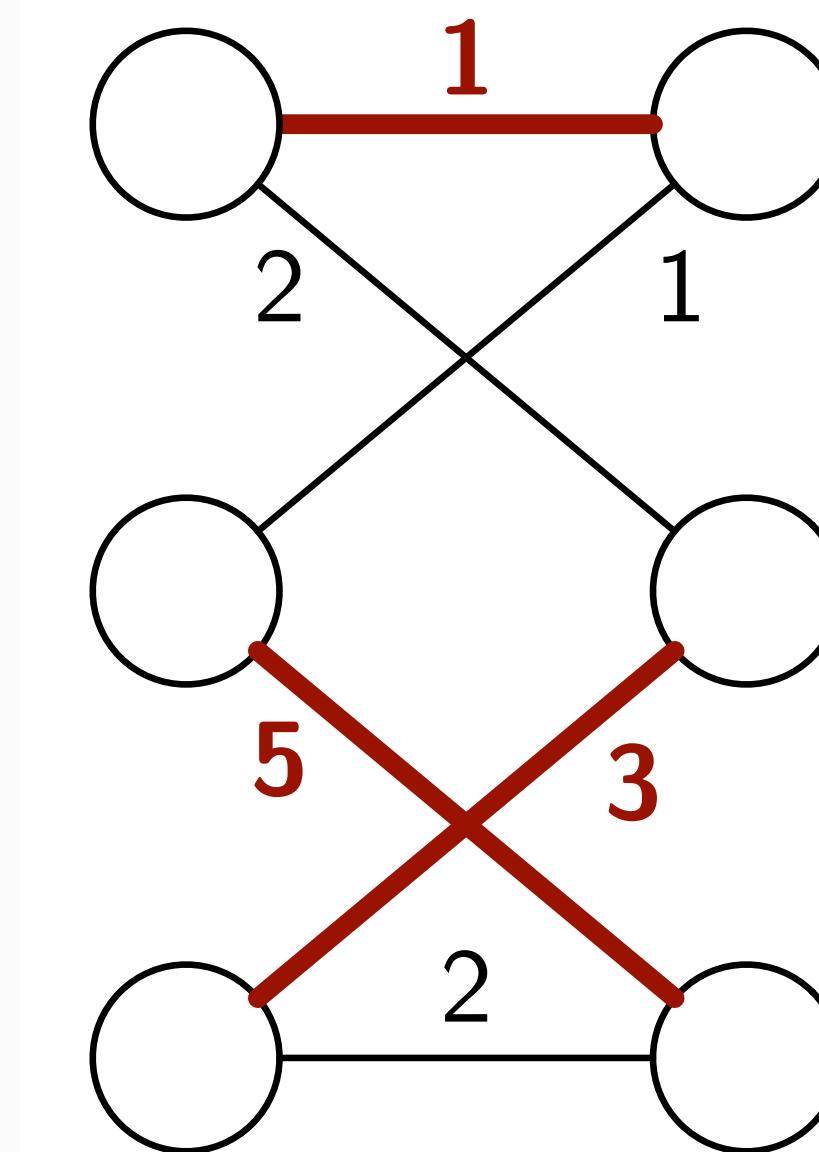
Or: Which consumer/producer pairing is the most effective?

What if the edges in a matching also have costs associated?

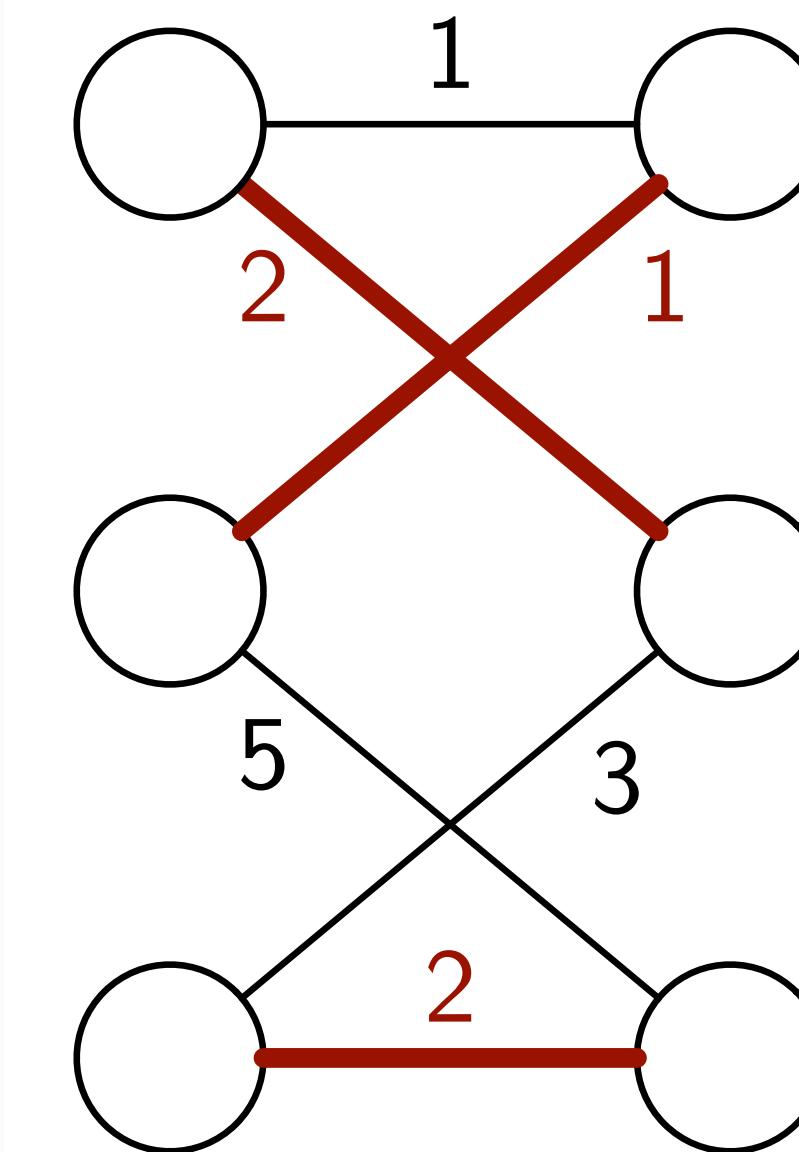
- ▶ cardinality of the matching is no longer the only objective
- ▶ second priority: minimize the total cost

We search for the cheapest among all maximum matchings.

two maximum matchings of different cost:



$$1 + 5 + 3 = 9$$



$$2 + 1 + 2 = 5$$

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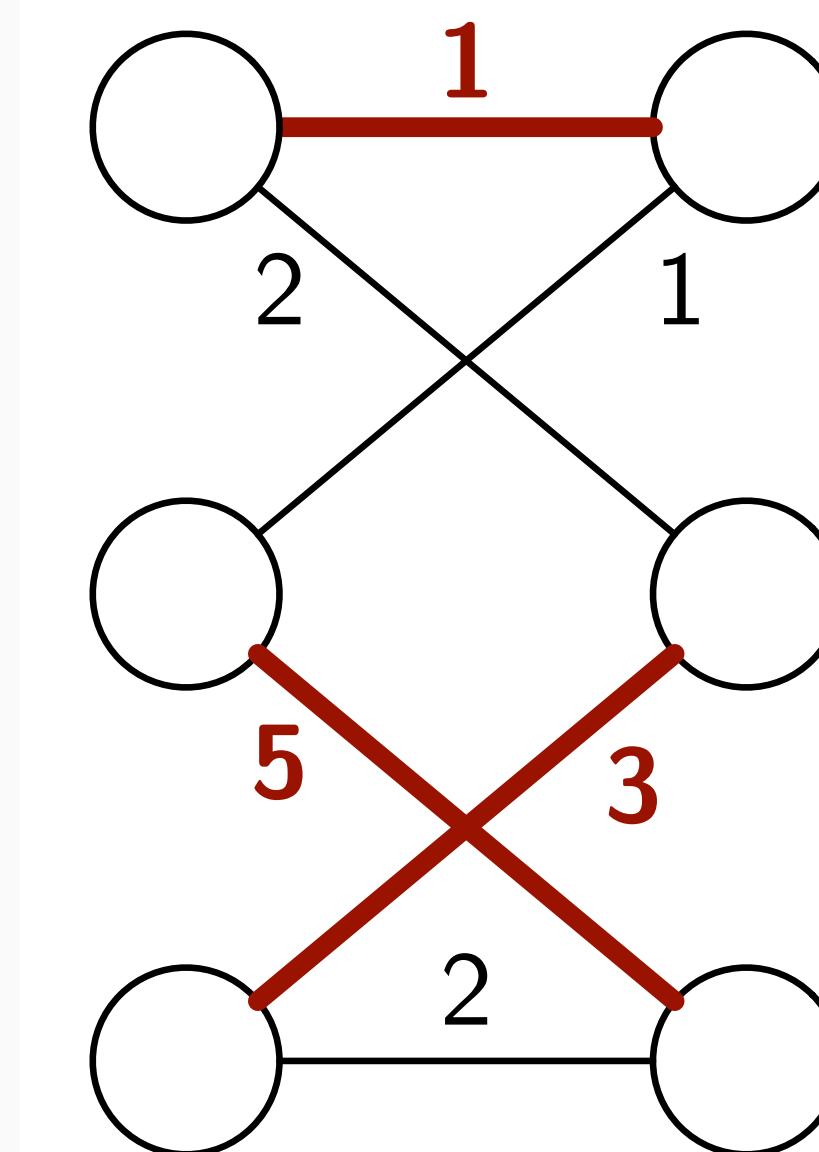
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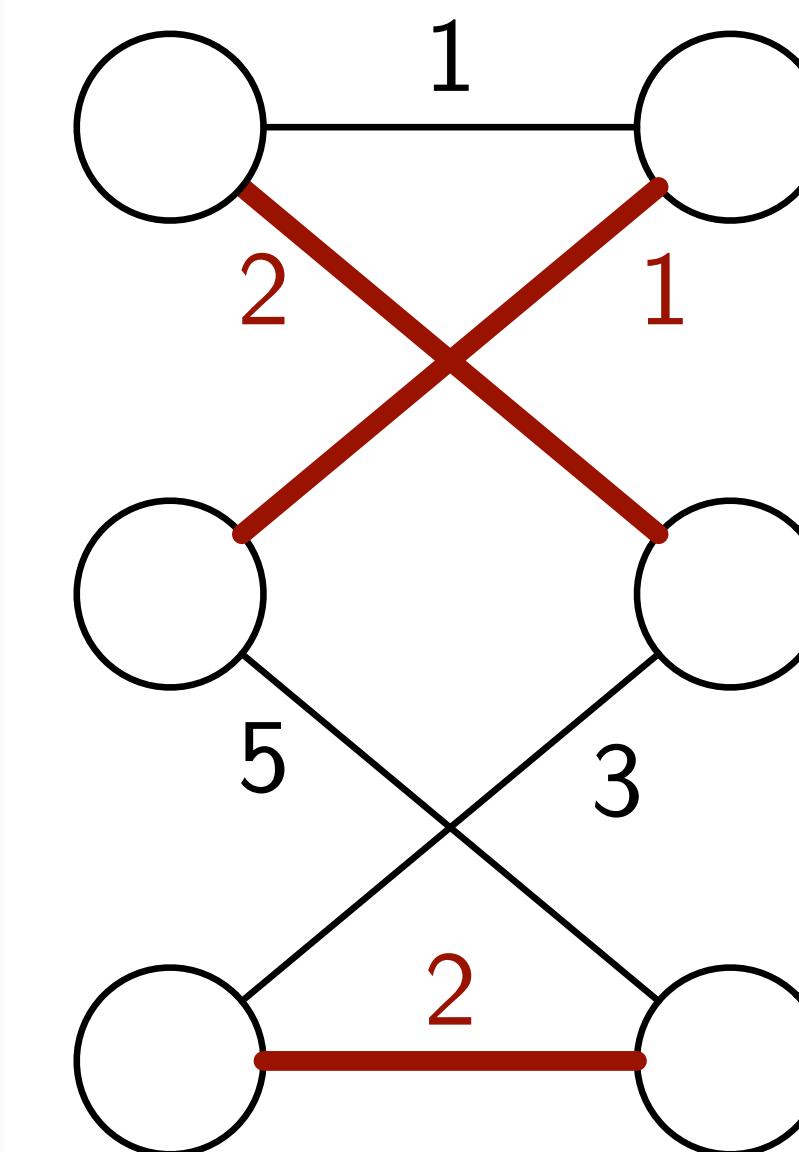
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This does not fit into our model of flows.

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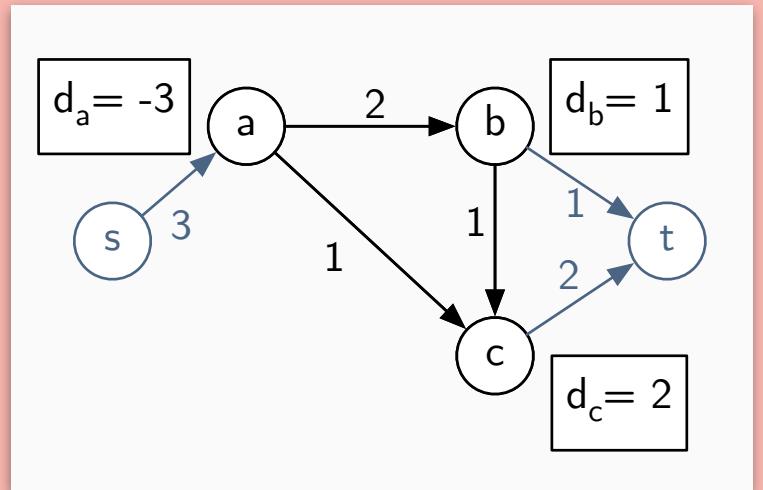
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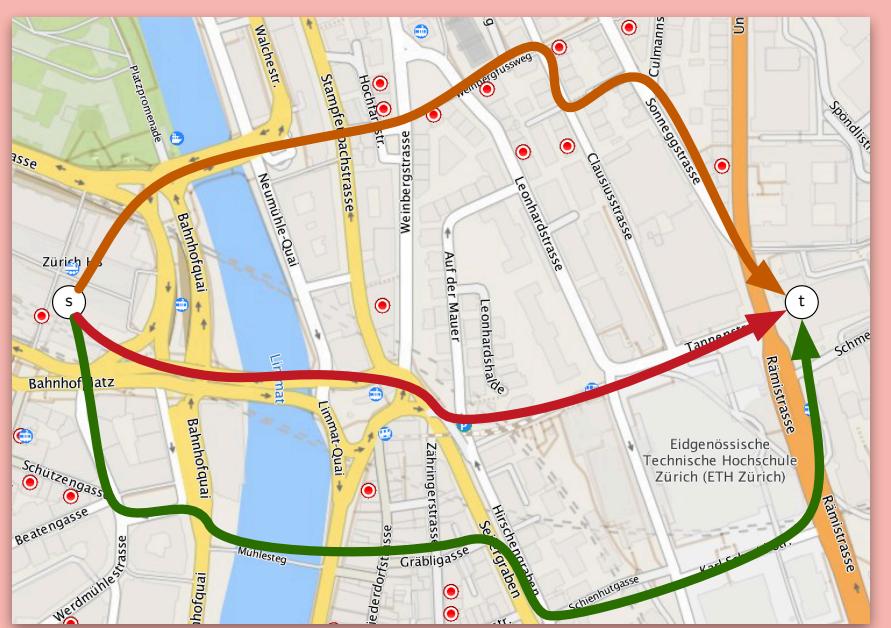
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Problem Landscape

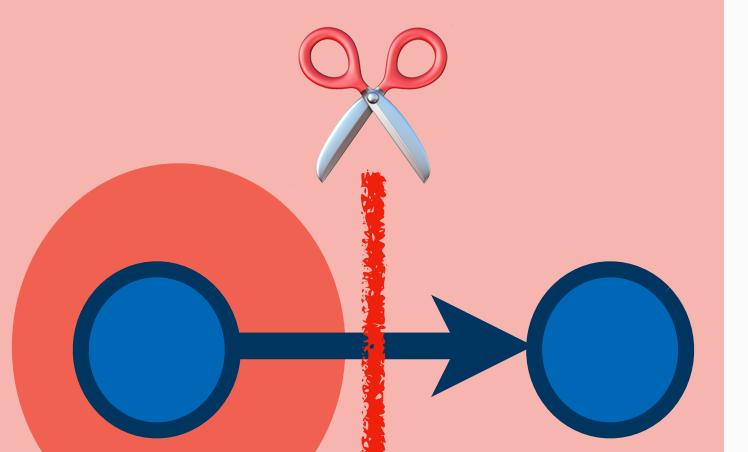
Circulation



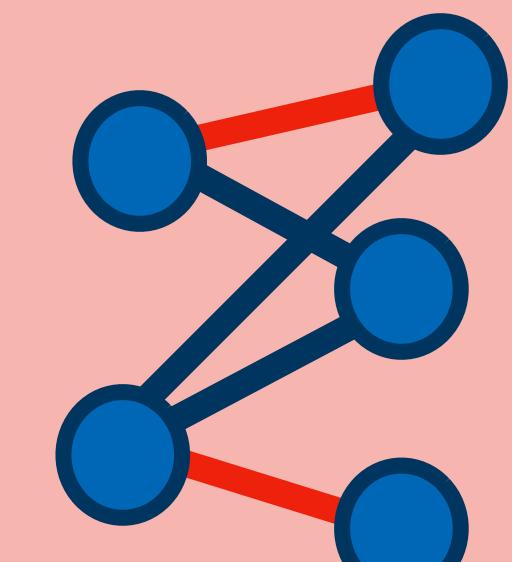
Edge-disjoint Paths



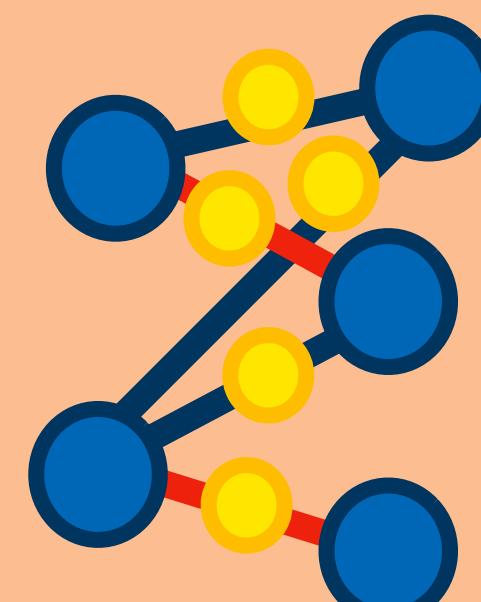
Minimum Cut



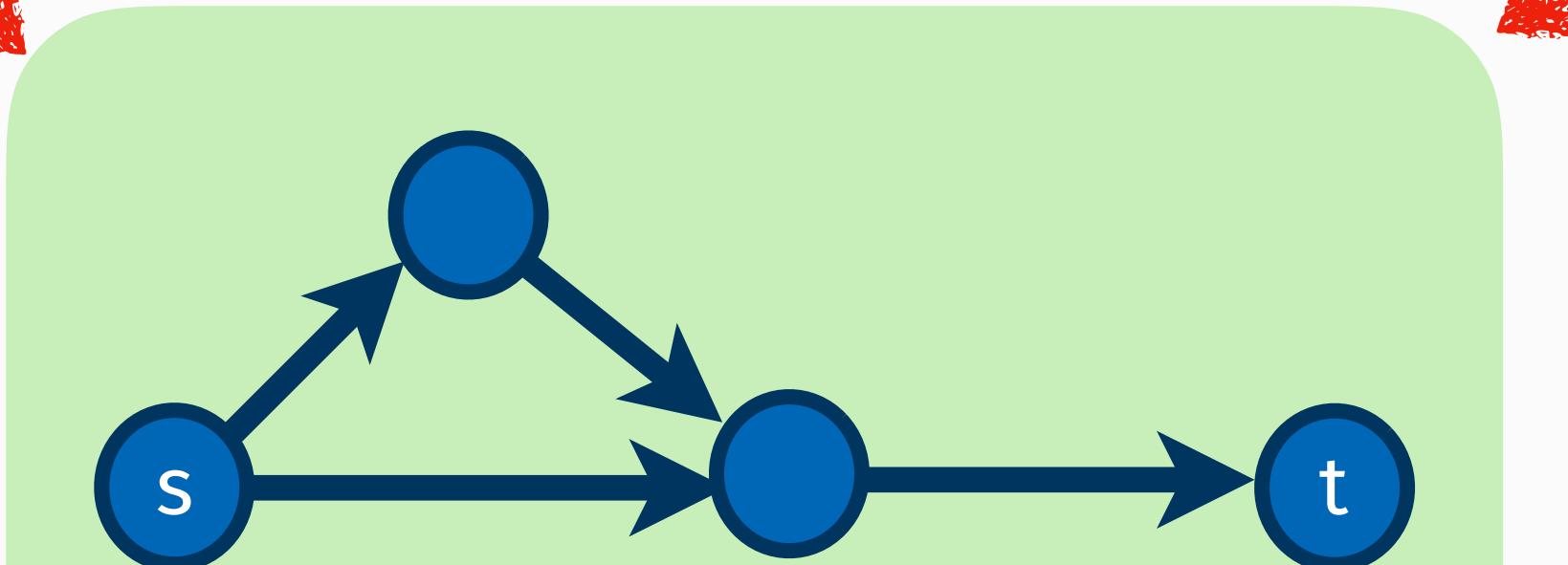
Bipartite Matching



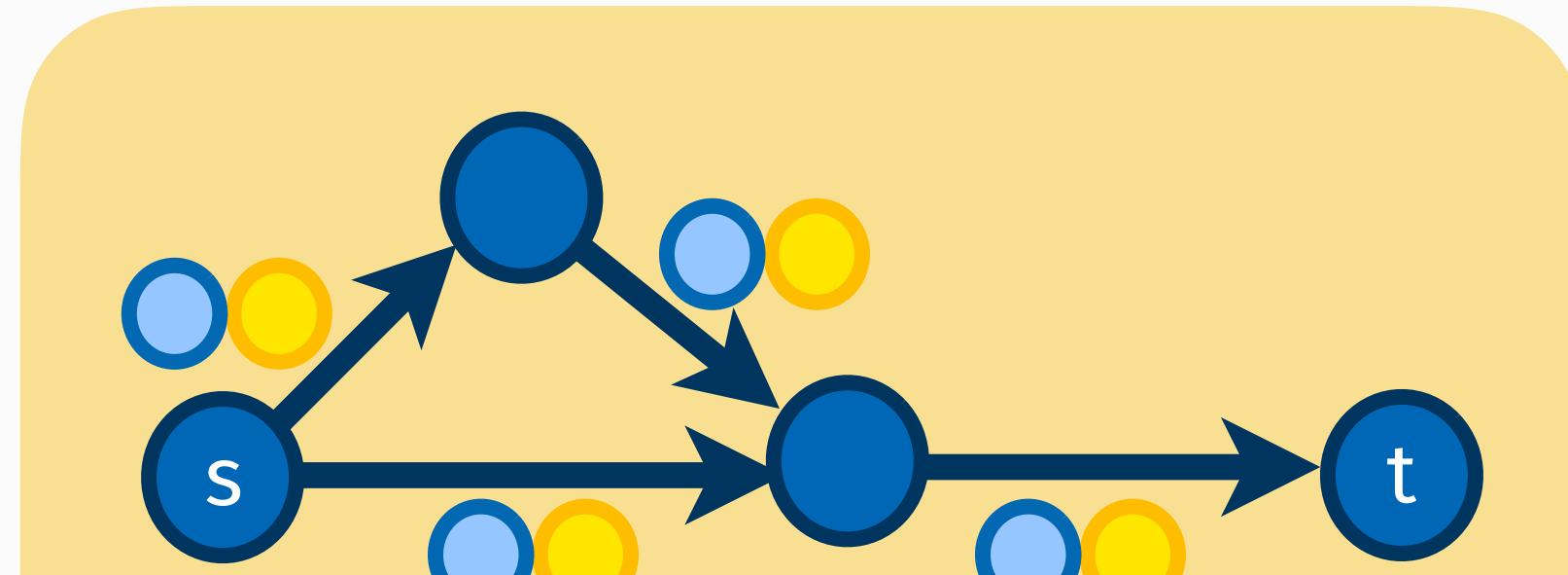
Bipartite Cost Matching



Maximum Flow



Min Cost Max Flow



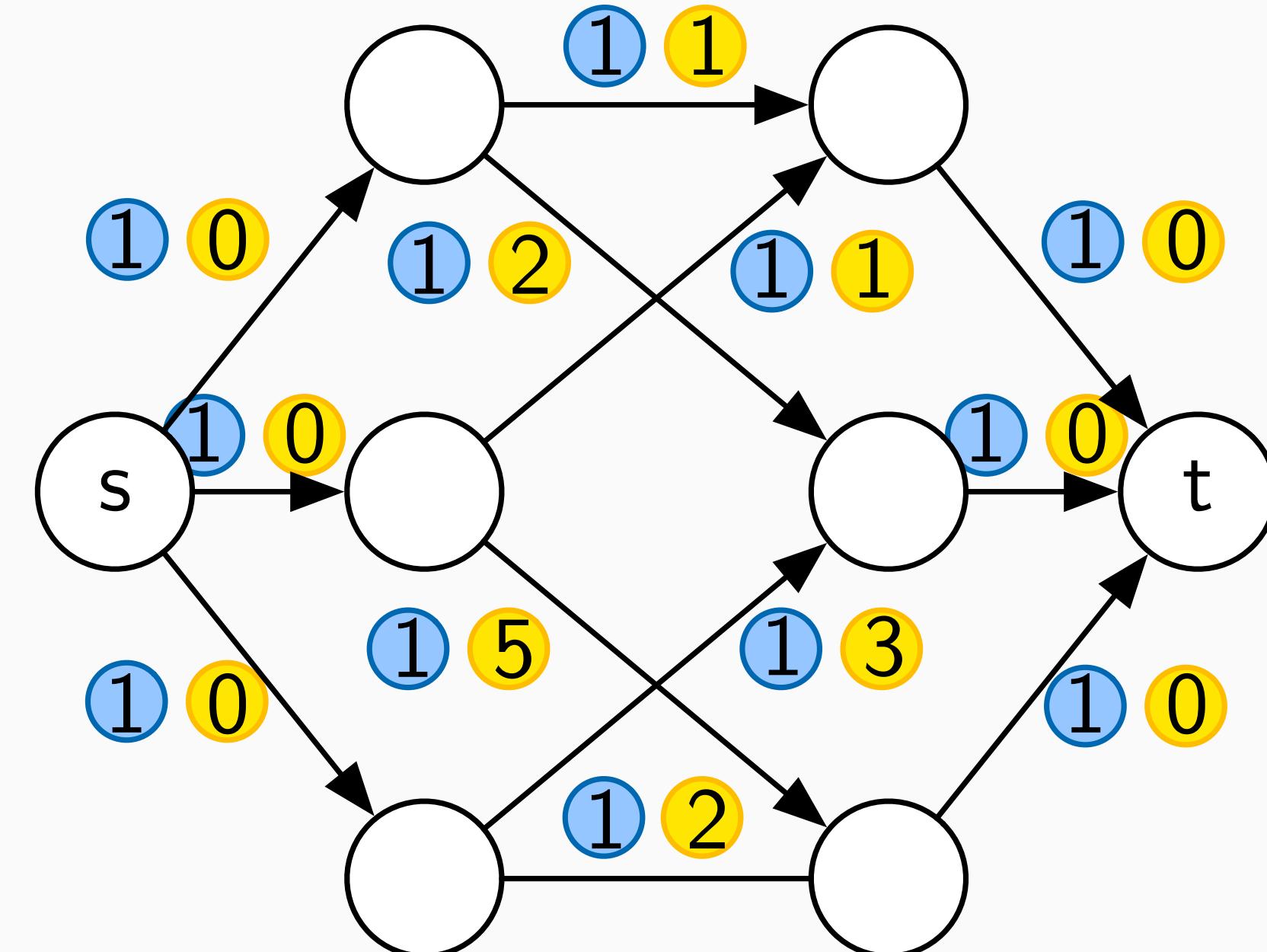
More General Model: Minimum Cost Maximum Flow

We extend the network flow problem by allowing edge costs that occur per unit of flow.

Input: A flow network consisting of

- ▶ directed graph $G = (V, E)$
- ▶ source and sink $s, t \in V$
- ▶ edge capacity $cap : E \rightarrow \mathbb{N}$
- ▶ edge cost $cost : E \rightarrow \mathbb{Z}$.

Output: A flow function f with minimal $cost(f) = \sum_{e \in E} f(e) \cdot cost(e)$ among all flows with maximal $|f|$.



This can model much more than just minimum cost bipartite matching.

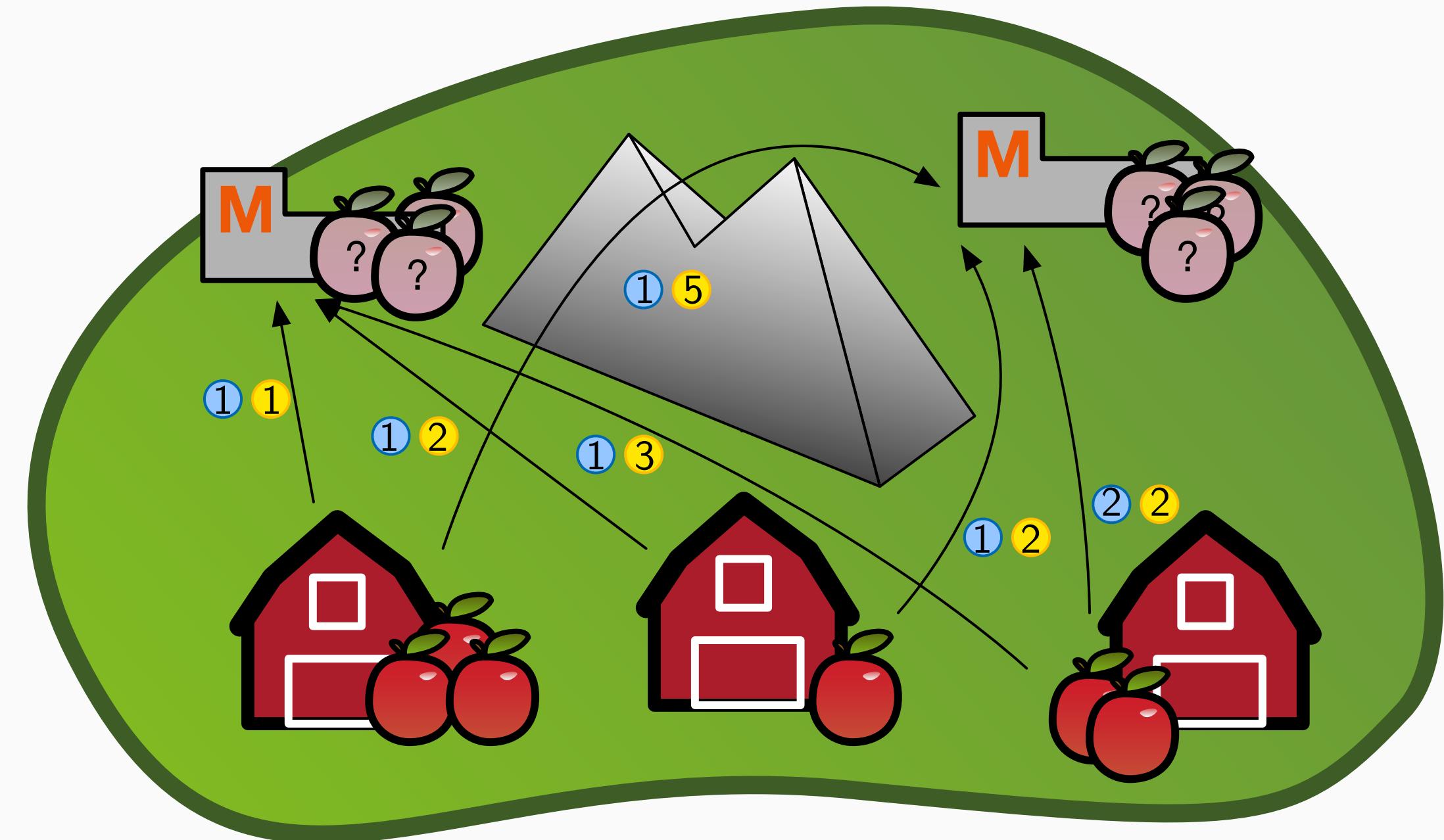
Legend: capacity ● vs ○ cost

Example: Fruit Delivery

Migros wants to schedule fruit deliveries from their farmers to their shops.

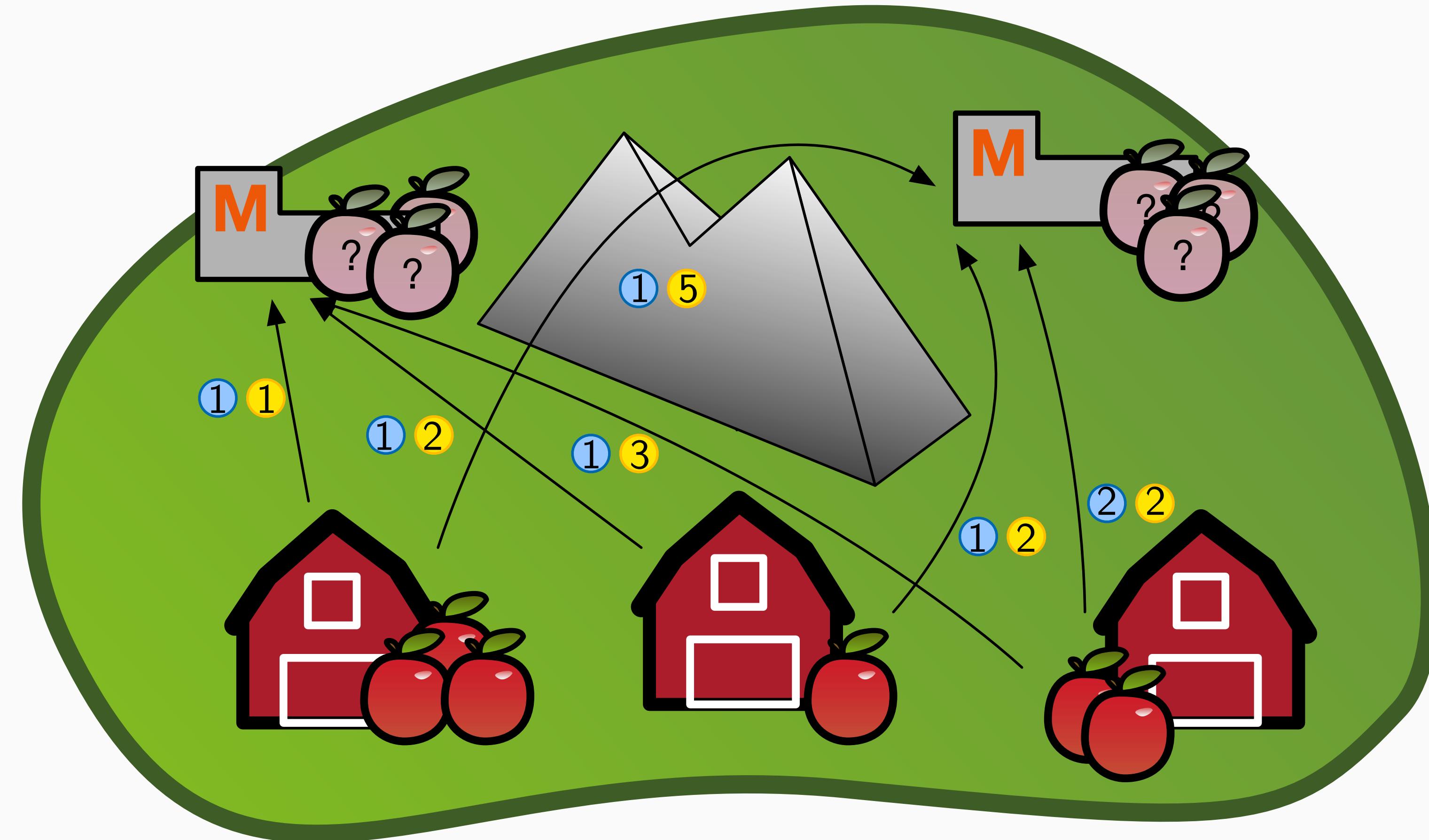
They know all important parameters;

- ▶ production per farm [in kg]
- ▶ demand per shop [in kg]
- ▶ transportation capacity [in kg] and
transportation cost [in Fr. pro kg]
for every farm-shop pair



Note: This is not just a bipartite matching, even though the graph is bipartite.
One farm might deliver to multiple shops (and vice versa).

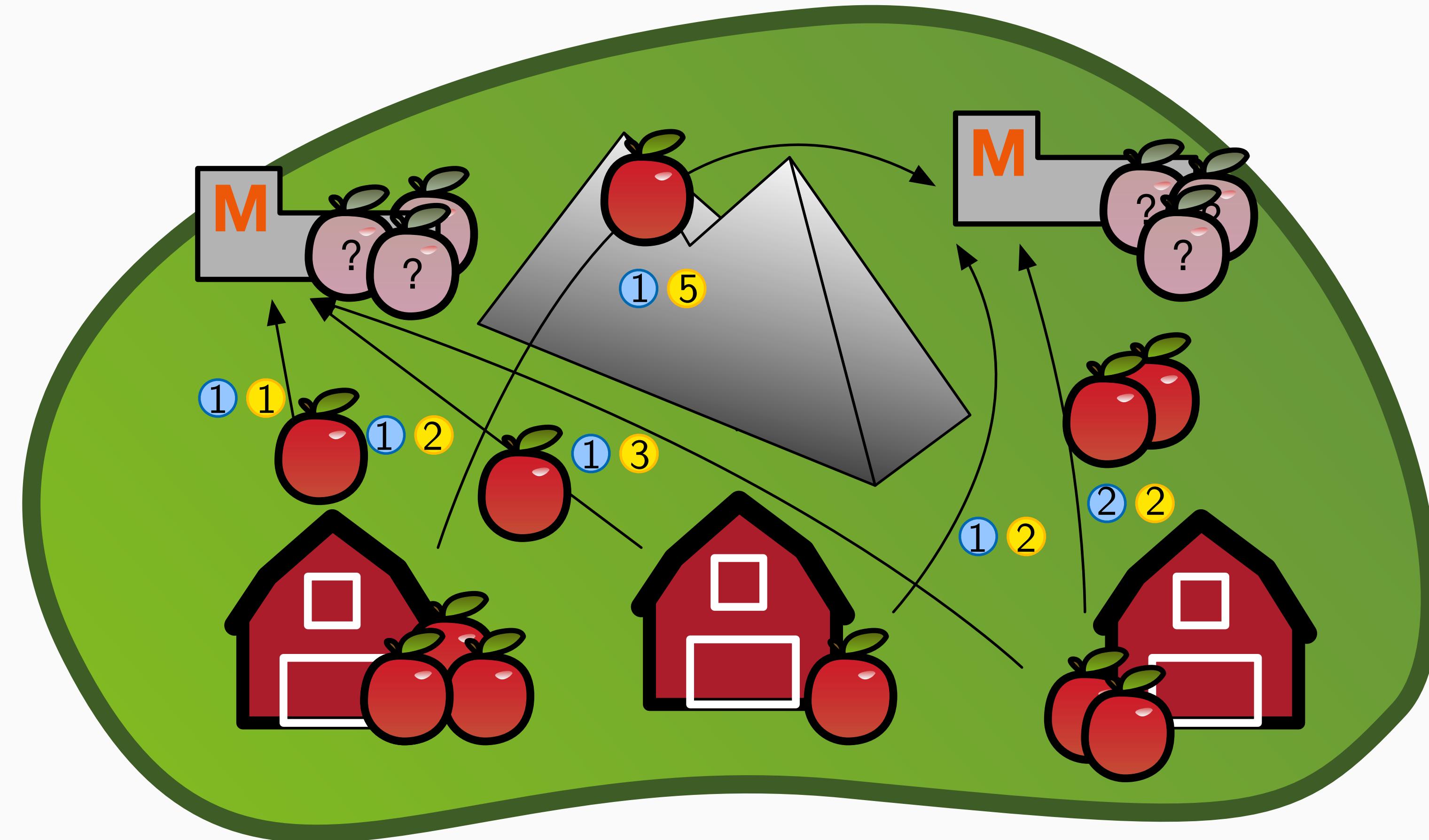
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Flow: $2 + 1 + 2 = 5$

Cost: $1 \cdot 1 + 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 2 = 12$

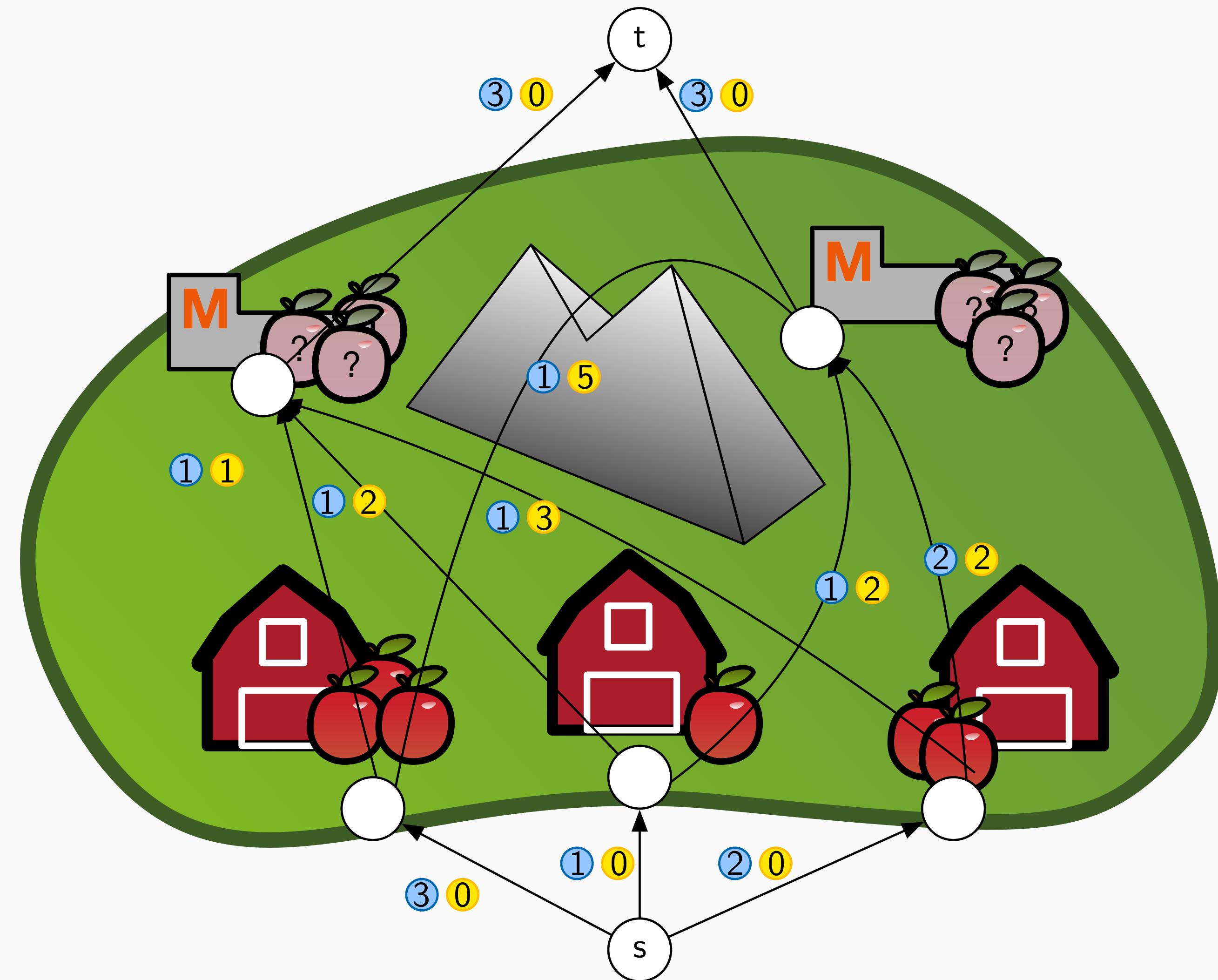
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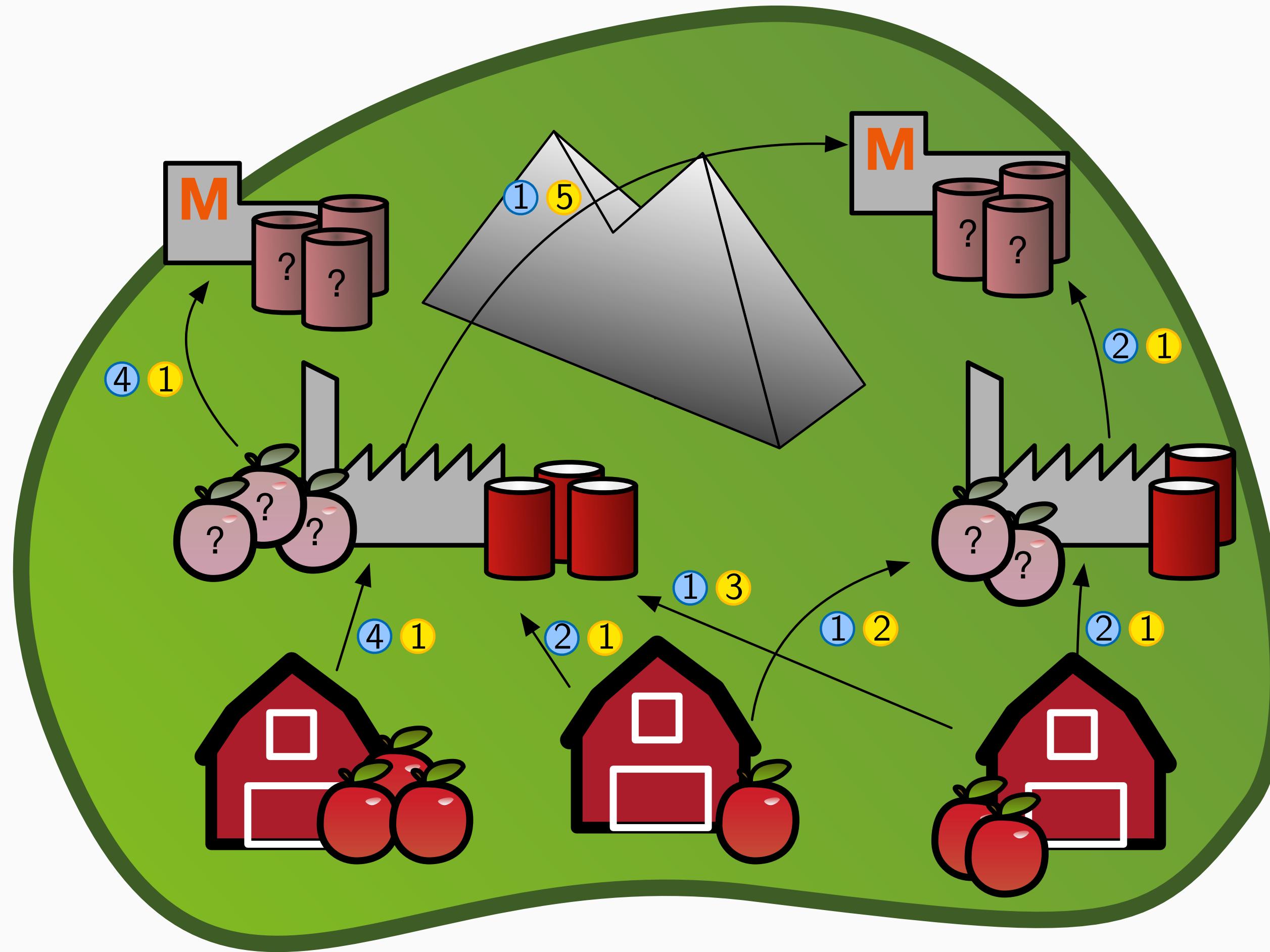
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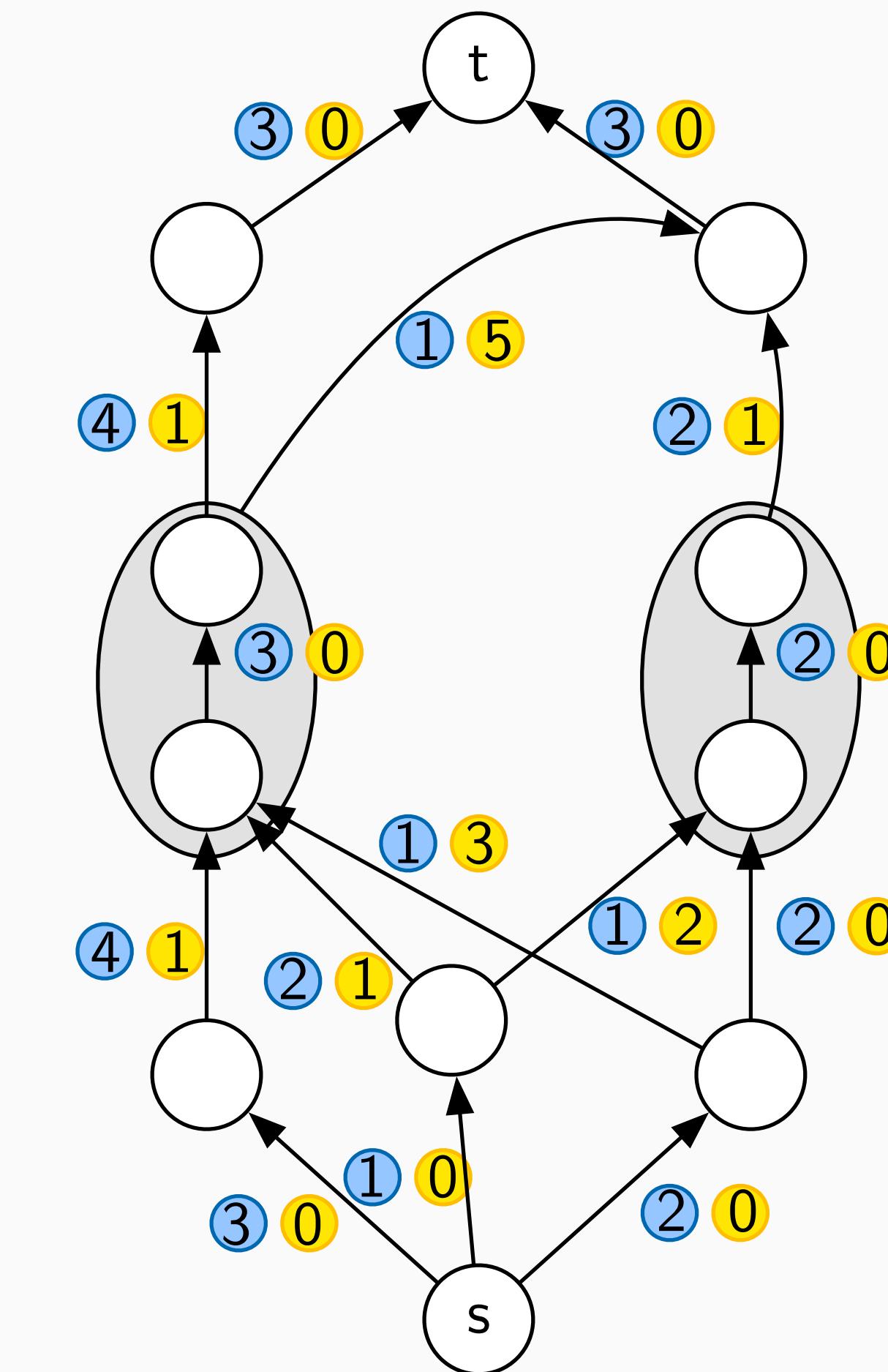
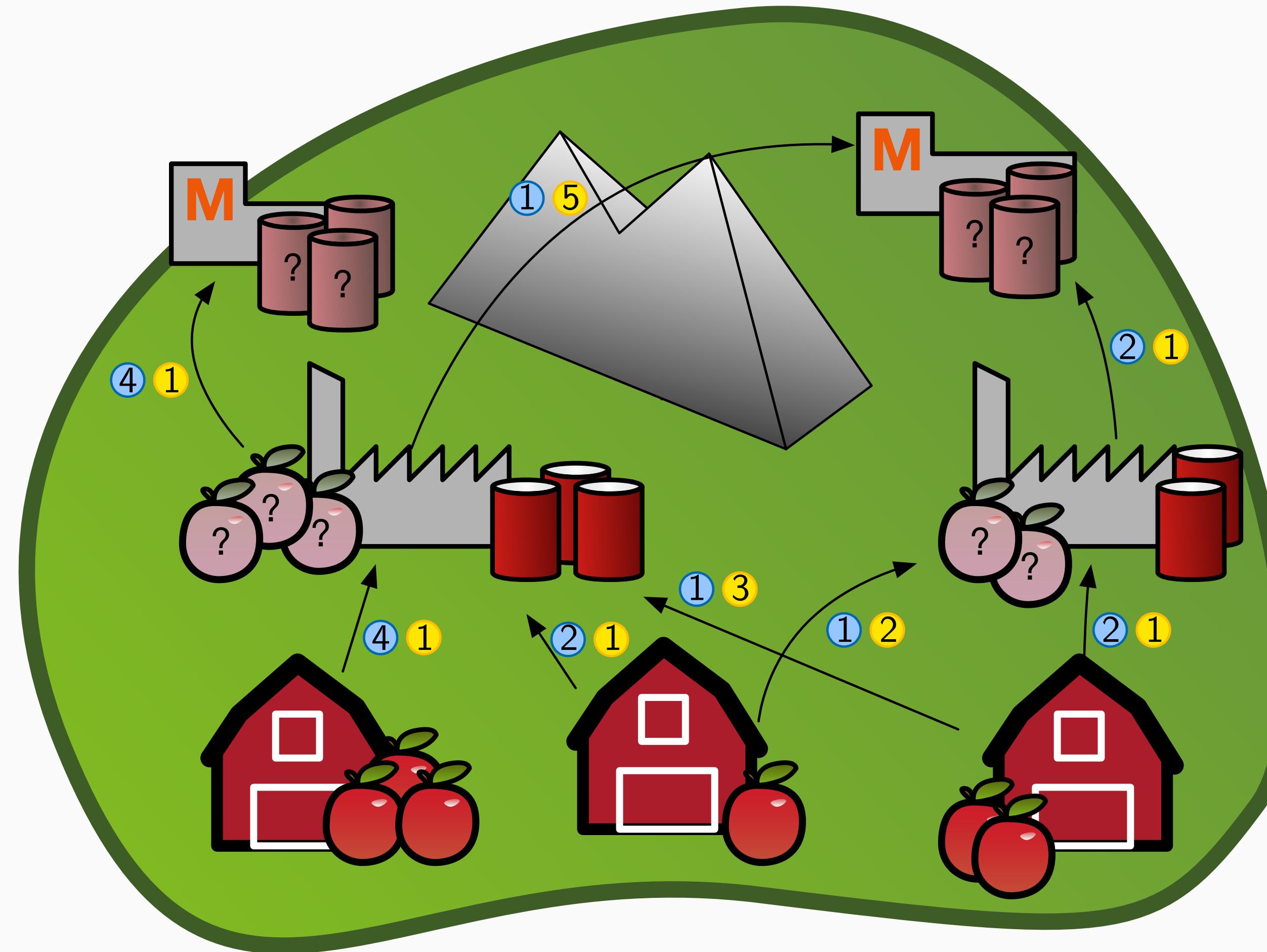
Extended Example: Canned Fruit Delivery

Extension: Canned fruit requires transportation to and from a canning factory.



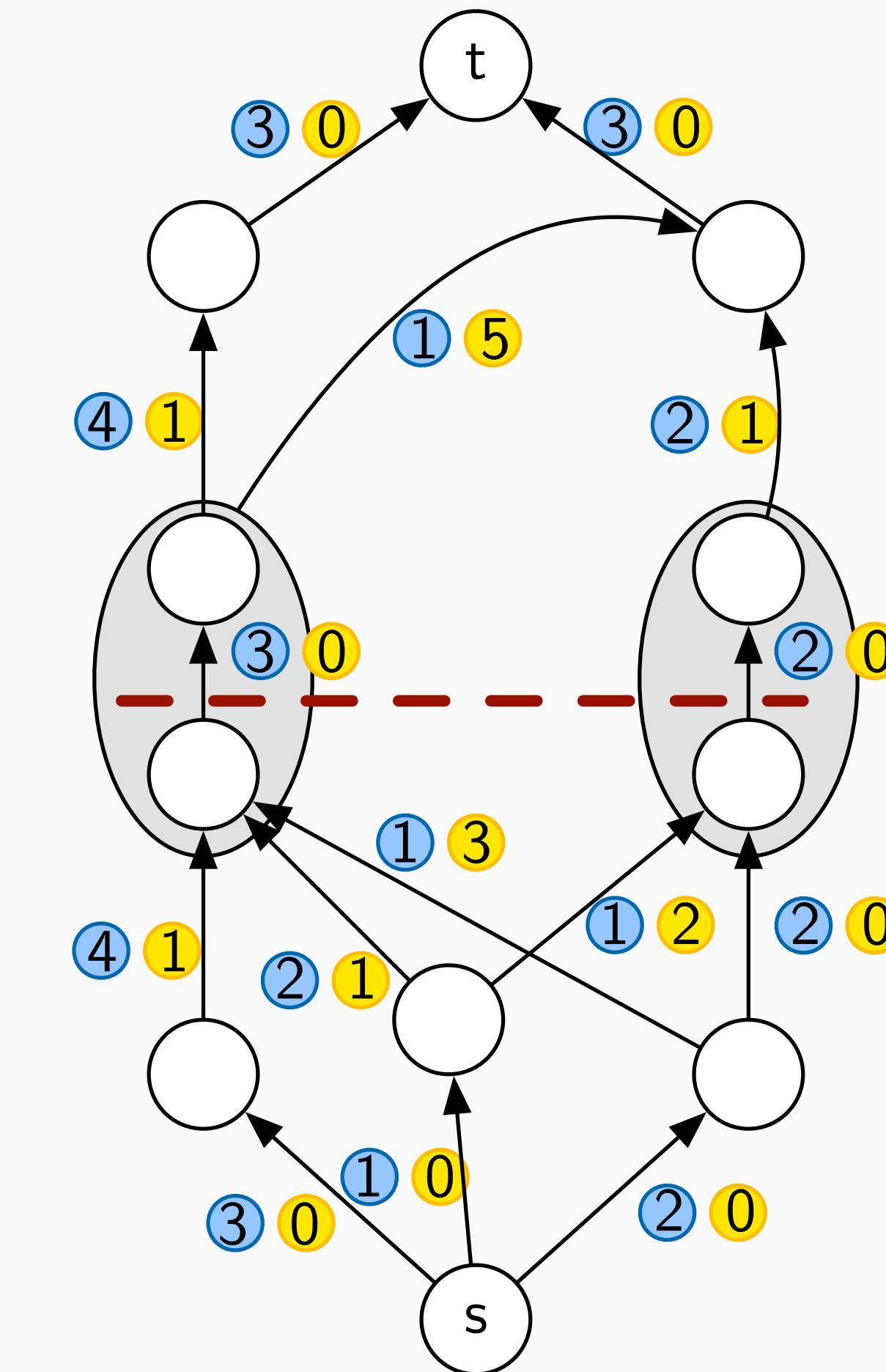
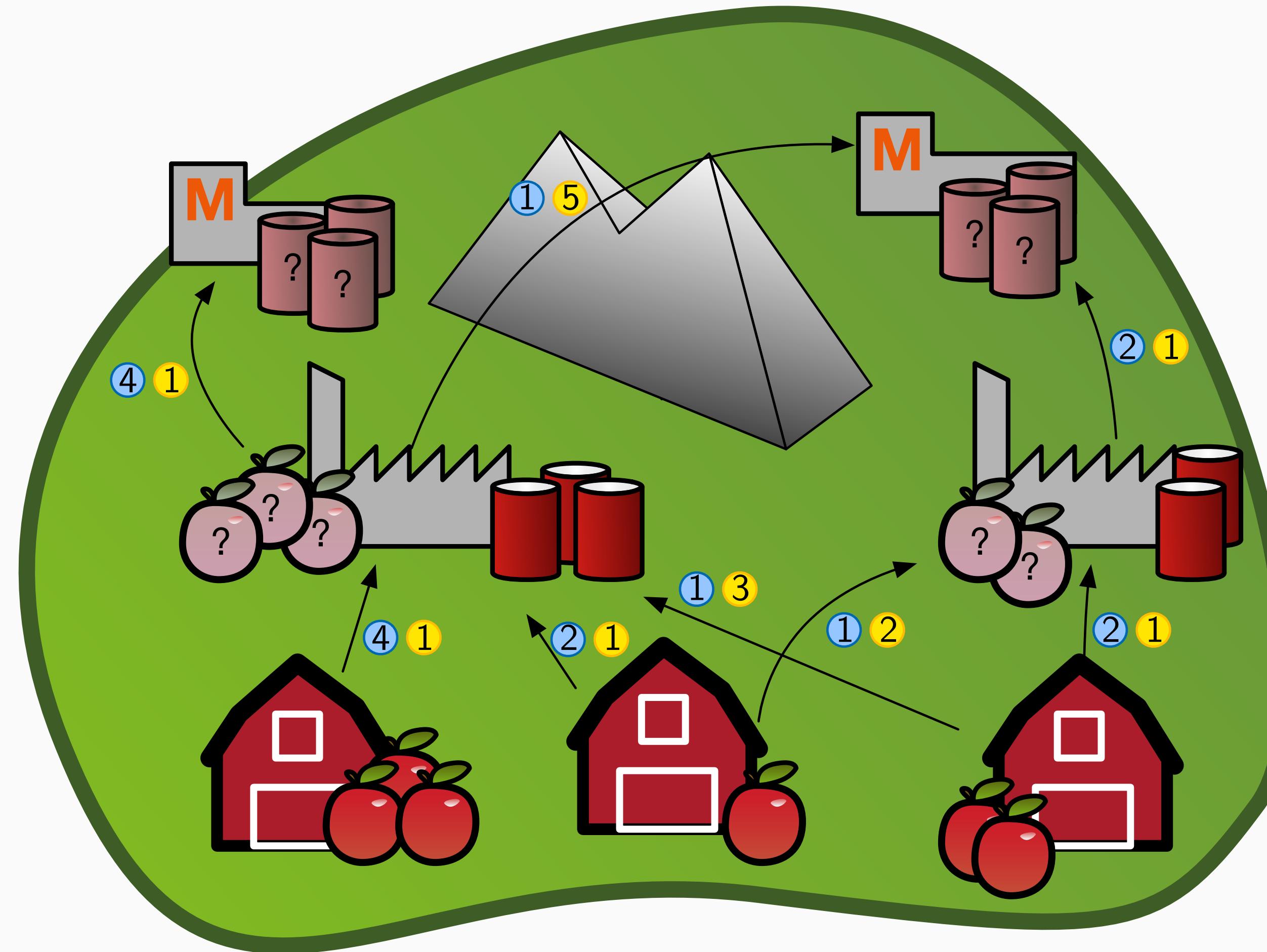
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Min Cost Max Flow with BGL

There are two algorithms available in BGL (available in BGL v1.55+):

- ▶ `cycle_canceling()`
 - ▶ slow, but can handle negative costs
 - ▶ needs a maximum flow to start with (call e.g. `push_relabel_max_flow` before)
 - ▶ runtime $\mathcal{O}(C \cdot (nm))$ where C is the cost of the initial flow
 - ▶ [\[BGL documentation\]](#), [\[BGL example\]](#).

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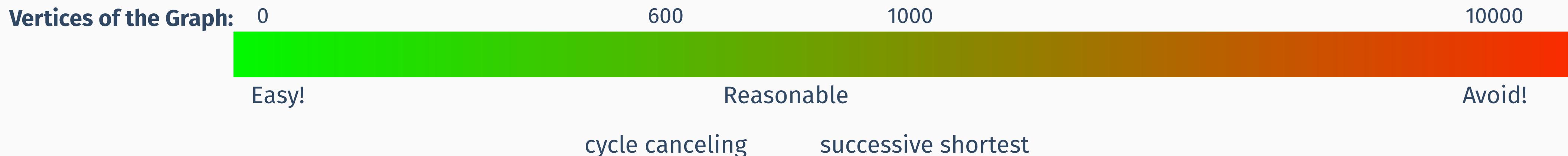
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- ▶ **`successive_shortest_path_nonnegative_weights()`**
 - ▶ faster, but works only for non-negative costs
 - ▶ sum up all residual capacities at the source to get the flow value
 - ▶ runtime $\mathcal{O}(|f| \cdot (m + n \log n))$
 - ▶ [\[BGL documentation\]](#), [\[BGL example\]](#).

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 - ▶ runtime $\mathcal{O}(C \cdot (nm))$ where C is the cost of the initial flow
 - ▶ [\[BGL documentation\]](#), [\[BGL example\]](#).
- ▶ **successive_shortest_path_nonnegative_weights()**
 - ▶ faster, but works only for non-negative costs
 - ▶ sum up all residual capacities at the source to get the flow value
 - ▶ runtime $\mathcal{O}(|f| \cdot (m + n \log n))$
 - ▶ [\[BGL documentation\]](#), [\[BGL example\]](#).



Min Cost Max Flow with BGL

There are two algorithms available in BGL (available in BGL v1.55+):

- ▶ **`cycle_canceling()`**
 - ▶ slow, but can handle negative costs
 - ▶ needs a maximum flow to start with (call e.g. `push_relabel_max_flow` before)
 - ▶ runtime $\mathcal{O}(C \cdot (nm))$ where C is the cost of the initial flow
 - ▶ [\[BGL documentation\]](#), [\[BGL example\]](#).
- ▶ **`successive_shortest_path_nonnegative_weights()`**
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Useful things to know:

- ▶ costs implemented as `edge_weight_t` property
- ▶ call `find_flow_cost()` to compute the cost of the flow

Min Cost Max Flow with BGL

Weights and capacities, just one more nesting level in the typedefs:

```
16 // Graph Type with nested interior edge properties for Cost Flow Algorithms
17 typedef boost::adjacency_list_traits<boost::vecS, boost::vecS, boost::directedS> traits;
18 typedef boost::adjacency_list<boost::vecS, boost::vecS, boost::directedS, boost::no_property,
19     boost::property<boost::edge_capacity_t, long,
20     boost::property<boost::edge_residual_capacity_t, long,
21         boost::property<boost::edge_reverse_t, traits::edge_descriptor,
22             boost::property <boost::edge_weight_t, long> >>> graph; // new!
23
24 typedef boost::graph_traits<graph>::edge_descriptor edge_desc;
25 typedef boost::graph_traits<graph>::out_edge_iterator out_edge_it; // Iterator
```

Code file: → [bgl_mincostmaxflow.cpp](#)

Min Cost Max Flow with BGL

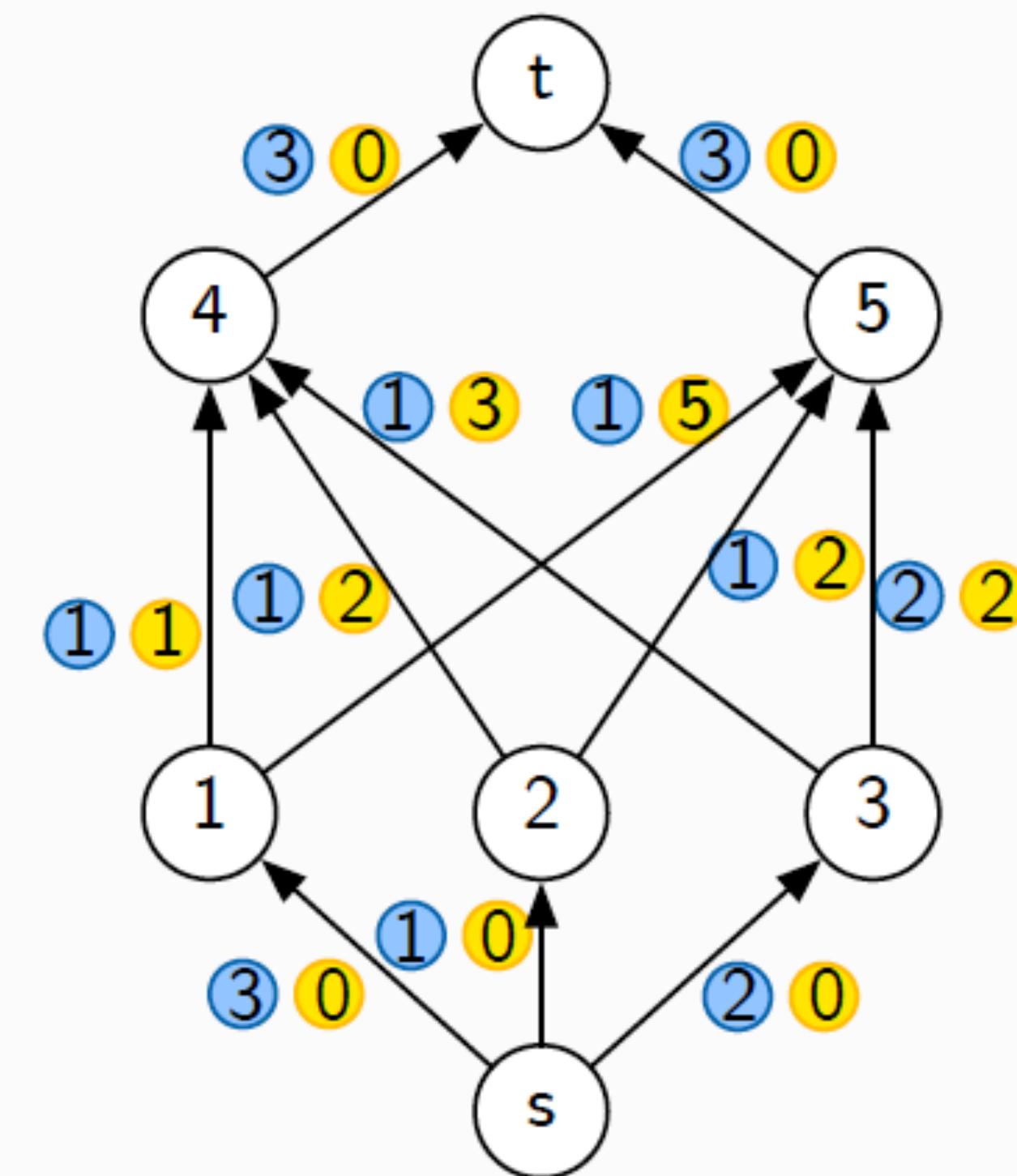
Extending the edge adder:

```
28 class edge_adder {
29     graph &G;
30
31 public:
32     explicit edge_adder(graph &G) : G(G) {}
33     void add_edge(int from, int to, long capacity, long cost) {
34         auto c_map = boost::get(boost::edge_capacity, G);
35         auto r_map = boost::get(boost::edge_reverse, G);
36         auto w_map = boost::get(boost::edge_weight, G); // new!
37         const edge_desc e = boost::add_edge(from, to, G).first;
38         const edge_desc rev_e = boost::add_edge(to, from, G).first;
39         c_map[e] = capacity;
40         c_map[rev_e] = 0;
41         r_map[e] = rev_e;
42         r_map[rev_e] = e;
43         w_map[e] = cost;    // new assign cost
44         w_map[rev_e] = -cost; // new negative cost
45     }
46 };
```

Min Cost Max Flow with BGL

Building the graph

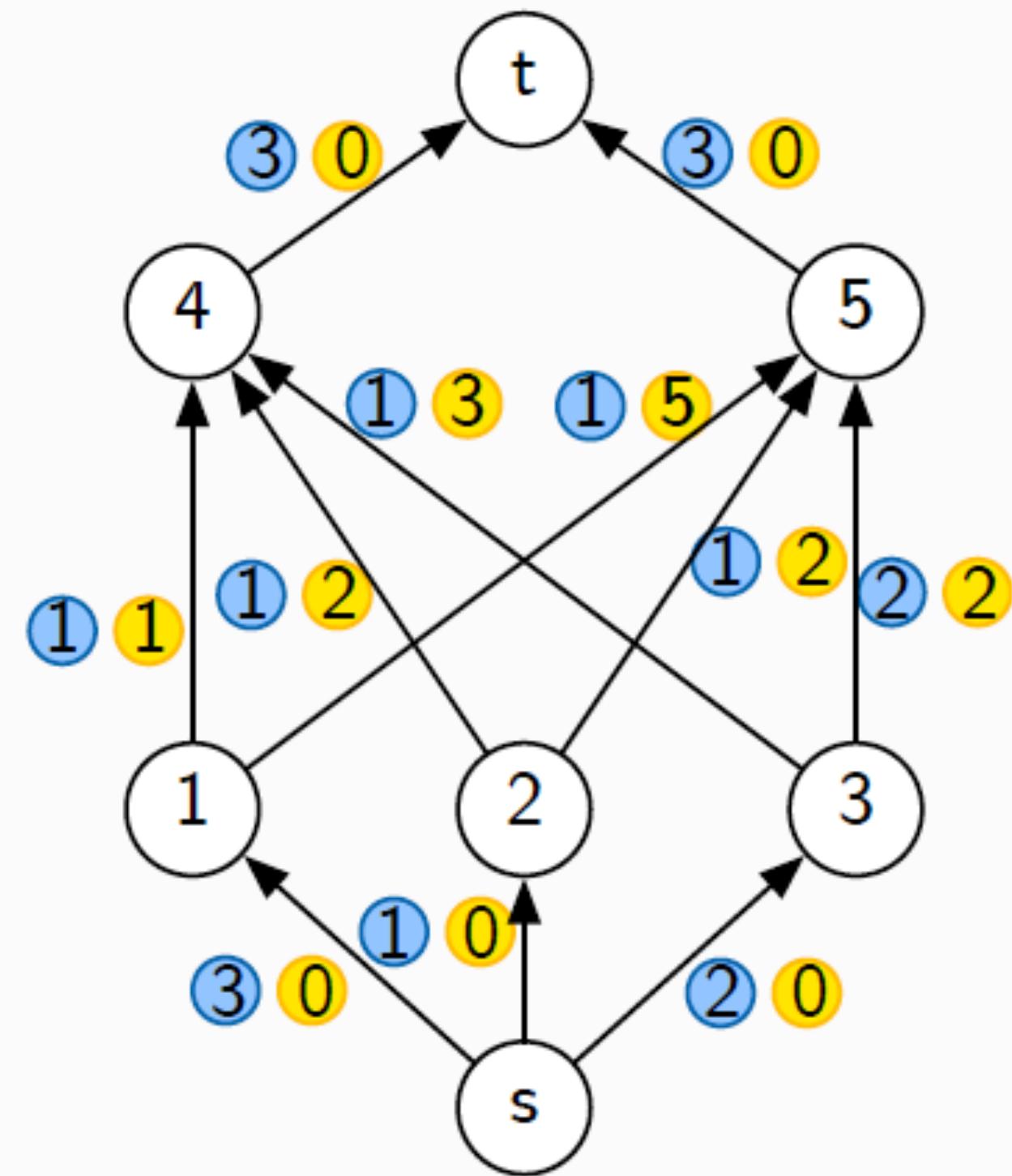
```
50 const int N=7;
51 const int v_source = 0;
52 const int v_farm1 = 1;
53 const int v_farm2 = 2;
54 const int v_farm3 = 3;
55 const int v_shop1 = 4;
56 const int v_shop2 = 5;
57 const int v_target = 6;
58
59 // Create graph, edge adder class and property maps
60 graph G(N);
61 edge_adder adder(G);
62 auto c_map = boost::get(boost::edge_capacity, G);
63 auto r_map = boost::get(boost::edge_reverse, G);
64 auto rc_map = boost::get(boost::edge_residual_capacity, G);
```



Min Cost Max Flow with BGL

Add the edges:

```
66 // Add the edges
67 adder.add_edge(v_source, v_farm1, 3, 0);
68 adder.add_edge(v_source, v_farm2, 1, 0);
69 adder.add_edge(v_source, v_farm3, 2, 0);
70
71 adder.add_edge(v_farm1, v_shop1, 1, 1);
72 adder.add_edge(v_farm1, v_shop2, 1, 5);
73 adder.add_edge(v_farm2, v_shop1, 1, 2);
74 adder.add_edge(v_farm2, v_shop2, 1, 2);
75 adder.add_edge(v_farm3, v_shop1, 1, 3);
76 adder.add_edge(v_farm3, v_shop2, 2, 2);
77
78 adder.add_edge(v_shop1, v_target, 3, 0);
79 adder.add_edge(v_shop2, v_target, 3, 0);
```



Min Cost Max Flow with BGL

Running the algorithm:

```
83 // Option 1: Min Cost Max Flow with cycle_canceling
84 int flow1 = boost::push_relabel_max_flow(G, v_source, v_target);
85 boost::cycle_canceling(G);
86 int cost1 = boost::find_flow_cost(G);
87 std::cout << "flow" << flow1 << "\n"; // 5
88 std::cout << "cost" << cost1 << "\n"; // 12
```

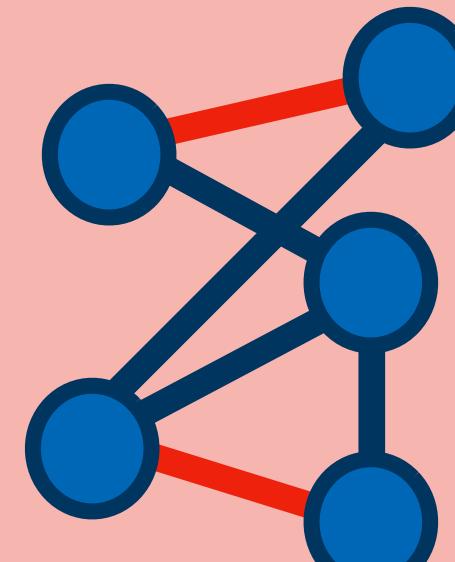
Min Cost Max Flow with BGL

Running the algorithm:

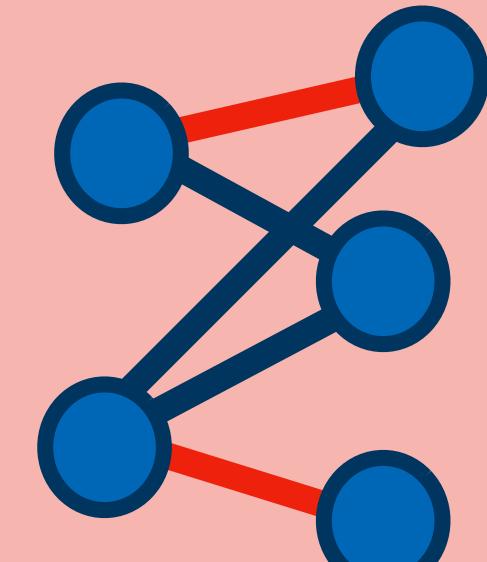
```
92 // Option 2: Min Cost Max Flow with successive_shortest_path_nonnegative_weights
93 boost::successive_shortest_path_nonnegative_weights(G, v_source, v_target);
94 int cost2 = boost::find_flow_cost(G);
95 std::cout << "cost" << cost2 << "\n"; // 12
96 // Iterate over all edges leaving the source to sum up the flow values.
97 int s_flow = 0;
98 out_edge_it e, eend;
99 for(boost::tie(e, eend) = boost::out_edges(boost::vertex(v_source,G), G); e != eend; ++e)
100     s_flow += c_map[*e] - rc_map[*e];
101 std::cout << "s-out_flow" << s_flow << "\n"; // 5
102 // Or equivalently, you can do the summation at the sink, but with reversed edge.
103 int t_flow = 0;
104 for(boost::tie(e, eend) = boost::out_edges(boost::vertex(v_target,G), G); e != eend; ++e)
105     t_flow += rc_map[*e] - c_map[*e];
106 std::cout << "t-in_flow" << t_flow << "\n"; // 5
```

Problem Landscape

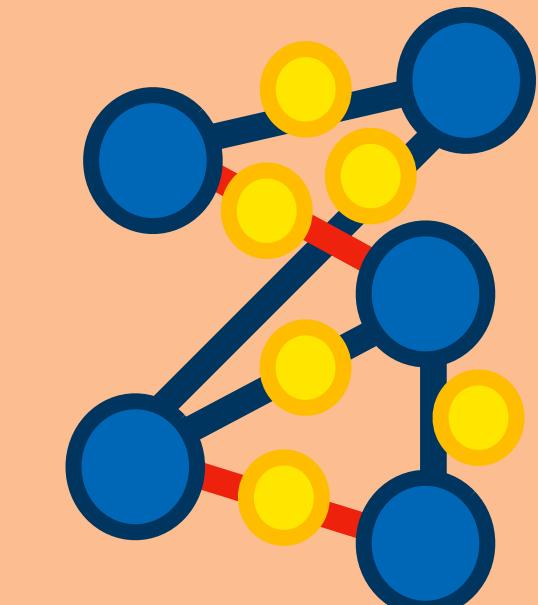
General Matching



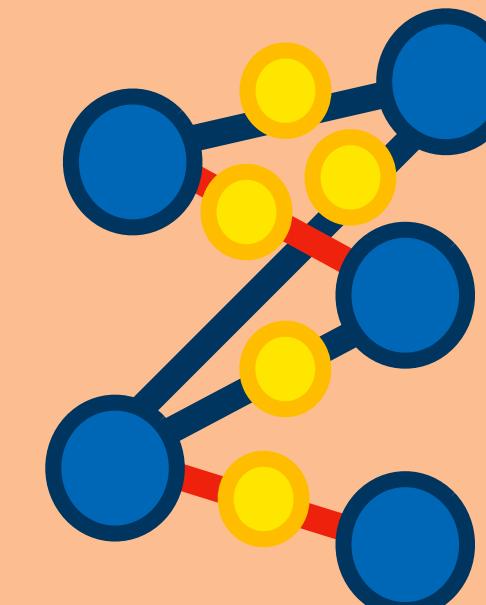
Bipartite Matching



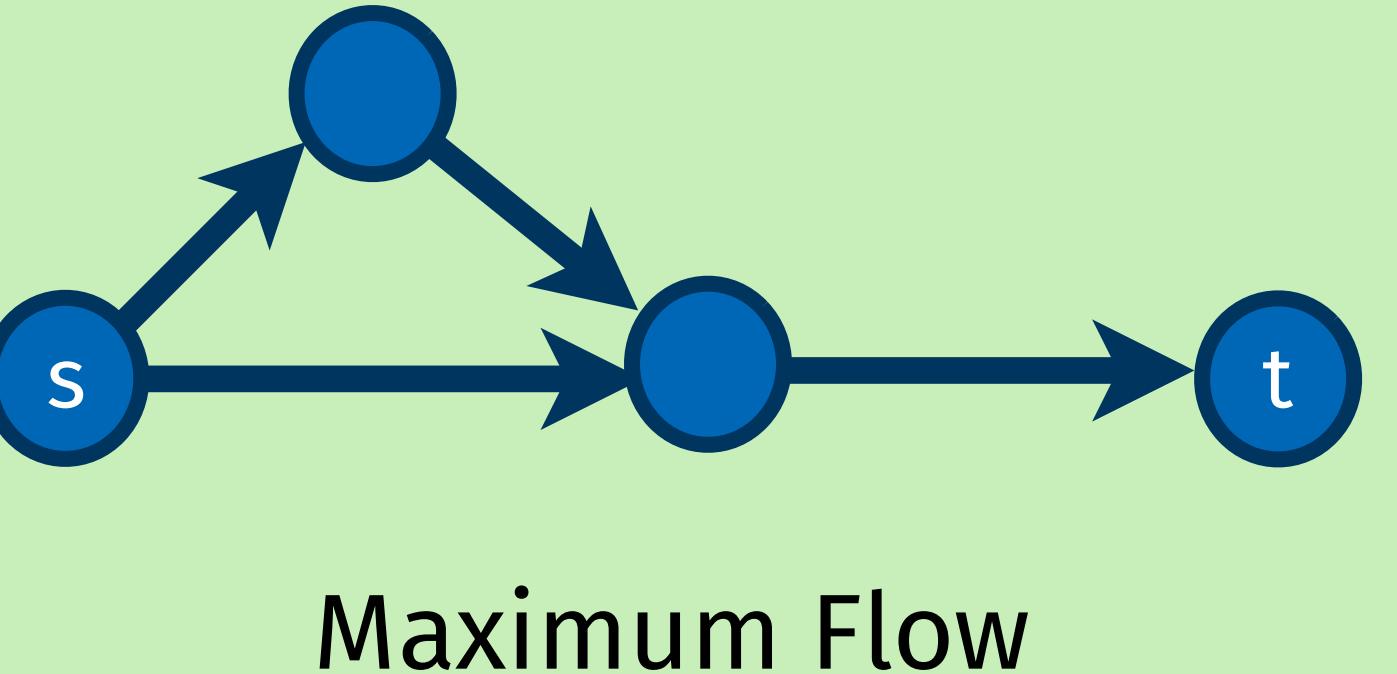
General Cost Matching



Bipartite Cost Matching

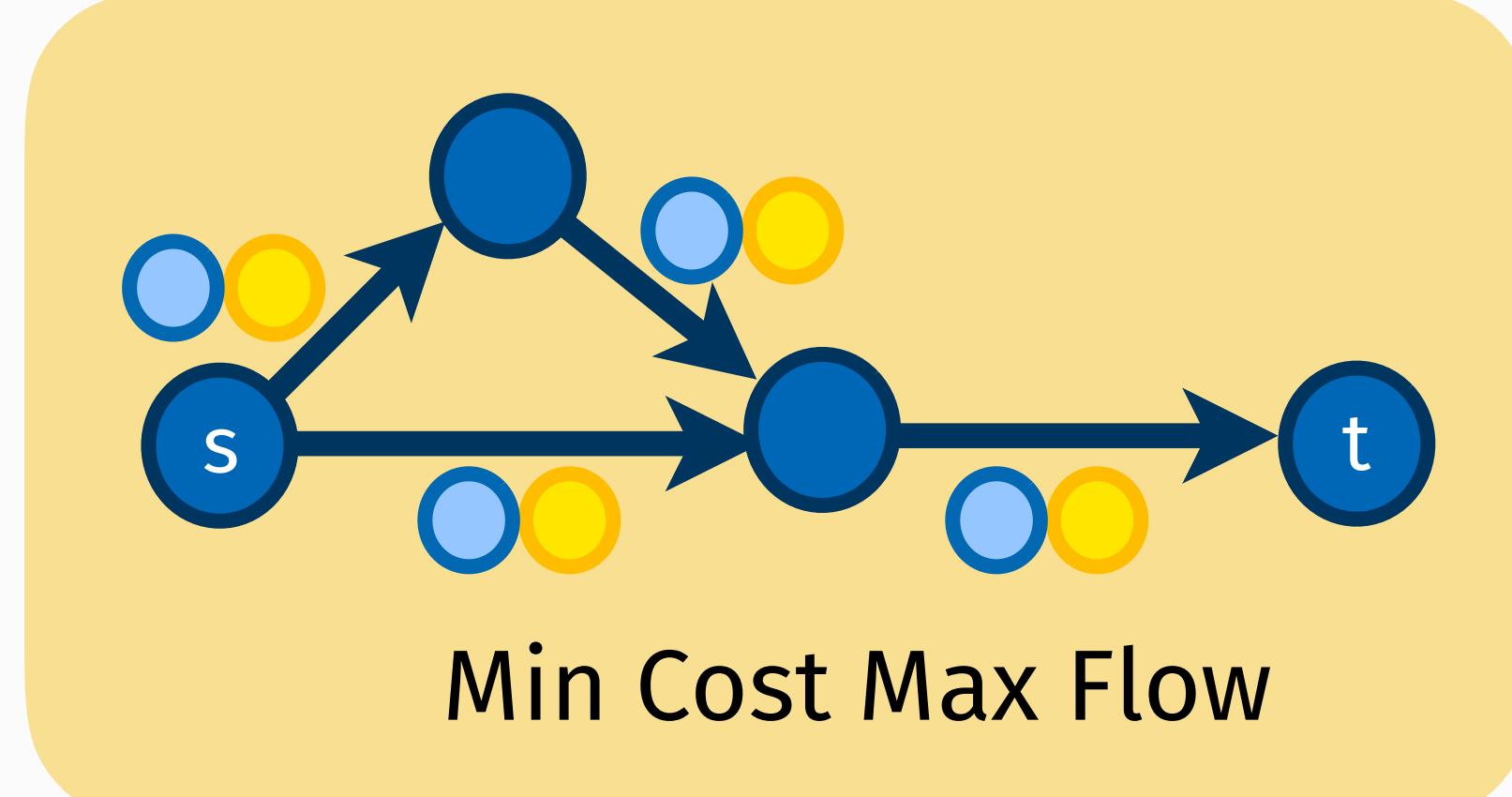


Edmonds
Maximum
Cardinality
Matching



Maximum Flow

???
not in
BGL



Min Cost Max Flow

Summary: Min Cost Max Flow with BGL

What you should remember from this part:

Minimum Cost Maximum Flow

- ▶ is a powerful and versatile modeling tool.
- ▶ is a tiebreaker among several maximum flows (but might still not be unique).
- ▶ = *maximum* cost maximum flow with negated costs.
- ▶ can often be reformulated without negative costs which allows us to use a faster algorithm in BGL (key step in many problems).
- ▶ can easily be implemented when starting with [our template on the Judge](#).
- ▶ is not harder to use in BGL than the regular network flow algorithms, see [BGL Docs](#).

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But be prepared to combine all your skills:

- ▶ On top of a flow problem, do binary search for the answer
- ▶ LP formulation vs. flow formulation?
- ▶ Some graph problems can be solved greedily (e.g. MST), others not (e.g. flow)
- ▶ Dijkstra and MinCostMaxFlows are just “special” dynamic programs
- ▶ Find a Min Cost Max Flow formulation where greedy fails for non-unit weights
- ▶ Do BFS on Delaunay triangulation or do Union-Find on Euclidean MST
- ▶ ...

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- ▶ ...

Starting next week:

- ▶ How to balance reading, solving, coding, debugging under time constraints
- ▶ No more problems labeled by topic – figure it out yourself
- ▶ In-class exercises on Wednesdays