Primal-Dual Formulation

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1 Simplified Problem

Let us take a look at a simplified problem of each residence scheduling only a single load of unit rating for one interval anywhere within the time window. This means that each residence has to schedule their load for only a single interval within the time window of length T. The residence n aims to obtain the optimal load schedule $\{p_n^t\}_{t=1}^T \in \{0,1\}$. The power schedules are coupled across space through the voltage constraints (1d). The overall optimization problem is shown below. For every interval t, the load is either switched on with $p_n^t = 1$ or it is turned off $p_n^t = 0$. (1c) ensures that the load at each residence n is switched on for only a single interval.

$$
\begin{aligned}
\text{min} \quad & \sum_{n=1}^{N} \sum_{t=1}^{T} c^t p_i^t && \text{(1a)} \\
\text{over} \quad & \left\{ \{p_i^t\}_{t=1}^T \right\}_{n=1}^N \in \{0,1\} && \text{(1b)} \\
\text{s.to.} \quad & \sum_{t=1}^{w} p_n^t = 1 \quad \forall n = 1,2,\cdots,N && \text{(1c)} \\
& \alpha \leq -\sum_{j=1}^{N} R_{nj} p_j^t \leq \beta \quad \forall n = 1,\cdots,N \quad \forall t = 1,\cdots,T && \text{(1d)}
\end{aligned}
$$

The above problem can be solved for the optimal solution in a centralized manner. However, in reality, the information regarding all residences need not be available to the central entity. This requires a distributed approach of finding the optimal solution. We use a primal-dual approach to address this problem [1, 2].

Let $\mathbf{P} \in \{0,1\}^{N \times T}$ be the power schedule where the entry at the n^{th} row and t^{th} column denotes the power schedule for the time interval t at residence n. Let the N Lagrangian multipliers corresponding to the equality constraints be stacked in the vector $\boldsymbol{\lambda} \in \mathbb{R}^N$. Let $\mathbf{U} \in \mathbb{R}^{N \times T}$ be the Lagrangian multipliers corresponding to the inequality constraints. Therefore, we can write the Lagrangian function as

$$
\mathcal{L}(\mathbf{P}, \mathbf{U}, \boldsymbol{\lambda}) = \mathbf{1}^T \mathbf{P} \mathbf{c} + \boldsymbol{\lambda} \left(\mathbf{1} - \mathbf{P} \mathbf{1} \right) + \mathsf{Tr} \left[\mathbf{U}^T \left(\alpha \mathbf{1} \mathbf{1}^T + \mathbf{R} \mathbf{P} \right) \right] \tag{2}
$$

Next steps

- Compute the partial derivative of the Lagrangian function $\mathcal{L}$ with respect to each row of $\mathbf{P}$, each row of $\mathbf{U}$ and each entry of $\boldsymbol{\lambda}$: $\frac{\partial \mathcal{L}}{\partial \mathbf{p}_n}, \frac{\partial \mathcal{L}}{\partial \mathbf{u}_n}, \frac{\partial \mathcal{L}}{\partial \lambda_n}$
- Update the primal and dual variables over successive iterations

$$
\begin{aligned}
\mathbf{p}_n^{(k+1)} &= \mathbf{p}_n^{(k)} + \gamma_{\mathbf{P}} \frac{\partial \mathcal{L}}{\partial \mathbf{p}_n} && \text{(3a)} \\
\mathbf{u}_n^{(k+1)} &= \mathbf{u}_n^{(k)} + \gamma_{\mathbf{u}} \frac{\partial \mathcal{L}}{\partial \mathbf{u}_n} && \text{(3b)} \\
\lambda_n^{(k+1)} &= \lambda_n^{(k)} + \gamma_{\lambda} \frac{\partial \mathcal{L}}{\partial \lambda_n} && \text{(3c)} \\
& && \text{(3d)}
\end{aligned}
$$

References

[1] Mark Eisen, Clark Zhang, Luiz F. O. Chamon, Daniel D. Lee, and Alejandro Ribeiro. Learning Optimal Resource Allocations in Wireless Systems. *IEEE Transactions on Signal Processing*, 67(10):2775–2790, May 2019.

[2] Sarthak Gupta, Vassilis Kekatos, and Walid Saad. Optimal real-time coordination of energy storage units as a voltage-constrained game, 2018.