CS 240: Reasoning Under Uncertainty (Fall 21)	Nov 8

Lecture 18-19

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1 Introduction

These lectures covered the Markov Chain.

2 Markov Chain

A Markov chain is a stochastic model describing a sequence of possible events in which the probability of each event depends only on the previous event. [Wikipedia] Intuitively, you can use a Markov chain to predict how the system will look in the future given its current state.

2.1 Discrete Time Markov Chains

The state changes at certain discrete time instances, indexed by an integer variable n.

2.2 Markov Property

It says that the probability of the system being at a particular state depends only on its previous state.

$$P(X_{n+1} = j | X_n = i, X_{n-1} = s_{n-1} \ldots) = P(X_{n+1} | X_n = i)$$

2.3 Transition Probability Graph

A Markov chain can be described using a transition probability graph whose nodes are the states and whose arrows are the possible transitions.

2.4 Chapman-Kolmogorov Equation

It formularizes how you can compute the distribution of X_n given the distribution of its previous states

$$v_n = [P(X_n = 0), P(X_n = 1), \ldots P(X_n = k)] = [\sum_i p_{i0} v_{n-1}[i], \ldots, \sum_i p_{ik} v_{n-1}[i]]$$

If the Markov chain is encoded as a **transition probability matrix** A:

$$v_n = v_{n-1} A = v_0 A^n$$

2.5 Steady State Distribution

If $v = vA$ we say v is a steady state distribution of the Markov Chain. It is essentially the left eigenvector of matrix A associated with eigenvalue 1.

2.6 Recurrent State

A state i is recurrent if for every state j that is accessible from i, state i is also accessible from j.

2.7 Trainsient State

A state that is not recurrent.

2.8 Recurrent Class

A recurrent class $A(i)$ contains the set of states that are accessible from the recurrent state i.

2.9 Periodic v Aperiodic

A recurrent Markov chain is called periodic if there is some $t \in \{2, 3, \ldots\}$ such that there exists a state s which can be visited only at time $\{t, 2t, 3t, \ldots\}$ steps. If t = 1 we call the chain aperiodic.

Theorem 1. *A Markov Chain with a single aperiodic recurrent class has a steady state regardless of the initial state.*

3 Problems

1. Consider a Markov chain with 2 states and a transition matrix:

$$A = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$$

When does A has a single recurrent class? When is it aperiodic? If this Markov Chain has a single aperiodic recurrent class, what are its steady states?

4 Answer

This Markov chain has a single recurrent class when S_1 can reach S_2 and S_2 can reach S_1 too. It occurs when $a > 0$ and $b > 0$.

This single recurrent class is aperiodic when either S_1 or S_2 has a self-loop (or both have self-loops). That is $1 - a > 0$ or $1 - b > 0$.

Suppose its steady state is $v =< c, 1 - c >$.

Then $vA =< c \times (1-a) + (1-c) \times b, c \times a + c \times (1-b) >=< c - ac + b - bc, ac + c - bc >$

Since $vA = v$, we have $< c - ac + b - bc, ac + c - bc >=< c, 1 - c >$.

This allows us to write c in terms of a and b: $c = \frac{b}{a+b}$, $1 - c = \frac{a}{a+b}$ and $v =< \frac{b}{a+b}, \frac{a}{a+b} >$

(a and b are variables in this case, but they might be constant in other questions).