CS 240: Reasoning Under Uncertainty (Fall 21)	Oct 20

Lecture 12

Instructor: *Profs Peter J. Hass and Jie Xiong* — SI Worksheet: *Juelin Liu*

1 Introduction

In this lecture, we discussed several specific continuous random variables. They are uniform continuous random variables, exponential random variables, normal (Gaussian) random variables, and the standard normal random variable.

2 Continuous Random Variables

2.1 Uniform Continuous Random Variable

A uniform continuous random variable has a uniform probability density in $[a, b]$. Its expected value:

$$E(X) = \frac{b+a}{2}$$

Its variance:

$$Var(X) = \frac{1}{12}(b-a)^2$$

Its PDF is:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & otherwise \end{cases}$$

Its CDF:

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

2.2 Exponential Random Variable

An exponential random variable characterizes the (time/space) interval between events in a Poisson point process. Suppose we have a Poisson random variable X. We know that $E(X) = \lambda$ which is the expected number of times an event occurs internally with a unit length is λ. Now the question is what is the expected interval between two events? Let's use the random variable Y to represent the size of the interval between the two events.

Its expected value should be:

$$E(Y) = \frac{1}{\lambda}$$

Its variance:

$$Var(Y) = \frac{1}{\lambda^2}$$

Its PDF is:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & 0 < x \\ 0 & otherwise \end{cases}$$

Its CDF:

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

2.3 Normal Random Variable

Let X be a normal random variable with $E(X) = \mu$ and $Var(X) = \sigma^2$.

Its probability density function is:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Its cumulative distribution function is:

$$F_X(x) = \frac{1}{2}(1 + erf(\frac{x-\mu}{\sigma\sqrt{2}}))$$

where $erf(x) = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-t^2} dt$

The **probability mass** of an interval $[a, b]$ is the definite integral:

$$P(a < X < b) = \int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = F_X(b) - F_X(a)$$

2.4 Standard Normal Random Variable

Let X be a normal random variable with $E(X) = 0$ and $Var(X) = \sigma^2 = 1$.

Its probability density function is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Its cumulative distribution function is:

$$F_X(x) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$$

2.5 Standardizing a Normal Variable

For a given normal random variable X with mean μ and variance σ^2, you can standardize it by defining a new random variable Y given by

$$Y = \frac{X - \mu}{\sigma}$$

Where Y is a standard normal variable with mean 0 and variance 1.

This is because Y is a linear function of X ($Y = aX + b$), we know Y perceives the normality.

2.6 Practice Problems

1. Suppose that Y has a uniform distribution over the interval (0, 1).
   a) Find F(y).
   b) Show that $P(a \leq Y \leq a + b)$ (for $a \geq 0$, $b \geq 0$, and $a + b \leq 1$) depends only upon the value of b.

2. A random variable Y has a uniform distribution over the interval (a, b). Derive the variance of Y.

3. Let X be a continuous random variable with PDF $f_X(x) = \frac{\lambda}{2} e^{-\lambda |x|}$.
   a) Verify that f_X is a valid PDF.
   b) Evaluate $E(X)$ and $Var(X)$

4. Let Z denote a normal random variable with mean 0 and standard deviation 1.
   a) Find $P(Z > 1)$
   b) Find $P(-1 < Z < 1)$
   c) Find $P(0.3 < Z < 1.5)$

5. If Z is a standard normal random variable, find the value z_0 such that
   a) $P(Z > z_0) = 0.5$
   b) $P(Z < z_0) = 0.8643$
   c) $P(-z_0 < Z < z_0) = 0.90$
   d) $P(-z_0 < Z < z_0) = 0.99$

6. A soft-drink machine can be regulated so that it discharges an average of μ ounces per cup. If the ounces of fill are normally distributed with a standard deviation of 0, compute the setting for μ so that 8-ounce cups will overflow only 1 percent of the time.

3 Answer

1. a)

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

   b) $P(a < X < a + b) = F_X(a + b) - F_X(a) = b$

2.

$$f_X(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a \leq x \leq b \\ 0 & x > b \end{cases}$$

$E(X) = \int_a^b x \frac{1}{b-a} dx = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}$
$Var(X) = E(X^2) - (E(X))^2 = \int_a^b x^2 \frac{1}{b-a} dx - (\frac{a+b}{2})^2 = \frac{b^3-a^3}{3(b-a)} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}$

3. a) $P(-\infty < X < \infty) = \int_{-\infty}^{\infty} \frac{\lambda}{2} e^{-\lambda|x|} dx = \int_0^{\infty} \lambda e^{-\lambda x} dx = 1$
b) $E(X) = \int_{-\infty}^{\infty} \frac{\lambda}{2} x e^{-\lambda|x|} dx = 0$ (symmetry)
Or you can use integration by parts:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} \frac{\lambda}{2} x e^{-\lambda|x|} dx \\ &= \int_{-\infty}^{0} \frac{\lambda}{2} x e^{\lambda x} dx + \int_0^{\infty} \frac{\lambda}{2} x e^{-\lambda x} dx \end{aligned}$$

Let $A = \int_{-\infty}^0 \frac{\lambda}{2} x e^{\lambda x} dx$, $B = \int_0^{\infty} \frac{\lambda}{2} x e^{-\lambda x}$
We now show that $A = -B$ using integration by parts ($\int uv' = uv - \int u'v$)
To compute A let $u = x$, $u' = 1$, $v' = \frac{\lambda}{2} e^{\lambda x}$, $v = \frac{1}{2} e^{\lambda x}$

$$\begin{aligned} A &= \int_{-\infty}^{0} \frac{\lambda}{2} x e^{\lambda x} dx \\ &= [\frac{1}{2} x e^{\lambda x}]_{-\infty}^{0} - \int_{-\infty}^{0} \frac{1}{2} e^{\lambda x} dx \\ &= - \int_{-\infty}^{0} \frac{1}{2} e^{\lambda x} dx \\ &= -[\frac{1}{2\lambda} e^{\lambda x}]_{-\infty}^{0} \\ &= -\frac{1}{2\lambda} \end{aligned}$$

To compute B let $u = x$, $u' = 1$, $v' = \frac{\lambda}{2} e^{-\lambda x}$, $v = -\frac{1}{2} e^{-\lambda x}$

$$\begin{aligned} B &= \int_0^{\infty} \frac{\lambda}{2} x e^{-\lambda x} dx \\ &= [-\frac{1}{2} x e^{-\lambda x}]_0^{\infty} - \int_0^{\infty} -\frac{1}{2} e^{-\lambda x} dx \\ &= \int_0^{\infty} \frac{1}{2} e^{-\lambda x} dx \\ &= [-\frac{1}{2\lambda} e^{-\lambda x}]_0^{\infty} \\ &= \frac{1}{2\lambda} \end{aligned}$$

Thus, $E(X) = A + B = 0$
$Var(X) = E(X^2) - E(X)^2 = \int_{-\infty}^{\infty} \frac{\lambda}{2} x^2 e^{-\lambda|x|} dx = \int_0^{\infty} \lambda x^2 e^{-\lambda|x|} dx = \frac{2}{\lambda^2}$
(use integrate by parts twice)

4. a) $P(Z > 1) = 0.15866$
   b) $P(-1 < Z < 1) = (0.5 - P(Z > 1)) + (0.5 - P(Z < -1)) = 1 - 2 * 0.15866 = 0.68268$
   c) $P(0.3 < Z < 1.5) = (0.5 - P(Z > 1.5)) - (0.5 - P(Z > 0.3)) = 0.38209 - 0.06681 = 0.31528$

5. a) $z_0 = 0$
   b) $z_0 = 1.1$
   c) $P(Z < z_0) = 0.9 + (1 - 0.9)/2 = 0.95,\ z_0 = 1.64$
   d) $P(Z < z_0) = 0.995,\ z_0 = 2.58$

6. $P(Z < \frac{8-\mu}{0.3}) = 0.99,\ \mu = 7.301$