CS 240: Reasoning Under Uncertainty (Fall 21)	Oct 4
Lecture 9	
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1 Introduction

This lecture discussed functions of random variables and standard deviation.

2 Functions of Random Variables

For a random variable X, we can define a new random variable $Y = f(X)$ where f is a function. Then we have:

$$E(Y) = \sum_y yP(Y = y) = \sum_x f(x)P(X = x)$$

2.1 Recap Linear Function

If $Y = aX + b$ where a and b are real numbers:

$$\begin{aligned}
Var(Y) &= Var(aX + b) \\
&= \sum_k (ak + b - E(aX + b))^2 P(X = k) \\
&= \sum_k (ak + b - aE(X) - b)^2 P(X = k) \\
&= \sum_k (ak - aE(X))^2 P(X = k) \\
&= \sum_k (a^2k^2 - 2a^2kE(X) + a^2E(X)^2)P(X = k) \\
&= a^2 \sum_k (k^2 - 2kE(X) + E(X)^2)P(X = k) \\
&= a^2 \sum_k k^2 P(X = k) - a^2 \sum_k 2kE(X)P(X = k) + a^2 \sum_k E(X)^2 P(X = k) \\
&= a^2E(X^2) - 2a^2E(X)^2 + a^2E(X)^2 \\
&= a^2E(X^2) - a^2E(X)^2 \\
&= a^2Var(X) \\
E(Y) &= E(aX + b) = aE(X) + b
\end{aligned}$$

Note: it is **not** generally true that $E[g(X)] = g(E(X))$ unless $g(X)$ is a linear function.

3 Standard Deviation

The standard deviation of a random variable is the non-negative square root of its variance.

$$std(X) = \sqrt{var(X)}$$

4 Practice Problems

1. The number of imperfections in the weave of a certain textile has a Poisson distribution with a mean of 4 per square yard. Find the probability that a
   a) A 1-square-yard sample will contain at least one imperfection.
   b) A 3-square-yard sample will contain at least one imperfection.
   c) The cost of repairing the imperfections in the weave is 10 dollars per imperfection. Find the mean and standard deviation of the repair cost for an 8-square-yard bolt of the textile.

2. Two assembly lines I and II have the same rate of defectives. Five regulators are sampled from each line and tested. Among the total of 10 tested regulators, 4 are defective. Find the probability that exactly 2 of the defective regulators came from line I.

5 Answers

1. a) $P(X \geq 1) = 1 - P(X = 0) = 1 - \frac{e^{-4} \times 4^0}{0!} = 0.982$
   b) Let Y be the number of imperfections in a 3-square-yard sample.
   Y has a Poisson distribution with a mean of 12.
   $$P(Y >= 1) = 1 - \frac{e^{-12} \times 12^0}{0!}$$
   c) Let Z be the number of imperfections in an 8-square-yard sample.
   Z has a Poisson distribution with a mean of 32.
   $Var(Z) = E(Z) = 4 \times 8 = 32$.
   The repairing cost can be represented as R = 10Z.
   $E(R) = E(10Z) = 320$, $std(R) = \sqrt{var(R)} = \sqrt{10^2 \times 32} = 40\sqrt{2}$

2. $P(X = 2) = \binom{5}{2} \times \binom{5}{2} \div \binom{10}{4} = 0.476$