CS 240: Reasoning Under Uncertainty (Fall 21) September 29

Lecture 8

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1 Introduction

This lecture discussed expected value and variance.

2 Expectation & Variance

For a discrete random variable, its expected value can be computed using the following formula:

$$E(X) = \sum_{x \in R} xP(X = x)$$

Its variance, which measures how far we expect this random variable to be from its expected value:

$$Var(X) = E[(X - E(X)^2)] = \sum_k (k - E(X))^2 \times P(X = k)$$

Or alternatively:

$$Var(X) = E(X^2) - E(X)^2$$

2.1 Linearity of Expectation

If $Y = aX + b$ where a and b are real numbers:

$$\begin{aligned}
E(Y) &= E(aX + b) = aE(X) + b \\
Var(Y) &= Var(aX + b) \\
&= E((aX + b)^2) - E(aX + b)^2 \\
&= E((a^2X^2 + 2abX + b^2) - (aE(X) + b)^2 \\
&= a^2E(X^2) + 2abE(X) + b^2 - a^2E(X)^2 - 2abE(X) - b^2 \\
&= a^2(E(X^2) - E(X)^2) \\
&= a^2Var(X)
\end{aligned}$$

Note: it is **not** generally true that $E[g(X)] = g(E(X))$ unless $g(X)$ is a linear function.

2.2 Discrete Uniform

A discrete uniform random variable with range [a, b] takes on any integer value between a and b inclusive. (For simplicity we assume a and b are integers.) Each value has the same probability. Its PMF is:

$$P(X = k) = \frac{1}{b - a + 1} \, for \; k = a, \ldots, b$$

Its expected value is:

$$E(X) = \frac{a + b}{2}$$

Its variance is:

$$Var(X) = \frac{(b - a)(b - a + 12)}{12}$$

2.3 Bernoulli

A Bernoulli random variable has two possible outcomes. We use 1 (success) or 0 (fail) to represent the two outcomes. Its PMF is often described as:

$$P(X = 1) = p \text{ and } P(X = 0) = 1 - p$$

Its expected value is:

$$E(X) = (1 - p) \times 0 + 1 \times p = p$$

Its variance is:

$$Var(X) = p - p^2$$

2.4 Binomial

A binomial random variable is associated with the number of successful trials in several independent Bernoulli trials. In this formula, n is the total number of independent Bernoulli trials, x is the number of successful trials and p is the probability that a Bernoulli trial is successful. Its PMF is:

$$P(X = k) = \binom{n}{x} \times p^k \times (1 - p)^{n-k}$$

Its expected value is:

$$E(X) = \sum_{k=0}^{n} k \times \binom{n}{k} p^k (1 - p)^{n-k} = np$$

Its variance is:

$$Var(X) = np(1 - p)$$

2.5 Gemometric

A geometric random variable is associated with the number of independent Bernoulli trials required to come up with the first successful trial.

$$P(X = k) = (1 - p)^{k-1} \times p$$

Its expected value is:

$$E(X) = \sum_{k=0}^{\infty} k(1-p)^{k-1}p = \frac{1}{p}$$

Its variance is:

$$Var(X) = \frac{1-p}{p^2}$$

2.6 Poisson

A Poisson random variable is associated with the number of times a random and independent event occurs in a fixed interval. In this formula, λ is the expected number of occurrences, k is the actual number of occurrences and e is the Euler's number.

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

Its expected value is:

$$E(X) = \sum_{k=0}^{\infty} k\frac{e^{-\lambda}\lambda^k}{k!} = \lambda$$

Its variance is:

$$Var(X) = \lambda$$

3 Practice Problems

1. Let X be a random variable with PMF:

$$P(X = x) = \begin{cases} \frac{x^2}{a} & \text{if x = -3,-2,-1,0,1,2,3} \\ 0 & \text{otherwise} \end{cases}$$

   a) Find a and $E(X)$
   b) What is the PMF of the random variable $Z = (X - E(X))^2$
   c) Using the result from part (b), find the variance of X.
   d) Find the variance of X using the formula $Var(X) = \sum_x (x - E(X))^2 P(X = x)$.

2. As an advertising campaign, a chocolate factory places golden tickets in some of its candy bars, with the promise that a golden ticket is worth a trip through the chocolate factory, and all the chocolate you can eat for life. If the probability of finding a golden ticket is p, find the mean and the variance of the number of candy bars you need to eat to find a ticket.

3. Show that the variance of a Poisson random variable with mean λ is also λ.

4 Answers

1. a) $\frac{1}{a} \times (1^2 + 2^2 + 3^2 + 0 + (-1)^2 + (-2)^2 + (-3)^2) = 1$, $a = 28$, $E(X) = \sum_{x=-3}^{3} \frac{x^3}{28} = 0$
   b) If $z \in \{1, 4, 9\}$, $P(Z = z) = \frac{z}{14}$; Otherwise $P(Z = z) = 0$.

c) $Var(X) = E(Z) = \sum_{z\in\{1,4,9\}} \frac{z^2}{14} = 7$
d) $Var(X) = \sum_x (x - E(X))^2 P(X = x) = 7$

2. The number C of candy bars you need to eat is a geometric random variable with parameter p. Thus the mean is $E(C) = \frac{1}{p}$, and the variance is $var(C) = \frac{1-p}{p^2}$.

3.

$$\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= E(X(X-1) + X) - E(X)^2 \\
&= E(X(X-1)) + E(X) - E(X)^2 \\
&= \sum_{x=0}^{\infty} x(x-1)\frac{e^{-\lambda}\lambda^x}{x!} + \lambda - \lambda^2 \\
&= \lambda^2 \sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!} + \lambda - \lambda^2 \ (y = x-2) \\
&= \lambda^2 + \lambda - \lambda^2 \\
&= \lambda
\end{aligned}$$