CS 240: Reasoning Under Uncertainty (Fall 21)	Oct 25
Lecture 13	
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1 Introduction

In this lecture, we further discussed the Von Neumann method for simulating random variables. We also discussed the joint PDF of multiple continuous random variables. The lecture ended with a discussion of Markov inequality and the Chebyshev inequality.

2 Von Neumann Method

The Von Neumann method can simulate any random variable (continuous or discrete) with a uniform random variable $U[0, 1]$. Suppose a random variable X has a strictly increasing CDF. We can simulate X by simulating its CDF. Let $X = F^{-1}(U)$, where F is a CDF function which we don't know yet:

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(U) \leq x) \\ &= P(F(F^{-1}(U)) \leq F(x)) \\ &= P(U \leq F(x)) \\ &= F(x) \end{aligned}$$

Since $P(X \leq x) = F_X(x)$ we see that $F = F_X$.

3 Joint PDF and CDF

Two continuous random variables associated with the same experiment have a **joint PDF** $f_{X,Y}(x, y)$. The joint PDF needs to satisfy two properties:

- **Non-negativity**: $f_{X,Y}(x, y) \geq 0$ for all x and y
- **Normalization**: $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx\, dy = 1$.

We can compute the marginal PDFs of f_X and f_Y as:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y) dx$$

The probabilities are computed by 2D integration:

$$P((X,Y) \in B) = \int\int_{(x,y)\in B} f_{X,Y}(x,y)dx\,dy$$

Where $B = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$.

4 Limit Theorems

4.1 Markov's Inequality

For any non-negative random variable X and any $t > 0$:

$$P(X \geq t) \leq \frac{E(X)}{t}$$

4.2 Chebyshev's Inequality

We can derive Chebyshev's inequality from Markov's inequality. First, we have:

$$P(|Y| \geq t) = P(Y^2 \geq t^2) \leq \frac{E(Y)^2}{t^2}$$

This is because Y^2 must be a non-negative random variable so we can apply Markov's inequality to it. Then, let $Y = X - E(X)$:

$$P(|X - E(X)| \geq t) = P((X - E(X))^2 \geq t^2) \leq \frac{(X - E(X))^2}{t^2}$$

which gives the Chebyshev's inequality:

$$P(|X - E(X)| \geq t) \leq \frac{Var(X)}{t^2}$$

5 Problems

1. Let X, Y denote two continuous uniform random variables at the interval $[0,1]$. The joint PDF of Y_1 and Y_2 can be described as:

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

a) Find the value of c.
b) Find $P(X < 0.2, Y < 0.5)$

2. Suppose the joint PDF of random variable X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} 3x & 0 \leq y \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

   Find $P(X < 0.5, Y > 0.25)$.

3. The joint PDF of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} 30xy^2 & x-1 \leq y \leq 1-x, 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

   a) Find $P(X < 0.5, Y < 0.5)$
   b) Find $P(X < 0.5, Y < 2)$
   c) Find $P(X > Y)$

4. Proof Markov's inequality.

6 Answer

1. a) $\int_0^1 \int_0^1 cdxdy = \int_0^1 cdy = c = 1$
   b) $P(X < 0.2, Y < 0.5) = \int_0^{0.5} \int_0^{0.2} 1\, dxdy = \int_0^{0.5} 0.2dy = 0.1$

2.

$$\begin{aligned} P(X < 0.5, Y > 0.25) &= \int_{0.25}^{0.5} \int_{0.25}^{x} 3x\, dydx \\ &= \int_{0.25}^{0.5} 3x(x - 0.25)dx \\ &= [x^3 - \frac{3x^2}{8}]_{0.25}^{0.5} \\ &= \frac{5}{128} \end{aligned}$$

3. a)

$$\begin{aligned} P(X < 0.5, Y < 0.5) &= \int_0^{0.5} \int_{x-1}^{0.5} 30xy^2\, dydx \\ &= \int_0^{0.5} 10x(0.5^3 - (x-1)^3)dx \\ &= \frac{9}{16} \end{aligned}$$

b)

$$
\begin{aligned}
P(X < 0.5, Y < 0.5) &= \int_0^{0.5} \int_{x-1}^{1-x} 30xy^2 \, dydx \\
&= \frac{13}{16}
\end{aligned}
$$

c)

$$
\begin{aligned}
P(X < 0.5, Y < 0.5) &= \int_0^{0.5} \int_{x-1}^{x} 30xy^2 \, dydx + \int_{0.5}^{1} \int_{x-1}^{1-x} 30xy^2 \, dydx \\
&= \frac{15}{32} + \frac{3}{16} \\
&= \frac{21}{32}
\end{aligned}
$$

4.

$$
\begin{aligned}
E(X) &= \int_0^a x f_X(x) \, dx + \int_a^\infty x f_X(x) \, dx \\
&\geq \int_a^\infty x f_X(x) \, dx \\
&\geq \int_a^\infty a f_X(x) \, dx = a \int_a^\infty f_X(x) \, dx = aP(X \geq a)
\end{aligned}
$$

Dividing a on both sides we get:

$$
P(X \geq a) \leq \frac{E(X)}{a}
$$