CS 240: Reasoning Under Uncertainty (Fall 21) Oct 6

Lecture 10

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1 Introduction

This lecture discussed multiple random variables and the functions of two random variables.

2 Multiple Random Variables

2.1 Two Random Variables

We start by discussing the case of two random variables. Consider two random variables X and Y associated with the same experiment. For $x, y \in R$, we can define the events of the form:

$$\{X = x, Y = y\} = \{X = x\} \cap \{Y = y\}$$

The probabilities of these events give the **joint PMF** of X and Y:

$$\begin{aligned} p_{X,Y}(x, y) &= P(X = x, Y = y) \\ &= P(X = x \, and \, Y = y) \\ &= P(X = x \cap Y = y) \end{aligned}$$

This is useful for describing multiple properties of a single experiment.

You can visualize the PMF of two random variables as a three-dimensional graph. In this graph, the z-axis represents the joint probability given the value of x and y of the two random variables.

One question is can we retrieve the PMF of a single random variable from the joint PMF? The answer is yes and:

$$P(X = x) = \sum_y P(X = x \cap Y = y)$$

$$P(Y = y) = \sum_x P(X = x \cap Y = y)$$

To compute the probability of X=x, we sum up all the probabilities in the joint PMF where X=x. It works because of the total probability theorem.

We say that $P(X = x)$ is the **marginal PMF** of X and $P(Y = y)$ is the **marginal PMF** of Y if we start with the joint PMF of X and Y.

2.2 Functions of Two Random Variables

Naturally, we can create a new random variable by applying a function on two random variables just like applying a function to a single random variable. For example, let $Z = f(X, Y) = X + Y$. Then the expected value of Z can be computed by:

$$\begin{aligned}
E(Z) &= \sum_z zP(Z = z) \\
&= \sum_{x,y} (x + y)P(X = x, Y = y) \\
&= \sum_{x,y} xP(X = x, Y = y) + \sum_{x,y} yP(X = x, Y = y) \\
&= \sum_x x \sum_y P(X = x, Y = y) + \sum_y y \sum_x P(X = x, Y = y) \\
&= \sum_x xP(X) + \sum_y yP(Y = y) \\
&= E(X) + E(Y)
\end{aligned}$$

2.3 Expectation of Products of Independent Variables

Lemma: If X and Y are independent then $E(XY) = E(X)E(Y)$

This is because $P(X = x \cap Y = y) = P(X = x)P(Y = y)$ if X and Y are independent

Proof.

$$\begin{aligned}
E(XY) &= \sum_a \sum_b abP(X = a, Y = b) \\
&= \sum_a \sum_b abP(X = a)P(Y = b) \\
&= \sum_a aP(X = a) \sum_b bP(Y = b) \\
&= E(X)E(Y)
\end{aligned}$$

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2.4 Variance of Products of Independent Variables

Lemma: If X and Y are independent then $Var(X + Y) = Var(X) + Var(Y)$

This is because $E(XY) = E(X)E(Y)$ if X and Y are independent as shown above

Proof.

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\begin{aligned}
Var(X+Y) &= E((X+Y)^2) - E(X+Y)^2 \\
&= E(X^2 + 2XY + Y^2) - (E(X) + E(Y))^2 \\
&= E(X^2) + 2E(XY) + E(Y^2) - E(X)^2 - 2E(X)E(Y) - E(Y)^2 \\
&= (E(X^2) - E(X)^2) + (E(Y^2) - E(Y)^2) + 2(E(XY) - E(X)E(Y)) \\
&= Var(X) + Var(Y) + 2 \times 0 \\
&= Var(X) + Var(Y)
\end{aligned}
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