1 Introduction

This lecture covers independence, conditional independence, and the counting principle (not covered in this worksheet).

2 Independence

Definition:

We say events A and B are independent events if and only if:

$$P(A \cap B) = P(A) \times P(B)$$

In addition, if $P(B) > 0$:

$$P(A|B) = P(A)$$

Note: The definition says that the occurrence of event A does not change the probability that event B has occurred (and vice versa.)

It is easy to think that two disjoint events are independent but this is **not true**. The fact is two disjoint events A, B are not independent if $P(A) > 0$ and $P(B) > 0$. The best practice is to always verify the equation in the above definition.

3 Independence of Many Events

Definition:

We say that the events $A_1, A_2, \ldots, A_n$ are independent if and only if:

$$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$$

for **every** subset S of $\{1, 2, \ldots, n\}$.

4 Conditional Independence

Definition:

We say A and B are conditionally independent given C ($P(C) > 0$) if and only if:

$$P(A \cap B|C) = P(A|C)P(B|C)$$

In addition, if $P(B \cap C) > 0$:
$$P(A|B \cap C) = P(A|C)$$

5 Practice Problems

1. Alice and Bob want to choose between the opera and the movies by tossing a fair coin. Unfortunately, the only available coin is biased (though the bias is not known exactly). How can they use the biased coin to decide so that either option (opera or the movies) is equally likely to be chosen?

Answer:

The trick is to toss the coin twice. If the first outcome is head and the second is tail, go for the opera. If the first outcome is the tail and the second is the head, go for the movie. In other cases, Alice and Bob need to toss the coin twice for another round.

The sample space of tossing the coin twice consists of four possible outcomes: $S = \{HH, TT, HT, TH\}$. We know that $P(HT) = P(TH)$ because the first toss and second toss are independent events. By limiting the sample space to $S' = \{HT, TH\}$, we make the two outcomes equally likely. Formally, let A_k be the event that a decision is made at the kth round, conditional on the event A_k we have:

$$P(opera) = \sum_{k=1}^{\infty} P(opera \mid A_k) P(A_k) = \sum_{k=1}^{\infty} \frac{1}{2} P(A_k) = \frac{1}{2}$$

We have assumed $P(H) > 0$ and $P(T) > 0$ so $\sum_{k=0}^{\infty} P(A_k) = 1$.