E1 244: Detection & Estimation Theory	Jan-May 2017
Lecture 9 — February 2	
Lecturer: Aditya Gopalan	*Scribe: Astha Singh & Varadaraj M.S.*

Signal Detection in Discrete Time

9.1 Introduction

In this lecture we will learn to apply the binary hypothesis-testing principles, to derive optimum procedures for detecting signals embedded in noise. Let's consider the case of discrete-time detection.

9.2 Signal Detection Models and Detector Structures

The statistical model we consider has the observation one of the two possible discrete time signals, (with n samples) corrupted by additive noise. Thus the observation vector $\underline{Y}$ is,

$$\underline{Y} = (Y_1, Y_2, \dots, Y_n)^T \quad Y \in \Re^n,$$

$\underline{N}$ is a vector of noise samples,

$$\underline{N} = (N_1, \cdots N_n)^T,$$

And, $\underline{S}_0$ and $\underline{S}_1$ are vectors of samples from the two possible signals,

$$\underline{S}_o = (S_{01}, \dots S_{0n})^T$$
$$\underline{S}_1 = (S_{11}, \dots S_{1n})^T.$$

The hypothesis pair is,

$$H_o : Y_k = S_{0k} + N_k \quad k \in [n]$$
$$\text{vs}$$
$$H_1 : Y_k = S_{1k} + N_k \quad k \in [n]$$
$$\text{where } [n] \equiv 1, 2, \dots, n.$$

Typically, the noise $\underline{N}$ is independent of the signals $\underline{S}_0$ and $\underline{S}_1$, and we work with this assumption throughout.
Now there are three cases:

- $\underline{S}_0$ and $\underline{S}_1$ are deterministic and known
- $\underline{S}_0$ and $\underline{S}_1$ are partially deterministic and partially random
- $\underline{S}_0$ and $\underline{S}_1$ are completely random

9.2.1 Detection of Deterministic Signals in Independent Noise

The two signals S_0 and S_1 are completely deterministic. We have $\underline{S}_j = \underline{s}_j$, with $\underline{s}_j \in \Re^n$ being known to the designer. This is also known as the *coherent* detection problem.

Assumption 9.2.1. N_1 , N_2 ,..., N_n are independent.

The likelihood ratio of an observation $\underline{y} \in \Re^n$ is:

$$\begin{aligned}
L(y) &= \frac{p_1(y)}{p_0(y)} \\
&= \frac{p_{\underline{N}}(\underline{y} - \underline{s}_1)}{p_{\underline{N}}(\underline{y} - \underline{s}_0)} \quad \text{where,} p_{\underline{N}}(x)\text{: Probability density of } \underline{\text{N}} \text{ at x} \\
&= \frac{p_{N_1}(y_1 - s_{11}).p_{N_2}(y_2 - s_{12}) \dots p_{N_n}(y_n - s_{1n})}{p_{N_1}(y_1 - s_{01}).p_{N_2}(y_2 - s_{02}) \dots p_{N_n}(y_n - s_{0n})} \\
&= \prod_{k=1}^{n} \frac{p_{N_k}(y_k - s_{1k})}{p_{N_k}(y_k - s_{0k})} \\
&:= \prod_{k=1}^{n} L_k(y_k)
\end{aligned}$$

An optimum detector (Bayes, Minmax, Neyman-Pearson) has the form:

$$\delta(y) = \begin{cases} 1, & \text{if} & > \\ \gamma, & \text{if } \sum_{k=1}^{n} logL_k(y_k) & = log\tau \\ 0, & \text{if} & < \end{cases}$$

As illustrated in Figure 9.1, this structure consists of a time-varying instantaneous non-linearity $logL_k$, followed by an accumulator, that is in turn followed by a threshold comparator.

Example 9.2.2 (Coherent Detection in iid (independent and identically distributed) Gaussian Noise).

$$(N_1, N_2, ..., N_n) \sim \mathcal{N}(0, \sigma^2)$$

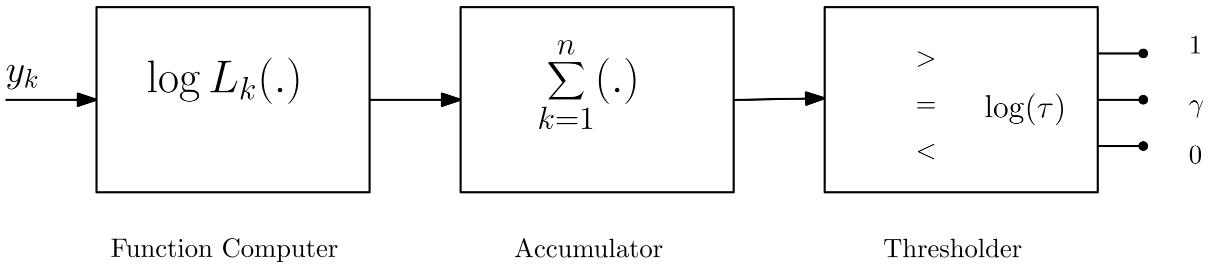

Figure 9.1: Detector Structure : Coherent Detection in Independent Noise

Assumption 9.2.3.

$$\begin{aligned} \underline{s}_0 &= \underline{0} \quad \in \Re^n \\ \underline{s}_1 &= \underline{s} \quad \in \Re^n \end{aligned}$$

This assumption does not result in any loss in generality since we could always redefine our observations as $\underline{y}' = \underline{y} - \underline{s}_0$ so that the signal would be $\underline{0}$ under H_0 and $\underline{s} = \underline{s}_1 - \underline{s}_0$ under H_1.

Computing $L_k(y_k)$:

$$\begin{aligned} logL_k(y_k) &= log\frac{\exp -\frac{(y_k - s_k)^2}{2\sigma^2}}{\exp -\frac{(y_k - 0)^2}{2\sigma^2}} \\ &= \frac{1}{2\sigma^2}(y_k^2 - (y_k - s_k)^2) \\ &= s_k(y_k - \frac{s_k}{2})\frac{1}{\sigma^2} \end{aligned}$$

The optimum detector is:

$$\delta(y) = \begin{cases} 1, & \text{if} \quad \sum_{k=1}^{n} s_k(y_k - \frac{s_k}{2}) > \tau' \\ \gamma, & \text{if} \quad \sum_{k=1}^{n} s_k(y_k - \frac{s_k}{2}) = \tau' \\ 0, & \text{if} \quad \sum_{k=1}^{n} s_k(y_k - \frac{s_k}{2}) < \tau' \end{cases}$$

where, $\tau' := \sigma^2 log\tau$

$$\Rightarrow \begin{cases} 1, & \text{if} \quad \sum_{k=1}^{n} s_k y_k > \tau'' \\ \gamma, & \text{if} \quad \sum_{k=1}^{n} s_k y_k = \tau'' \\ 0, & \text{if} \quad \sum_{k=1}^{n} s_k y_k < \tau'' \end{cases}$$

where, $\tau'' := \tau' + \frac{1}{2}\sum_{k=1}^{n} s_k^2$

This optimum detector structure is depicted in Figure 9.2 and is known as Matched Filter Detector.

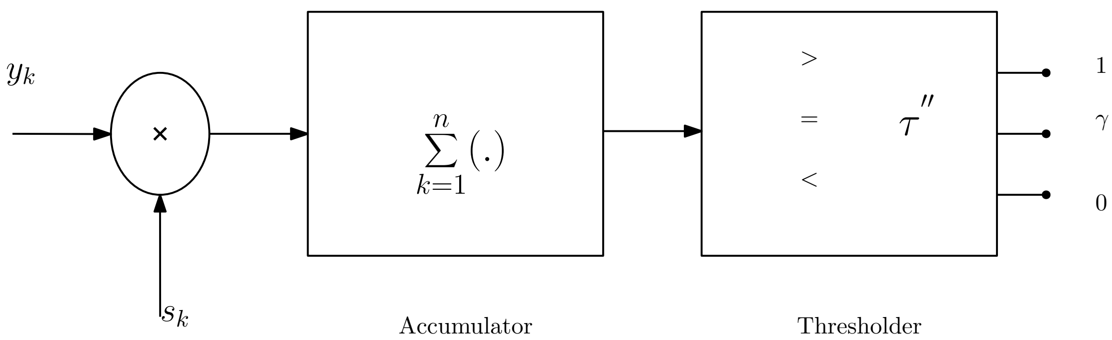

Figure 9.2: Optimum detector for coherent signals i.i.d Gaussian noise.

Example 9.2.4 (Coherent Detection in iid Laplace Noise). Here also the noise samples $N_1, \ldots, N_2$ are i.i.d, but with Laplacian marginal probability density,

$$p_{N_k}(x) = \frac{\alpha}{2} e^{-\alpha|x|}$$

where, $\alpha > 0$: a constant

This model is sometimes used to represent the behaviour of impulsive noises in communication receivers.

Computing $L_k(y_k)$:

$$\begin{aligned} logL_k(y_k) &= log\frac{e^{-\alpha|y_k - s_k|}}{e^{-\alpha|y_k|}} \\ &= \alpha(|y_k| - |y_k - s_k|) \end{aligned}$$

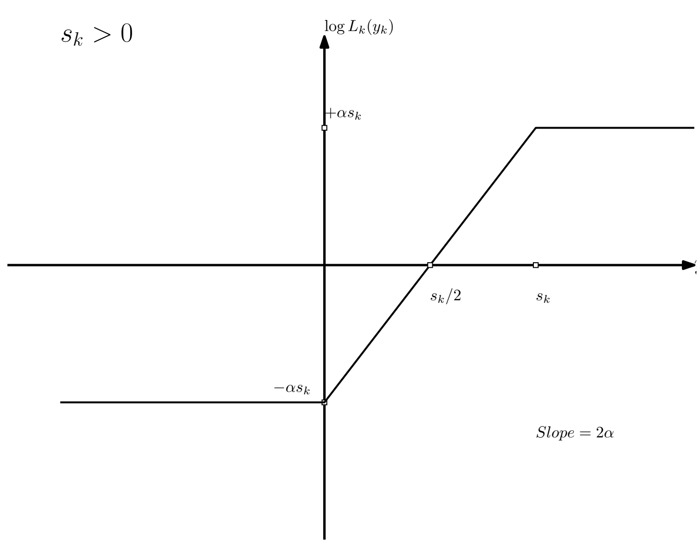

Figure 9.3: $S_k > 0$

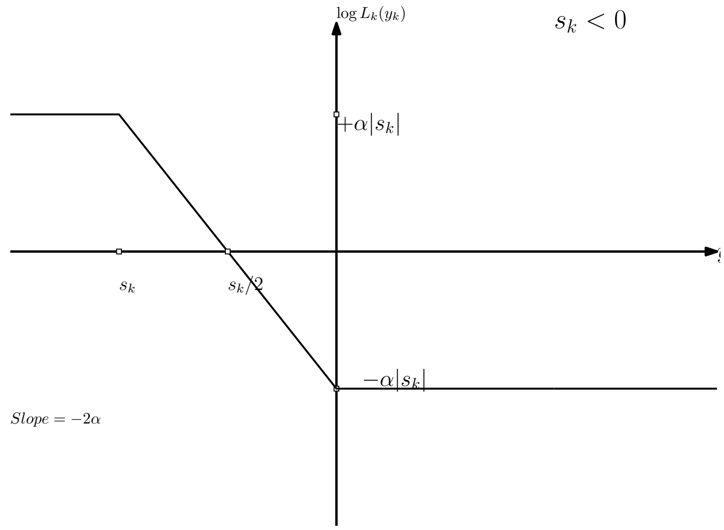

Figure 9.4: $S_k < 0$

This function $logL_k(y_k)$ is depicted in Figure 9.3 and Figure 9.4 for both cases $s_k < 0$ and $s_k > 0$ respectively.

We now define $l_k(x)$ as:

$$l_k(x) = \begin{cases} -\frac{|s_k|}{2} & \text{if} \quad x \leq -\frac{|s_k|}{2} \\ x & \text{if} \quad \frac{-|s_k|}{2} < x \leq \frac{|s_k|}{2} \\ \frac{|s_k|}{2} & \text{if} \quad x > \frac{|s_k|}{2} \end{cases}$$

This function is sometimes known as a *Soft Limiter*, Figure 9.5.

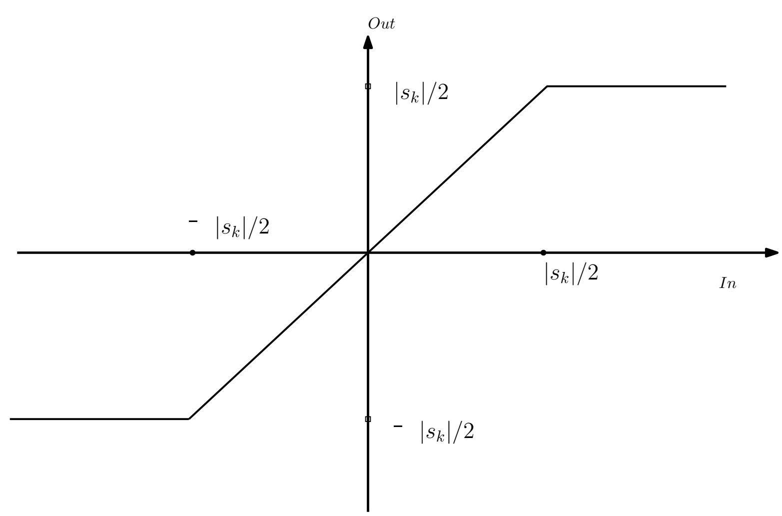

Figure 9.5: Soft Limiter

$$logL_k(y_k) = 2\alpha sign(S_k) l_k(y_k - \frac{S_k}{2}) \tag{9.1}$$

Thus the optimal detector is:

$$\delta(y) = \begin{cases} 1, & \text{if} \quad & > \\ \gamma, & \text{if} \quad \sum_{k=1}^{n} sign(s_k) l_k(y_k - \frac{s_k}{2}) & = \tau' \\ 0, & \text{if} \quad & < \end{cases}$$

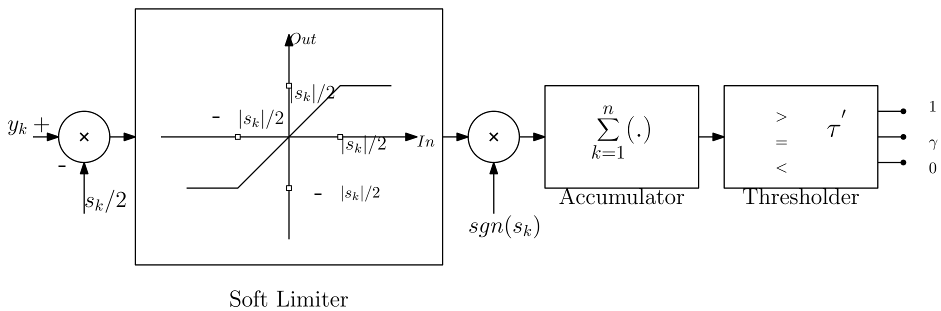

Figure 9.6: Soft Limiter Detector

Thus this detector system "centers" the observations by subtracting $s_k/2$ from each y_k. It then correlates the centered(soft-limited) data with the known signal and compares the output of this correlation with a threshold. It has the effect of making the system more tolerant to large noise values.

9.2.2 Locally Optimum Detection of Coherent Signals in i.i.d. Noise

While detecting signals, the expected structure or form of the received signal is often known and it's amplitude is unknown. Such a problem is modelled using composite hypothesis-testing described by

$$H_0 : Y_k = N_k, \quad k \in [n],$$
$$H_1 : Y_k = N_k + \theta s_k, \quad k \in [n], \quad \theta > 0,$$

Assumption 9.2.5. $\underline{s} = (s_1, ..., s_n)^T$ is a known signal.

Assumption 9.2.6. $\underline{N} = (N_1, ..., N_n)^T$ is a continuous random vector with i.i.d. components and marginal probability density functions p_{N_k}, where θ is the parameter generally associated with attenuation.

Hence the distribution is

$$\Lambda = [0, \infty)$$
$$\Lambda_0 = \{0\}, \ \Lambda_1 = (0, \infty)$$

Definition 9.2.7. The critical region for testing H_0 v/s H_1 is: $\Gamma_\theta = \{\underline{y} \in \mathbb{R}^n | L_\theta(\underline{y}) > \tau\}$.

Note 1. You can always subtract one from another. Hence testing H_0 v/s H_1 simplifies to testing $\{0\}$ v/s $\{\theta\}$.

Note 2. If the critical region depends on $\theta \in \Lambda_1$ then there cannot exist a UMP (Uniformly Most Powerful) test. Hence UMP test exists only for particular noise models.

Note 3. However LMP (Locally Most Powerul) tests have a simple and inherently reasonable structure, and thus it is of interest to consider locally optimum detection for this case.

9.2.3 LMP test

A LMP test is of the form:

$$\delta_{LMP}(y) = \begin{cases} 1 & > \\ \gamma \quad \text{if } \frac{\partial}{\partial\theta} P_\theta(\underline{y})\big|_{\theta=0} & = \tau P_0(\underline{y}). \\ 0 & < \end{cases} \tag{9.2}$$

$$\frac{\partial}{\partial\theta} L_\theta(\underline{y})\big|_{\theta=0} \quad \gtreqless \quad \tau \tag{9.3}$$

Upon differentiation, we have

$$\begin{aligned}
\frac{\partial}{\partial\theta} L_\theta(\underline{y})\,|_{\theta=0} &= \frac{\partial}{\partial\theta}\left(\prod_{k=1}^{n} \frac{p_{N_1}(y_k - \theta s_k)}{p_{N_1}(y_k)}\right)\Bigg|_{\theta=0} \\
&= \left(\prod_{k=1}^{n} \frac{p_{N_1}(y_k - \theta s_k)}{p_{N_1}(y_k)}\right)\left(\sum_{k=1}^{n} \frac{\frac{\partial}{\partial\theta} p_{N_1}(y_k - \theta s_k)}{p_{N_1}(y_k - \theta s_k)}\right)\Bigg|_{\theta=0} \\
&= \sum_{k=1}^{n} \frac{-s_k p'_{N_1}(y_k - \theta s_k)}{p_{N_1}(y_k - \theta s_k)}\Bigg|_{\theta=0} \\
&= \sum_{k=1}^{n} s_k \left(\frac{-p'_{N_1}(y_k)}{p_{N_1}(y_k)}\right) \\
&= \sum_{k=1}^{n} s_k g_{lo}(y_k)
\end{aligned} \tag{9.4}$$

where the second equality is obtained by the modified version of chain rule of differentiation

$$\begin{aligned}
\frac{dv}{d\theta}(f(\theta).g(\theta)) &= f(\theta).g^{'}(\theta) + f^{'}(\theta).g(\theta) \\
&= f(\theta).g(\theta)\left\{\frac{g^{'}(\theta)}{g(\theta)} + \frac{f^{'}(\theta)}{f(\theta)}\right\}
\end{aligned} \tag{9.5}$$

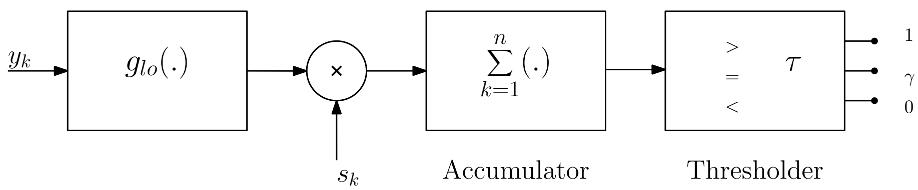

Figure 9.7: Locally optimum detector structure for coherent signals in i.i.d noise

where $g_{lo}(y) \triangleq -\, p'_{N_1}(y)/p_{N_1}(y)$, where $p'_{N_1}(y) = dp_{N_1}(y)/dy$. Structure depicted in Figure(9.7) (Note:Similar to likelihood ratio the locally optimum nonlinearity g_{lo} shapes the observation to reduce the ill effects of noise.)

Example 9.2.8. For Standard Gaussian noise $\mathcal{N}(0, \sigma^2)$, we have

$$g_{lo}(y) = \frac{-\frac{1}{\sigma.\sqrt{2\pi}} e^{-(y^2)/2\sigma^2} . \frac{-2y}{2\sigma^2}}{\frac{1}{\sigma.\sqrt{2\pi}} e^{-(y^2)/2\sigma^2}} \tag{9.6}$$

$$= \frac{y}{\sigma^2} \tag{9.7}$$

Hence the locally optimum detector structure is simply the correlation detector.

Example 9.2.9. For Standard laplacian noise with density $p_{N1}(y) = \frac{\alpha}{2} e^{-\alpha|y|}$, we have

$$g_{lo}(y) = \frac{p'_{N_1}(y)}{p_{N_1}(y)} \tag{9.8}$$

$$= \begin{cases} \alpha & if \quad y > 0 \\ 0 & if \quad y = 0 \\ -\alpha & if \quad y < 0 \end{cases} \tag{9.9}$$

we have $g_{lo}(y)$ = $\alpha sign(y)$, So the locally optimum detector correlates the signal with the sequence of signs of the observations. The function $g_{lo}(y)$ in this case is known as a ***hard limiter***(Figure9.8).

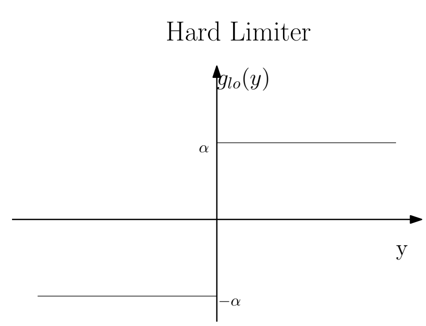

Figure 9.8: Locally optimum non linearity for standard Laplacian noise