Inf 674: Galton Watson processes

Nidhi Hegde Fabien Mathieu

December 17, 2015

1 Using Python

Within Inf674, we will try to benchmark theoretical results against numerical values. We will use Python 3[1]. No strong developing skill is required besides defining a function. If you don't know anything about Python, try one of the numerous online tutorials that should bring you up to speed in an hour or less.

The following modules may be used : `numpy` (for a minimum of maths primitives); `matplotlib` (for conveniently displaying the results). These modules will be loaded using, for example,

```
import numpy as np
import matplotlib.pyplot as plt
```

2 Galton-Watson branching process

We consider the Galton-Watson process: the system begins with one alive node. At each elementary step, one alive node spawns a certain number of children then dies. The number of children is drawn i.i.d. according to some given distribution $(p_k)_{k\in\mathbb{N}}$ s.t. $\sum_{k=0}^{\infty} p_k = 1$.

We call μ the average number of children, supposed finite: $\mu = \sum_{k=0}^{\infty} kp_k < +\infty$.

The focus for this first session will be to study the probability of extinction P_{ext}, that is the probability that eventually no alive node remains. We will admit that if $\mu > 1$, $P_{ext} < 1$.

We first assume a very simple children distribution, where a node can only have 0 or 2 children: $p_0 = 1 - \mu/2, p_2 = \mu/2, p_k = 0$ for $k \notin \{0, 2\}$.

Question 1

Give an equation that relates P_{ext} and $(p_k)_{k\in\mathbb{N}}$.

Question 2

Relate P_{ext} and μ. Write a (very) small function that gives P_{ext} for a list of μ's.

[1] To begin with.

Question 3

Give a function that tries to numerically estimate P_{ext} by observing the state (dead or alive) after t elementary steps for n samples. Suggested values: $t = 10, t = 100, t = 1000$. $n = 10000$. Display the results against the value of previous question.

Question 4

n is boring. Try to compute exactly the probability that all nodes are dead after t elementary steps. Display the results and compare. Hint: for $t < \infty$, write a function *pop_ after_ t* that computes as a function of p the (alive) population *distribution* after t elementary steps.

Question 5

We now consider a geometric distribution $p_k = (1-a)a^k$ for some $0 \leq a < 1$. Relate a and μ and go back to question 2.

Question 6

We now consider a Poisson distribution of parameter μ ($p_k = e^{-\mu}\frac{\mu^k}{k!}$). Go back to question 2.

Question 7

Bonus: start with two nodes.