Power Laws. Exercises

Master ACN: Random graphs and epidemics

January 29, 2016

1. Generative models. We considered the preferential attachement model of graph generation in class. Now consider the following model of creating a graph.

   - N nodes are created in order, $1, \ldots, N$.
   - When node i is created, it links to an existing node as follows:
     - With probability α node i chooses a node uniformly at random among those nodes already created, and creates a link to that node.
     - With probability $1 - \alpha$ node i choose a node uniformly at random, say node j, and creates a link to the node k that j points to.

   (a) Is this similar to the preferential attachement model, i.e. does it generate a power-law for the degree distribution? Explain your answer.

2. Is it because of age?
   Consider the the preferential attachment model with $\alpha = 1$. This corresponds to an attachment model where each new node chooses links to attach uniformly at random. Older nodes then will have a higher degree. Does this result in a power law? Prove it.

3. Lognormal
   A random variable X has a lognormal distribution if the random variable $Y = \ln X$ has a normal distribution. The normal distribution is given by the density function

$$f(y) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(y-\mu)^2/2\sigma^2}$$

   where μ is the mean, σ is the standard deviation and $-\infty < y < \infty$. The density function for the lognormal distribution is then

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \mathrm{e}^{-(\ln x-\mu)^2/2\sigma^2}.$$

   The lognormal has mean $\mathrm{e}^{\mu+\frac{1}{2}\sigma^2}$.

   (a) On a log-log plot, plot Pareto for $\alpha = 1$ and $\alpha = 1.5$.

   (b) On a log-log plot, plot a Lognormal with (approximately) the same slope as the above two plots.

4. Collaboration Networks. Let's take another look at the collaboration networks we used last time from the snap dataset. Plot the degree distribution and fit it to a Pareto and lognorml. Which fit is better?

Mini-project

This is the third option in your list of mini-projects.

3 Utility maximization under linear constraints [1] [1]
Consider another generative model for power-law distributions, a case of nodes communicating to each other over a set of shared links. Each node's utility is a function of the rate it receives. Let's choose a so-called $\alpha-$fair utility function, where $\alpha \geq 0$ is a fairness parameter. The objective is to assign rates for the links such that the total utility is maximized, as follows:

$$\begin{aligned} \underset{\mathbf{x}}{\text{maximize}} \quad & \sum_j \frac{x_j^{1-\alpha}}{1-\alpha} \\ \text{subject to} \quad & \mathbf{Ax} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

where b_i is the capacity of link i, x_j is the rate of session j, and $\mathbf{A}$ is a binary routing matrix between sessions and links:

$$A_{ij} = \begin{cases} 1 & \text{if session } j \text{ is present on link } i \\ 0 & \text{otherwise} \end{cases}$$

Show that x_j follows a power law (for fixed α), and give an intuition for your answer.

4 Bonus option. Yule process. A precursor to the preferential attachment model is the Yule process, named for the model suggested by Yule in 1925 for the evolution of the number of species within genera of plants. Previously, Willis had shown empirically that the distribution of such species follows a power law.

Species are grouped according to genera. Each species gives birth, at a fixed rate, to a new species due to a mutation. With probability α, this new species is so different that it creates a new genus. With probability $1-\alpha$, it remains in its genus of origin. Denote by $X_i(t)$ the number of genera with exactly i species after the t-th species appears. Write down the transition in X_i at a new species arrival. Note that the equations for X_1 will be different than those for $X_i, i > 1$. Does the distribution follow power law? Show it.

References

[1] Mung Chiang, *Networked Life. 20 Questions and Answers*, Cambridge University Press, 2012.

[1] This exercise comes from [1], but there are no answers in that book!