COSI 131 (Spring 2013): Homework Assignment #1

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Problem 1

The probability of selecting apple:

$$\begin{aligned}P(F = apple) = &\sum_{B=r,b,g} P(F = apple, B)\\ =&\frac{3}{10} * 0.2 + \frac{1}{2} * 0.2 + \frac{3}{10} * 0.6\\ =&0.34\end{aligned}$$

The probability of selecting an orange given it came from the green box:

$$\begin{aligned}P(B = green \mid F = orange) =&\frac{P(F = orange, B = green)}{P(F = orange)}\\ =&\frac{P(F = orange, B = green)}{\sum_{B=r,b,g} P(F = orange, B)}\\ =&\frac{0.3 * 0.6}{\frac{4}{10} * 0.2 + \frac{1}{2} * 0.2 + \frac{3}{10} * 0.6} = 0.5\end{aligned}$$

Problem 2

The covariance equation is:

$$\begin{aligned}\sigma(x, y) =&\mathbb{E}[(x - \mathbb{E}[x])(y - \mathbb{E}[y])]\\ =&\mathbb{E}[xy - x\mathbb{E}[y] - y\mathbb{E}[x] + \mathbb{E}[x]\mathbb{E}[y]]\\ =&\mathbb{E}[xy] - \mathbb{E}[x]\mathbb{E}[y]\end{aligned}$$

since x,y is independent, so $\mathbb{E}[xy] = \mathbb{E}[x]\mathbb{E}[y]$*, so* $\sigma(x, y) = 0$

Problem 3

Assume x,y are continuous variables, so

$$\begin{aligned}\mathbb{E}[x+z] =& \iint (x+z)p(x,z) \\ =& \iint (x+z)p(x)p(z) = \int xp(x)\mathrm{d}x + \int zp(z)\mathrm{d}z \\ =& \mathbb{E}[x] + \mathbb{E}[z]\end{aligned}$$

$$\begin{aligned}var[x+z] =& \iint (x+z-\mathbb{E}[x+z])^2 p(x)p(z)\mathrm{d}x\mathrm{d}z \\ =& \int (x-\mathbb{E}[x])^2 p(x)\mathrm{d}x + \int (z-\mathbb{E}[z])^2 p(z)\mathrm{d}z \\ =& var[x] + var[z]\end{aligned}$$

For discrete variables the integrals are replaced by summations, and we get the same results.

Problem 4

Since for every matrix $\mathbf{A}$, *can be writtern in the form* $\frac{1}{2}(\mathbf{A}+\mathbf{A}^\top)+\frac{1}{2}(\mathbf{A}-\mathbf{A}^\top)$, *and* $\mathbf{A}+\mathbf{A}^\top$ *is symmetric and* $\mathbf{A}-\mathbf{A}^\top$ *is antisymmetric.*

$$\begin{aligned}2 * \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}x_i x_j =& \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}x_i x_j + \sum_{j=1}^{D}\sum_{i=1}^{D} w_{ji}x_i x_j \\ =& \sum_{i=1}^{D}\sum_{j=1}^{D} (w_{ij}^S + w_{ji}^S + w_{ij}^A + w_{ji}^A) x_i x_j \\ =& 2 * \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^S x_i x_j\end{aligned}$$

So we get $\sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}x_i x_j = \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij}^S x_i x_j$

Since w_{ij} *is dependent on* w_{ji} *when* $i \neq j$, *the number of independent parameter is the number of elements of matrix's upper triangular, which is* $\frac{D(D+1)}{2}$.

Problem 5

$$h(x,y) = h(x) + h(y)$$

Thus,

$$h(p^2) = 2h(p)$$

$$h(p^{k+1}) = h(p^k) + h(p) = (k+1)h(p)$$

So $h(p^n) = nh(p)$*,*

$$h(p^{\frac{n}{m}}) = nh(p^{\frac{1}{m}}) = \frac{mn}{m}h(p^{\frac{1}{m}}) = \frac{n}{m}h(p)$$

for $q = p^k$*,*

$$\frac{h(q)}{\ln q} = \frac{kh(p)}{k\ln p} = \frac{h(p)}{\ln p}$$

it's proved.

Problem 6

Since $\left|\int p(\mathbf{x})\mathrm{d}\mathbf{x}\right| = \left|\int p(\mathbf{y})\mathrm{d}\mathbf{y}\right| = 1$*, we have* $p(\mathbf{y})\left|\mathbf{A}\right| = p(\mathbf{x})$

$$\begin{aligned}
H[\mathbf{y}] =& -\int p(\mathbf{y})\ln p(\mathbf{y})\mathrm{d}\mathbf{y} \\
=& -\int \left|\mathbf{A}\right| p(\mathbf{x})\left|\mathbf{A}\right|^{-1}\ln\left(p(\mathbf{x})\left|\mathbf{A}\right|^{-1}\right)\mathrm{d}\mathbf{x} \\
=& -\int p(\mathbf{x})\ln p(\mathbf{x}) + \ln\left|\mathbf{A}\right|\int p(\mathbf{x})\mathrm{d}\mathbf{x} \\
=& -\int p(\mathbf{x})\ln p(\mathbf{x}) + \ln\left|\mathbf{A}\right| \\
=& H[\mathbf{x}] + \ln\left|\mathbf{A}\right|
\end{aligned}$$

Problem 7

Given that,

$$\begin{aligned}
p(x=1) =& p(x=1, y=0) + p(x=1, y=1) = \frac{1}{3} \\
p(x=0) =& p(x=0, y=0) + p(x=0, y=1) = \frac{2}{3} \\
p(y=1) =& \frac{2}{3} \\
p(y=0) =& \frac{1}{3}
\end{aligned}$$

(a)

$$\begin{aligned}H[x] &= -\sum_x p(x)\ln p(x)\\ &= -\left(p(x=0)\ln p(x=0) + p(x=1)\ln p(x=1)\right)\\ &= -\left(\frac{2}{3}\ln\frac{2}{3} + \frac{1}{3}\ln\frac{1}{3}\right)\end{aligned}$$

(b)

Using the same method,we get:

$$H[y] = -\left(\frac{2}{3}\ln\frac{2}{3} + \frac{1}{3}\ln\frac{1}{3}\right)$$

(c)

$$\begin{aligned}H[y \mid x] &= -\sum_x\sum_y p(x,y)\ln p(y \mid x)\\ &= \frac{2}{3}\ln 2\end{aligned}$$

(d)

$$H[x \mid y] = \frac{2}{3}\ln 2$$

(e)

$$H[x,y] = H[y \mid x] + H[x] = \ln 3$$

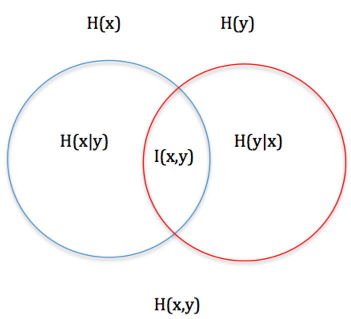

Figure 1: Relation

(f)

$$I[x,y] = H[x] - H[x \mid y] = \ln 3 - \frac{4}{3}\ln 2$$

Problem 8

(a)

$$\frac{\partial f(x)}{\partial x} = C - [\ln \sum_{n=1}^{N} e^{-\frac{d_n}{x}} + \frac{x\sum_{n=1}^{N}(e^{-\frac{d_n}{x}} \cdot \frac{d_n}{x^2})}{\cdot \sum_{n=1}^{N} e^{-\frac{d_n}{x}}}]$$

(b),(c)

$$\begin{aligned}
\frac{\partial^2 f(x)}{\partial x^2} =& -2g^{'}(x) - xg^{''}(x) \\
g^{'}(x) =& \frac{\sum_{n=1}^{N}(e^{-\frac{d_n}{x}}\frac{d_n}{x^2})}{\sum_{n=1}^{N} e^{-\frac{d_n}{x}}} \\
g^{''}(x) =& \frac{\sum_{n=1}^{N} e^{-\frac{d_n}{x}}(\sum_{n=1}^{N}(e^{-\frac{d_n}{x}}\frac{(d_n)^2}{x^4}) - 2\sum_{n=1}^{N}(e^{-\frac{d_n}{x}}\frac{d_n}{x^3})) + [\sum_{n=1}^{N}(e^{-\frac{d_n}{x}}\frac{d_n}{x^2})]^2}{(\sum_{n=1}^{N} e^{-\frac{d_n}{x}})^2} \\
\frac{\partial^2 f(x)}{\partial x^2} =& -\frac{\sum_{n=1}^{N} e^{-\frac{d_n}{x}}(\sum_{n=1}^{N}(e^{-\frac{d_n}{x}}\frac{(d_n)^2}{x^4})) + [\sum_{n=1}^{N}(e^{-\frac{d_n}{x}}\frac{d_n}{x^2})]^2}{(\sum_{n=1}^{N} e^{-\frac{d_n}{x}})^2} < 0
\end{aligned}$$

So it is a convex function.