Trigonometry Final Exam

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Topic: Introduction to the unit circle

Question: Using a unit circle, what are the coordinates for the given angle?

$$\frac{2\pi}{3}$$

Answer choices:

A $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

B $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

C $\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

D $\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

Solution: B

The unit circle allows for easy conversion between degrees and radians, and it provides the coordinates on the cartesian plane for a circle centered at the origin $(0,0)$ with radius 1.

Looking at the unit circle shows that the coordinate point associated with $2\pi/3$ is

$$\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$$

Topic: Finding sine and cosine of an angle theta

Question: Which of the following is true about the angle?

$$\theta = \frac{15\pi}{8} \text{ radians}$$

Answer choices:

A $\sin(\theta) \approx -0.56$ and $\cos(\theta) \approx 0.83$

B $\sin(\theta) \approx -0.38$ and $\cos(\theta) \approx 0.92$

C $\sin(\theta) \approx -0.47$ and $\cos(\theta) \approx 0.88$

D $\sin(\theta) \approx -0.25$ and $\cos(\theta) \approx 0.97$

Solution: B

We'll sketch the sides of an angle of $15\pi/8$ radians. First, note that

$$\frac{3\pi}{2} = \frac{12\pi}{8} < \frac{15\pi}{8} < \frac{16\pi}{8} = 2\pi$$

Therefore, the terminal side of θ lies in the fourth quadrant. Also,

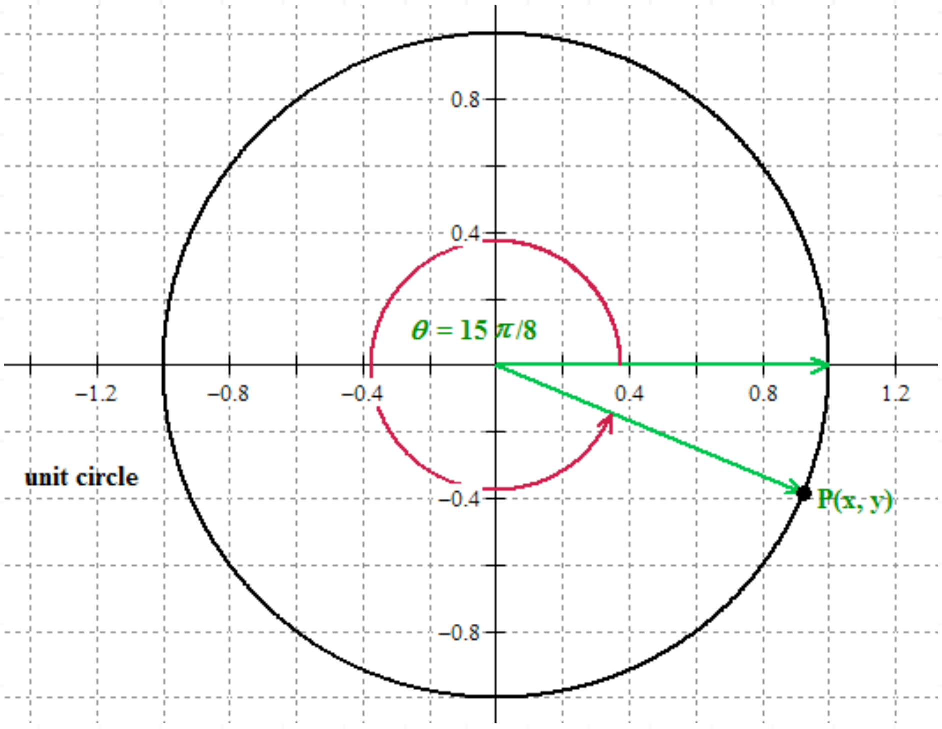
$$\frac{3\pi}{2} - \frac{12\pi}{8} = \frac{12\pi}{8} - \frac{12\pi}{8} < \frac{15\pi}{8} - \frac{12\pi}{8} < \frac{16\pi}{8} - \frac{12\pi}{8} = 2\pi - \frac{12\pi}{8}$$

Thus

$$\frac{3\pi}{2} - \frac{12\pi}{8} = 0 < \frac{3\pi}{8} < \frac{4\pi}{8} = 2\pi - \frac{12\pi}{8}$$

That is, in terms of angle measure, the terminal side of an angle of $15\pi/8$ radians is located $3/4$ of the way from the terminal side of an angle of $3\pi/2$ radians to the terminal side of an angle of 2π radians.

On the same set of coordinate axes, we'll graph the unit circle and then determine the values of the x- and y-coordinates of the point P where that circle intersects the terminal side of an angle of $15\pi/8$ radians.



According to the graph, $0.9 < x < 1$ and $-0.4 < y < -0.3$. Since

$$\sin\frac{15\pi}{8} = y$$

and

$$\cos\frac{15\pi}{8} = x$$

we find that the correct answer is B.

$\sin\frac{15\pi}{8} \approx -0.38$ and $\cos\frac{15\pi}{8} \approx 0.92$

Topic: Finding sine given cosine, or vice versa

Question: Find the value.

If the measure of an angle θ, in radians, is $6\pi/5$ and $\sin\theta = -0.588$, what is the value of $\cos\theta$?

Answer choices:

A -0.809

B 0.998

C 0.809

D -0.998

Solution: A

We're given the value of $\sin(6\pi/5)$, and we want to compute the value of $\cos(6\pi/5)$ from it, so we'll use the equation

$$\cos^2\left(\frac{6\pi}{5}\right) = 1 - \sin^2\left(\frac{6\pi}{5}\right)$$

Substituting the value of $\sin(6\pi/5)$, we have

$$\cos^2\left(\frac{6\pi}{5}\right) = 1 - (-0.588)^2$$

Now $(-0.588)^2 \approx 0.346$, so

$$\cos^2\left(\frac{6\pi}{5}\right) \approx 1 - 0.346$$

$$\cos^2\left(\frac{6\pi}{5}\right) \approx 0.654$$

Thus either

$$\cos\left(\frac{6\pi}{5}\right) \approx \sqrt{0.654}$$

or

$$\cos\left(\frac{6\pi}{5}\right) \approx -\sqrt{0.654}$$

The terminal side of an angle of $6\pi/5$ (radians) lies in the third quadrant, because

$$\pi = \frac{5\pi}{5} < \frac{6\pi}{5} = \frac{12\pi}{10} < \frac{15\pi}{10} = \frac{3\pi}{2}$$

Since the x-coordinate of every point in the third quadrant is negative,

$$\cos\left(\frac{6\pi}{5}\right) \approx -\sqrt{0.654}$$

Taking the square root of 0.654, we obtain

$$\cos\left(\frac{6\pi}{5}\right) \approx -0.809$$

That is,

$$\cos\theta \approx -0.809$$

Topic: All possible values for the angle theta

Question: Which of the following is a set of angles (in radians) that satisfy the equation?

$$\cos\theta = \cos\left(\frac{11\pi}{8}\right)$$

where $\frac{11\pi}{8}$ is in radians

Answer choices:

A $\left\{-\frac{57\pi}{8} \pm 2\pi k: \ k = 0,1,2,\ldots\right\}$

B $\left\{\frac{45\pi}{8} \pm 2\pi k: \ k = 0,1,2,\ldots\right\}$

C $\left\{-\frac{75\pi}{8} \pm 2\pi k: \ k = 0,1,2,\ldots\right\}$

D $\left\{\frac{29\pi}{8} \pm 2\pi k: \ k = 0,1,2,\ldots\right\}$

Solution: C

One possibility is that an angle of $11\pi/8$ is included in one of the given sets of angles. Since any two angles in each of the given sets differ from each other by some multiple of 2π, we can easily determine if this possibility occurs, by comparing the difference between just one angle in each set and an angle of $11\pi/8$. We will compute the differences between the $k=0$ angle in each set (which we'll call θ) and $11\pi/8$.

For $\theta = -\frac{57\pi}{8}$:

$$\theta - \frac{11\pi}{8} = \frac{(-57-11)\pi}{8} = -\frac{68\pi}{8} = -\frac{17\pi}{2}$$

For $\theta = \frac{45\pi}{8}$:

$$\theta - \frac{11\pi}{8} = \frac{(45-11)\pi}{8} = \frac{34\pi}{8} = \frac{17\pi}{4}$$

For $\theta = -\frac{75\pi}{8}$:

$$\theta - \frac{11\pi}{8} = \frac{(-75-11)\pi}{8} = -\frac{86\pi}{8} = -\frac{43\pi}{4}$$

For $\theta = \frac{29\pi}{8}$:

$$\theta - \frac{11\pi}{8} = \frac{(29-11)\pi}{8} = \frac{18\pi}{8} = \frac{9\pi}{4}$$

None of the results of this subtraction process yields an angle which is an integer multiple of 2π. Therefore, we must use a different approach to determine the answer to the question. We will use symmetry.

Note that

$$\pi = \frac{8\pi}{8} < \frac{(8+3)\pi}{8} = \frac{11\pi}{8} < \frac{12\pi}{8} = \frac{3\pi}{2}$$

This tells us that $11\pi/8$ is in the third quadrant, and in particular that $11\pi/8$ is in the interval $(\pi, 3\pi/2)$. Moreover,

$$\frac{11\pi}{8} = \frac{(8+3)\pi}{8} = \frac{8\pi}{8} + \frac{3\pi}{8} = \pi + \frac{3\pi}{8}$$

Therefore, an angle of measure $11\pi/8$ is $3\pi/8$ radians “below” the negative x-axis, so one of the given sets of angles must contain an angle in the interval $(\pi/2, \pi)$ which is $3\pi/8$ radians “above” the negative x-axis. That is, one of the given sets of angles must contain an angle of measure

$$\pi - \frac{3\pi}{8} = \frac{8\pi - 3\pi}{8} = \frac{5\pi}{8}$$

Then the measure of every angle in the same set as the angle of measure $5\pi/8$ would differ from $5\pi/8$ by some integer multiple of 2π.

Here's what we're going to do: We'll subtract $5\pi/8$ from the $k = 0$ angle (which we'll again call θ) in each of the four given sets of angles. Then the correct answer to our question will be the set of angles that contains the angle θ for which $\theta - (5\pi/8)$ is an integer multiple of 2π.

For $\theta = -\frac{57\pi}{8}$:

$$-\frac{57\pi}{8}-\frac{5\pi}{8}=\frac{(-57-5)\pi}{8}=-\frac{62\pi}{8}=-\frac{31\pi}{4}$$

For $\theta=\frac{45\pi}{8}$:

$$\frac{45\pi}{8}-\frac{5\pi}{8}=\frac{(45-5)\pi}{8}=\frac{40\pi}{8}=5\pi$$

For $\theta=-\frac{75\pi}{8}$:

$$-\frac{75\pi}{8}-\frac{5\pi}{8}=\frac{(-75-5)\pi}{8}=-\frac{80\pi}{8}=-10\pi$$

For $\theta=\frac{29\pi}{8}$:

$$\frac{29\pi}{8}-\frac{5\pi}{8}=\frac{(29-5)\pi}{8}=\frac{24\pi}{8}=3\pi$$

What we have found is that the only subtraction that resulted in an integer multiple of 2π is the one with $\theta=-(75/8)\pi$. Thus the answer to our question is the set of angles that contains an angle of $-(75/8)\pi$, so the correct answer is C.

Topic: Completing a right triangle

Question: Which of the following most closely approximates the length of the hypotenuse of the right triangle?

The length of one of the legs is 2.4 and the measure of the acute interior angle which is opposite the other leg is 28°.

Answer choices:

A 2.72

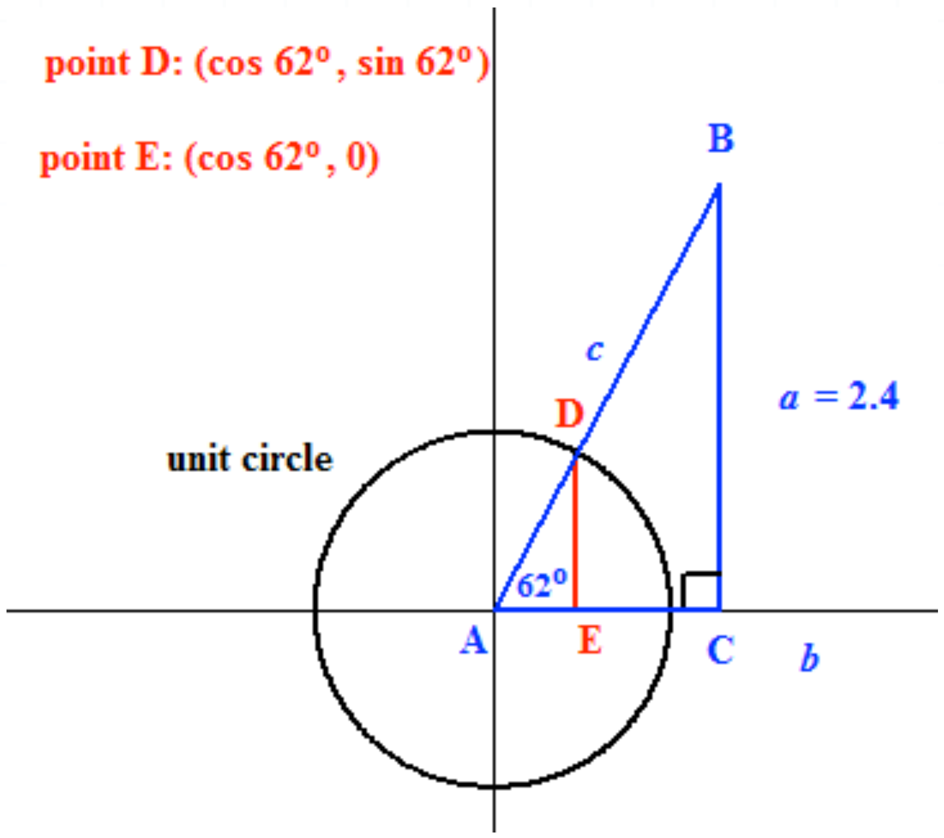
B 4.51

C 1.28

D 5.11

Solution: A

We'll draw this right triangle (triangle ABC) with vertex A at the origin, vertex B in the first quadrant, and vertex C on the positive x-axis. We'll let BC be the leg whose length is known, and we'll let a be the length of that leg (i.e., the length of side BC is $a = 2.4$). Then the measure of the acute interior angle which is opposite that leg (i.e., angle BAC) is $90^\circ - 28^\circ = 62^\circ$. We'll let b be the length of the other leg (side AC) and c the length of the hypotenuse (side AB). Then we'll draw a new right triangle (triangle ADE) which is similar to ABC, with the sides of triangle ADE oriented along the corresponding sides of triangle ABC and the "outer" endpoint of the hypotenuse (point D) on the unit circle.



By the similarity of triangles ABC and ADE, we have the following:

$$\frac{a}{DE} = \frac{c}{AD}$$

Substituting $a = 2.4$, we have

$$\frac{2.4}{\overline{DE}} = \frac{c}{\overline{AD}}$$

Since point D is on the unit circle, its coordinates (x, y) are $(\cos 62^\circ, \sin 62^\circ)$ and the length of AD is 1. Also, the coordinates (x, y) of point E are $(\cos 62^\circ, 0)$.

Note that DE is parallel to the positive y-axis, so the length of DE is equal to the difference between the y-coordinate of point D (which is $\sin 62^\circ$) and the y-coordinate of point E (which is 0):

$$\overline{DE} = \sin 62^\circ - 0 = \sin 62^\circ$$

Substituting these results, we find that

$$\frac{2.4}{\sin 62^\circ} = \frac{c}{1}$$

Simplifying the right-hand side and then turning this equation around, we have

$$c = \frac{2.4}{\sin 62^\circ}$$

Using a calculator, we can compute $\sin 62^\circ$:

$$\sin 62^\circ \approx 0.883$$

Thus

$$c \approx \frac{2.4}{0.883}$$

$$c \approx 2.72$$

Topic: Reference angle theorem

Question: If θ is an angle such that $\sin\theta = 0.439$, what are the two possible values of $\cos(\theta + 540^\circ)$?

Answer choices:

A $0.193, \ -0.193$

B $0.807, \ -0.807$

C $0.327, \ -0.327$

D $0.898, \ -0.898$

Solution: D

We're given the value of $\sin\theta$, so we'll first find the value of $\cos\theta$:

$$\sin^2\theta + \cos^2\theta = 1$$

Substituting the value of $\sin\theta$, we have

$$(0.439)^2 + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - (0.439)^2$$

Now $(0.439)^2 \approx 0.193$, so

$$\cos^2\theta \approx 1 - 0.193$$

$$\cos^2\theta \approx 0.807$$

Well,

$$\sqrt{0.807} \approx 0.898$$

so either

$$\cos\theta \approx 0.898$$

or

$$\cos\theta \approx -0.898$$

Since $\sin\theta$ is positive, θ is in either the first quadrant or the second quadrant. We'll first consider the case where θ is in the first quadrant, which implies that $\cos\theta$ is positive, hence

$$\cos\theta \approx 0.898$$

Since θ is in the first quadrant, we can "think" of θ as being in the interval $(0^\circ, 90^\circ)$, so θ is above the positive x-axis (which we can represent by an angle of measure 0). Therefore, the reference angle for θ is $\theta - 0 = \theta$.

Now note that

$$\theta + 540^\circ = \theta + (360^\circ + 180^\circ) = (\theta + 360^\circ) + 180^\circ$$

Since θ is in the first quadrant (and we're "thinking" of θ as having positive measure), an angle of measure $\theta + 540^\circ$ is equivalent to (starting on the positive x-axis and) undergoing a rotation of θ in the positive direction (which brings us to θ), followed by one complete turn around the origin in the positive direction (to get from θ to $\theta + 360^\circ$) and an additional rotation of 180° in the positive direction (to get from $\theta + 360^\circ$ to $\theta + 540^\circ$).

Denote $\theta + 360^\circ$ by θ', and note that the terminal side of θ' coincides with the terminal side of θ (because θ and $\theta + 360^\circ$ differ by an integer multiple of 360°). Therefore, the terminal side of $\theta' + 180^\circ$ coincides with the terminal side of $\theta + 180^\circ$. Note that

$$\theta' + 180^\circ = (\theta + 360^\circ) + 180^\circ = \theta + 540^\circ$$

That is, the terminal side of $\theta + 540^\circ$ coincides with the terminal side of $\theta + 180^\circ$, so the reference angle for $\theta + 540^\circ$ is equal to the reference angle for $\theta + 180^\circ$.

Since θ is in the first quadrant, we know that $\theta + 180^\circ$ is in the third quadrant, so $\theta + 180^\circ$ is below the negative x-axis (which we can represent by an angle of measure 180°). Now recall that we're "thinking" of θ as being in the interval $(0^\circ, 90^\circ)$:

$$0^\circ < \theta < 90^\circ$$

Adding 180° throughout, we obtain

$$0^\circ + 180^\circ < \theta + 180^\circ < 90^\circ + 180^\circ$$

$$180^\circ < \theta + 180^\circ < 270^\circ$$

Therefore, the reference angle for $\theta + 180^\circ$ (and hence the reference angle for $\theta + 540^\circ$) is

$$(\theta + 180^\circ) - 180^\circ = \theta + 180^\circ - 180^\circ = \theta$$

This is equal to the reference angle for θ, which we found earlier. Thus by the reference angle theorem, together with the fact that θ is in the first quadrant and $\theta + 540^\circ$ is in the third quadrant, we find that

$$\cos(\theta + 540^\circ) = -\cos(\theta) \approx -0.898$$

Now let's consider the case where θ is in the second quadrant, which implies that $\cos\theta$ is negative, hence

$$\cos\theta \approx -0.898$$

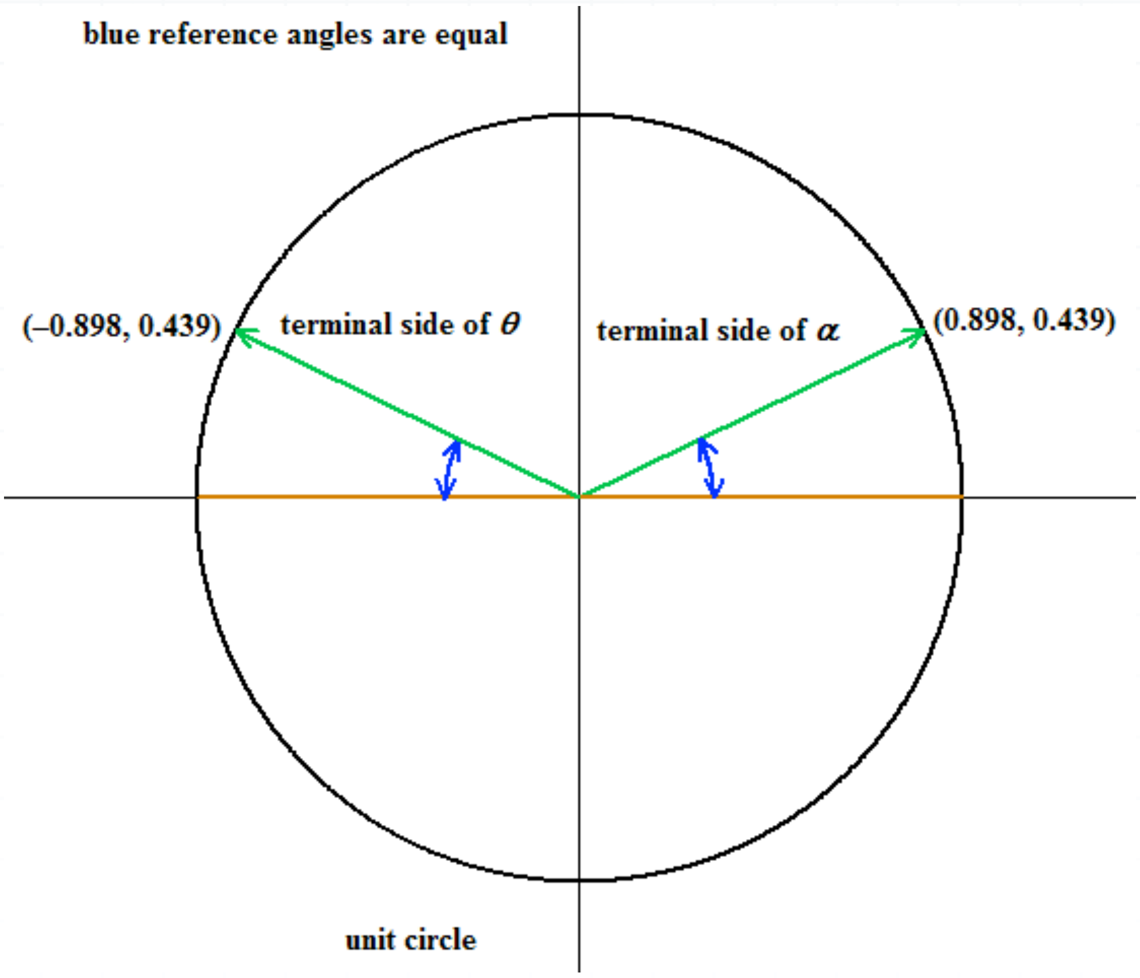
Therefore, the terminal side of θ intersects the unit circle at the point

$$(x, y) = (\cos\theta, \sin\theta) \approx (-0.898, 0.439)$$

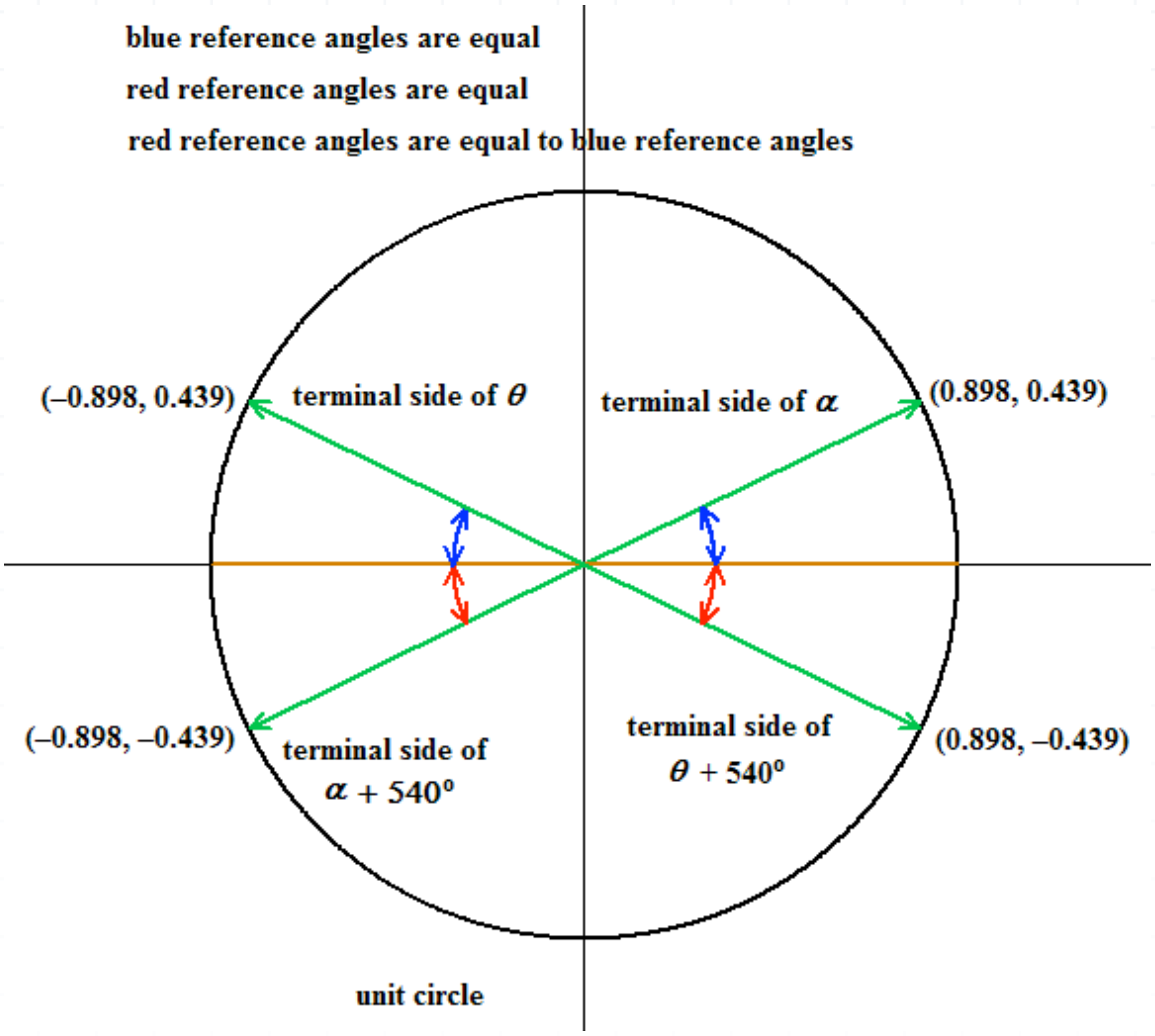
Now let α denote the first-quadrant angle θ that we discussed earlier, and note that α intersects the unit circle at the point

$$(x, y) = (\cos\alpha, \sin\alpha) \approx (0.898, 0.439)$$

Let's draw θ and α on the same set of coordinate axes.



By symmetry, we can see that the reference angle for θ is equal to the reference angle for α (which was α itself, since α is in the first quadrant and we "thought" of α as being in the interval $(0^\circ, 90^\circ)$).



If we were to go through the corresponding process for θ that we did for α, we would find that the terminal side of $\theta + 540^\circ$ is in the fourth quadrant. We would also find that the reference angle for $\theta + 540^\circ$ is equal to the reference angle for $\theta + 180^\circ$, and that the reference angle for $\theta + 180^\circ$ is equal to the reference angle for θ (which we found to be α). Since θ is in the second quadrant and $\theta + 540^\circ$ is in the fourth quadrant, the reference angle theorem tells us that

$$\cos(\theta + 540^\circ) = -\cos\theta = 0.898$$

Topic: Given a point on the terminal side of the angle

Question: Find the value of each angle.

Let α be an angle whose terminal side contains the point $(12,5)$, and let $\theta = \alpha + \pi$, where α is measured in radians. What are the values of $\sin\theta$ and $\cos\theta$?

Answer choices:

A $\sin\theta = -\frac{5}{13}, \ \cos\theta = -\frac{12}{13}$

B $\sin\theta = -\frac{12}{17}, \ \cos\theta = -\frac{5}{17}$

C $\sin\theta = \frac{5}{17}, \ \cos\theta = -\frac{12}{17}$

D $\sin\theta = \frac{12}{13}, \ \cos\theta = \frac{5}{13}$

Solution: A

First, we'll find $\sin\alpha$ and $\cos\alpha$. We can do this by substituting $x=12$ and $y=5$ into the following formulas for $\sin\alpha$ and $\cos\alpha$:

$$\sin\alpha=\frac{y}{\sqrt{x^2+y^2}}$$

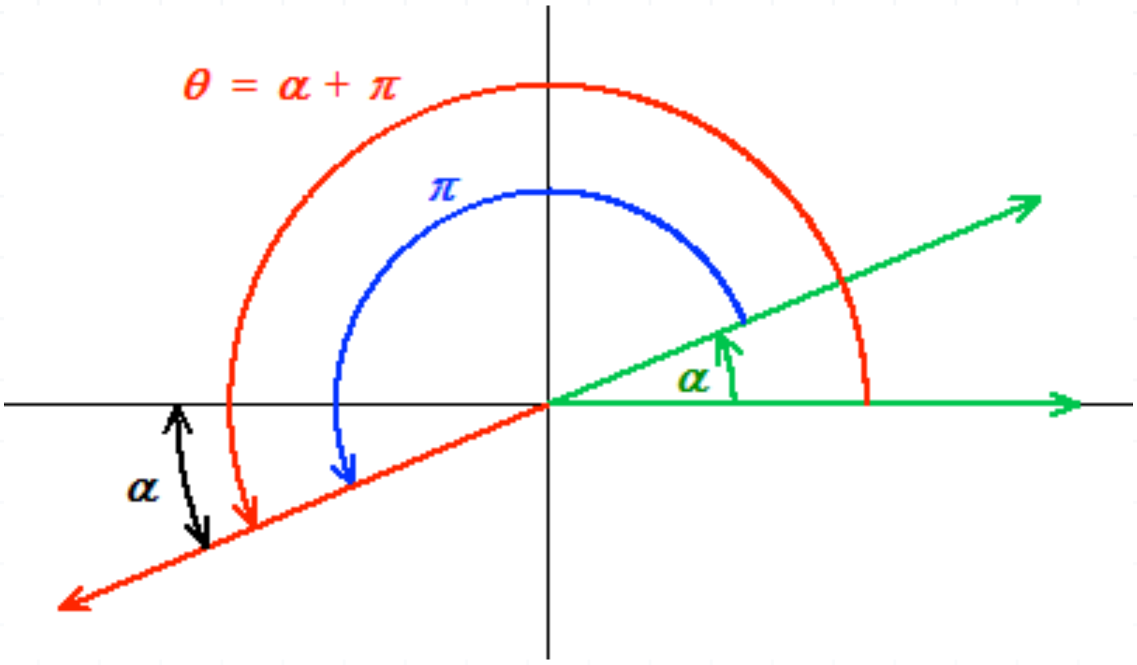
$$\cos\alpha=\frac{x}{\sqrt{x^2+y^2}}$$

Therefore,

$$\sin\alpha=\frac{5}{\sqrt{12^2+5^2}}=\frac{5}{\sqrt{144+25}}=\frac{5}{\sqrt{169}}=\frac{5}{13}$$

$$\cos\alpha=\frac{12}{\sqrt{12^2+5^2}}=\frac{12}{\sqrt{144+25}}=\frac{12}{\sqrt{169}}=\frac{12}{13}$$

Since the point $(12,5)$ is on the terminal side of α, we know that α is in the first quadrant. Thus $\alpha+\pi$ is in the third quadrant. Moreover, the reference angle for θ is equal to α, which is also the reference angle for α itself.



By the reference angle theorem, we obtain the following:

$$\sin\theta = -\sin\alpha = -\frac{5}{13}$$

$$\cos\theta = -\cos\alpha = -\frac{12}{13}$$

Topic: Period and amplitude

Question: Which of the following statements is false?

Answer choices:

A The period of the function $-\pi\sin(\pi^2\theta)$ is $2/\pi$.

B The period of the function $\pi^3\tan(4\theta/3)$ is $3\pi/4$.

C The period of the function $-1.7\left[\frac{\sec\left(\frac{\theta}{3}\right)}{\pi}\right]$ is 6π.

D The period of the function $2\cot(4\pi\theta/3)$ is $3/2$.

Solution: D

First, we'll show that the statement in answer choice D is false.

Note that the function

$$2\cot\left(\frac{4\pi\theta}{3}\right)$$

is in the form $a\cot(b\theta)$ with $a=2$ and $b=4\pi/3$. Since b is positive and the period of the "basic" cotangent function ($\cot\theta$) is π, the period of the function

$$2\cot\left(\frac{4\pi\theta}{3}\right)$$

is

$$\frac{\pi}{b}=\frac{\pi}{\left(\frac{4\pi}{3}\right)}=\pi\left(\frac{3}{4\pi}\right)=\frac{3}{4}\neq\frac{3}{2}$$

Now we'll show that the other three statements are true.

The function in answer choice A is $-\pi\sin(\pi^2\theta)$, which is in the form $a\sin(b\theta)$ with $a=-\pi$ and $b=\pi^2$. Since b is positive and the period of the "basic" sine function ($\sin\theta$) is 2π, the period of the function $-\pi\sin(\pi^2\theta)$ is

$$\frac{2\pi}{b}=\frac{2\pi}{\pi^2}=\frac{2}{\pi}$$

The function in answer choice B is

$$\pi^3 \tan\left(\frac{4\theta}{3}\right)$$

This is in the form $a\tan(b\theta)$ with $a = \pi^3$ and $b = 4/3$. Since b is positive and the period of the "basic" tangent function ($\tan\theta$) is π, the period of the function

$$\pi^3 \tan\left(\frac{4\theta}{3}\right)$$

is

$$\frac{\pi}{b} = \frac{\pi}{\left(\frac{4}{3}\right)} = \pi\left(\frac{3}{4}\right) = \frac{3\pi}{4}$$

The function in answer choice C is

$$-1.7\left[\frac{\sec\left(\frac{\theta}{3}\right)}{\pi}\right]$$

This is in the form $a\sec(b\theta)$ with $a = -1.7/\pi$ and $b = 1/3$. The period of the "basic" secant function ($\sec\theta$) is 2π, so the period of the function

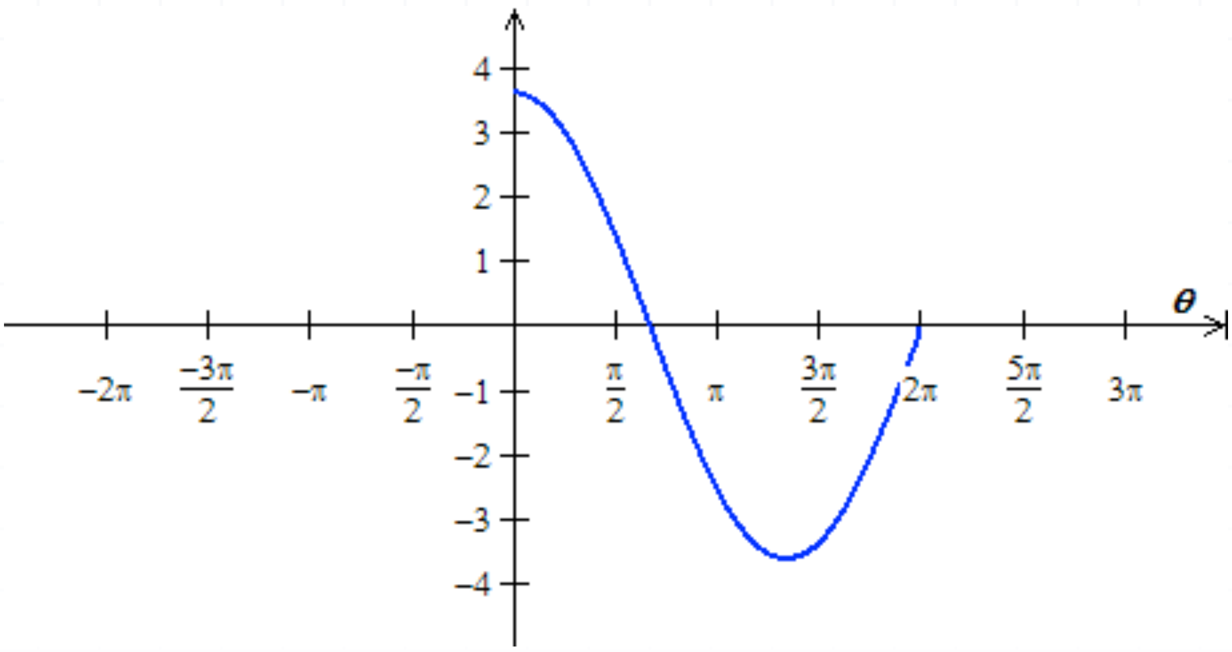
$$-1.7\left[\frac{\sec\left(\frac{\theta}{3}\right)}{\pi}\right]$$

is

$$\frac{2\pi}{b} = \frac{2\pi}{\left(\frac{1}{3}\right)} = 2\pi\left(\frac{3}{1}\right) = 6\pi$$

Topic: Sketching sine and cosine

Question: The following curve is the graph of which of the following functions?



Answer choices:

A $\quad 3.6\cos\left(\frac{3\theta}{4}\right)$

B $\quad 3.6\sin\left(\frac{6\theta}{5}\right)$

C $\quad 3.6\cos\left(\frac{4\theta}{3}\right)$

D $\quad -3.6\sin\left(\frac{3\theta}{2}\right)$

Solution: A

Since the value of the function at $\theta = 0$ isn't equal to 0, the curve can't be the graph of either of the sine functions given in answer choices B and D. Thus all that remains is to show that the curve is the graph of the cosine function given in answer choice A (and not the graph of the cosine function given in answer choice C).

First, let's consider the function given in answer choice A: $3.6\cos(3\theta/4)$. This function is in the form $a\cos(b\theta)$, where $a = 3.6$ and $b = 3/4$. Thus the period of this function is

$$\frac{2\pi}{b} = \frac{2\pi}{\left(\frac{3}{4}\right)} = 2\pi\left(\frac{4}{3}\right) = \frac{8\pi}{3}$$

This tells us that the value of $3.6\cos(3\theta/4)$ is equal to 0 at the angles θ which are one-fourth and three-fourths of the way through the interval $[0,8\pi/3)$. That is, $3.6\cos(3\theta/4)$ is equal to 0 at

$$\theta = \left(\frac{1}{4}\right)\left(\frac{2\pi}{b}\right) = \left(\frac{1}{4}\right)\left(\frac{8\pi}{3}\right) = \frac{2\pi}{3}$$

and at

$$\theta = \left(\frac{3}{4}\right)\left(\frac{2\pi}{b}\right) = \left(\frac{3}{4}\right)\left(\frac{8\pi}{3}\right) = 2\pi$$

Now let's consider the function given in answer choice C: $3.6\cos(4\theta/3)$. This function is in the form $a\cos(b\theta)$, where $a = 3.6$ and $b = 4/3$. Thus its period is

$$\frac{2\pi}{b}=\frac{2\pi}{\left(\frac{4}{3}\right)}=2\pi\left(\frac{3}{4}\right)=\frac{3\pi}{2}$$

Thus the value of $3.6\cos(4\theta/3)$ is equal to 0 at the angles θ which are one-fourth and three-fourths of the way through the interval $[0,3\pi/2)$. That is, $3.6\cos(4\theta/3)$ is equal to 0 at

$$\theta=\left(\frac{1}{4}\right)\left(\frac{2\pi}{b}\right)=\left(\frac{1}{4}\right)\left(\frac{3\pi}{2}\right)=\frac{3\pi}{8}$$

and

$$\theta=\left(\frac{3}{4}\right)\left(\frac{2\pi}{b}\right)=\left(\frac{3}{4}\right)\left(\frac{3\pi}{2}\right)=\frac{9\pi}{8}$$

Inspection of the curve shows that it includes neither the point $(3\pi/8,0)$ nor the point $(9\pi/8,0)$. However, the curve does include both the point $(2\pi/3,0)$ and the point $(2\pi,0)$. Therefore, the curve is the graph of the function given in answer choice A over the interval $[0,2\pi)$.

Topic: Horizontal and vertical shifts

Question: Which of the following is accurate?

Consider the following statements:

I The graph of the function $\sin(\theta - (\pi/2))$ over the interval $[0,2\pi)$ is identical to the graph of the function $\cos\theta$ over that interval.

II The graph of the function $\sin(\theta - (\pi/2))$ over the interval $[0,2\pi)$ is identical to the graph of the function $-\cos\theta$ over that interval.

Answer choices:

A Statement I is true, and statement II is false.

B Statement I is false, and statement II is true.

C Statements I and II are both true.

D Statements I and II are both false.

Solution: B

Answer choice C can be eliminated, because if statements I and II were both true, then the graph of the function $\cos\theta$ over the interval $[0,2\pi)$ would be identical to the graph of the function $-\cos\theta$ over that interval, which is clearly not the case. We will show that statement II is true (hence that statement I is false).

First, we'll compare the values of $\sin(\theta-(\pi/2))$ and $-\cos\theta$ at "key angles" θ in the interval $[0,2\pi)$:

θ	$\theta-\frac{\pi}{2}$	$\sin\left(\theta-\frac{\pi}{2}\right)$	$\cos\theta$	$-\cos\theta$
0	$-\frac{\pi}{2}$	-1	1	-1
$\frac{\pi}{2}$	0	0	0	0
π	$\frac{\pi}{2}$	1	-1	1
$\frac{3\pi}{2}$	π	0	0	0

To verify that $\sin(\theta-(\pi/2))=-\cos\theta$ at every angle θ in the interval $[0,2\pi)$, we can apply the difference identity for sine to the angles θ and $\pi/2$:

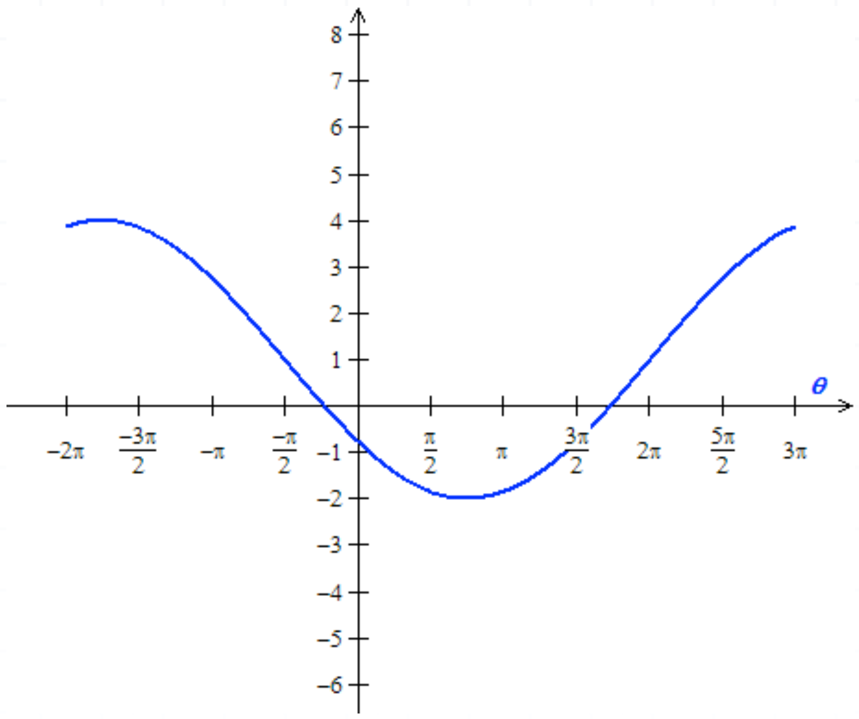
$$\sin\left(\theta-\frac{\pi}{2}\right)=\sin\theta\left(\cos\frac{\pi}{2}\right)-\cos\theta\left(\sin\frac{\pi}{2}\right)$$

$$\sin\left(\theta-\frac{\pi}{2}\right)=\sin\theta(0)-\cos\theta(1)$$

$$\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$$

Topic: Sequence of transformations

Question: The following curve is the graph of some sine or cosine function over the interval $[-2\pi, 3\pi)$, and its period is 5π. What is the function?



Answer choices:

A $3\cos\left(\frac{\theta}{5}+\frac{\pi}{10}\right)-1$

B $-3\sin\left(\frac{2\theta}{5}+\frac{\pi}{5}\right)+1$

C $2\sin\left(\frac{2\theta}{5}+\frac{\pi}{10}\right)+2$

D $-2\cos\left(\frac{\theta}{10}+\frac{2\pi}{5}\right)-1$

Solution: B

The values of the function whose graph is shown in the curve range from -2 to 4, so the amplitude of the function is

$$\left(\frac{1}{2}\right)[4-(-2)]=\left(\frac{1}{2}\right)(6)=3$$

Therefore, we can rule out answer choices C and D, because the functions given in those answer choices have an amplitude of 2.

Also, note that $4=3+1$ and $-2=-3+1$. This means that the curve has a vertical shift of 1. This rules out choice A, because the function given in that answer choice has a vertical shift of -1.

To verify that the function given in answer choice B,

$$-3\sin\left(\frac{2\theta}{5}+\frac{\pi}{5}\right)+1$$

has a period of 5π, note that

$$\frac{2\theta}{5}+\frac{\pi}{5}=\left(\frac{2}{5}\right)\left[\left(\frac{5}{2}\right)\left(\frac{2\theta}{5}\right)+\left(\frac{5}{2}\right)\left(\frac{\pi}{5}\right)\right]=\left(\frac{2}{5}\right)\left(\theta+\frac{\pi}{2}\right)$$

Therefore, the function given in answer choice B is

$$-3\sin\left(\left(\frac{2}{5}\right)\left(\theta+\frac{\pi}{2}\right)\right)+1$$

This indicates that the period of this function is

$$\frac{2\pi}{\left(\frac{2}{5}\right)} = 2\pi\left(\frac{5}{2}\right) = 5\pi$$

As a final piece of evidence that the given curve is the graph of the function given in answer choice B, let's determine the values of θ at which that function is equal to 1. Note that those are the values of θ for which

$$\sin\left(\frac{2\theta}{5} + \frac{\pi}{5}\right) = 0$$

Recall that the sine of an angle is equal to 0 if and only if that angle is equal to $n\pi$ for some integer n. Therefore,

$$\frac{2\theta}{5} + \frac{\pi}{5} = n\pi$$

Solving this equation for θ:

$$\frac{2\theta}{5} = n\pi - \frac{\pi}{5}$$

$$\frac{2\theta}{5} = \frac{5(n\pi) - 1(\pi)}{5}$$

$$\frac{2\theta}{5} = \frac{(5n-1)\pi}{5}$$

$$\theta = \left(\frac{5}{2}\right)\left[\frac{(5n-1)\pi}{5}\right]$$

$$\theta = \frac{(5n-1)\pi}{2}$$

Let's determine the integers n for which $(5n-1)\pi/2$ lies in the interval $[-2\pi,3\pi)$:

$$-2\pi \leq \frac{(5n-1)\pi}{2} \leq 3\pi$$

$$-4\pi \leq (5n-1)\pi \leq 6\pi$$

$$-4\pi \leq 5n\pi - \pi \leq 6\pi$$

$$-4\pi + \pi \leq 5n\pi \leq 6\pi + \pi$$

$$-3\pi \leq 5n\pi \leq 7\pi$$

$$-3 \leq 5n \leq 7$$

$$-\frac{3}{5} \leq n \leq \frac{7}{5}$$

The only integers n that satisfy this pair of inequalities are 0 and 1. Setting n to 0 and 1 in the expression we found for θ, namely $(5n-1)\pi/2$, gives $\theta = -\pi/2$ and $\theta = 4\pi/2(=2\pi)$, respectively. Inspection of the given curve shows that the function whose graph it represents is indeed equal to 1 at $\theta = -\pi/2$ and $\theta = 2\pi$.

Topic: Transformations algebraically

Question: Choose the correct transformation.

$y = x^2 - 2$ is

shifted to the left 4 units

reflected across the y-axis

shrunk vertically by a factor of 7

Answer choices:

A $\quad y = \frac{1}{7}x^2 + \frac{26}{7}$

B $\quad y = -\frac{1}{7}x^2 + \frac{26}{7}$

C $\quad y = \frac{1}{7}x^2 - \frac{8}{7}x + 2$

D $\quad y = \frac{1}{7}x^2 + \frac{8}{7}x - 2$

Solution: C

To transform a function algebraically, the following rules apply.

Shift $y = f(x)$ up by a units	$y = f(x) + a$
Shift $y = f(x)$ down by a units	$y = f(x) - a$
Shift $y = f(x)$ to the right by a units	$y = f(x - a)$
Shift $y = f(x)$ to the left by a units	$y = f(x + a)$
Reflect $y = f(x)$ across the x-axis	$y = -f(x)$
Reflect $y = f(x)$ across the y-axis	$y = f(-x)$
Stretch vertically by a factor of a	$y = af(x)$
Shrink vertically by a factor of a	$y = \frac{1}{a}f(x)$

To solve, first transform $y = x^2 - 2$ by shifting to the left 4 units. Remember to shift $y = f(x)$ to the left by a units, $y = f(x + a)$. This means that $y = x^2 - 2$ will become

$$y = (x + 4)^2 - 2$$

Next reflect across the y-axis. Remember to reflect $y = f(x)$ across the y-axis, $y = f(-x)$. This means that $y = (x + 4)^2 - 2$ will become

$$y = (-x + 4)^2 - 2$$

Finally, shrink vertically by a factor of 7. Remember to shrink vertically by a factor of a, $y = (1/a)f(x)$. This means that $y = (-x + 4)^2 - 2$ will become

$$y = \frac{1}{7}\left[(-x+4)^2 - 2\right]$$

$$y = \frac{1}{7}\left(x^2 - 8x + 16 - 2\right)$$

$$y = \frac{1}{7}\left(x^2 - 8x + 14\right)$$

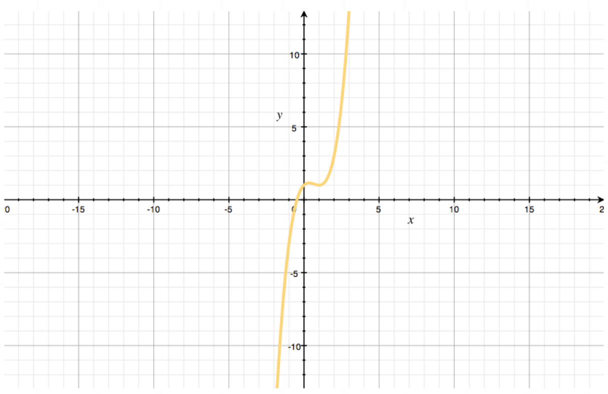
$$y = \frac{1}{7}x^2 - \frac{8}{7}x + 2$$

Topic: Sketching graphs of transformations

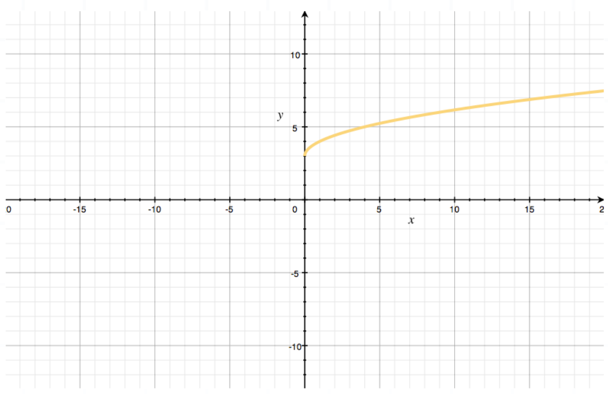
Question: If each graph below represents one equation, match each equation to its graph.

I	$y = x^3 - 2x^2 + x + 1$	II	$y = x^3 - 2$	III	$y = e^x$
IV	$y = \sqrt{x} + 3$	V	$y = x - 2$	VI	$y = \ln x$

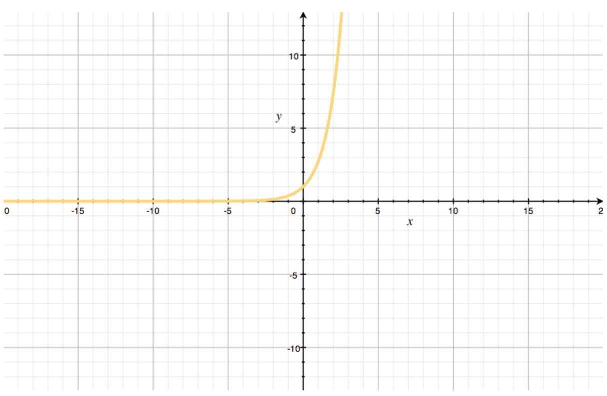
A



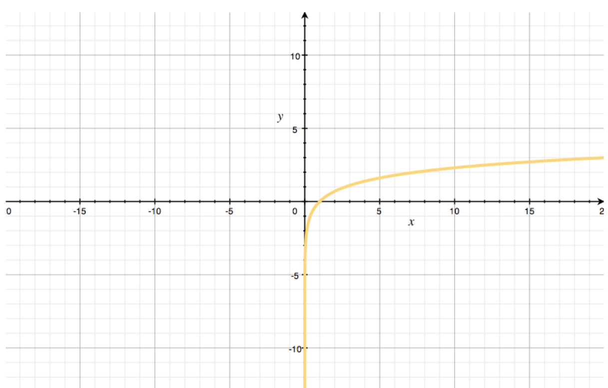
D



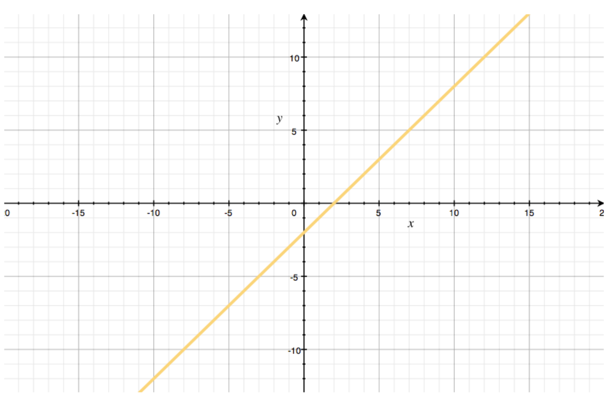
B



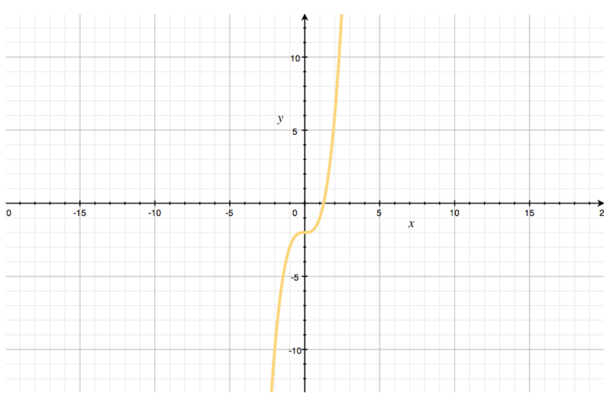
E



C



F



Answer choices:

A I is B, II is D, III is A, IV is C, V is E, VI is F

B I is D, II is A, III is B, IV is C, V is E, VI is F

C I is A, II is F, III is B, IV is E, V is D, VI is C

D I is A, II is F, III is B, IV is D, V is C, VI is E

Solution: D

I is graph A

II is graph F

III is graph B

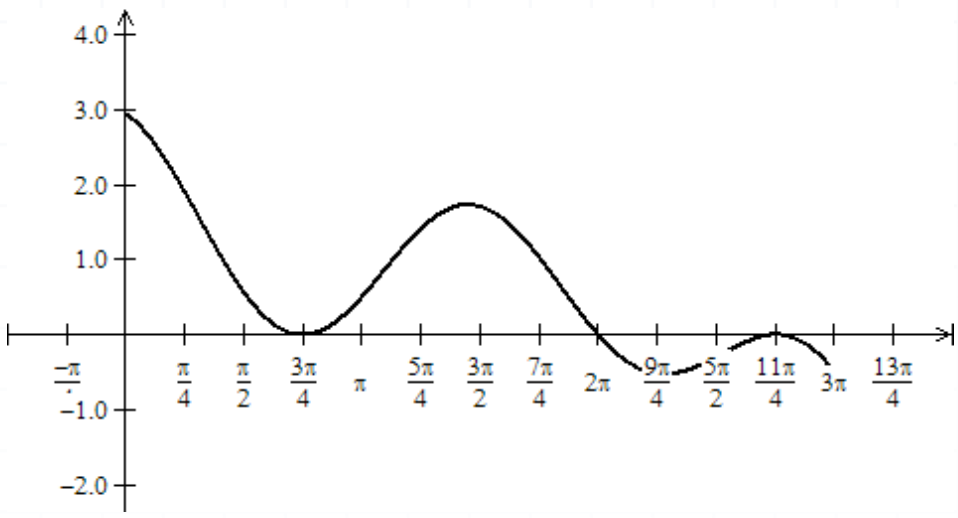
IV is graph D

V is graph C

VI is graph E

Topic: Graphing combinations of functions

Question: The following curve is the graph, over the interval $[0,3\pi)$, of a function which is the product of a sine function and a cosine function. Which product does the graph represent.



Answer choices:

A $\left(3\cos\left(\frac{2\theta}{3}-\frac{\pi}{4}\right)\right)\left(\sin\left(\theta-\frac{\pi}{3}\right)+1\right)$

B $\left(\sin\left(\frac{2\theta}{3}+\frac{\pi}{6}\right)\right)\left(4\cos\left(\theta-\frac{\pi}{2}\right)-1\right)$

C $\left(2\cos\left(\frac{\theta}{3}-\frac{\pi}{6}\right)\right)\left(\sin\left(\theta+\frac{3\pi}{4}\right)+1\right)$

D $\left(\sin\left(\theta+\frac{2\pi}{3}\right)\right)\left(2\cos\left(3\theta+\frac{\pi}{4}\right)-1\right)$

Solution: C

Let's consider the angles θ at which the function that corresponds to the given graph is equal to 0. Inspection of the graph tells us that in the interval $[0,3\pi)$, those angles are $3\pi/4$, 2π, and $11\pi/4$.

Now we'll show that the function given in answer choice C,

$$\left(2\cos\left(\frac{\theta}{3}-\frac{\pi}{6}\right)\right)\left(\sin\left(\theta+\frac{3\pi}{4}\right)+1\right)$$

is indeed equal to 0 at all three of the angles $3\pi/4$, 2π, and $11\pi/4$.

First, we'll show that at $\theta = 3\pi/4$, the second function in the product of functions given in answer choice C,

$$\sin\left(\theta+\frac{3\pi}{4}\right)+1$$

is equal to 0:

$$\theta = \frac{3\pi}{4} \qquad \to \qquad \theta+\frac{3\pi}{4}=\frac{3\pi}{4}+\frac{3\pi}{4}=\frac{3\pi}{2}$$

$$\to \qquad \sin\left(\theta+\frac{3\pi}{4}\right)+1=\sin\left(\frac{3\pi}{2}\right)+1=-1+1=0$$

Next, we'll show that at $\theta = 2\pi$, the first function in the product of functions given in answer choice C,

$$2\cos\left(\frac{\theta}{3}-\frac{\pi}{6}\right)$$

is equal to 0:

$$\theta = 2\pi \quad \to \quad \frac{\theta}{3} - \frac{\pi}{6} = \frac{2\pi}{3} - \frac{\pi}{6} = \frac{2(2\pi) - 1(\pi)}{6} = \frac{3\pi}{6} = \frac{\pi}{2}$$

$$\to \quad 2\cos\left(\frac{\theta}{3} - \frac{\pi}{6}\right) = 2\cos\left(\frac{\pi}{2}\right) = 2(0) = 0$$

Finally, we'll show that at $\theta = 11\pi/4$, the second function in the product of functions given in answer choice C,

$$\sin\left(\theta + \frac{3\pi}{4}\right) + 1$$

is equal to 0:

$$\theta = \frac{11\pi}{4} \quad \to \quad \theta + \frac{3\pi}{4} = \frac{11\pi}{4} + \frac{3\pi}{4} = \frac{14\pi}{4} = \frac{8\pi}{4} + \frac{6\pi}{4} = 2\pi + \frac{3\pi}{2}$$

$$\to \quad \sin\left(\theta + \frac{3\pi}{4}\right) + 1 = \sin\left(2\pi + \frac{3\pi}{2}\right) + 1 = \sin\left(\frac{3\pi}{2}\right) + 1 = -1 + 1 = 0$$

Now we'll test the function given in each of the other three answer choices, to see if they're equal to 0 at $\theta = 3\pi/4$. In each case, we'll show that neither of the two functions in the given product is equal to 0 at $\theta = 3\pi/4$.

Setting θ to $3\pi/4$ in the function given in answer choice A,

$$\left(3\cos\left(\frac{2\theta}{3} - \frac{\pi}{4}\right)\right)\left(\sin\left(\theta - \frac{\pi}{3}\right) + 1\right)$$

we have the following:

$$\theta = \frac{3\pi}{4} \qquad \to \qquad \frac{2\theta}{3} - \frac{\pi}{4} = \frac{2\left(\frac{3\pi}{4}\right)}{3} - \frac{\pi}{4} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{2(\pi) - 1(\pi)}{4} = \frac{\pi}{4}$$

$$\to \qquad 3\cos\left(\frac{2\theta}{3} - \frac{\pi}{4}\right) = 3\cos\left(\frac{\pi}{4}\right) = 3\left(\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2} \neq 0$$

$$\theta = \frac{3\pi}{4} \qquad \to \qquad \theta - \frac{\pi}{3} = \frac{3\pi}{4} - \frac{\pi}{3} = \frac{3(3\pi) - 4(\pi)}{12} = \frac{5\pi}{12}$$

$$\to \qquad \sin\left(\theta - \frac{\pi}{3}\right) = \sin\left(\frac{5\pi}{12}\right) \neq -1$$

$$\to \qquad \sin\left(\theta - \frac{\pi}{3}\right) + 1 \neq 0$$

Since neither of the functions of the product given in answer choice A is equal to 0 at $\theta = 3\pi/4$, we can eliminate answer choice A.

Setting θ to $3\pi/4$ in the function given in answer choice B,

$$\left(\sin\left(\frac{2\theta}{3} + \frac{\pi}{6}\right)\right)\left(4\cos\left(\theta - \frac{\pi}{2}\right) - 1\right)$$

we have the following:

$$\theta = \frac{3\pi}{4} \qquad \to \qquad \frac{2\theta}{3} + \frac{\pi}{6} = \frac{2\left(\frac{3\pi}{4}\right)}{3} + \frac{\pi}{6} = \frac{\pi}{2} + \frac{\pi}{6} = \frac{3(\pi) + 1(\pi)}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

$$\to \qquad \sin\left(\frac{2\theta}{3} + \frac{\pi}{6}\right) = \sin\left(\frac{2\pi}{3}\right) \neq 0$$

$$\theta = \frac{3\pi}{4} \quad \to \quad \theta - \frac{\pi}{2} = \frac{3\pi}{4} - \frac{\pi}{2} = \frac{1(3\pi) - 2(\pi)}{4} = \frac{\pi}{4}$$

$$\to \quad 4\cos\left(\theta - \frac{\pi}{2}\right) = 4\cos\left(\frac{\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2} \neq 1$$

$$\to \quad 4\cos\left(\theta - \frac{\pi}{2}\right) - 1 \neq 0$$

Neither of the functions of the product given in answer choice B is equal to 0 at $\theta = 3\pi/4$, so we can eliminate answer choice B.

Setting θ to $3\pi/4$ in the function given in answer choice D,

$$\left(\sin\left(\theta + \frac{2\pi}{3}\right)\right)\left(2\cos\left(3\theta + \frac{\pi}{4}\right) - 1\right)$$

we have the following:

$$\theta = \frac{3\pi}{4} \quad \to \quad \theta + \frac{2\pi}{3} = \frac{3\pi}{4} + \frac{2\pi}{3} = \frac{3(3\pi) + 4(2\pi)}{12} = \frac{17\pi}{12}$$

$$\to \quad \sin\left(\theta + \frac{2\pi}{3}\right) = \sin\left(\frac{17\pi}{12}\right) \neq 0$$

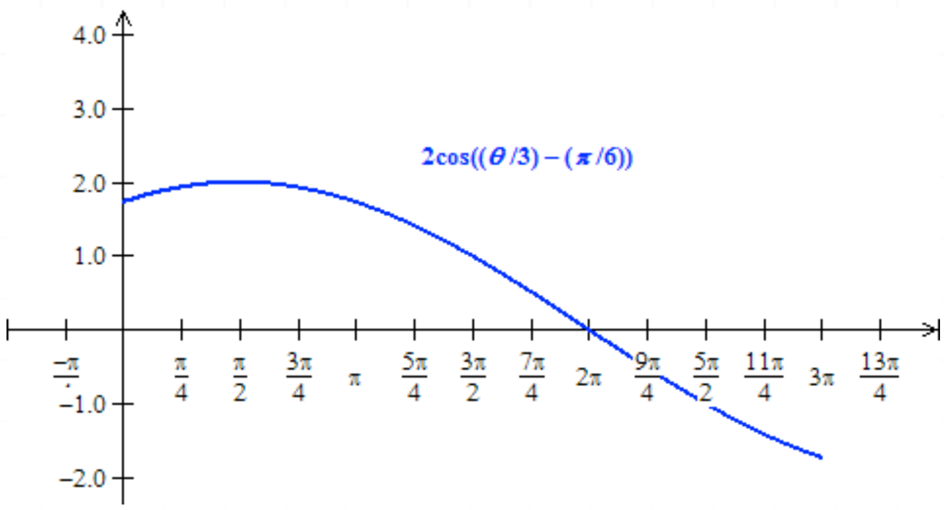
$$\theta = \frac{3\pi}{4} \quad \to \quad 3\theta + \frac{\pi}{4} = 3\left(\frac{3\pi}{4}\right) + \frac{\pi}{4} = \frac{9\pi}{4} + \frac{\pi}{4} = \frac{10\pi}{4} = \frac{8\pi}{4} + \frac{2\pi}{4} = 2\pi + \frac{\pi}{4}$$

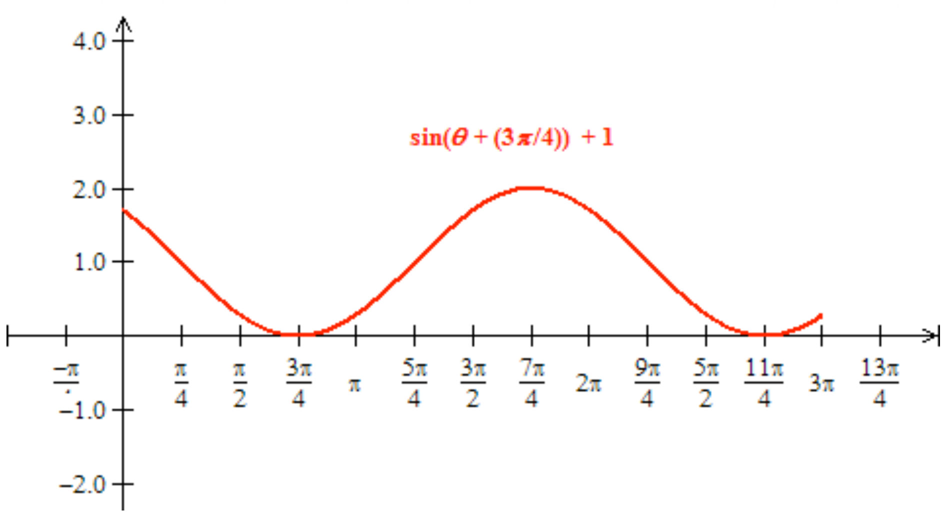
$$\to \quad 2\cos\left(3\theta + \frac{\pi}{4}\right) = 2\cos\left(2\pi + \frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \neq 1$$

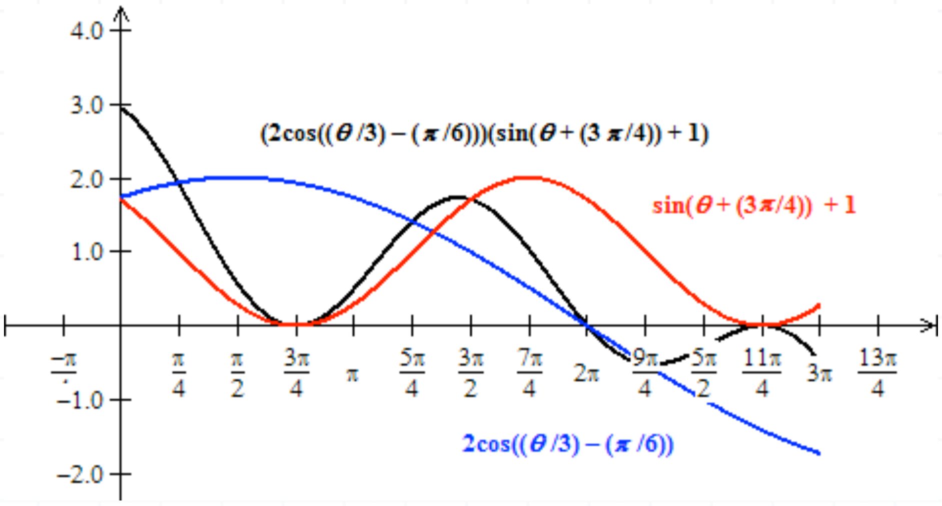
$$\rightarrow \quad 2\cos\left(3\theta+\frac{\pi}{4}\right)-1\neq 0$$

Neither of the functions of the product given in answer choice D is equal to 0 at $\theta = 3\pi/4$, so we can eliminate answer choice D.

Let's graph the cosine function and the sine function whose product is the function given in answer choice C, and their product, so that we can see the genesis of the graph of their product.







4.0
3.0
2.0
1.0
−1.0
−2.0
(2cos((θ/3) − (π/6)))(sin(θ + (3π/4)) + 1)
sin(θ + (3π/4)) + 1
2cos((θ/3) − (π/6))
π/4
π/2
3π/4
π
5π/4
3π/2
7π/4
2π
9π/4
5π/2
11π/4
3π
13π/4

Topic: Even-odd identities

Question: Find the values.

$$\csc\left(-\frac{49\pi}{3}\right) \text{ and } \tan\left(-\frac{49\pi}{3}\right)$$

Answer choices:

A $\csc\left(-\frac{49\pi}{3}\right) = -\frac{2}{\sqrt{3}}$ and $\tan\left(-\frac{49\pi}{3}\right) = -\sqrt{3}$

B $\csc\left(-\frac{49\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $\tan\left(-\frac{49\pi}{3}\right) = -\frac{1}{\sqrt{3}}$

C $\csc\left(-\frac{49\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\tan\left(-\frac{49\pi}{3}\right) = -\sqrt{3}$

D $\csc\left(-\frac{49\pi}{3}\right) = \frac{2}{\sqrt{3}}$ and $\tan\left(-\frac{49\pi}{3}\right) = \frac{1}{\sqrt{3}}$

Solution: A

Note that

$$\frac{49\pi}{3}=\frac{(48+1)\pi}{3}=16\pi+\frac{\pi}{3}$$

Thus an angle of $49\pi/3$ radians differs from an angle of $\pi/3$ radians by an integer multiple of 2π, so we have the following:

$$\sin\left(\frac{49\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

and

$$\cos\left(\frac{49\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$$

Using even-odd identities, we obtain the following:

$$\csc\left(-\frac{49\pi}{3}\right)=-\csc\left(\frac{49\pi}{3}\right)=-\frac{1}{\sin\left(\frac{49\pi}{3}\right)}=-\frac{2}{\sqrt{3}}$$

and

$$\tan\left(-\frac{49\pi}{3}\right)=-\tan\left(\frac{49\pi}{3}\right)=-\frac{\sin\left(\frac{49\pi}{3}\right)}{\cos\left(\frac{49\pi}{3}\right)}=-\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}=-\sqrt{3}$$

Topic: Sum-difference identities

Question: What are the exact values of $\tan(\theta + \alpha)$ and $\tan(\theta - \alpha)$ if θ is an angle in the second quadrant whose cosine is $-3/5$ and α is an angle in the third quadrant whose sine is $-2/\sqrt{5}$?

Answer choices:

A $\tan(\theta + \alpha) = \sqrt{\frac{11}{3}}$ and $\tan(\theta - \alpha) = 3$

B $\tan(\theta + \alpha) = \frac{2}{11}$ and $\tan(\theta - \alpha) = 2$

C $\tan(\theta + \alpha) = -\sqrt{\frac{3}{11}}$ and $\tan(\theta - \alpha) = -2$

D $\tan(\theta + \alpha) = \frac{11}{3}$ and $\tan(\theta - \alpha) = -3$

Solution: B

We'll apply sum-difference identities, so we first need to find $\sin\theta$ and $\cos\alpha$.

The basic Pythagorean identity tells us that

$$\sin^2\theta + \cos^2\theta = 1$$

Therefore,

$$\sin^2\theta = 1 - \cos^2\theta$$

Substituting $-3/5$ for $\cos\theta$, we obtain

$$\sin^2\theta = 1 - \left(-\frac{3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

Since θ is in the second quadrant, we know that $\sin\theta$ is positive, so

$$\sin\theta = \sqrt{\frac{16}{25}} = \frac{\sqrt{16}}{\sqrt{25}} = \frac{4}{5}$$

Similarly, the basic Pythagorean identity tells us that

$$\sin^2\alpha + \cos^2\alpha = 1$$

Therefore,

$$\cos^2\alpha = 1 - \sin^2\alpha$$

Substituting $-2/\sqrt{5}$ for $\sin\alpha$, we obtain

$$\cos^2\alpha = 1 - \left(-\frac{2}{\sqrt{5}}\right)^2 = 1 - \frac{4}{5} = \frac{1}{5}$$

Since α is in the third quadrant, we know that $\cos\alpha$ is negative, so

$$\cos\alpha = -\sqrt{\frac{1}{5}} = -\frac{1}{\sqrt{5}}$$

Next, we'll find $\tan\theta$ and $\tan\alpha$.

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\left(\frac{4}{5}\right)}{\left(-\frac{3}{5}\right)} = -\frac{4}{3}$$

and

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\left(-\frac{2}{\sqrt{5}}\right)}{\left(-\frac{1}{\sqrt{5}}\right)} = 2$$

By the sum identity for the tangent function,

$$\tan(\theta+\alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha}$$

$$\tan(\theta+\alpha) = \frac{\left(-\frac{4}{3}+2\right)}{1-\left(-\frac{4}{3}\right)(2)}$$

$$\tan(\theta + \alpha) = \frac{\left(\frac{2}{3}\right)}{1 + \left(\frac{8}{3}\right)}$$

$$\tan(\theta + \alpha) = \frac{\left(\frac{2}{3}\right)}{\left(\frac{11}{3}\right)}$$

$$\tan(\theta + \alpha) = \frac{2}{11}$$

By the difference identity for the tangent function,

$$\tan(\theta - \alpha) = \frac{\tan\theta - \tan\alpha}{1 + \tan\theta\tan\alpha}$$

$$\tan(\theta - \alpha) = \frac{\left(-\frac{4}{3} - 2\right)}{1 + \left(-\frac{4}{3}\right)(2)}$$

$$\tan(\theta - \alpha) = \frac{\left(-\frac{10}{3}\right)}{1 - \left(\frac{8}{3}\right)}$$

$$\tan(\theta - \alpha) = \frac{\left(-\frac{10}{3}\right)}{\left(-\frac{5}{3}\right)}$$

$$\tan(\theta - \alpha) = 2$$

Topic: Half-angle identities

Question: If θ is the angle in the interval $(17\pi/2,9\pi)$ with $\cos\theta = -3/7$, what are the values of $\sin(\theta/2)$ and $\cos(\theta/2)$?

Answer choices:

A $\sin\frac{\theta}{2} = -\sqrt{\frac{1}{7}}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{6}{7}}$

B $\sin\frac{\theta}{2} = \sqrt{\frac{5}{7}}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{2}{7}}$

C $\sin\frac{\theta}{2} = \sqrt{\frac{3}{7}}$ and $\cos\frac{\theta}{2} = \sqrt{\frac{4}{7}}$

D $\sin\frac{\theta}{2} = \sqrt{\frac{2}{7}}$ and $\cos\frac{\theta}{2} = -\sqrt{\frac{5}{7}}$

Solution: B

By the half-angle identity for cosine,

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$$

Substituting $-3/7$ for $\cos\theta$ gives

$$\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\left(-\frac{3}{7}\right)}{2}}=\pm\sqrt{\frac{\frac{7(1)+1(-3)}{7}}{2}}=\pm\sqrt{\frac{7-3}{14}}=\pm\sqrt{\frac{4}{14}}=\pm\sqrt{\frac{2}{7}}$$

By the half-angle identity for sine,

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$$

Substituting $-3/7$ for $\cos\theta$ gives

$$\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\left(-\frac{3}{7}\right)}{2}}=\pm\sqrt{\frac{\frac{7(1)+1(3)}{7}}{2}}=\pm\sqrt{\frac{7+3}{14}}=\pm\sqrt{\frac{10}{14}}=\pm\sqrt{\frac{5}{7}}$$

We were given that

$$\frac{17\pi}{2}<\theta<9\pi$$

Dividing through by 2:

$$\frac{17\pi}{4}<\frac{\theta}{2}<\frac{9\pi}{2}$$

Note that

$$\frac{17\pi}{4} = \frac{16\pi}{4} + \frac{\pi}{4} = 4\pi + \frac{\pi}{4}$$

and

$$\frac{9\pi}{2} = \frac{8\pi}{2} + \frac{\pi}{2} = 4\pi + \frac{\pi}{2}$$

Substituting these results, we have

$$4\pi + \frac{\pi}{4} < \frac{\theta}{2} < 4\pi + \frac{\pi}{2}$$

Since 4π is an integer multiple of 2π, an angle of 4π radians lies on the positive x-axis, so any angle between $4\pi + (\pi/4)$ and $4\pi + (\pi/2)$ lies strictly “between” the positive x-axis and the positive y-axis. Therefore, $\theta/2$ is in the first quadrant, hence both $\sin(\theta/2)$ and $\cos(\theta/2)$ are positive:

$$\sin\frac{\theta}{2} = \sqrt{\frac{5}{7}}$$

$$\cos\frac{\theta}{2} = \sqrt{\frac{2}{7}}$$

Topic: Double-angle identities

Question: Pick the equal value.

$$(\sin -65^\circ)[2(\cos 65^\circ)]$$

Answer choices:

A $\cos 130^\circ$

B $\cos(-65^\circ)$

C $\sin(-65^\circ)$

D $\sin(-130^\circ)$

Solution: D

By the even identity for the cosine function,

$$\cos 65^\circ = \cos(-65^\circ)$$

Therefore,

$$(\sin -65^\circ)[2\cos 65^\circ] = (\sin -65^\circ)[2\cos(-65^\circ)]$$

Changing the order of the factors on the right-hand side of this equation, we obtain

$$(\sin -65^\circ)[2\cos 65^\circ] = 2(\sin -65^\circ)(\cos -65^\circ)$$

By the double-angle identity for the sine function,

$$\sin 2\theta = 2\sin\theta\cos\theta$$

Now let $\theta = -65^\circ$. Then

$$\sin[2(-65^\circ)] = 2(\sin -65^\circ)(\cos -65^\circ)$$

Simplifying the left-hand side of this equation, we have

$$\sin(-130^\circ) = 2(\sin -65^\circ)(\cos -65^\circ)$$

We have shown that the given expression, $(\sin -65^\circ)[2(\cos 65^\circ)]$, is equal to

$$2(\sin -65^\circ)(\cos -65^\circ)$$

and we have applied the double-angle identity for the sine function to show that

$$2(\sin - 65^\circ)(\cos - 65^\circ) = \sin(-130^\circ)$$

Combining these results, we get

$$(\sin - 65^\circ)[2(\cos - 65^\circ)] = \sin(-130^\circ)$$

Topic: Product-to-sum identities

Question: Which of the following pairs is a solution of the equation?

$$\cos\theta\cos\alpha = -\left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)$$

Answer choices:

A $\left(\theta = \frac{17\pi}{8}, \alpha = \frac{25\pi}{8}\right)$

B $\left(\theta = \frac{11\pi}{8}, \alpha = \frac{7\pi}{8}\right)$

C $\left(\theta = \frac{5\pi}{8}, \alpha = \frac{7\pi}{8}\right)$

D $\left(\theta = \frac{9\pi}{8}, \alpha = \frac{11\pi}{8}\right)$

Solution: A

By one of the product-to-sum identities,

$$\cos\theta\cos\alpha = \frac{1}{2}\left[\cos(\theta+\alpha)+\cos(\theta-\alpha)\right]$$

Setting θ to $17\pi/8$ and α to $25\pi/8$, we have

$$\left(\cos\frac{17\pi}{8}\right)\left(\cos\frac{25\pi}{8}\right)=\frac{1}{2}\left[\cos\left(\frac{17\pi}{8}+\frac{25\pi}{8}\right)+\cos\left(\frac{17\pi}{8}-\frac{25\pi}{8}\right)\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\cos\frac{25\pi}{8}\right)=\frac{1}{2}\left[\cos\frac{(17+25)\pi}{8}+\cos\frac{(17-25)\pi}{8}\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\cos\frac{25\pi}{8}\right)=\frac{1}{2}\left[\cos\left(\frac{42\pi}{8}\right)+\cos\left(\frac{-8\pi}{8}\right)\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\cos\frac{25\pi}{8}\right)=\frac{1}{2}\left[\cos\left(\frac{42\pi}{8}\right)+\cos(-\pi)\right]$$

Note that

$$\frac{42\pi}{8}=\frac{(40+2)\pi}{8}=5\pi+\frac{2\pi}{8}=5\pi+\frac{\pi}{4}$$

Therefore,

$$\cos\left(\frac{42\pi}{8}\right)=\cos\left(5\pi+\frac{\pi}{4}\right)$$

Now an angle of 5π radians is on the negative x-axis, so an angle of $5\pi + (\pi/4)$ radians is halfway "between" the negative x-axis and the negative y-axis. Thus

$$\cos\left(5\pi + \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

Also, we know that $\cos(-\pi) = -1$.

Substituting these results, we obtain

$$\left(\cos\frac{17\pi}{8}\right)\left(\frac{25\pi}{8}\right) = \frac{1}{2}\left[-\frac{1}{\sqrt{2}} + (-1)\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\frac{25\pi}{8}\right) = \frac{1}{2}\left[\frac{1(-1) + \sqrt{2}(-1)}{\sqrt{2}}\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\frac{25\pi}{8}\right) = \frac{1}{2}\left[\frac{-\left(1 + \sqrt{2}\right)}{\sqrt{2}}\right]$$

$$\left(\cos\frac{17\pi}{8}\right)\left(\frac{25\pi}{8}\right) = -\left(\frac{1 + \sqrt{2}}{2\sqrt{2}}\right)$$

To show that none of the other answer choices is correct, note that

$$-\left(\frac{1 + \sqrt{2}}{2\sqrt{2}}\right) < 0$$

Therefore, it suffices to show that the sign of $\cos\theta\cos\alpha$ for each of the other three answer choices is positive.

First, we'll look at answer choice B:

$$\left(\theta = \frac{11\pi}{8}, \alpha = \frac{7\pi}{8}\right)$$

Note that

$$\frac{11\pi}{8} = \frac{8\pi + 3\pi}{8} = \frac{8\pi}{8} + \frac{3\pi}{8} = \pi + \frac{3\pi}{8}$$

An angle of π radians is on the negative x-axis, and an angle of $3\pi/8$ radians is acute (it lies "between" 0 and $\pi/2$). Therefore, an angle of $11\pi/8$ radians is in the third quadrant, so its cosine is negative.

An angle of $7\pi/8$ radians is in the second quadrant (it lies "between" $\pi/2$ and π), so its cosine is also negative. Thus $\cos\theta\cos\alpha$ is positive.

Next, we'll consider answer choice C:

$$\left(\theta = \frac{5\pi}{8}, \alpha = \frac{7\pi}{8}\right)$$

An angle of $5\pi/8$ radians is in the second quadrant (it lies "between" $\pi/2$ and π), so its cosine is negative. We have already noted that an angle of $7\pi/8$ radians is in the second quadrant, hence that its cosine is negative as well. Therefore, $\cos\theta\cos\alpha$ is positive.

Finally, we'll look at answer choice D:

$$\left(\theta = \frac{9\pi}{8}, \alpha = \frac{11\pi}{8}\right)$$

An angle of $9\pi/8$ radians lies "between" π and $11\pi/8$. Now we have already found that an angle of $11\pi/8$ radians is in the third quadrant. Thus $9\pi/8$ is also in the third quadrant. From this it follows that the cosines of both $9\pi/8$ and $11\pi/8$ are negative, hence that $\cos\theta\cos\alpha$ is positive.

Topic: Sum-to-product identities

Question: Which of the following is a trigonometric identity?

Answer choices:

A $\sin\left(\theta + \frac{3\pi}{2}\right) = \frac{1}{2} - \sin\theta$

B $\sin\left(\theta + \frac{3\pi}{2}\right) = -2\cos\left(\theta - \frac{3\pi}{4}\right) - \sin\theta$

C $\sin\left(\theta + \frac{3\pi}{2}\right) = \left(1 - \frac{1}{\sqrt{2}}\right) - \sin\theta$

D $\sin\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right) - \sin\theta$

Solution: D

We'll prove that

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right)$$

is an identity. The identity

$$\sin\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right) - \sin\theta$$

follows immediately from that.

By one of the sum-to-product identities,

$$\sin\theta + \sin\alpha = 2\sin\left(\frac{\theta+\alpha}{2}\right)\cos\left(\frac{\theta-\alpha}{2}\right)$$

Setting α to $\theta + (3\pi/2)$, we get

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left[\frac{\theta + \left(\theta + \frac{3\pi}{2}\right)}{2}\right]\cos\left[\frac{\theta - \left(\theta + \frac{3\pi}{2}\right)}{2}\right]$$

Regrouping within the numerators of the expressions in square brackets:

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left[\frac{(\theta+\theta) + \left(\frac{3\pi}{2}\right)}{2}\right]\cos\left[\frac{(\theta-\theta) - \left(\frac{3\pi}{2}\right)}{2}\right]$$

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left[\frac{2\theta + \left(\frac{3\pi}{2}\right)}{2}\right]\cos\left[\frac{0 - \left(\frac{3\pi}{2}\right)}{2}\right]$$

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left[\frac{2\theta + \left(\frac{3\pi}{2}\right)}{2}\right]\cos\left[\frac{-\left(\frac{3\pi}{2}\right)}{2}\right]$$

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left(\theta + \frac{3\pi}{4}\right)\cos\left(-\frac{3\pi}{4}\right)$$

We know that $\cos(-3\pi/4) = -1/\sqrt{2}$, so

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\sin\left(\theta + \frac{3\pi}{4}\right)\left(-\frac{1}{\sqrt{2}}\right)$$

Changing the order of the last two factors, we get

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = 2\left(-\frac{1}{\sqrt{2}}\right)\sin\left(\theta + \frac{3\pi}{4}\right)$$

$$\sin\theta + \sin\left(\theta + \frac{3\pi}{2}\right) = -\sqrt{2}\sin\left(\theta + \frac{3\pi}{4}\right)$$

Note that the expression on the left-hand side of all four answer choices is the same. Therefore, to prove that answer choices A, B, and C are not identities, it suffices to exhibit one angle θ for which the value of the expression on the right-hand side of the identity in answer choice D is unequal to the value of the expression on the right-hand side of each of the other three answer choices.

Well, let $\theta = 0$. Then the right-hand side of answer choice D is

$$-\sqrt{2}\sin\left(0+\frac{3\pi}{4}\right)-\sin(0)$$

$$-\sqrt{2}\sin\left(0+\frac{3\pi}{4}\right)-\sin(0)=-\sqrt{2}\sin\left(\frac{3\pi}{4}\right)-\sin(0)$$

Since $\sin(3\pi/4)=1/\sqrt{2}$ and $\sin 0 = 0$,

$$-\sqrt{2}\sin\left(0+\frac{3\pi}{4}\right)-\sin(0)=-\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)-0=-1-0=-1$$

With $\theta = 0$, the right-hand side of answer choice A is

$$\frac{1}{2}-\sin(0)$$

Since $\sin 0 = 0$,

$$\frac{1}{2}-\sin(0)=\frac{1}{2}-0=\frac{1}{2}\neq-1$$

With $\theta = 0$, the right-hand side of answer choice B is

$$-2\cos\left(0-\frac{3\pi}{4}\right)-\sin(0)$$

$$-2\cos\left(0-\frac{3\pi}{4}\right)-\sin(0)=-2\cos\left(-\frac{3\pi}{4}\right)-\sin(0)$$

Since $\sin(-3\pi/4) = -1/\sqrt{2}$ and $\sin 0 = 0$,

$$-2\cos\left(0 - \frac{3\pi}{4}\right) - \sin(0) = -2\left(-\frac{1}{\sqrt{2}}\right) - 0 = \sqrt{2} - 0 = \sqrt{2} \neq -1$$

With $\theta = 0$, the right-hand side of answer choice C is

$$\left(1 - \frac{1}{\sqrt{2}}\right) - \sin(0)$$

Since $\sin 0 = 0$,

$$\left(1 - \frac{1}{\sqrt{2}}\right) - \sin(0) = \left(1 - \frac{1}{\sqrt{2}}\right) - 0 = 1 - \frac{1}{\sqrt{2}} \neq -1$$

Topic: Using trig identities to prove the equation

Question: Which of the following is an identity?

Answer choices:

A $\quad \sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta+\sin\left(\frac{\theta}{2}\right)\right]$

B $\quad \sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta-\sin\left(\frac{\theta}{2}\right)\right]$

C $\quad \cos\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\cos\theta-\cos\left(\frac{\theta}{2}\right)\right]$

D $\quad \sin\left(\frac{\theta}{4}\right)\sin\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\cos\theta+\cos\left(\frac{\theta}{2}\right)\right]$

Solution: B

First, we'll prove that the equation given in answer choice B is an identity:

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta-\sin\left(\frac{\theta}{2}\right)\right]$$

Well, by one of the product-to-sum identities,

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\left(\frac{\theta}{4}+\frac{3\theta}{4}\right)+\sin\left(\frac{\theta}{4}-\frac{3\theta}{4}\right)\right]$$

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\left(\frac{\theta+3\theta}{4}\right)+\sin\left(\frac{\theta-3\theta}{4}\right)\right]$$

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\left(\frac{4\theta}{4}\right)+\sin\left(\frac{-2\theta}{4}\right)\right]$$

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta+\sin\left(-\frac{\theta}{2}\right)\right]$$

By the odd identity for sine (i.e., that $\sin(-\theta/2)=-\sin(\theta/2)$), we have

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta-\sin\left(\frac{\theta}{2}\right)\right]$$

Now we'll show that none of the other three answer choices is an identity.

Answer choice A is

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\sin\theta+\sin\left(\frac{\theta}{2}\right)\right]$$

We'll show that this is false for $\theta=\pi$:

$$\sin\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right)=\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)=-\frac{1}{2}$$

but

$$\frac{1}{2}\left[\sin\theta+\sin\left(\frac{\theta}{2}\right)\right]=\frac{1}{2}\left[\sin\pi+\sin\left(\frac{\pi}{2}\right)\right]=\frac{1}{2}(0+1)=\frac{1}{2}(1)=\frac{1}{2}\neq-\frac{1}{2}$$

Answer choice C is

$$\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)=\frac{1}{2}\left[\cos\theta-\cos\left(\frac{\theta}{2}\right)\right]$$

We'll show that this is false for $\theta=2\pi$:

$$\cos\left(\frac{\theta}{4}\right)\cos\left(\frac{3\theta}{4}\right)$$

$$\cos\left(\frac{2\pi}{4}\right)\cos\left[\frac{3(2\pi)}{4}\right]$$

$$\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{6\pi}{4}\right)$$

$$\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{3\pi}{2}\right)$$

$$0(0) = 0$$

but

$$\frac{1}{2}\left[\cos\theta - \cos\left(\frac{\theta}{2}\right)\right]$$

$$\frac{1}{2}\left[\cos(2\pi) - \cos\left(\frac{2\pi}{2}\right)\right]$$

$$\frac{1}{2}\left[\cos(2\pi) - \cos(\pi)\right]$$

$$\frac{1}{2}[1 - (-1)]$$

$$\frac{1}{2}(2) = 1 \neq 0$$

Answer choice D is

$$\sin\left(\frac{\theta}{4}\right)\sin\left(\frac{3\theta}{4}\right) = \frac{1}{2}\left[\cos\theta + \cos\left(\frac{\theta}{2}\right)\right]$$

We'll show that this is false for $\theta = 2\pi$:

$$\sin\left(\frac{\theta}{4}\right)\sin\left(\frac{3\theta}{4}\right)$$

$$\sin\left(\frac{2\pi}{4}\right)\sin\left[\frac{3(2\pi)}{4}\right]$$

$$\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{6\pi}{4}\right)$$

$$\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{3\pi}{2}\right)$$

$$1(-1) = -1$$

but

$$\frac{1}{2}\left[\cos\theta + \cos\left(\frac{\theta}{2}\right)\right]$$

$$\frac{1}{2}\left[\cos(2\pi) + \cos\left(\frac{2\pi}{2}\right)\right]$$

$$\frac{1}{2}\left[\cos(2\pi) + \cos(\pi)\right]$$

$$\frac{1}{2}[1 + (-1)]$$

$$\frac{1}{2}(0) = 0 \neq -1$$

Topic: Listing the solutions in the interval

Question: What are the solutions of the equation that lie in the interval?

$\tan(4\theta + \pi) = -1$ on $[0,2\pi)$

Answer choices:

A $\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$

B $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

C $\theta = \frac{3\pi}{16}, \frac{7\pi}{16}, \frac{11\pi}{16}, \frac{15\pi}{16}, \frac{19\pi}{16}, \frac{23\pi}{16}, \frac{27\pi}{16}, \frac{31\pi}{16}$

D $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$

Solution: C

By the sum identity for tangent,

$$\tan(4\theta + \pi) = \frac{\tan(4\theta) + \tan(\pi)}{1 - \tan(4\theta)\tan(\pi)}$$

Since $\tan \pi = 0$, we have

$$\tan(4\theta + \pi) = \frac{\tan(4\theta) + 0}{1 - \tan(4\theta)(0)} = \frac{\tan(4\theta)}{1} = \tan(4\theta)$$

Thus we need to solve the equation

$$\tan(4\theta) = -1$$

The only quadrants in which the tangent is negative are the second and fourth quadrants. The acute angle whose tangent is $+1$ is $\pi/4$, so any angle 4θ whose tangent is equal to -1 has reference angle $\pi/4$ and is in the second or fourth quadrant. One such angle in the second quadrant is

$$4\theta = \pi - \frac{\pi}{4} = \frac{4(\pi) - 1(\pi)}{4} = \frac{4\pi - \pi}{4} = \frac{3\pi}{4}$$

Similarly, one angle 4θ in the fourth quadrant whose tangent is equal to -1 is

$$4\theta = 2\pi - \frac{\pi}{4} = \frac{4(2\pi) - 1(\pi)}{4} = \frac{8\pi - \pi}{4} = \frac{7\pi}{4}$$

Thus every angle θ that is a solution of the equation $\tan(4\theta + \pi) = -1$ has the property that there is some integer n that satisfies one of the following two equations:

$$4\theta = \frac{3\pi}{4} + 2n\pi$$

$$4\theta = \frac{7\pi}{4} + 2n\pi$$

Dividing both sides of each of these two equations by 4, we find that $\tan(4\theta + \pi) = -1$ if and only if there is some integer n that satisfies one of the following two equations:

$$\theta = \frac{3}{16} + \frac{n\pi}{2}$$

$$\theta = \frac{7\pi}{16} + \frac{n\pi}{2}$$

Note that when $n = 0$, the angles that satisfy these two equations are $3\pi/16$ and $7\pi/16$, respectively. Now

$$0 < \frac{3\pi}{16} < \frac{32\pi}{16} = 2\pi$$

and

$$0 < \frac{7\pi}{16} < \frac{32\pi}{16} = 2\pi$$

Therefore, both of these angles are in the interval $[0,2\pi)$.

Now let $n = 1$. Then we get the angles

$$\frac{3\pi}{16} + \frac{1(\pi)}{2} = \frac{1(3\pi) + 8(\pi)}{16} = \frac{3\pi + 8\pi}{16} = \frac{11\pi}{16}$$

and

$$\frac{7\pi}{16} + \frac{1(\pi)}{2} = \frac{1(7\pi) + 8(\pi)}{16} = \frac{7\pi + 8\pi}{16} = \frac{15\pi}{16}$$

respectively. Note that

$$0 < \frac{11\pi}{16} < \frac{32\pi}{16} = 2\pi$$

and

$$0 < \frac{15\pi}{16} < \frac{32\pi}{16} = 2\pi$$

Thus the angles $11\pi/16$ and $15\pi/16$ are also solutions that lie in the interval $[0,2\pi)$.

How about $n = 2$? Well,

$$\frac{3\pi}{16} + \frac{2(\pi)}{2} = \frac{1(3\pi) + 8(2\pi)}{16} = \frac{3\pi + 16\pi}{16} = \frac{19\pi}{16}$$

and

$$\frac{7\pi}{16} + \frac{2(\pi)}{2} = \frac{1(7\pi) + 8(2\pi)}{16} = \frac{7\pi + 16\pi}{16} = \frac{23\pi}{16}$$

respectively. Note that

$$0 < \frac{19\pi}{16} < \frac{32\pi}{16} = 2\pi$$

and

$$0 < \frac{23\pi}{16} < \frac{32\pi}{16} = 2\pi$$

Thus both $19\pi/16$ and $23\pi/16$ are solutions that lie in the interval $[0,2\pi)$.

Dare we even think that the angles with $n = 3$ will lie in that interval? Well, here goes:

$$\frac{3\pi}{16} + \frac{3(\pi)}{2} = \frac{1(3\pi) + 8(3\pi)}{16} = \frac{3\pi + 24\pi}{16} = \frac{27\pi}{16}$$

and

$$\frac{7\pi}{16} + \frac{3(\pi)}{2} = \frac{1(7\pi) + 8(3\pi)}{16} = \frac{7\pi + 24\pi}{16} = \frac{31\pi}{16}$$

respectively. Now

$$0 < \frac{27\pi}{16} < \frac{32\pi}{16} = 2\pi$$

and

$$0 < \frac{31\pi}{16} < \frac{32\pi}{16} = 2\pi$$

We have found that both $27\pi/16$ and $31\pi/16$ are also in the interval $[0,2\pi)$.

Let's go on to $n = 4$:

$$\frac{3\pi}{16} + \frac{4(\pi)}{2} = \frac{1(3\pi) + 8(4\pi)}{16} = \frac{3\pi + 32\pi}{16} = \frac{35\pi}{16}$$

and

$$\frac{7\pi}{16} + \frac{4(\pi)}{2} = \frac{1(7\pi) + 8(4\pi)}{16} = \frac{7\pi + 32\pi}{16} = \frac{39\pi}{16}$$

respectively. Note that

$$\frac{35\pi}{16} > \frac{32\pi}{16} = 2\pi$$

and

$$\frac{39\pi}{16} > \frac{32\pi}{16} = 2\pi$$

Thus neither of the solutions for $n = 4$ lies in the interval $[0,2\pi)$.

How about negative integers n? First, we'll check $n = -1$:

$$\frac{3\pi}{16} + \frac{(-\pi)}{2} = \frac{1(3\pi) + 8(-\pi)}{16} = \frac{3\pi - 8\pi}{16} = -\frac{5\pi}{16} < 0$$

and

$$\frac{7\pi}{16} + \frac{(-\pi)}{2} = \frac{1(7\pi) + 8(-\pi)}{16} = \frac{7\pi - 8\pi}{16} = -\frac{\pi}{16} < 0$$

respectively. Since both of these angles are less than 0, they are not in the interval $[0,2\pi)$. Also, it is easy to see that the same is true for all $n \le -2$, since those angles are even "more negative" than the ones for $n = -1$.

Collecting all our results, we find that the solutions of the equation

$$\tan(4\theta + \pi) = -1$$

that lie within the interval $[0,2\pi)$ are

$$\frac{3\pi}{16}, \frac{7\pi}{16}, \frac{11\pi}{16}, \frac{15\pi}{16}, \frac{19\pi}{16}, \frac{23\pi}{16}, \frac{27\pi}{16}, \frac{31\pi}{16}$$

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