Trigonometry Midterm Exam

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Topic: Complementary and supplementary angles

Question: Find the angle θ which is $1/3$ as large as the angle that's supplementary to an angle of 87°.

Answer choices:

A $\theta = 1^\circ$

B $\theta = 31^\circ$

C $\theta = 37\frac{2}{3}^\circ$

D $\theta = 13^\circ$

Solution: B

Let α be the angle that's supplementary to an angle of 87°. Then

$$\alpha + 87^\circ = 180^\circ$$

$$\alpha = 180^\circ - 87^\circ$$

$$\alpha = 93^\circ$$

Now $\theta = (1/3)\alpha$, so

$$\theta = \frac{1}{3}(93^\circ)$$

$$\theta = 31^\circ$$

Topic: Sketching angles in standard position

Question: Between which two axes is the terminal side of an angle of -2.6 radians (in standard position) located?

Answer choices:

A Between the positive horizontal axis and the negative vertical axis

B Between the negative horizontal axis and the positive vertical axis

C Between the negative horizontal axis and the negative vertical axis

D Between the positive vertical axis and the positive horizontal axis

Solution: C

If we divide -2.6 by π, we get approximately -0.828, so $-2.6 \approx -0.828\pi$. Note that

$$-\frac{1}{2}(2\pi) = -\pi < -0.828\pi < -0.5\pi = -\frac{1}{2}\pi = -\left(\frac{1}{2}\right)\left(\frac{2}{2}\right)\pi = -\frac{1}{4}(2\pi)$$

Since the terminal side of an angle of measure $-(1/2)(2\pi)$ radians is on the negative horizontal axis, and the terminal side of an angle of measure $-(1/4)(2\pi)$ radians is on the negative vertical axis, the terminal side of an angle of -2.6 (approximately -0.828) radians is located between the negative horizontal axis and the negative vertical axis.

Topic: Quadrant of the angle

Question: Where is the angle located?

10.5π radians

Answer choices:

A On the positive vertical axis

B On the negative vertical axis

C In the second quadrant

D In the fourth quadrant

Solution: A

First, note that

$$10.5\pi = \left(10\frac{1}{2}\right)\pi = \frac{21}{2}\pi$$

An angle of $(21/2)\pi$ radians isn't in the range 0 to 2π. Since it's a positive angle, we can subtract positive integer multiples of 2π from $(21/2)\pi$ until we find an angle that's in the range 0 to 2π radians.

$$\frac{21}{2}\pi - 2\pi = \frac{21}{2}\pi - \frac{4}{2}\pi = \frac{17}{2}\pi$$

$$\frac{21}{2}\pi - 2(2\pi) = \frac{17}{2}\pi - 2\pi = \frac{17}{2}\pi - \frac{4}{2}\pi = \frac{13}{2}\pi$$

$$\frac{21}{2}\pi - 3(2\pi) = \frac{13}{2}\pi - 2\pi = \frac{13}{2}\pi - \frac{4}{2}\pi = \frac{9}{2}\pi$$

$$\frac{21}{2}\pi - 4(2\pi) = \frac{9}{2}\pi - 2\pi = \frac{9}{2}\pi - \frac{4}{2}\pi = \frac{5}{2}\pi$$

$$\frac{21}{2}\pi - 5(2\pi) = \frac{5}{2}\pi - 2\pi = \frac{5}{2}\pi - \frac{4}{2}\pi = \frac{1}{2}\pi$$

An angle of $(1/2)\pi$ radians is on the positive vertical axis. Since 10.5π differs from $(1/2)\pi$ by an integer multiple of 2π, an angle of 10.5π radians is also on the positive vertical axis.

Topic: Converting between degrees and DMS

Question: Consider the following problem.

If a disc undergoes a rotation through an angle of $190^\circ 41' 58''$ about its center, followed immediately by a rotation of $135^\circ 56' 37''$ (again about its center, and in the same direction as the first rotation), what is the (total) angle through which the disc is rotated?

Answer choices:

A $326^\circ 38' 35''$

B 327.483°

C 325.95°

D $325^\circ 79' 35''$

Solution: A

First, let's compute the sum of the degrees parts of the two rotations, and then do the same for the minutes parts and the seconds parts:

$$190^\circ + 135^\circ = 325$$

$$41' + 56' = 97'$$

$$58'' + 37'' = 95''$$

We have to adjust these in such a way that the minutes part of the total angle of rotation ends up being in the interval $[0',59']$ and the seconds part ends up being in the interval $[0'',60'')$.

Starting with the sum of the seconds parts of the two rotations, $95''$, we have to successively subtract $60''$ (i.e., $1'$) from $95''$ until we reach an angle that's in the interval $[0'',60'')$. This will take just one subtraction:

$$95'' - 60'' = 35''$$

Therefore, the seconds part of the total angle of rotation is $35''$.

Next, we deal with the minutes. Note, however, that to make up for the $60''(=1')$ that we subtracted from the sum of the seconds parts of the two rotations, we have to add $1'$ to the sum of the minutes parts of the two angles of rotation, which gives

$$97' + 1' = 98'$$

Now we have to successively subtract $60'$ (i.e., 1°) from $98'$ until we reach an angle that's in the interval $[0',59']$. A single subtraction of $60'$ yields an angle in that interval:

$$98' - 60' = 38'$$

Thus the minutes part of the total angle of rotation is $38'$.

Finally, to make up for the $60'\,(= 1^\circ)$ that we subtracted from the sum of the minutes parts, we have to add 1° to the sum of the degrees parts. Therefore, the degrees part of the total angle of rotation is

$$325^\circ + 1^\circ = 326^\circ$$

What we have found is that the total angle through which the disc is rotated is $326^\circ 38' 35''$.

Topic: Finding coterminal angles

Question: Which of the following angles (in radians) is not coterminal with the angle?

$$-\frac{8}{3}\pi$$

Answer choices:

A $\frac{4}{3}\pi$

B $-\frac{4}{3}\pi$

C $-\frac{26}{3}\pi$

D $\frac{22}{3}\pi$

Solution: B

Let $\theta = -(8/3)\pi$, and recall that if two angles are coterminal, then their measures (in radians) differ by an integer multiple of 2π. Let α denote an angle which is coterminal with θ. Then there is some integer m such that

$$\alpha = \theta + m(2\pi)$$

Putting this another way, there is some integer m such that

$$\frac{\alpha - \theta}{2\pi} = m$$

Let's check each of the answer choices in turn, by substituting each of them as a value of α and determining whether there is some integer m that satisfies the equation $(\alpha - \theta)/(2\pi) = m$.

$$\alpha = \frac{4}{3}\pi: \quad \alpha - \theta = \frac{4}{3}\pi - \left(-\frac{8}{3}\pi\right) = \left(\frac{4}{3} + \frac{8}{3}\right)\pi = \frac{12}{3}\pi = 4\pi \qquad \frac{4\pi}{2\pi} = 2$$

$$\alpha = -\frac{4}{3}\pi: \quad \alpha - \theta = -\frac{4}{3}\pi - \left(-\frac{8}{3}\pi\right) = \left(-\frac{4}{3} + \frac{8}{3}\right)\pi = \frac{4}{3}\pi \qquad \frac{\left(\frac{4}{3}\pi\right)}{2\pi} = \frac{2}{3}$$

$$\alpha = -\frac{26}{3}\pi: \quad \alpha - \theta = -\frac{26}{3}\pi - \left(-\frac{8}{3}\pi\right) = \left(-\frac{26}{3} + \frac{8}{3}\right)\pi = -\frac{18}{3}\pi = -6\pi \quad \frac{-6\pi}{2\pi} = -3$$

$$\alpha = \frac{22}{3}\pi: \quad \alpha - \theta = \frac{22}{3}\pi - \left(-\frac{8}{3}\pi\right) = \left(\frac{22}{3} + \frac{8}{3}\right)\pi = \frac{30}{3}\pi = 10\pi \qquad \frac{10\pi}{2\pi} = 5$$

The only value of α for which $(\alpha - \theta)/(2\pi)$ is not equal to an integer is $-(4/3)\pi$.

Topic: Functions of negative angles

Question: Which of the following angles (in radians) lies in the interval is coterminal with the angle?

$$\frac{67}{5}\pi$$

on the interval $\left(-\frac{3}{5}\pi, \frac{7}{5}\pi\right]$

Answer choices:

A $-\frac{2}{5}\pi$

B $-\frac{3}{5}\pi$

C $\frac{7}{5}\pi$

D $-\frac{1}{5}\pi$

Solution: C

Let $\theta = (67/5)\pi$, and let α be the angle that is coterminal with θ and lies in the interval $(-(3/5)\pi, (7/5)\pi]$. Note that the length of the interval $(-(3/5)\pi, (7/5)\pi]$ is

$$\frac{7}{5}\pi - \left(-\frac{3}{5}\pi\right) = \left(\frac{7}{5} + \frac{3}{5}\right)\pi = \left(\frac{7+3}{5}\right)\pi = \frac{10}{5}\pi = 2\pi$$

Thus there is a unique integer n such that

$$\alpha = \theta + n(2\pi)$$

Therefore,

$$-\frac{3}{5}\pi < \alpha \le \frac{7}{5}\pi$$

$$-\frac{3}{5}\pi < \theta + n(2\pi) \le \frac{7}{5}\pi$$

Substituting $\theta = (67/5)\pi$:

$$-\frac{3}{5}\pi < \frac{67}{5}\pi + n(2\pi) \le \frac{7}{5}\pi$$

Subtracting $(67/5)\pi$ from all three quantities in these two inequalities, we obtain

$$-\frac{3}{5}\pi - \frac{67}{5}\pi < n(2\pi) \le \frac{7}{5}\pi - \frac{67}{5}\pi$$

$$-\left(\frac{3}{5} + \frac{67}{5}\right)\pi < n(2\pi) \le \left(\frac{7}{5} - \frac{67}{5}\right)\pi$$

$$-\left(\frac{3+67}{5}\right)\pi < n(2\pi) \le \left(\frac{7-67}{5}\right)\pi$$

$$-\frac{70}{5}\pi < n(2\pi) \le -\frac{60}{5}\pi$$

$$-14\pi < n(2\pi) \le -12\pi$$

Dividing through by 2π,

$$\frac{-14\pi}{2\pi} < n \le \frac{-12\pi}{2\pi}$$

$$-7 < n \le -6$$

Thus n is an integer that satisfies $-7 < n \le -6$, so n must be equal to -6.

To obtain the measure of α, we'll substitute $n = -6$ in the expression $\theta + n(2\pi)$:

$$\alpha = \theta + (-6)(2\pi)$$

$$\alpha = \frac{67}{5}\pi - 12\pi$$

$$\alpha = \left(\frac{67}{5} - 12\right)\pi$$

$$\alpha = \left[\frac{67 + 5(-12)}{5}\right]\pi$$

$$\alpha = \left(\frac{67-60}{5}\right)\pi$$

$$\alpha = \frac{7}{5}\pi$$

Topic: Converting between degrees and radians

Question: Which of the following is the best approximation (in radians) to the measure of the angle?

163°

Answer choices:

A 2.84 radians

B 3.45 radians

C 1.76 radians

D 2.66 radians

Solution: A

Since there are π radians in 180°, we will multiply 163° by 1, written in the form $\pi/(180^\circ)$:

$$163^\circ = 163^\circ(1)$$

$$163^\circ = 163^\circ\left(\frac{\pi}{180^\circ}\right)$$

$$163^\circ = \left(\frac{163}{180}\right)\pi$$

Thus 163° is equivalent to $(163/180)\pi$ radians. If we substitute the numerical value of π ($3.1415...$), we find that 163° is approximately 2.84 radians.

Topic: Oriented arc for a real number

Question: Which of the following most closely approximates the length of an oriented arc (of the unit circle) that corresponds to an angle in standard position whose measure is in DMS (degrees, minutes, and seconds)?

$$-502^\circ 15' 0''$$

Answer choices:

A -4.523

B -5.029

C -9.318

D -8.766

Solution: D

Let $\theta = -502^\circ 15' 0''$, and recall that what this means is that

$$\theta = (-502^\circ) + (-15') + (0'')$$

Recall that the length of an oriented arc (of the unit circle) is numerically equal to the measure of θ (in radians). Thus what we need to do is convert the measure of θ in DMS to its measure in (decimal) degrees, and then convert that to its measure in radians.

To get the measure of θ in (decimal) degrees, and given that the seconds part of this angle is equal to 0, it suffices to convert its minutes part, $-15'$, to degrees, and then add the result to -502°. For this, we'll use the conversion factor $(1^\circ)/(60')$:

$$(-15') = (-15')(1)$$

$$(-15') = (-15')\left(\frac{1^\circ}{60'}\right)$$

$$(-15') = -\left(\frac{15}{60}\right)^\circ$$

$$(-15') = -\left(\frac{1}{4}\right)^\circ$$

Substituting this result into the expression for θ:

$$\theta = (-502^\circ) + (-15') + (0'')$$

$$\theta = (-502^\circ) + \left[-\left(\frac{1}{4}\right)^\circ\right] + 0^\circ$$

$$\theta = -\left[502 + \left(\frac{1}{4}\right)\right]^\circ$$

$$\theta = -\left[\frac{4(502) + 1}{4}\right]^\circ$$

$$\theta = -\left(\frac{2{,}008 + 1}{4}\right)^\circ$$

$$\theta = -\left(\frac{2{,}009}{4}\right)^\circ$$

To get the measure of θ in radians, we use the conversion factor $\pi/(180^\circ)$:

$$\theta = -\left(\frac{2{,}009}{4}\right)^\circ (1)$$

$$\theta = -\left(\frac{2{,}009}{4}\right)^\circ \left(\frac{\pi}{180^\circ}\right)$$

$$\theta = -\left[\frac{2{,}009}{4(180)}\right]\pi$$

$$\theta = -\frac{2{,}009}{720}\pi$$

$$\theta \approx -8.766$$

From this it follows that the length of an oriented arc (of the unit circle) that corresponds to an angle in standard position whose measure in DMS is $-502^\circ 15' 0''$ is -8.766.

Topic: Area of a circular sector

Question: Find the area of the circular sector.

Find the area of a sector of a circle that passes through the point $(-2,4)$ and has its center at the point $(-6,1)$ if the arc which bounds that sector subtends a central angle of 150 degrees.

Answer choices:

A $A = 20\pi$

B $A = \pi\left(\frac{125}{6}\right)$

C $A = 125$

D $A = \pi\left(\frac{125}{12}\right)$

Solution: D

Since the center of the circle is at $(-6,1)$, every point of this circle satisfies the equation

$$[x-(-6)]^2+(y-1)^2=r^2$$

where r is the radius. Also, this circle passes through the point $(-2,4)$, so letting $(x,y)=(-2,4)$, we can find the radius:

$$[-2-(-6)]^2+(4-1)^2=r^2$$

$$4^2+3^2=r^2$$

$$16+9=r^2$$

$$25=r^2$$

Since r must be positive, we see that $r=\sqrt{25}=5$.

Now we can compute the area A of a sector of this circle which is bounded by an arc that subtends a central angle of 150 degrees:

$$A=\pi r^2\left(\frac{\theta}{360}\right)$$

$$A=\pi\left(5^2\right)\left(\frac{150}{360}\right)$$

$$A=\pi(25)\left(\frac{5}{12}\right)$$

$$A = \pi \left[\frac{25(5)}{12} \right]$$

$$A = \pi \left(\frac{125}{12} \right)$$

Topic: Linear and angular velocity

Question: Find the linear speed.

If a wheel of diameter 21.0 inches is rotating at a rate of 0.543π radians per second, what is the linear speed v of a point on the outside edge of the wheel?

Answer choices:

A $v = 1.49$ feet per second

B $v = 2.98$ feet per second

C $v = 17.9$ feet per second

D $v = 3.66$ feet per second

Solution: A

To compute v, we'll use the standard formula: $v = r\omega$. However, we have to take into account two other aspects of this problem. One of these is that we've been given the diameter (not the radius) of the wheel, so we have to compute the radius. The other is that all the answer choices are given in units of feet per second, but the diameter of the wheel is given in units of inches (not feet), so we have to convert inches to feet.

To get the radius of the wheel (which is equal to the distance of any point on the outside edge of the wheel from the center of the wheel), we simply divide the diameter by 2 (recall that the radius of any circle is one-half the diameter), so

$$r = \frac{21.0}{2} = 10.5 \text{ inches}$$

In order to take into account the different units of length (inches vs. feet), we'll use the fact that there are 12 inches in a foot. (In our calculation, we'll use "in" and "ft" as abbreviations for inches and feet, respectively.) Also, recall that we don't write "radians" or "rad" when we substitute the value of ω into the formula $v = r\omega$, since ω is understood to be in units of radians per unit time.

$$v = r\omega$$

$$v = (10.5 \text{ in})\left(\frac{0.543\pi}{\text{sec}}\right)$$

$$v = (10.5 \text{ in})(1)\left(\frac{0.543\pi}{\text{sec}}\right)$$

$$v = (10.5 \text{ in})\left(\frac{1 \text{ ft}}{12 \text{ in}}\right)\left(\frac{0.543\pi}{\text{sec}}\right)$$

$$v = \left[\frac{10.5(0.543)}{12}\pi\right] \text{ feet per second}$$

$$v \approx 0.475\pi \text{ feet per second}$$

Substituting the numerical value of π $(3.1415\ldots)$, we obtain

$$v \approx 1.49 \text{ feet per second}$$

Topic: Domain and range of the six circular functions

Question: Choose the best approximation.

If θ is an angle in the second quadrant such that $\cos\theta = -0.412$, which of the following most closely approximates the value of $\tan\theta$?

Answer choices:

A -2.21

B 2.01

C -0.452

D 0.496

Solution: A

We're given the value of $\cos\theta$, so we'll first find the value of $\sin\theta$:

$$\sin^2\theta + \cos^2\theta = 1$$

Substituting the value of $\cos\theta$, we have

$$\sin^2\theta + (-0.412)^2 = 1$$

$$\sin^2\theta = 1 - (-0.412)^2$$

Now $(-0.412)^2 \approx 0.170$, so

$$\sin^2\theta \approx 1 - 0.170$$

$$\sin^2\theta \approx 0.830$$

Well,

$$\sqrt{0.830} \approx 0.911,$$

so either

$$\sin\theta \approx 0.911$$

or

$$\sin\theta \approx -0.911$$

Since θ is in the second quadrant, $\sin\theta$ is positive, so $\sin\theta \approx 0.911$. Therefore,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \approx \frac{0.911}{-0.412} \approx -2.21$$

Topic: Find the value or say where it's undefined

Question: What is the value of the function?

$$\csc \frac{19\pi}{4}$$

Answer choices:

A $-\sqrt{2}$

B $-\frac{1}{\sqrt{2}}$

C $\sqrt{2}$

D $\frac{1}{\sqrt{2}}$

Solution: C

Note that

$$\frac{19\pi}{4}=\frac{(16+3)\pi}{4}=\frac{16\pi}{4}+\frac{3\pi}{4}=4\pi+\frac{3\pi}{4}$$

Therefore, an angle of $19\pi/4$ radians differs from an angle of $3\pi/4$ radians by 4π, which is an integer multiple of 2π, so

$$\csc\frac{19\pi}{4}=\csc\frac{3\pi}{4}$$

Also,

$$\pi-\frac{3\pi}{4}=\frac{\pi}{4}$$

so the reference angle for both $19\pi/4$ and $3\pi/4$ is $\pi/4$. Since $3\pi/4$ is in the second quadrant and $\pi/4$ is in the first quadrant (hence $\sin(3\pi/4)=\sin(\pi/4)$), the reference angle theorem gives

$$\csc\frac{19\pi}{4}=\csc\frac{3\pi}{4}=\frac{1}{\sin\frac{3\pi}{4}}=\frac{1}{\sin\frac{\pi}{4}}=\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}=\sqrt{2}$$

Topic: Find all six trig functions given the quadrant of the angle

Question: What are the possible values of the trig function?

$\tan\theta$ if $\cos\theta = -0.218$

Answer choices:

A 3.27 and -3.27

B 4.48 and -4.48

C 2.14 and -2.14

D 5.46 and -5.46

Solution: B

We'll ultimately use the definition of $\tan\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

In order to do that, we need to compute the possible values of $\sin\theta$:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta$$

Substituting -0.218 for $\cos\theta$ gives

$$\sin^2\theta = 1 - (-0.218)^2 \approx 1 - 0.0475 = 0.953$$

Therefore, either

$$\sin\theta = \sqrt{0.953} \approx 0.976$$

or

$$\sin\theta = -\sqrt{0.953} \approx -0.976$$

Since $\cos\theta$ is negative, θ is in either the second quadrant or the third quadrant. If θ is in the second quadrant, $\sin\theta$ is positive (hence $\sin\theta \approx 0.976$). If θ is in the third quadrant, $\sin\theta$ is negative (hence $\sin\theta \approx -0.976$).

By definition,

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Substituting the values of $\sin\theta$ and $\cos\theta$, we find that if θ is in the second quadrant, then

$$\tan\theta \approx \frac{0.976}{-0.218} \approx -4.48$$

If θ is in the third quadrant, then

$$\tan\theta \approx \frac{-0.976}{-0.218} \approx 4.48$$

Thus the possible values for $\tan\theta$ are 4.48 and -4.48.

Topic: Angle of depression and elevation

Question: How far away is the fish?

A man is trying to catch some fish and spots one with an angle of depression of 51° with respect to the end of his fishing rod. The fish is located 8.2 feet lower than the end of the fishing rod.

Answer choices:

A 13.0 feet

B 19.1 feet

C 9.16 feet

D 10.6 feet

Solution: D

Let a be the vertical distance between the fish and the end of the fishing rod, and let c be the (overall) distance between them. What we need to do is compute c from a and the angle of depression.

Note that

$$\frac{a}{\sin 51^\circ} = \frac{c}{1}$$

That is,

$$\frac{a}{\sin 51^\circ} = c$$

Turning this equation around, we have

$$c = \frac{a}{\sin 51^\circ}$$

We are given that $a = 8.2$ feet. Substituting 8.2 for a, we get

$$c = \frac{8.2}{\sin 51^\circ}$$

Now

$$\sin 51^\circ \approx 0.777$$

Thus

$$c \approx \frac{8.2}{0.777} \approx 10.6 \text{ feet}$$

Topic: Proving the trig equation

Question: Which of the following is not a trigonometric identity?

Answer choices:

A $\sec\theta\sin\theta=\tan\theta$

B $\sin\theta(\cot\theta+\tan\theta)=\cos\theta$

C $\sec^4\theta-\sec^2\theta=\tan^2\theta+\tan^4\theta$

D $\cos^2\theta\tan^2\theta+\sin^2\theta\cot^2\theta=1$

Solution: B

First, we'll show that the equation in B, $\sin\theta(\cot\theta+\tan\theta)=\cos\theta$, is not a trigonometric identity. Let $\theta=\pi/6$, and recall that

$$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$$

and

$$\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$$

By the definitions of $\tan\theta$ and $\cot\theta$,

$$\tan\left(\frac{\pi}{6}\right)=\frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}=\frac{\left(\frac{1}{2}\right)}{\left(\frac{\sqrt{3}}{2}\right)}=\frac{1}{\sqrt{3}}$$

and

$$\cot\left(\frac{\pi}{6}\right)=\frac{\cos\left(\frac{\pi}{6}\right)}{\sin\left(\frac{\pi}{6}\right)}=\frac{\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)}=\sqrt{3}$$

Therefore,

$$\sin\left(\frac{\pi}{6}\right)\left[\cot\left(\frac{\pi}{6}\right)+\tan\left(\frac{\pi}{6}\right)\right]$$

$$= \left(\frac{1}{2}\right)\left(\sqrt{3} + \frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{1}{2}\right)\left[\frac{\sqrt{3}(\sqrt{3}) + 1}{\sqrt{3}}\right]$$

$$= \left(\frac{1}{2}\right)\frac{4}{\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \neq \frac{\sqrt{3}}{2} = \cos\left(\frac{\pi}{6}\right)$$

Now we'll prove that the other three equations are indeed trigonometric identities. First, the equation in A: $\sec\theta\sin\theta = \tan\theta$.

$\sec\theta\sin\theta$	$= \frac{1}{\cos\theta}(\sin\theta)$	Use $\sec\theta = \frac{1}{\cos\theta}$
	$= \tan\theta$	Use $\frac{\sin\theta}{\cos\theta} = \tan\theta$

Next, the equation in C: $\sec^4\theta - \sec^2\theta = \tan^2\theta + \tan^4\theta$.

$\sec^4\theta - \sec^2\theta$	$= (\sec^2\theta)^2 - \sec^2\theta$	Write $\sec^4\theta$ as $(\sec^2\theta)^2$
	$= (1 + \tan^2\theta)^2 - (1 + \tan^2\theta)$	Use $\sec^2\theta = 1 + \tan^2\theta$
	$= (1 + 2\tan^2\theta + \tan^4\theta) - (1 + \tan^2\theta)$	Expand $(1 + \tan^2\theta)^2$
	$= (1 - 1) + (2\tan^2\theta - \tan^2\theta) + \tan^4\theta$	Regroup
	$= \tan^2\theta + \tan^4\theta$	Simplify

Finally, the equation in D: $\cos^2\theta\tan^2\theta+\sin^2\theta\cot^2\theta=1$.

$$\cos^2\theta\tan^2\theta+\sin^2\theta\cot^2\theta=\cos^2\theta\left(\frac{\sin\theta}{\cos\theta}\right)^2+\sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\right)^2$$

Use $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta=\dfrac{\cos\theta}{\sin\theta}$

$$=\cos^2\theta\left(\frac{\sin^2\theta}{\cos^2\theta}\right)+\sin^2\theta\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

Carry out indicated squaring.

$$=\sin^2\theta+\cos^2\theta$$

Cancel $\cos^2\theta$ in first term, and $\sin^2\theta$ in second term.

$$=1$$

Use $\sin^2\theta+\cos^2\theta=1$

Topic: Law of sines

Question: How many triangles with these properties are there?

Consider a triangle that has one side of length $a = 20$ and another side of length $c = 16$, and where the interior angle opposite the side of length 16 (call it angle C) has measure 35°.

Answer choices:

A There are two triangles with the stated properties.

B There is only one triangle with the stated properties.

C There is no triangle with the stated properties.

D The number of triangles with the stated properties cannot be determined.

Solution: A

By the law of sines,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

where angle A is the interior angle which is opposite the side of length 20, b is the length of the third side of the triangle, and angle B is the interior angle which is opposite the side of length b.

Substituting the known values, we have

$$\frac{20}{\sin A}=\frac{b}{\sin B}=\frac{16}{\sin 35^\circ}$$

Multiplying both sides of the equation

$$\frac{20}{\sin A}=\frac{16}{\sin 35^\circ}$$

by

$$(\sin A)\left(\frac{\sin 35^\circ}{16}\right)$$

we get

$$\frac{20(\sin 35^\circ)}{16}=\sin A$$

Turning this equation around, we have

$$\sin A=\frac{20(\sin 35^\circ)}{16}=\left(\frac{5}{4}\right)(\sin 35^\circ)$$

Using a calculator, we find that $\sin 35^\circ \approx 0.574$, so

$$\sin A \approx \left(\frac{5}{4}\right)(0.574) \approx 0.718$$

What this tells us is that if angle A is acute, then its measure is approximately 45.9°; and if angle A is obtuse, then its measure is (approximately)

$$180^\circ - 45.9^\circ = 134.1^\circ$$

Suppose the measure of angle A is (approximately) 134.1°. Then the sum of the measures of angles A and C is

$$134.1^\circ + 35^\circ = 169.1^\circ < 180$$

Therefore, there are two triangles with the stated properties.

Next, let's set angle A to 45.9°, compute the measure of angle B, and then use the law of sines to determine b.

In this case, the measure of angle B is

$$180^\circ - (45.9^\circ + 35^\circ) = 180^\circ - 80.9^\circ = 99.1^\circ$$

To get $\sin B$, we'll solve the equation

$$\frac{b}{\sin B} = \frac{16}{\sin 35^\circ}$$

Substituting 99.1° for angle B gives

$$b = \frac{16(\sin 99.1^\circ)}{\sin 35^\circ}$$

Using a calculator, we get that $\sin 99.1^\circ \approx 0.987$, so

$$b = \frac{16(\sin 99.1^\circ)}{\sin 35^\circ} \approx \frac{16(0.987)}{0.574} \approx 27.5$$

Finally, let's set angle A to 134.1°. Then the measure of angle B is

$$180^\circ - (134.1^\circ + 35^\circ) = 180^\circ - 169.1^\circ = 10.9^\circ$$

To get $\sin B$, we'll solve the equation

$$\frac{b}{\sin B} = \frac{16}{\sin 35^\circ}$$

Substituting 10.9° for angle B gives

$$b = \frac{16(\sin 10.9^\circ)}{\sin 35^\circ}$$

Using a calculator, we get that $\sin 10.9^\circ \approx 0.189$, so

$$b = \frac{16(\sin 10.9^\circ)}{\sin 35^\circ} \approx \frac{16(0.189)}{0.574} \approx 5.27$$

Thus there are two triangles with the indicated properties:

- a triangle with interior angles of measure 45.9°, 99.1°, and 35°, and with sides opposite those angles which are of length 20, 27.5, and 16, respectively

- a triangle with interior angles of measure 134.1°, 10.9°, and 35°, and with sides opposite those angles which are of length 20, 5.27, and 16, respectively

Topic: Law of cosines

Question: Only one of the following triples of numbers consists of lengths of the sides of some triangle. Which triple is it?

Answer choices:

A $(24,4,19)$

B $(10,25,32)$

C $(34,16,16)$

D $(47,35,11)$

Solution: B

First, we'll show that there is indeed a triangle with sides of length 10, 25, and 32.

Let $a=10$, $b=25$, and $c=32$.

$$|a-b|=|10-25|=15<32=c=32<35=10+25=a+b$$

$$|a-c|=|10-32|=22<25=b=25<42=10+32=a+c$$

$$|b-c|=|25-32|=7<10=a=10<57=25+32=b+c$$

Now we'll show that none of the other answer choices works.

For answer choice A, let $a=24$, $b=4$, and $c=19$. Then

$$|a-b|=|24-4|=20\not<19=c$$

For answer choice C, let $a=34$, $b=16$, and $c=16$. Then

$$a=34\not<32=16+16=b+c$$

For answer choice D, let $a=47$, $b=35$, and $c=11$. Then

$$|a-c|=|47-11|=36\not<35=b$$

Topic: Area of a triangle

Question: Find the area of a triangle.

The measures of two of the interior angles are 77° and 56° and the length of the side that's common to those two angles is 39.

Answer choices:

A The area is approximately 841.

B The area is approximately 492.

C The area is approximately 571.

D The area is approximately 708.

Solution: A

Since we know the length of only one of the sides of this triangle, we can't find its area by immediately applying the law of sines for the area of a triangle (which requires, among other things, knowing the lengths of two of the sides), much less can we find the area by immediately applying Heron's formula (which requires knowing the lengths of all three sides). However, we can easily determine the measure of the third interior angle of this triangle, and we can apply the (usual) law of sines to find the length of either of the two unknown sides. Once we've done both of those things, we'll be able to apply the law of sines for the area of a triangle.

Let angle A be the angle of measure 77°, and let angle B be the angle of measure 56°. Then the measure of the third interior angle (angle C) is

$$180^\circ - (77^\circ + 56^\circ) = 180^\circ - 133^\circ = 47^\circ$$

Now let a, b, and c be the lengths of the sides opposite angles A, B, and C, respectively. Then $c = 39$, and (at this point) the values of both a and b are unknown.

By the (usual) law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Substituting the data:

$$\frac{a}{\sin 77^\circ} = \frac{b}{\sin 56^\circ} = \frac{39}{\sin 47^\circ}$$

Multiplying both sides of the equation

$$\frac{a}{\sin 77^\circ} = \frac{39}{\sin 47^\circ}$$

by $\sin 77^\circ$, we get

$$a = (\sin 77^\circ)\left(\frac{39}{\sin 47^\circ}\right)$$

Using a calculator, we find that $\sin 77^\circ \approx 0.974$ and $\sin 47^\circ \approx 0.731$, so

$$a \approx (0.974)\left(\frac{39}{0.731}\right) \approx 52.0$$

Now we're ready to apply the law of sines for the area of a triangle:

$$\text{area} = \frac{1}{2}ac \sin B$$

Substituting the data:

$$\text{area} \approx \frac{1}{2}(52.0)(39)\sin 56^\circ \approx \frac{(52.0)(39)}{2} \sin 56^\circ \approx \frac{2{,}028}{2} \sin 56^\circ$$

Again with the use of a calculator, we find that $\sin 56^\circ \approx 0.829$, so

$$\text{area} \approx \frac{2{,}028}{2}(0.829) \approx 841$$

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