Machine Learning Assignment 10-313652008

313652008 黃睿帆

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Problem 1– Derivation of the Probability Flow ODE

Consider the forward SDE:

$$dx_t = f(x_t, t)\,dt + g(x_t, t)\,dW_t.$$

In class, we know that its corresponding probability density $p(x, t)$ satisfies the Fokker—Planck equation:

$$\partial_t p(x, t) = -\partial_x\big[f(x, t)\,p(x, t)\big] + \tfrac{1}{2}\,\partial_x^2\big[g^2(x, t)\,p(x, t)\big]. \tag{1}$$

Then like we did in the class, defining a deterministic ODE:

$$dx_t = v(x_t, t)\,dt,$$

whose flow preserves the same marginal density $p(x, t)$. Then $p(x, t)$ satisfies (continuity) equation:

$$\partial_t p(x, t) = -\partial_x\big[v(x, t)\,p(x, t)\big]. \tag{2}$$

Comparing right-hand sides of equations (1) and (2), we get

$$-\partial_x\big[v\,p\big] = -\partial_x\big[f\,p\big] + \tfrac{1}{2}\,\partial_x^2\big[g^2 p\big].$$

Simplify:

$$\partial_x\big[(f - v)p\big] = \tfrac{1}{2}\,\partial_x^2\big[g^2 p\big].$$

Integrating once with respect to x (and assuming vanishing boundary terms),

$$(f - v)p = \tfrac{1}{2}\,\partial_x\big[g^2 p\big].$$

Hence,

$$v(x, t) = f(x, t) - \frac{1}{2p(x, t)}\,\partial_x\big[g^2(x, t)\,p(x, t)\big].$$

Then expanding the derivative term $\partial_x[g^2 p]$ at right hand side,

$$\partial_x\big[g^2 p\big] = (\partial_x g^2)\,p + g^2\,\partial_x p,$$

we obtain

$$v(x, t) = f(x, t) - \frac{1}{2p(x, t)}\,\partial_x\big[g^2(x, t)\,p(x, t)\big] = f(x, t) - \frac{1}{2p(x, t)}\,[(\partial_x g^2)\,p + g^2\,\partial_x p].$$

By calculation we have

$$v(x, t) = f(x, t) - \tfrac{1}{2}\,\partial_x g^2(x, t) - \tfrac{g^2(x,t)}{2}\,\frac{\partial_x p(x, t)}{p(x, t)}.$$

Since $\frac{\partial_x p}{p} = \partial_x \log p$, the final expression for the probability flow ODE is

$$\boxed{dx_t = \left[f(x_t, t) - \frac{1}{2}\frac{\partial}{\partial x}g^2(x_t, t) - \frac{g^2(x_t, t)}{2}\,\frac{\partial}{\partial x}\log p(x_t, t)\right]\,dt.}$$

Remark:

In Friday's (11/7) class, we knew the multidimensional SDE, with $G(x, t) = g(x, t)g(x, t)^\top$, the general form becomes

$$v = f - \tfrac{1}{2}\nabla\!\cdot G - \tfrac{1}{2}G\,\nabla \log p.$$

Problem 2: AI 的未來與機器學習的基石

我認爲目前 AI 無法做到，但 20 年後有可能做到的事情是 **AI 理解與重構人類記憶**. (P.S. 此作業有部分經由 Chat GPT 潤飾語句和分析需用到那些機器學習之原理) 細節請見另一個文件檔...

Unanswered questions

- How exactly do the probability-flow ODE and the score PDE relate: under what assumptions does the deterministic PF-ODE reproduce the same time-evolving marginals as the SDE, and how does the score function $\nabla_x \log p(x,t)$ enter and determine that equivalence (especially when the diffusion coefficient $g(x,t)$ is state-dependent) ?
- What are the existence, uniqueness, and boundary-condition issues for the PF-ODE, the Fokker Planck and Score PDEs (e.g., choices of spatial domain, decay at infinity, and how nonzero integration constants or nonvanishing flux at boundaries change the drift correction term)?