CC 501 - HOMEWORK 2 — STUDENT ID: 20204817

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2.1.8 (a) $X = \{-6, -4, -2, 0, 2, 4, 6\}$

Probability mass function (p.m.f) is:

x_i	-6	-4	-2	0	2	4	6
p_i	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

(b) Cumulative distribution function:

$$F(x) = \begin{cases} 0, & x < -6 \\ \frac{1}{8}, & -6 \le x < -4 \\ \frac{2}{8}, & -4 \le x < -2 \\ \frac{3}{8}, & -2 \le x < 0 \\ \frac{5}{8}, & 0 \le x < 2 \\ \frac{6}{8}, & 2 \le x < 4 \\ \frac{7}{8}, & 4 \le x < 6 \\ 1, & x \ge 6 \end{cases}$$

(c) The most likely value of x^* is the one for which $P(x^*) \ge P(x_i)$ $\forall x_i$

→ so the value $x^* = 0$.

2.2.10 $F(x) = \begin{cases} \frac{0.25}{500} x & 0 \le x < 500 \\ \frac{0.25}{500} x + 0.5 & 500 \le x \le 100 \end{cases}$

So, (a) $P(X \le 200) = F(200) = 200 \cdot \frac{0.25}{500} = \boxed{0.1}$

(b) $F(700) - F(400) = 700 \cdot \frac{0.25}{500} + 0.5 - 400 \cdot \frac{0.25}{500} = \boxed{0.65}$

The probability on the second range appears much higher because it contains the probability of winning 500 \$, which is 0.5.

2.3.14

$$\begin{cases} F(0) = A + \frac{1}{125} B = 0 \\ F(50) = A + \frac{1}{55^3} B = 1 \end{cases} \Rightarrow \text{we obtain: } \begin{cases} A = 1.00075 \\ B = -125.093 \end{cases}$$

$\Rightarrow F(r) = 1.00075 - \frac{125.093}{(r+5)^3}$

$f(r) = \frac{dF(r)}{dr} = \frac{375.36}{(r+5)^4}$ with $r \in [0, 50]$

so $E(r) = \int_{-\infty}^{\infty} f(x) \times dx = \int_{0}^{50} \frac{375.36}{(r+5)^4} dr \approx \boxed{2.49}$

- Median deviation is obtained by:

$F(r) = 0.5 \rightarrow 1.00075 - \frac{125.093}{(r+5)^3} = 0.5 \Rightarrow \boxed{r \approx 1.3}$

$\boxed{2.4.8}$ $\int_0^{10} f(\theta)\, d\theta = 1$ because it is a probability distribution

$\Rightarrow \int_0^{10} A(e^{10-\theta} - 1)\, d\theta = 1 \Rightarrow A = \frac{1}{e^{10} - 11}$

(a) $\sigma^2 = E(x^2) - \mu^2$, so:

$E(x^2) = \int_0^{10} \frac{(e^{10-\theta} - 1)}{e^{10} - 11} \theta^2\, d\theta \approx 1.98$

$\mu^2 = \left[\int_0^{10} \frac{(e^{10-\theta} - 1)\theta}{e^{10} - 11}\right]^2 = 0.99773^2 \approx 0.995$

so: $\sigma^2 = 1.98 - 0.995 = \boxed{0.985}$

(b) $\sigma = \sqrt{\sigma^2} = \sqrt{0.985} = \boxed{0.992}$

(c) - upper quartile: $F(\theta) = 0.75$, so:

$\int_0^{\theta} \frac{e^{10-\theta} - 1}{e^{10} - 11}\, d\theta = 0.75 \Rightarrow \boxed{\theta \approx 1.385}$

- lower quartile: $F(\theta) = 0.25$:

$\int_0^{\theta} \frac{e^{10-\theta} - 1}{e^{10} - 11}\, d\theta = 0.25 \Rightarrow \boxed{\theta \approx 0.288}$

(d) IQR (Interquartile range) $= Q_3 - Q_1 = 1.385 - 0.288 = \boxed{1.097}$

$\boxed{2.5.2}$ (a) Probability mass function:

y_i \ x_i	0	1	2	3
0	$\frac{1}{16}$	$\frac{1}{16}$	0	0
1	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	0
2	0	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{1}{16}$
3	0	0	$\frac{1}{16}$	$\frac{1}{16}$

(b) Marginal probability m.f. of X:

x_i	0	1	2	3
p_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Marginal probability m.f. of Y:

y_i	0	1	2	3
p_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(c) Independence if $f(x,y) = f_X(x) \cdot f_Y(y)$

But i.e. $\rightarrow f(0,0) = \frac{1}{16} \neq f_X(0) \cdot f_Y(0) = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}$

Since $f(0,0) \neq f_X(0) \cdot f_Y(0)$, then X, Y are NOT independent.

(d) • $E(x) = \sum_{i=0}^{3} p_i x_i = \left[\frac{3}{2}\right] = \mu_x$

$E(x^2) = \sum_{i=0}^{3} p_i x_i^2 = 3$

• $\sigma_x^2 = E(x^2) - \mu_x^2 = 3 - \left(\frac{3}{2}\right)^2 = \left[\frac{3}{4}\right]$

• $E(y) = \sum_{i=0}^{3} p_i y_i = \left[\frac{3}{2}\right] = \mu_y$

$E(y^2) = \sum_{i=0}^{3} p_i y_i^2 = 3$

• $\sigma_x^2 = E(y^2) - \mu_y^2 = 3 - \left(\frac{3}{2}\right)^2 = \left[\frac{3}{4}\right]$

(e) $\mathrm{cov}(x, y) = E(xy) - E(x)E(y)$

$E(xy) = \sum_{i=0}^{3} \sum_{j=0}^{3} p_{ij} \cdot x_i \cdot x_j = \frac{44}{16}$

$\Rightarrow \mathrm{cov}(x, y) = \frac{44}{16} - \frac{3}{2} \cdot \frac{3}{2} = \left[\frac{1}{2}\right]$

(f) $P(x=0 \mid y=1) = \frac{p_{01}}{p_{+1}} = \frac{1}{16} / \frac{6}{16} = \left[\frac{1}{6}\right]$

$P(x=1 \mid y=1) = \frac{p_{11}}{p_{+1}} = \frac{3}{16} / \frac{6}{16} = \left[\frac{1}{2}\right]$

$P(x=2 \mid y=1) = \frac{p_{21}}{p_{+1}} = \frac{1}{8} / \frac{6}{16} = \left[\frac{1}{3}\right]$

$P(x=3 \mid y=1) = \frac{p_{31}}{p_{+1}} = 0 / \frac{6}{16} = [0]$

$E(x \mid y=1) = 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} = \left[\frac{7}{6}\right]$

$E(x^2 \mid y=1) = 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{3} = \frac{11}{6}$

$\Rightarrow \sigma^2(x \mid y=1) = E(x^2 \mid y=1) - [E(x \mid y=1)]^2 = \left[\frac{17}{36}\right]$

2.6.4 Since they are independent,

$\sigma^2(0.5 x_\alpha + 0.3 x_\beta + 0.2 x_\delta) = \sigma^2(0.5 x_\alpha) + \sigma^2(0.3 x_\beta) + \sigma^2(0.2 x_\delta)$

$\Rightarrow \sigma = \sqrt{0.5^2 \cdot 1.2^2 + 0.3^2 \cdot 2.4^2 + 0.2^2 \cdot 3.1^2} = [1.24]$

2.6.16

(a) $E(A) = E\left(\frac{x_1 + x_2}{2}\right) = \frac{1}{2} E(x_1) + \frac{1}{2} E(x_2) =$

$= \frac{1}{2} w + \frac{1}{2} w = [w]$

$\sigma(A) = \sqrt{\sigma^2\left(\frac{x_1}{2}\right) + \sigma^2\left(\frac{x_2}{2}\right)} = \sqrt{\frac{1}{4} \cdot 3^2 + \frac{1}{4} \cdot 4^2} = \left[\frac{5}{2}\right]$

(b)

$\sigma^2(B) = \delta^2 \sigma^2(x_1) + (1-\delta)^2 \sigma^2(x_2) = 25\delta^2 - 32\delta + 16$

we minimize $f(\delta) \to f'(\delta) = 0 = 50\delta - 32 \to \left[\delta = \frac{16}{25}\right]$

So, $\sigma^2(B)$ given $\delta = \frac{16}{25} = \frac{144}{25}$; The minimum standard deviation will be:

$\sigma(B)\big|_{\delta = \frac{16}{25}} = \left[\frac{12}{5}\right]$

9.6

a) $\begin{cases} E(Y) = 100 = E(dx+c) = dE(X) + c = 250d + c \\ \sigma^2(Y) = \sigma^2(dx+c) = d^2\sigma^2(X) = 16\delta^2 = 1 \end{cases}$

$\Rightarrow$ we obtain: $\begin{cases} d = 0.25 \\ c = 37.5 \end{cases}$

b) $E(\underbrace{X + X + \dots + X}_{10}) = \underbrace{E(X) + E(X) + \dots + E(X)}_{10} = 10E(X) = \boxed{2500}$

$\sigma^2(\underbrace{X + X + \dots + X}_{10}) = \underbrace{\sigma(X) + \sigma(X) + \dots + \sigma(X)}_{10} = 10\sigma(X) = 160$

Thus, $\sigma(10X) = \sqrt{10\sigma^2(X)} = \sqrt{10 \cdot 16} = \boxed{4\sqrt{10}}$

9.16

$f(x) = Ax$

a) $\int_5^6 Ax\,dx = 1 \rightarrow A\left[\frac{x^2}{2}\right]_5^6 = 1;\ \boxed{A = \frac{2}{11}}$

b) c.d.f. $= \int_5^x \frac{2}{11}x\,dx = \boxed{\frac{x^2-25}{11} = F(x)}$

c) $E(X) = \int_5^6 \frac{2x}{11} \cdot x\,dx = \frac{182}{33} = \mu$

d) $\sigma = \sqrt{E(x^2) - \mu^2}$

$E(x^2) = \int_5^6 \frac{2}{11}x \cdot x^2\,dx = \frac{671}{22}$

So, $\sigma = \sqrt{\frac{671}{22} - \left(\frac{182}{33}\right)^2} = \sqrt{0.0831} \approx \boxed{0.288}$

9.24

a) Marginal p.d.f. of X:

$f_X(x) = \int_1^2 4x(2-y)\,dy = \boxed{2x},\quad x \in [0,1]$

b) Marginal p.d.f. of Y:

$f_Y(y) = \int_0^1 4x(2-y)\,dx = 2(2-y)$

Independence: if $f(x,y) = f_X(x) \cdot f_Y(y)$, so:

$4x(2-y) = 2x \cdot 2(2-y) = 4x(2-y)$ ✓

They are independent.

c) $\text{cov}(X,Y) = 0$, because independence implies zero covariance.

d) $f_{X|Y=1.5}(x) = f_X(x) = 2x$, because X and Y are independent.