[CC511] Homework 12 20204817 Federico Berto

December 8, 2020

1 Homework 12 - Federico Berto

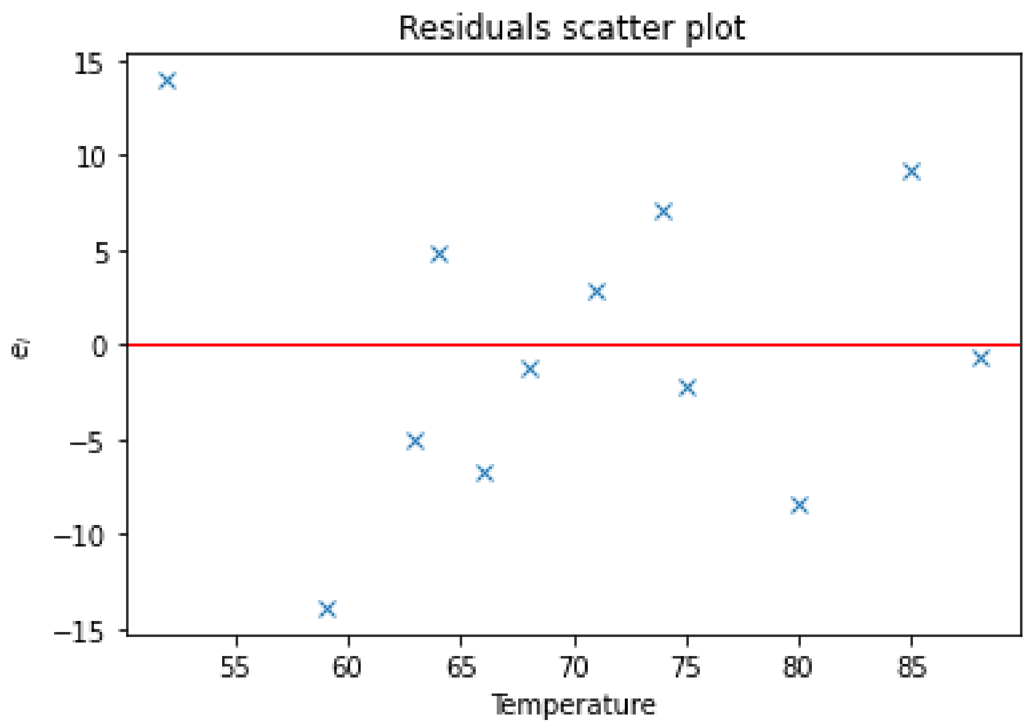
In [1]:
```
# Importing useful libraries
import math
import numpy as np
from numpy import log
import pandas as pd
import matplotlib.pyplot as plt
from scipy.stats import norm
from scipy.stats import t
from scipy.stats import f
from statsmodels.stats.anova import anova_lm
from statsmodels.formula.api import ols
from statsmodels.stats.multicomp import pairwise_tukeyhsd
from statsmodels.graphics.regressionplots import abline_plot
from statsmodels.api import qqplot
from IPython.display import display, Math
from tabulate import tabulate
```

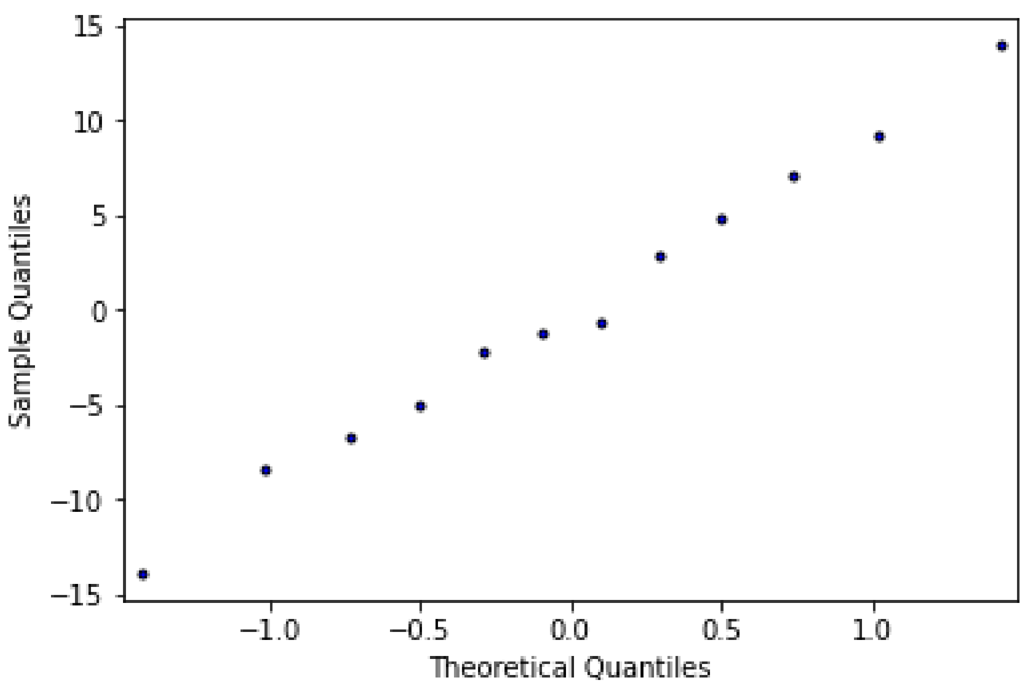
1.1 Exercise 12.7.2

We plot by using Python:

In [2]:
```
# Data loader
data = pd.read_excel('DS 12.2.2.xls')
x = data.iloc[:, 1].to_numpy() # In this case, the columns are exchanged
y = data.iloc[:, 0].to_numpy()

# Data fitting
fit = ols('y~x', data).fit()
res = fit.resid # Obtain residuals

# Plotting
# fig, ax = plt.figure(figsize=(10, 6))
plt.plot(x, res, 'x')
plt.xlabel(r'Temperature')
plt.ylabel(r'$e_i$')
```

```
plt.axhline(linewidth=1, color='r')
plt.title("Residuals scatter plot")
qqplot(res, color='black', markersize=3);
```



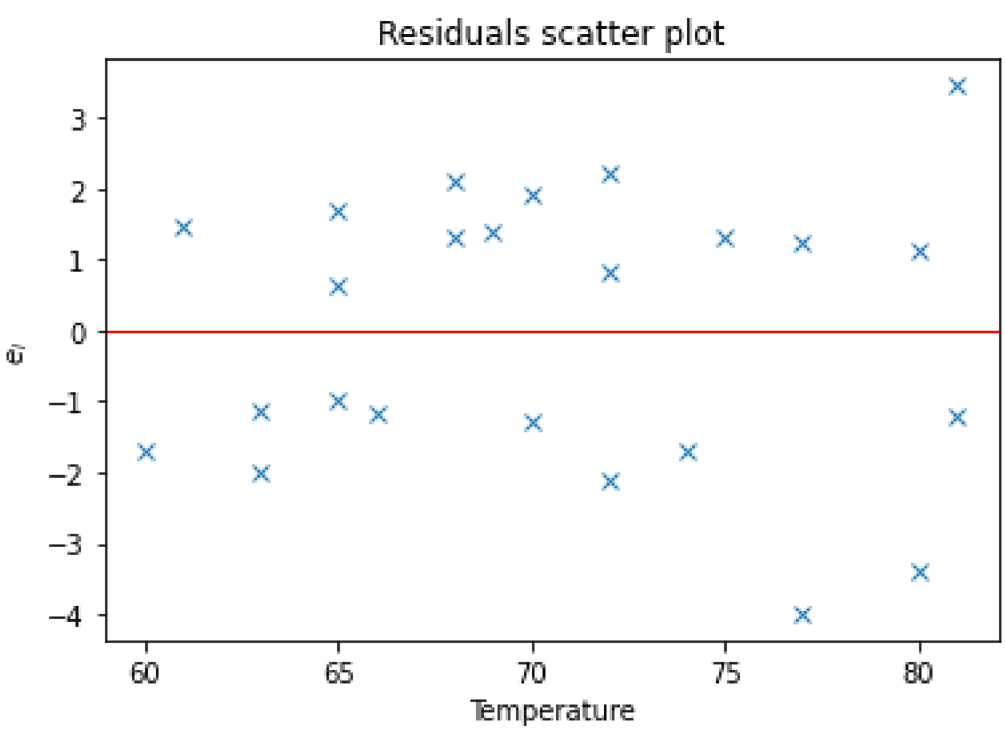


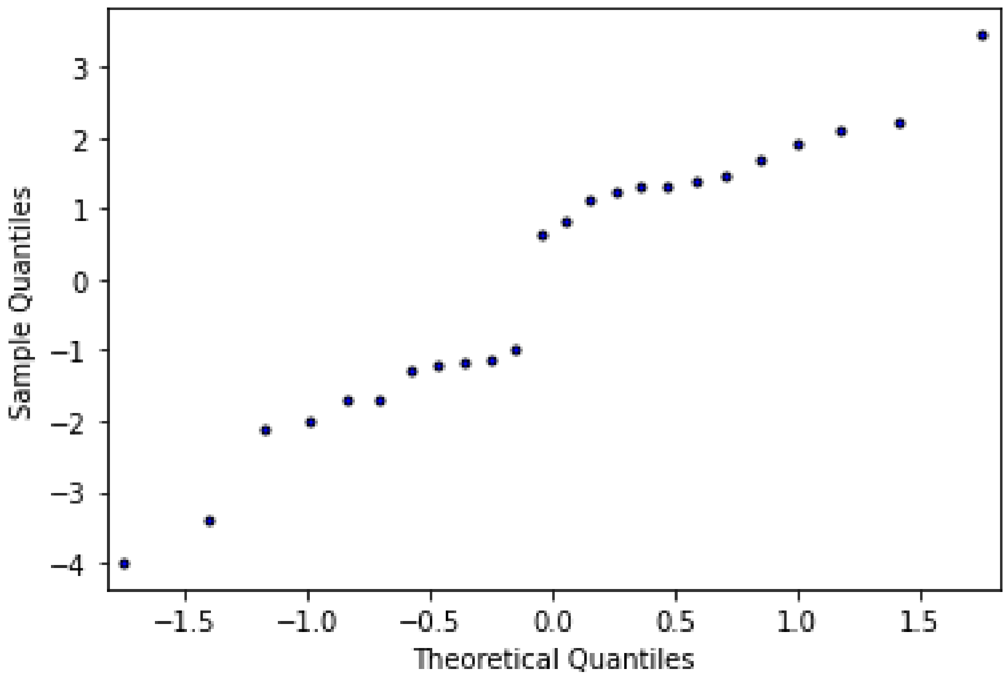
Both the plots of the residuals against the temperature and the normal probability plot do not show any outlier and there is no suggestion that the fitted regression model is not appropriate.

1.2 Exercise 12.7.6

We use Python:

```
In [3]: # Data loader
        data = pd.read_excel('DS 12.2.6.xls')
        x = data.iloc[:, 0].to_numpy()
        y = data.iloc[:, 1].to_numpy()

        # Data fitting
        fit = ols('y~x', data).fit()
        res = fit.resid # Obtain residuals

        # Plotting
        # fig, ax = plt.figure(figsize=(10, 6))
        plt.plot(x, res, 'x')
        plt.xlabel(r'Temperature')
        plt.ylabel(r'$e_i$')
        plt.axhline(linewidth=1, color='r')
        plt.title("Residuals scatter plot")
        qqplot(res, color='black', markersize=3);
```





Higher temperature resistances higher residuals in absolute values; these outliers actually suggest that, also by looking at the normal probability plot, there could be a small increase in the variability of resistance values.

1.3 Exercise 12.8.4

We want to show that $P = \gamma_0 A^{\gamma_1}$ provides a good fit to the data set. We can rewrite the model as:

$$ln(P) = ln\gamma_0 + \gamma_1 lnA$$

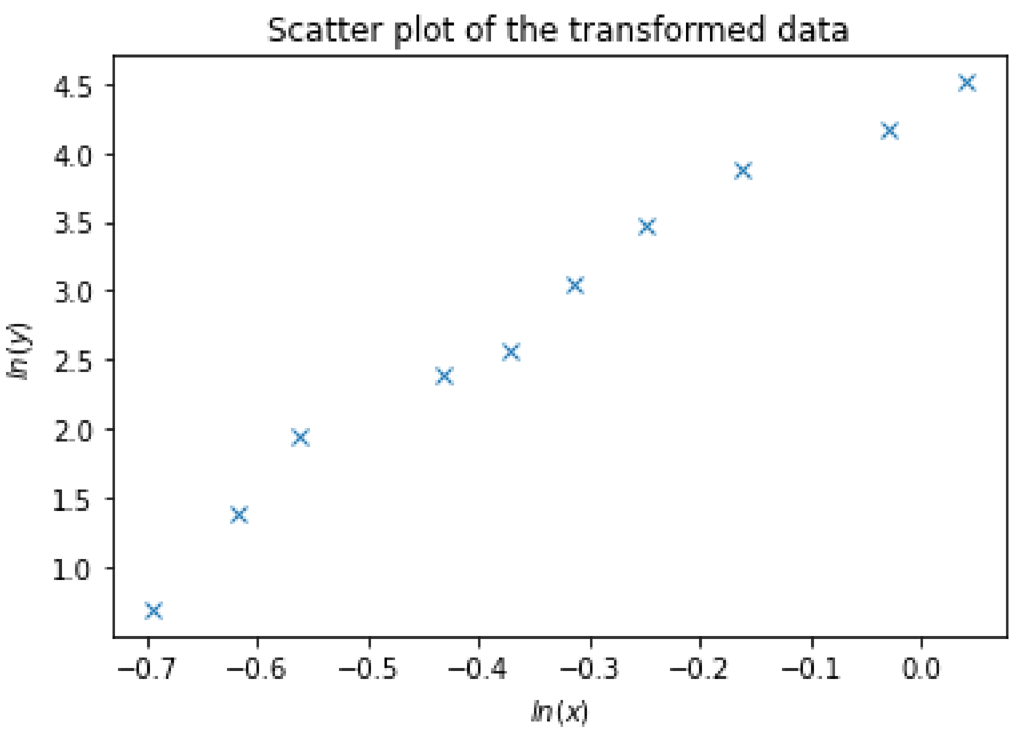
We make suitable transformations by Python:

```
In [4]: # Data loader
        data = pd.read_excel('DS 12.8.4.xls')
        x = data.iloc[:, 1].to_numpy() # Columns are changed
        y = data.iloc[:, 0].to_numpy()

        # We transform the data via a log transformation
        x = log(x)
        y = log(y)

        print(data)
        # Plot for showing the transformed data
        plt.plot(x, y, 'x')
```

```
        plt.xlabel(r'$ln(x)$')
        plt.ylabel(r'$ln(y)$')
        plt.title('Scatter plot of the transformed data')
```

```
   Pressure Differential  Cross-Sectional Area
0                      2                  0.50
1                      4                  0.54
2                      7                  0.57
3                     11                  0.65
4                     13                  0.69
5                     21                  0.73
6                     32                  0.78
7                     48                  0.85
8                     64                  0.97
9                     91                  1.04
```

```
Out[4]: Text(0.5, 1.0, 'Scatter plot of the transformed data')
```



a) As we can see from the plot, the model seems to provide a good fit to the data set.

b) We now fit the data to obtain the parameters:

```
In [5]: # Data fitting
        fit = ols('y~x', data).fit()
        print(fit.summary())
```

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.974
Model:                            OLS   Adj. R-squared:                  0.971
Method:                 Least Squares   F-statistic:                     304.7
Date:                Tue, 08 Dec 2020   Prob (F-statistic):           1.18e-07
Time:                        15:08:08   Log-Likelihood:                 2.5270
No. Observations:                  10   AIC:                            -1.054
Df Residuals:                       8   BIC:                           -0.4488
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
Intercept      4.4967      0.118     38.253      0.000       4.226       4.768
x              4.9931      0.286     17.457      0.000       4.334       5.653
==============================================================================
Omnibus:                        0.785   Durbin-Watson:                   1.201
Prob(Omnibus):                  0.675   Jarque-Bera (JB):                0.638
Skew:                          -0.285   Prob(JB):                        0.727
Kurtosis:                       1.902   Cond. No.                         4.82
==============================================================================

Notes:
[1] Standard Errors assume that the covariance matrix of the errors is correctly specified.


/home/fedebotu/.local/lib/python3.8/site-packages/scipy/stats/stats.py:1603: UserWarning: kurto
  warnings.warn("kurtosistest only valid for n>=20 ... continuing "
```

In this case, $\beta_0 = ln(\gamma_0) = 4.4967$, so:

$$\gamma_0 = e^{4.4967} = 89.72$$

While $\beta_1 = \gamma_1 = 4.9931$ so:

$$\gamma_1 = 4.9931$$

c) We can obtain the 95% confidence interval from the built-in function:

```
In [6]: print(fit.conf_int())
```

```
                  0         1
Intercept  4.225643  4.767790
x          4.333523  5.652685
```

We adjust the intercept result with with:

$$lower = e^{4.225643} = 68.418$$

$$upper = e^{4.767790} = 117.659$$

Therefore, the confidence intervals will be:

$$\gamma_0 \in (68.418, 117.659)$$

$$\gamma_1 \in (4.333, 5.652)$$

1.4 Exercise 12.8.6

We have the following model:

$$e^{y/\gamma_0} = \gamma_1 / x^2$$

If we apply the logarithm, we get:

$$\frac{y}{\gamma_0} = ln(\frac{\gamma_1}{x^2})$$

$$\implies y = \gamma_0 ln(\gamma_1) - 2\gamma_0 ln(x)$$

If we apply the linear regression, we get:

$$\hat{\beta}_0 = \hat{\gamma}_0 ln(\hat{\gamma}_1)$$

$$\hat{\beta}_1 = -2\hat{\gamma}_0$$

by which we get

$$\hat{\gamma}_0 = \frac{\hat{\beta}_1}{2}$$

$$\hat{\gamma}_1 = e^{-2(\frac{\hat{\beta}_0}{\hat{\beta}_1})}$$

1.5 Exercise 12.9.4

Pearson's correlation coefficient can be calculated as:

$$r = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum(x_i - \overline{x})^2 \sum(y_i - \overline{y})^2}} = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

Then, we can calculate both manually and with the Numpy package:

```
In [7]: # Data loader
        data = pd.read_excel('DS 12.2.2.xls')
        x = data.iloc[:, 1].to_numpy() # Columns are changed
        y = data.iloc[:, 0].to_numpy()

        # Covariance
        def cov(X, Y):
            x_bar = X.mean()
            y_bar = Y.mean()
            return sum( (X - x_bar)*(Y - y_bar) )

        # Standard deviation
        def sigma(X):
```

```
        x_bar = X.mean()
        return math.sqrt(sum( (X - x_bar)**2 ))

    r = cov(x, y)/(sigma(x)*sigma(y))
    print('Correlation coefficient (manual calculation): {:.4f}'.format(r))
    print('Correlation coefficient (via Numpy): {:.4f}'.format(np.corrcoef(x, y)[0, 1]))
```

```
Correlation coefficient (manual calculation): 0.3328
Correlation coefficient (via Numpy): 0.3328
```

We want to show that the sample correlation coefficient

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

is equal to the t-statistics

$$t = \frac{\hat{\beta}_1}{s.e.(\hat{\beta}_1)}$$

Let us verify by Python:

```
In [8]: # First method
        n = len(x)
        t_stat_1 = r*math.sqrt(n - 2)/ math.sqrt(1 - r**2)
        print('t-statistics with method 1: {:.4f}'.format(t_stat_1))

        # Second method: we get beta_1 and se(beta_1) from the fitted model
        fit = ols('y~x', data).fit()
        beta_1 = fit.params[1]
        se_beta_1 = fit.bse[1]
        t_stat_2 = beta_1/se_beta_1
        print('t-statistics with method 2: {:.4f}'.format(t_stat_2))
```

```
t-statistics with method 1: 1.1160
t-statistics with method 2: 1.1160
```

And thus we have demonstrated the two statistics are equal

1.6 Exercise 12.9.8

Pearson's correlation coefficient can be calculated as:

$$r = \frac{\sum(x_i - \overline{x})(y_i - \overline{y})}{\sqrt{\sum(x_i - \overline{x})^2 \sum(y_i - \overline{y})^2}} = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$$

Then, we can calculate both manually and with the Numpy package:

In [15]:
```python
# Data loader
data = pd.read_excel('DS 12.2.6.xls')
x = data.iloc[:, 0].to_numpy()
y = data.iloc[:, 1].to_numpy()

# Covariance
def cov(X, Y):
    x_bar = X.mean()
    y_bar = Y.mean()
    return sum( (X - x_bar)*(Y - y_bar) )

# Standard deviation
def sigma(X):
    x_bar = X.mean()
    return math.sqrt(sum( (X - x_bar)**2 ))

r = cov(x, y)/(sigma(x)*sigma(y))
print('Correlation coefficient (manual calculation): {:.4f}'.format(r))
print('Correlation coefficient (via Numpy): {:.4f}'.format(np.corrcoef(x, y)[0, 1]))
```

```
Correlation coefficient (manual calculation): 0.9365
Correlation coefficient (via Numpy): 0.9365
```

We want to show that the sample correlation coefficient

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

is equal to the t-statistics

$$t = \frac{\hat{\beta}_1}{s.e.(\hat{\beta}_1)}$$

Let us verify by Python:

In [10]:
```python
# First method
n = len(x)
t_stat_1 = r*math.sqrt(n - 2)/ math.sqrt(1 - r**2)
print('t-statistics with method 1: {:.4f}'.format(t_stat_1))

# Second method: we get beta_1 and se(beta_1) from the fitted model
fit = ols('y~x', data).fit()
beta_1 = fit.params[1]
se_beta_1 = fit.bse[1]
t_stat_2 = beta_1/se_beta_1
print('t-statistics with method 2: {:.4f}'.format(t_stat_2))
```

```
t-statistics with method 1: 12.5265
t-statistics with method 2: 12.5265
```

And thus we have demonstrated the two statistics are equal

1.7 Exercise 12.12.38

We first need to calculate the F-statistics as:

$$F = \frac{MSR}{MSE} = \frac{(n-2)R^2}{1-R^2} = \frac{18 \times 0.853}{1-0.853} = 104.45$$

We to test $H_0 : \beta_0 = 0$, we first calculate the p-value via Python:

```
In [11]: print('p-value = {:.4f}'.format(f.sf(104.45, 1, n-2)))

p-value = 0.0000
```

The p-value for the two-sided hypothesis test shows that there is no evidence that the value of β_0 is actually equal to 0

1.8 Exercise 12.12.40

a) A $100(1-\alpha)$ confidence interval for a single response $Y_{x_0}^{new}$ can be obtained as following:

$$\hat{y}_{x_0} \pm t_{\frac{\alpha}{2},n-2}\hat{\sigma}\sqrt{1 + \frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}}$$

We can find the two-sided prediction interval in Python:

```
In [12]: # Data
         n = 20
         beta_0 = 123.57
         beta_1 = -3.90
         sigma = 11.52
         sum_xi = 856
         sum_xi2 = 37636
         sum_yi = -869
         sum_yi2 = 55230
         alpha = 0.05
         input_value = 40

         # Calculate mean
         x_bar = sum_xi/n

         # Calculate Sxx
         s_xx = sum_xi2 - (n*x_bar**2)

         # Calculate the critical point
         crit = t.ppf(1-alpha/2, n-2)

         # Calculate the wing span
         wing_span = crit*sigma*math.sqrt(1 + 1/n + (input_value - x_bar)**2/s_xx)
```

```python
# Prediction
pred = beta_0 + beta_1*input_value

# Printing
print('{:.0f}% Confidence interval: ({:.4f}, {:.4f})'.format((1-alpha)*100, pred-wing
```

```
95% Confidence interval: (-57.3228, -7.5372)
```

b) We first manually compute the ANOVA table in Python, and then calculate the coefficient of determination

```python
In [13]: # Calculate the SS
         SST = sum_yi2 - (sum_yi)**2/n
         SSE = sigma**2*(n-2)
         SSR = SST - SSE

         # Calculate MS
         MSR = SSR
         MSE = SSE/(n-2)

         # F-statistics and p-value
         f_stat = MSR/MSE
         p_value = f.sf(f_stat, 1, n-2)

         # Anova table creation
         anova_table = [['Sources', 'Df', 'SS', 'MS', 'F', 'p-value'],
                 ['Regression', 1, SSR, MSR, f_stat, p_value ],
                 ['Error',n-2, SSE, MSE, float('nan'), float('nan') ],
                 ['Total',n-1, SST, float('nan'), float('nan'), float('nan')]]

         # print with tabulate (pip install tabulate)
         print(tabulate(anova_table, headers='firstrow', tablefmt='presto'))

         # Calculate the coefficient of determination
         R2 = SSR/SST
         display(Math('R^2 = {:.0f} \%'.format(R2*100)))
```

```
 Sources    |   Df |       SS |       MS |       F |       p-value
------------+------+----------+----------+---------+---------------
 Regression |    1 | 15083.2  | 15083.2  | 113.655 |   3.30631e-09
 Error      |   18 |  2388.79 |   132.71 | nan     | nan
 Total      |   19 | 17471.9  |   nan    | nan     | nan
```

$R^2 = 86\%$