Homework 2.

(Due Sep. 15)

2.1.8 A fair coin is tossed three times. A player wins \$1 if the first toss is a head, but loses \$1 if the first toss is tail. Similarly, the player wins \$2 if the second toss is a head. but loses \$2 if the second toss is a tail, and wins or loses \$3 according to the result of the third toss. Let the random variable X be the total winnings after the three tosses (possibly a negative value if losses are incurred).

(a) Construct the probability mass function.

(b) Construct the cumulative distribution function.

(c) What is the most likely value of the random variable X?

2.2.10 Sometimes a random variable is a mix of discrete and continuous components. For example, suppose that the dial-spinning game is modified in the following way. First a fair coin is tossed and if a head is obtained, the player wins \$500 and the dial is not spun. However, if a tail is obtained, the player spins the dial and receives winning of

$$\$1000 \times \frac{\theta}{180} \tag{1}$$

as before. In this game there is a probability of 0.5 of winning \$500, with all the other possible winnings between \$0 and \$1000 beging equally likely. The coin toss provides a discrete element to the winnings, and the dial spin provides a continuous element. The best way to describe the probabilistic properties of *mixed* random variables such as this through a cumulative distribution function. The cumulative distribution function of the winnings from this game is given in Figure 2.32.

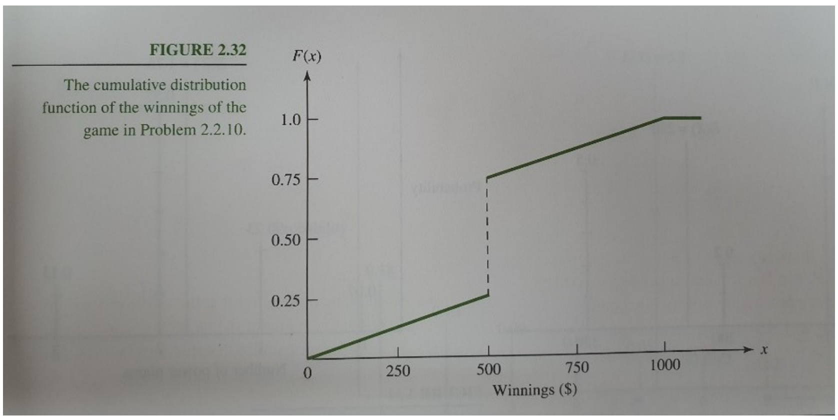

FIGURE 2.32 The cumulative distribution function of the winnings of the game in Problem 2.2.10.

(a) What is the probability of winning less than \$ 200?

(b) What is the probability of winning between \$400 and \$700?
Interpret your answers.

2.3.14 Consider again the archery problem discussed in Problem 2.2.9. What is the expected deviation from the center of the target? What is the median deviation? (You have to derive A and B in the problem 2.2.9 by yourself.)

2.2.9 [Not a homework problem] An archer shoots an arrow at a circular target with a radius of 50 cm. If the arrow hits the target, the distance r between the point of impact and the center of the target is measured. Suppose that this distance has a cumulative distribution function

$$F(r) = A + \frac{B}{(r+5)^3} \tag{2}$$

for $0 \leq r \leq 50$.

2.4.8 Consider again the plastic bending capabilities discussed in Problems 2.2.8 and 2.3.13. (You have to find A in Problem 2.2.8 by yourself.)

(a) What is the variance of the deformity angle?

(b) What is the standard deviation of the deformity angle?

(c) Find the upper and lower quartiles of the deformity angle.

(d) What is the interquartile range?

2.2.8 [Not a homework problem] The bending capabilities of plastic sheets are investigated by bending sheets at increasingly large angles until a deformity appears in the sheet. The angle θ at which the deformity first appears is then recorded. Suppose that this angle takes values between $0°$ and $10°$ with a probability density function

$$f(\theta) = A(e^{10-\theta} - 1) \tag{3}$$

for $0 \leq \theta \leq 10$ and $f(\theta) = 0$ elsewhere.

2.5.2 A fair coin is tossed four times, and the random variable X is the number of heads in the *first three* tosses and the random variable Y is the number of heads in the *last three tosses*.

(a) What is the joint probability mass function of X and Y?

(b) What are the marginal probability mass functions of X and Y?

(c) Are the random variables X and Y independent?

(d) What are the expectations and variances of the random variables X and Y?

(e) What is the covariance of X and Y?

(f) If there is one head in the last three tosses, what is the conditional probability mass function of X? What are the conditional expectation and variance of X?

2.6.4 A person's cholesterol level C can be measured by three different tests. Test-α returns a value X_α with a mean C and a standard deviation 1.2, test-β returns a value X_β with a mean C and a standard deviation 2.4, and test-γ returns a value X_γ with a mean C and a standard deviation 3.1. Suppose that the three test results are independent. If a doctor decides to use the weighted average $0.5X_\alpha + 0.3X_\beta + 0.2X_\gamma$ what is the standard deviation of the cholesterol level obtained by the doctor?

2.6.16 An object has a weight W. When it is weighed with machine 1, a value X_1 is obtained. When it is weighed with machine 2, a value X_2 is obtained. The value X_1 has a mean W and a standard deviation 3. The value X_2 has a mean W and a standard deviation 4. The values X_1 and X_2 are independent.

(a) Suppose that an engineer uses the value $A = (X_1+X_2)/2$ to estimate the weight. what are the expectation and the standard deviation of A?

(b) Suppose that an engineer uses the value $B = \delta X_1 + (1 - \delta)X_2$ to estimate the weight. What value of δ gives an estimate B with the smallest standard deviation? What is this smallest standard deviation?

9.6 Bricks have weights that have a mean 250 and a standard deviation 4.

(a) Suppose X is the weight of a randomly chosen brick. Let $Y = c+dX$. What are the values of c and d such that Y has a mean 100 and a standard deviation 1.

(b) Suppose 10 bricks are chosen at random. What are the mean and the standard deviation of the total weight of these 10 bricks?

9.16 A continuous random variable has a probability density function

$$f(x) = Ax \tag{4}$$

for $5 \leq x \leq 6$.

(a) What is the value of A?

(b) What is the cumulative distribution function of the random variable?

(c) What is the expectation of the random variable?

(d) What is the standard deviation of the random variable?

9.24 Suppose that the random variables X and Y have a joint probability density function

$$f(x, y) = 4x(2 - y) \tag{5}$$

for $0 \leq x \leq 1$ and $1 \leq y \leq 2$.

(a) What is the marginal probability density function of X?

(d) Are the random variables X and Y independent?

(c) What is $\mathrm{Cov}(X, Y)$?

(d) What is the probability density function of X conditional on $Y = 1.5$?