HOMEWORK 5 - PROBABILITY AND STATISTICS [CC511]

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1.6 ① $P(X \le 5) = 0.8 \to \Phi\left(\frac{5-\mu}{\sigma}\right) = 0.8$

② $P(X \ge 0) = 0.6 \Rightarrow = 1 - \Phi\left(\frac{0-\mu}{\sigma}\right) = 0.6 \to \Phi\left(\frac{0-\mu}{\sigma}\right) = 0.4$

Looking at the tables, the values yield to:

$$\begin{cases} \frac{5-\mu}{\sigma} = 0.8416 \\ \frac{0-\mu}{\sigma} = -0.2533 \end{cases} \Rightarrow \begin{cases} \boxed{\mu = 1.1567 \\ \sigma = 4.5666} \end{cases}$$

1.8 We use the online tables to obtain the results:

- Lower quartile "Q_1": $\Phi(x) = 0.25 \to \boxed{x = -0.6745}$
- Upper quartile "Q_3": $\Phi(x) = 0.75 \to \boxed{x = 0.6745}$
- Interquartile range $N(0,1)$ $0.6745 - (-0.6745) = 1.3490 = \Phi(x)$

If $N(\mu, \sigma^2) \Rightarrow x = \frac{z-\mu}{\sigma} \overset{Q_1}{\Rightarrow} z_1 - \mu = -0.6745 \times \sigma$

Q_3: $z_3 - \mu = 0.6745 \times \sigma$

$\Rightarrow Q_3 - Q_1 = (z_3 - \cancel{\mu}) - (z_1 - \cancel{\mu}) = 0.6745\sigma - (-0.6745\sigma) = \boxed{1.3490 \times \sigma}$

1.16 We have a standard Normal distribution with:

$$N(\mu, \sigma^2) = N(1320, 15^2)$$

Thus:

1) $P(N(1320, 15^2) \le 1300) \Rightarrow P\left(N(0,1) \le \frac{1300-1320}{15}\right)$

$= \boxed{P\left(N(0,1) \le -\frac{4}{3}\right)} = \boxed{0.0912}$ "P_1"

Each one is e.g.: $P(N(0,1) \le -\frac{4}{3}) = \Phi(-\frac{4}{3})$

2) $P(1300 \le N(1320, 15^2) \le 1330) \Rightarrow P\left(N(0,1) \le \frac{1330-1320}{15}\right) - P_1 = \quad = P_2$

$= 0.7475 - 0.0912 = \boxed{0.6563}$

3) $P(N(1320, 15^2) \ge 1330) = 1 - P_2 - P_1 = \boxed{0.2525} = P_3$

Multinomial distribution:

$$P(3, 4, 3, P_1, P_2, P_3) = \left(\frac{10!}{3!\,4!\,3!}\right)(0.0912)^3 \cdot (0.6563)^4 (0.2525)^3 =$$

$$= \boxed{0.0095}$$

$\boxed{2.12}$ $N(8, 2^2) \Rightarrow$ Given independent Times for The students, Then :

(a) $X \sim N(8, 2^2)$; $Y = 5X \Rightarrow N(8+8+8+8+8, 2^2+2^2+2^2+2^2+2^2)$
$= N(40, 5\cdot 2^2)$

$\Rightarrow P(N(40, 5\cdot 2^2) \geq 45) =$
$= 1 - P\left(N(0,1) \leq \frac{45-40}{\sqrt{5\cdot 2^2}}\right) = 1 - \Phi(1.118) = [0.1318]$

(b) Probability to have a response $\sim N(28, 5^2)$

$\rightarrow P(N(28, 5^2) \geq N(8+8+8, 2^2+2^2+2^2)) \Rightarrow$ we can create a single distribution by summing The previous two

$= P(N(28-24, 25+12) \geq 0)$

$= P(N(4, 37) \geq 0)$

$\rightarrow P\left(N(0,1) \geq \frac{-4}{\sqrt{37}}\right) = 1 - \Phi\left(\frac{-4}{\sqrt{37}}\right)$
$= 1 - \Phi(-0.6576) = [0.7446]$

$\boxed{2.14}$ $X \sim N(-1.9, 2.2)$ $Y \sim N(3.3, 1.7)$, $Z \sim N(0.8, 0.2)$

(a) $X - Y \geq -3$; $P(N(-1.9-3.3, 2.2+1.7) \geq -3)$
$\rightarrow P\left(N(0,1) \geq \frac{2.2}{\sqrt{3.9}}\right) = 1 - \Phi(1.1140) = [0.1326]$

(b) $2X+3Y+4Z \leq 10$; $\mu = 2\cdot(-1.9) + 3(3.3) + 4\cdot(0.8) = 9.3$
$\sigma^2 = 2^2\cdot 2.2 + 3^2\cdot 1.7 + 4^2\cdot 0.2 = 27.3$

$\rightarrow P(N(9.3, 27.3) \leq 10) = P\left(N(0,1) \leq \frac{0.7}{\sqrt{27.3}}\right) = [0.55$

(c) $3Y - Z \leq 8$; $P(N(3\cdot 3.3 - 0.8, 3^2\cdot 1.7 + .2) \leq 8)$.
$\Rightarrow P(N(9.1, 15.5) \leq 8) = P\left(N(0,1) \leq \frac{-1.1}{\sqrt{15.5}}\right) = [0.3900]$

(d) $2X - 2Y + 3Z \leq -6$; $P(N(2\cdot(-1.9) - 2\cdot(3.3) + 3\cdot 0.8), 2^2\cdot 2.2 + 2^2\cdot 1.7 + 3^2\cdot 0.2) \leq -6)$
$\Rightarrow P(N(-8, 17.4) \leq -6$; $P\left(N(0,1) \leq \frac{2}{\sqrt{17.4}}\right) = [0.6842]$

(e) $|X+Y-Z| \geq 1.5$; $P(|N(-1.9+3.3-0.8, 2.2+1.7+0.2)| \geq 1.5$
$= P(|N(0.6, 4.1)| \geq 1.5) = P(N(0.6, 4.1) \geq 1.5) + P(N(0.6, 4.1) \leq -1.5$

$= P\left(N(0,1) \geq \frac{0.9}{\sqrt{5.1}}\right) + P\left(N(0,1) \leq \frac{-2.1}{\sqrt{5.1}}\right) = 0.3284 + 0.1498$
$= \boxed{0.4782}$

(f) $|4X - Y + 10| \leq 0.5$; $P(|N(4\cdot(-1.8) - 3.3 + 10, 4^2 \quad +1.7)| \leq 0.5$;
$P(|N(-0.9, 36.9)| \leq 0.5|) = P(-0.5 \leq N(-0.9, 36.9) \leq 0.5) =$
$= P\left(N(0,1) \leq \frac{1.4}{\sqrt{36.9}}\right) - P\left(N(0,1) \leq \frac{0.4}{\sqrt{36.9}}\right) = 0.5911 - 0.5263 =$
$= \boxed{0.0648}$

$\boxed{3.4}$ (a) We have that $X \sim B(n,p)$ and $Z \sim N(0,1)$, $\begin{cases} n = 60 \\ p = \frac{1}{4} \\ q = \frac{3}{4} \end{cases}$

so, $P(X \geq 30) \approx P\left(Z \geq \frac{30 - 0.5 - 60\cdot\frac{1}{4}}{\sqrt{60\cdot\frac{1}{4}\cdot\frac{3}{4}}}\right)$

$= P(Z \geq 4.3231) \approx \boxed{3.4882\cdot 10^{-5}} \rightarrow$ extremely low !!!

(b) 99% confident means → 0.99 probability that X success is less or equal than 0.35. Given $Z \sim N(0,1)$, we need to find n = number of questions. X will be $\leq 0.35n$

$\rightarrow P(X \leq 0.35n) \approx P\left(Z \leq \frac{0.35n + 0.5 - n\cdot\frac{1}{4}}{\sqrt{n\cdot\frac{1}{4}\cdot\frac{3}{4}}}\right) = 0.99$

$\Rightarrow$ we solve the equation finding Z.
It is satisfied for $\boxed{n \geq 92.}$

$\boxed{3.8}$ (a) $n = 1000$ probability $X = 6 \rightarrow \frac{1}{6}$

$P(150 \leq X \leq 180)$ given $P \sim B(1000, \frac{1}{6})$. Thus,

$= P\left(Z \leq \frac{180 + 0.5 - 1000\cdot\frac{1}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}}\right) - P\left(Z \leq \frac{149 + 0.5 - 1000\cdot\frac{1}{6}}{\sqrt{1000\cdot\frac{1}{6}\cdot\frac{5}{6}}}\right) =$

$= P(Z \leq 1.1738) - P(Z \leq -1.4566) = 0.8798 - 0.0726 =$
$= \boxed{0.8072}$

(b) $P(X \geq 50)$ given $P \sim B(n, \frac{1}{6})$

$\hookrightarrow P(X \geq 50) = 0.99 \longrightarrow \Phi\left(\frac{50 - 0.5 - n\cdot\frac{1}{6}}{\sqrt{n\cdot\frac{1}{6}\cdot\frac{5}{6}}}\right) = 0.99$

$\Rightarrow 2.3263 = \frac{49.5 - \frac{n}{6}}{\sqrt{\frac{5n}{36}}} \Rightarrow 0.7516n = \frac{n^2}{36} + 2450.25 - 16.5n$

$$\Rightarrow \frac{n^2}{36} - 17.2516n + 2450.25 = 0$$

$$\to \frac{17.2516 \pm \sqrt{25.395}}{(2/36)} = 401.23 \Rightarrow$$

which is satisfied for $\boxed{n \geq 402}$

3.16

a) • Exact calculation:

$$P(x \geq 7) = \binom{10}{7} 0.3^7 0.7^3 + \binom{10}{8} 0.3^8 0.7^2 + \binom{10}{9} 0.3^9 0.7 + \binom{10}{10} 0.3^{10} 0.7^0 = \boxed{0.0106}$$

• Normal approximation: $1 - \Phi\left(\frac{7 - 0.5 - 10 \cdot 0.3}{\sqrt{10 \cdot 0.3 \cdot 0.7}}\right) =$

$$= 1 - \Phi(2.4152) = 1 - 0.9921 = \boxed{0.0079}$$

b) From now on, let's avoid writing every single step:

• Exact calculation: $P(9 \leq x \leq 12) = \boxed{0.6160}$

• Normal approximation: $\Phi\left(\frac{12 + 0.5 - 21 \cdot 0.5}{\sqrt{21 \cdot 0.5 \cdot 0.5}}\right) - \Phi\left(\frac{8 + 0.5 - 21 \cdot 0.5}{\sqrt{21 \cdot 0.5 \cdot 0.5}}\right) =$

$$= \boxed{0.6172}$$

c) • Exact: $P(x \leq 3) = \boxed{0.9667}$

• Normal approximation $= \Phi\left(\frac{3 + 0.5 - 7 \cdot 0.2}{\sqrt{7 \cdot 0.2 \cdot 0.8}}\right) = \boxed{0.9764}$

d) Exact: $P(9 \leq x \leq 11) = \boxed{0.3410}$

• Normal approximation: $\Phi\left(\frac{11 + 0.5 - 12 \cdot 0.65}{\sqrt{12 \cdot 0.65 \cdot 0.35}}\right) - \Phi\left(\frac{8 + 0.5 - 12 \cdot 0.65}{\sqrt{12 \cdot 0.65 \cdot 0.35}}\right)$

$$= \boxed{0.3233}$$