PROBABILITY AND STATISTICS - HOMEWORK 4

Berto Federico - 2020 4817

Ex 4.1.4 (a) $F(x) = \frac{1}{b-a}x$ where $b = 100$, $a = 60$

$\rightarrow P(X \leq 85) = \frac{85-60}{100-60} = \frac{5}{8}$

so, $P(B(6, \frac{5}{8}) = 3) = \binom{6}{3}\left(\frac{5}{8}\right)^3 \cdot \left(1-\frac{5}{8}\right)^3 = \boxed{0.2575}$

(b) $P(X \leq 80) = \frac{80-60}{100-60} = \frac{1}{2}$

$P(80 \leq x \leq 90) = \frac{90-80}{100-60} = \frac{1}{4}$

$P(X \geq 90) = \frac{100-90}{100-60} = \frac{1}{4}$

$\Rightarrow$ multinomial distribution

P(2 score less than 80, 2 between 80 and 90, 2 more than 90) =

$= \frac{6!}{2!2!2!} * \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{4}\right)^2 \cdot \left(\frac{1}{4}\right)^2 = \boxed{0.0879}$

Ex 4.2.6 (a) $E(x) = \frac{1}{\lambda} = \frac{1}{2m^{-1}} = \boxed{0.5\,m}$

(b) $P(X \geq 1) = 1 - F(1) = \cancel{1} - (\cancel{1} - e^{-2 \cdot 1}) = e^{-2} = \boxed{0.1353}$

(c) Distribution for 3 meters: given $\lambda = 2m^{-1}$ for a meter, then $\lambda' = 3 \cdot \lambda = 3 \cdot 2 = 6$

The distribution becomes: $\boxed{F(x) = 6 \cdot e^{-6x}}$

(d) $P(X \leq 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) =$

given the distribution: $F(x) = \frac{e^{-\lambda}\lambda^x}{x!}$

$= \frac{e^{-6}6^0}{0!} + \frac{e^{-6}6^1}{1!} + \frac{e^{-6}6^2}{2!} + \frac{e^{-6}6^3}{3!} + \frac{e^{-6}6^4}{4!} = \boxed{0.2851}$

Ex 4.2.12 $\lambda = 0.31 \rightarrow F(x;\lambda) = 0.31\,e^{-0.31x}$

(a) $P(X \leq 5) = 1 - e^{-0.31 \cdot 5} = \boxed{0.7878}$

(b) Expectation and variance: with binomial distribution

$n = 12$, $p = 0.7878$

$\Rightarrow E(X) = np = 12 \cdot 0.7878 = \boxed{9.454}$

$Var(X) = np(1-p) = 12 \cdot 0.7878(1-0.7878) = \boxed{2.006}$

(c) $P(B(12, 0.7878) \leq 9) = 1 - [P(X=10) + P(X=11) + P(X=12)] =$

$= 1 - \left[\binom{12}{10} 0.7878^{10}(1-0.7878)^2 + \binom{12}{11} 0.7878^{11} \cdot (1-0.7878)^1 + \binom{12}{12} \cdot 0.7878^{12} \cdot (1-0.7878)^0\right] = \boxed{0.4845}$

Ex 4.3.6

(a) The distribution is a Gamma one, with parameters $k=3$ and $\lambda=1.8$.

$\Rightarrow f(x) = \dfrac{1.8^3 \cdot x^2 \cdot e^{-1.8x}}{\Gamma(3)}$ $\qquad \Gamma(3) = (3-1)! = 2$

(b) $E(x) = \dfrac{k}{\lambda} = \dfrac{3}{1.8} = \boxed{1.6667}$

(c) $var(x) = \dfrac{k}{\lambda^2} = \dfrac{3}{1.8^2} = \boxed{0.9259}$

(d) • probability with the gamma distribution:

$P(x \geq 3) = 1 - P(x<3) = 1 - \int_0^3 \dfrac{1.8^3}{2!} \cdot x^2 e^{-1.8x} \, dx =$

$= 1 - 0.9052 = \boxed{0.0948}$ → solved via integration by parts

• probability with the poisson distribution:

Rate occurrence of events in 3 hours: $3 \cdot \lambda = 3 \cdot 1.8 = 5.4 = \lambda'$

we want the number of events to be more than 3 in this time. so:

$P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - [P(Y=0) + P(Y=1) + P(Y=2)] =$

$= 1 - \left[e^{-5.4}\left(\dfrac{5.4^0}{0!} + \dfrac{5.4^1}{1!} + \dfrac{5.4^2}{2!}\right)\right] = \boxed{0.0948}$

Ex 4.4.8 $\quad a=3, \lambda=0.5$

(a) Median failure time: $F(x) = 0.5$

So: $\int_0^x a\lambda(\lambda x)^{a-1} e^{-(\lambda x)^a} dx = 0.5$

$= 1 - e^{-(0.5x)^3} = 0.5 \Rightarrow x = \dfrac{-\ln(0.5)^{1/3}}{0.5} = \boxed{1.77}$

(c) $E(x) = \dfrac{1}{0.5} \Gamma\left(1+\dfrac{1}{3}\right) = \boxed{1.786}$

$var(x) = \dfrac{1}{0.5^2}\left[\Gamma\left(1+\dfrac{2}{3}\right) - \Gamma\left(1+\dfrac{1}{3}\right)^2\right] = \dfrac{1}{0.5^2}\left[0.9027 - 0.893^2\right] = \boxed{0.421}$

(d) P (at least one circuit working after three hours): $P(x \geq 3) = 1 - P(x \leq 3) =$

$= 1 - F(3) = e^{-(0.5x)^3} = 0.0342$

we have 1 circuit + 3 backup ones. So, to find the probability of least one circuit is still working, we can set:

$P(x \geq 1) = 1 - P(x=0)$ where $P(x=0) = P(B(x=0))$

$\Rightarrow = 1 - \binom{4}{0}(0.0342)^0 (1-0.0342)^4 = 1 - 0.9658^4 = \boxed{0.13}$

4.5.6 given $f(x) = A x^9 (1-x)^3$

we know the Beta distribution to be:

$$f(x; a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1} (1-x)^{b-1} \quad \text{for } 0 \le x \le 1$$

Therefore:

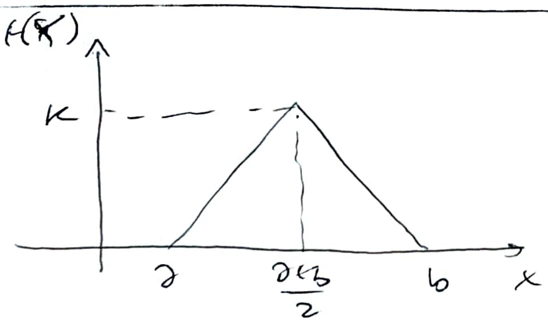
a) $a - 1 = 9 \rightarrow$ $a = 10$
$b - 1 = 3 \rightarrow$ $b = 4$

b) $A = \dfrac{\Gamma(10+4)}{\Gamma(10) \cdot \Gamma(4)}$. knowing that $\Gamma(n) = \Gamma(n-1)!$, then:

$$A = \frac{13!}{9!\,4!} = [2860]$$

c) $E(x) = \dfrac{a}{a+b} = \dfrac{10}{10+4} = \dfrac{5}{7} = [0.7143]$

4.8.5

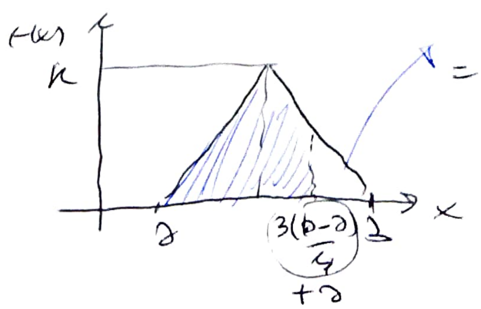


a) we want to find the height k at $x = \frac{a+b}{2}$. Given that $T(a,b)$ is a triangle, the area below it must be $= 1$. So,

$$A = \frac{(b-a)k}{2} = 1 \Rightarrow \left[k = \frac{2}{b-a}\right]$$

b) $P\left(X \le \frac{a}{4} + \frac{3b}{4}\right) = P\left(X \le \frac{a}{4} + 3\frac{(a+b-a)}{4}\right) = P\left(X \le a + \frac{3(b-a)}{4}\right)$

So, this probability is: $1 - P\left(X \ge \frac{3(b-a)}{4}\right)$ +a, but contribution of a = 0



$$= 1 - \frac{\frac{(b-a)}{4} \cdot \frac{k}{2}}{2} = 1 - \frac{\frac{(b-a)}{4} \cdot \frac{2}{(b-a)2}}{2} =$$

$1/8$

$$= 1 - \frac{1}{8} = \left[\frac{7}{8}\right]$$

c) Variance

$Var(X) = E(X^2) - (E(X))^2$

$E(X) = \frac{a+b}{2}$ being the distribution simmetric

$E(x)^2 = \left(\frac{a+b}{2}\right)^2$

For $E(x^2)$, we have to calculate the function, which is:

$$T(x; a, b) = \begin{cases} \frac{4}{(b-a)^2}(x-a) & a \le x \le \frac{a+b}{2} \\ -\frac{4}{(b-a)^2}(x-b) & \frac{a+b}{2} \le x \le b \end{cases}$$

$$E(x^2) = \int_a^{\frac{a+b}{2}} x^2 \left(\frac{4}{(b-a)^2}\right)(x-a)\,dx + \int_{\frac{a+b}{2}}^{b} -\frac{x^2 \cdot 4}{(b-a)^2}(x-b)\,dx =$$

$$= \frac{7a^2+7b^2+10ab}{24}$$

$$\Rightarrow var(X) = \frac{7a^2+7b^2+10ab}{24} - \frac{(a+b)^2}{4} = \boxed{\frac{(b-a)^2}{24}}$$

(d) cdf first part, $\left[a \le x \le \frac{a+b}{2}\right]$

$$\Rightarrow F(x) = \int_a^x \frac{4}{(b-a)^2}(x-a)\,dx = \boxed{\frac{2(x-a)^2}{(b-a)^2}}$$

second part, $\left[\frac{a+b}{2} \le x \le b\right]$

$$\Rightarrow F(x) = 1 - \int_x^b \frac{-4}{(b-a)^2}(x-b)\,dx = \boxed{1 - \frac{2(x-b)^2}{(b-a)^2}}$$

$\boxed{4.8.8}$ weibull distribution with $\lambda = 0.09$ and $a = 2.3$

- $P(X \le 8) = F(8) = 1 - e^{-(\lambda x)^a} = 1 - e^{-(0.09 \cdot 8)^{2.3}} = 0.3748 = P_1$
- $P(8 < X < 12) = F(12) - F(8) = 1 - e^{-(0.09 \cdot 12)^{2.3}} - 0.3748 = 0.3220 = P_2$
- $P(X \ge 12) = 1 - P(X \le 8) - P(8 < X < 12) = 1 - 0.3748 - 0.3032 = 0.3032 = P_3$

$\Rightarrow$ we can model this with a ~~binomial~~ multinomial distribution with $n = 10$:

$$F(x_1 = 3, x_2 = 4, x_3 = 3;\ P_1, P_2, P_3) =$$

$$= \frac{10!}{3!\,4!\,3!} \cdot 0.3748^3 \cdot 0.322^4 \cdot 0.3032^3 = \boxed{0.0667}$$