Appendices

Some useful properties of the Laplace transform

$$
\begin{aligned}
L\left(\alpha x(t)+\beta y(t)\right) =& \alpha X(s)+\beta Y(s)\\
L\left(x(t)*y(t)\right) =& X(s)\times Y(s)\\
L\left(t\,x(t)\right) =& -\frac{dX(s)}{ds}\\
L\left(x(t-t_0)\right) =& e^{-t_0 s}X(s)\\
L\left(e^{-at}x(t)\right) =& X(s+a)\\
L\left(\dot{x}(t)\right) =& sX(s)-x(0)\\
L\left(\ddot{x}(t)\right) =& s^2X(s)-sx(0)-\dot{x}(0)\\
L\left(\int_0^t x(\tau)d\tau\right) =& \frac{X(s)}{s}\\
\lim_{t\to 0+} x(t) =& \lim_{s\to+\infty} sX(s)\\
\lim_{t\to+\infty} x(t) =& \lim_{s\to 0} sX(s) \quad \text{if the limit exists}
\end{aligned}
$$

Some Laplace transform pairs

Signal	Laplace transform
$\delta(t)$	1
$\Gamma(t)$	$\frac{1}{s}$
$r(t)=t\Gamma(t)$	$\frac{1}{s^2}$
$t^2\Gamma(t)$	$\frac{2}{s^3}$
$e^{-at}\Gamma(t)$	$\frac{1}{s+a}$
$t^n e^{-at}\Gamma(t)$	$\frac{n!}{(s+a)^{n+1}}$
$\cos(\omega_0 t)\Gamma(t)$	$\frac{s}{s^2+\omega_0^2}$
$\sin(\omega_0 t)\Gamma(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at}\cos(\omega_0 t)\Gamma(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin(\omega_0 t)\Gamma(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

A few important transfer functions

First-order systems

$$G(s) = \frac{K}{1+Ts}$$

The 2 characteristic parameters of a first order system are:

- K: steady-state gain
$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)} = \lim_{s \to 0} G(s)$$
- T: (system) time-constant (link with the cut-off frequency $\omega_c = \frac{1}{T}$)

Characteristic values of a first-order system step response

Rise-time at 63%	$T_m^{63\%} = T$
Rise-time at 95%	$T_m^{95\%} \approx 3T$
Settling-time at 5 %	$T_r^{5\%} \approx 3T$

Second-order systems

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2} + 2\frac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

The 3 characteristic parameters of a second-order system are:

- K: steady-state gain
- z : damping coefficient ($z > 0$)
- ω_0 : undamped natural frequency

Characteristic values of a underdamped second-order system step response

Value of the first overshoot in %	$D_{1\%} = \dfrac{y(T_{D_1}) - y(+\infty)}{y(+\infty) - y(0)} \times 100 = e^{\frac{-\pi z}{\sqrt{1-z^2}}} \times 100$
Time-instant of the first overshoot	$T_{D_1} = \dfrac{\pi}{\omega_0\sqrt{1-z^2}}$
Value of the n^{th} overshoot in %	$D_{n\%} = -(-D_1)^n \times 100$
Time-instant of the n^{th} overshoot	$T_{D_n} = n\, T_{D_1}$
Pseudo-period	$T_p = \frac{2\pi}{\omega_0\sqrt{1-z^2}}$
Settling-time at 5 %	$T_r^{5\%} \approx \dfrac{3}{\omega_0 z}$
Settling-time at x %	$T_r^{x\%} = \dfrac{\ln\left(\dfrac{100}{x\sqrt{1-z^2}}\right)}{\omega_0 z}$
Rise-time (100%)	$T_m^{100\%} = \dfrac{\pi - \mathrm{acos}(z)}{\omega_0\sqrt{1-z^2}}$

Model identification from step responses

Identification of a first-order model

$$G(s) = \frac{K}{1 + Ts}$$

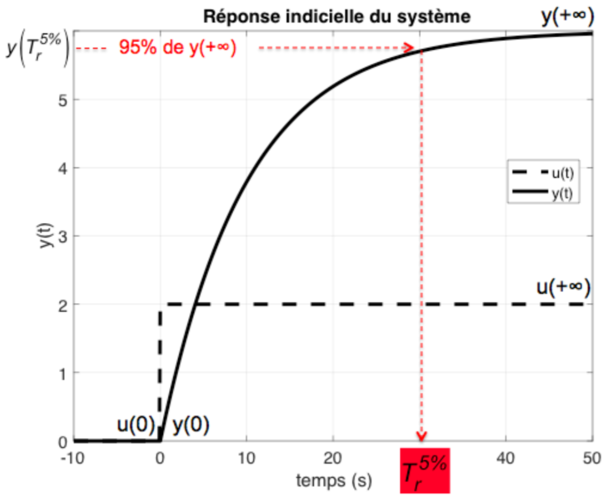

Figure 4.2: Step response of a first-order system

From the step response displayed in Figure 4.2, it is necessary to determine the steady-state gain K and the time constant T. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce K from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find $y(T_r^{5\%})$, deduce from it $T_r^{5\%}$ then T:

$$T = \frac{T_r^{5\%}}{3}$$

Identification of a second-order underdamped model

$$G(s) = \frac{K}{\frac{s^2}{\omega_0^2} + 2\frac{z}{\omega_0}s + 1} = \frac{K\omega_0^2}{s^2 + 2z\omega_0 s + \omega_0^2}$$

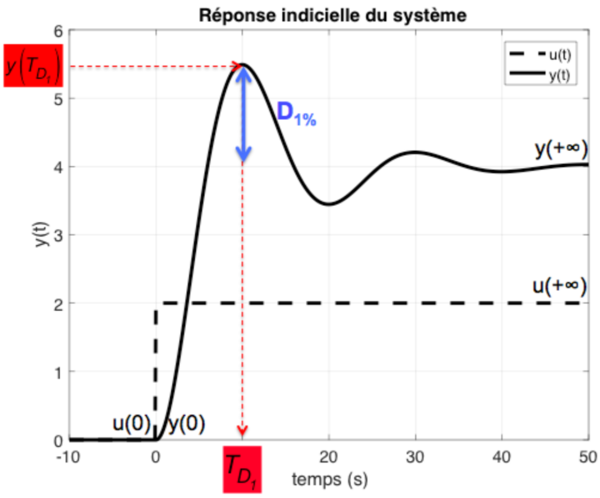

Figure 4.3: Step response of a second-order system

From the step response displayed in Figure 4.3, it is necessary to determine the steady-state gain K, the damping coefficient z and the undamped natural frequency ω_0. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce K from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find the final and initial values of the response and that of the first overshoot $y(t_{D_1})$. Deduce from it D_1, then z:

$$D_1 = \frac{y(T_{D_1}) - y(+\infty)}{y(+\infty) - y(0)}$$

$$z = \sqrt{\frac{(\ln(D_1))^2}{(\ln(D_1))^2 + \pi^2}}$$

3. Find the time-instant of the first overshoot T_{D_1}. Deduce from it ω_0:

$$\omega_0 = \frac{\pi}{T_{D_1}\sqrt{1 - z^2}}$$

Identification of a first-order plus time-delay model by the Broïda method

Broïda has suggested to approximate the underdamped step response of any n-th order system by a first-order plus time-delay model

$$G(s) = \frac{Ke^{-\tau s}}{1+Ts}$$

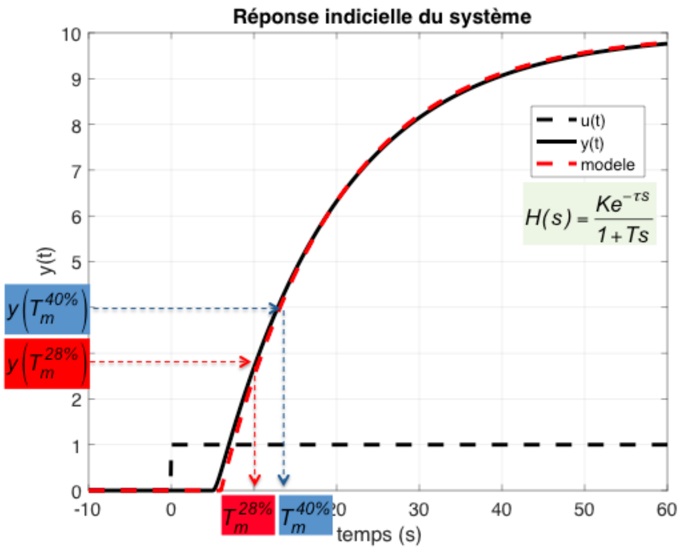

Figure 4.4: Underdamped step response of any n-th order system approximated by a first-order plus time-delay model

From the step response displayed in Figure 4.4, it is necessary to determine the steady-state gain K, the time-constant T and the pure time-delay τ. The procedure is as follows:

1. Find the final and initial values of the response and of the step. Deduce K from:

$$K = \frac{y(+\infty) - y(0)}{u(+\infty) - u(0)}$$

2. Find $y(T_m^{28\%})$ and $y(T_m^{40\%})$, deduce $T_m^{28\%}$ and $T_m^{40\%}$ then calculate :

$$\tau = 2,8T_m^{28\%} - 1,8T_m^{40\%}$$
$$T = 5,5\left(T_m^{40\%} - T_m^{28\%}\right)$$

English to French glossary

bandwidth	:	bande passante
closed-loop system	:	système bouclé
cut-off frequency	:	fréquence (ou pulsation) de coupure
damped frequency	:	pulsation amortie
damping ratio	:	coefficient d'amortissement
feedback	:	contre-réaction
feedback system	:	système à contre-réaction
impulse response	:	réponse impulsionnelle
integral wind-up	:	emballement (de l'action) intégral
input	:	entrée
gain	:	gain
linear time-invariant (LTI)	:	linéaire invariant dans le temps
output	:	sortie
overshoot	:	dépassement
rise time	:	time de montée
root locus	:	lieu des racines
setpoint	:	consigne
settling time	:	temps de réponse
steady-state gain	:	gain statique
steady-state response	:	réponse en régime permanent
step response	:	réponse indicielle
time-delay	:	retard pur
time-invariant	:	invariant dans le temps
transient response	:	réponse transitoire
undamped natural frequency	:	pulsation propre non amortie