Typing a π-calculus with linear logic

Julien Gabet

Mars-June, 2018

1 Elimination of the CUT rule in a simplified system

We study the following annotated rules for typing simplified π-calculus with MLL:

Rules for neutral elements:

$$\frac{}{0_x \vdash x : 1} \qquad \frac{P \vdash \Gamma}{\epsilon_x.P \vdash \Gamma, x : \bot}$$

Rules for atoms:

$$\frac{}{A_x \vdash x : a} \qquad \frac{}{x \to y \vdash x : E^\perp, y : E}$$

Construction rules:

$$\frac{P \vdash \Gamma, x : E \quad Q \vdash \Delta, x : F}{P|_x Q \vdash \Gamma, \Delta, x : E \otimes F} \qquad \frac{P \vdash \Gamma, x : E, y : F}{\lambda_x y.P \vdash \Gamma, x : E ⅋ F}$$

And give a translation for the left terms:

$$\begin{aligned}
\lfloor 0_x \rfloor &= 0 \\
\lfloor x \to y \rfloor &= 0 \\
\lfloor A_x \rfloor &= A \\
\lfloor \epsilon_x.P \rfloor &= \lfloor P \rfloor \\
\lfloor \lambda_x y.P \rfloor &= \lfloor P \rfloor \\
\lfloor P|_x Q \rfloor &= \lfloor P \rfloor | \lfloor Q \rfloor
\end{aligned}$$

Remark: the terms we use in our annotated context are terms decorated with fresh names that are completely separate from the names communicated by the corresponding π-terms, and do not interact with them. As such, we effectively have $\lfloor P\sigma \rfloor = \lfloor P \rfloor$ for all permutation σ on the decoration names. That is important as our deduction rules interact over at most one such name, and the remainder of both contexts must be disjoint. This allows for a form of α-equivalence: two decorated terms are equivalent if only the names of the free variables differ.

Let's introduce a new CUT rule to cut against terms on the right side:

$$\frac{P \vdash \Gamma, x : E \quad Q \vdash \Delta, x : E^\perp}{P||_x Q \vdash \Gamma, \Delta}$$

as well as a translation rule for it:

$$\lfloor P||_x Q \rfloor = \lfloor P \rfloor | \lfloor Q \rfloor$$

Proposition

This CUT rule is admissible in our system, ie.
if $P \vdash \Gamma, x : E$ and $Q \vdash \Delta, x : E^\perp$ then there exists R such that $R \vdash \Gamma, \Delta$
with $\lfloor R \rfloor \equiv \lfloor P||_x Q \rfloor$, and the proof of R does not use the CUT rule.

$\triangleright$ By induction on the couple (n_1, n_2) of the sizes of the proof trees against which the cut is applied:
If one the last rules is not the one that introduced the variable against which we use the CUT rule, without loss of generality, we can assume this rule was the left rule of the cut (the other case is a perfect symmetry):
Case 1: the last rule is a tensor rule:

i: the variable was introduced in the right subtree of the last rule

$$\dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : E} \quad \dfrac{(\pi_2)}{Q \vdash \Delta, x : F, y : G}}{P|_x Q \vdash \Gamma, \Delta, x : E \otimes F, y : G} \quad \dfrac{(\pi_3)}{R \vdash \Theta, y : G^\perp}}{(P|_x Q)||_y R \vdash \Gamma, \Delta, \Theta, x : E \otimes F}\ \text{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : E} \quad \dfrac{\dfrac{(\pi_2)}{Q \vdash \Delta, x : F, y : G} \quad \dfrac{(\pi_3)}{R \vdash \Theta, y : G^\perp}}{Q||_y R \vdash \Delta, \Theta, x : F}\ \text{cut}}{P|_x (Q||_y R) \vdash \Gamma, \Delta, \Theta, x : E \otimes F}$$

and we have the structural congruence
$\lfloor (P|_x Q)||_y R \rfloor = (\lfloor P \rfloor || \lfloor Q \rfloor) || \lfloor R \rfloor \equiv \lfloor P \rfloor || (\lfloor Q \rfloor || \lfloor R \rfloor) = \lfloor P|_x (Q||_y R) \rfloor$.

ii: the variable was introduced in the left subtree of the last rule

$$\dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : E, y : G} \quad \dfrac{(\pi_2)}{Q \vdash \Delta, x : F}}{P|_x Q \vdash \Gamma, \Delta, x : E \otimes F, y : G} \quad \dfrac{(\pi_3)}{R \vdash \Theta, y : G^\perp}}{(P|_x Q)||_y R \vdash \Gamma, \Delta, \Theta, x : E \otimes F}\ \text{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : E, y : G} \quad \dfrac{(\pi_3)}{R \vdash \Theta, y : G^\perp}}{P||_y R \vdash \Gamma, \Theta, x : E}\ \text{cut} \quad \dfrac{(\pi_2)}{Q \vdash \Delta, x : F}}{(P||_y R)|_x Q \vdash \Gamma, \Delta, \Theta, x : E \otimes F}$$

and we have
$\lfloor (P|_x Q)||_y R \rfloor = (\lfloor P \rfloor || \lfloor Q \rfloor) || \lfloor R \rfloor \equiv \lfloor P \rfloor || (\lfloor Q \rfloor || \lfloor R \rfloor) \equiv \lfloor P \rfloor || (\lfloor R \rfloor || \lfloor Q \rfloor) \equiv (\lfloor P \rfloor || \lfloor R \rfloor) || \lfloor Q \rfloor = \lfloor (P||_y R)|_x Q \rfloor$.

Case 2: the last rule is not a tensor rule:

i: the last rule is an epsilon rule

$$\dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : E}}{\epsilon_y.P \vdash \Gamma, x : E, y : \bot} \quad \dfrac{(\pi_2)}{Q \vdash \Delta, x : E^\perp}}{(\epsilon_y.P)||_x Q \vdash \Gamma, \Delta, y : \bot}\ \text{cut}$$

$$\rightsquigarrow \quad \cfrac{\cfrac{\cfrac{(\pi_1)}{P \vdash \Gamma, x : E} \quad \cfrac{(\pi_2)}{Q : \Delta, x : E^\perp}}{P||_x Q \vdash \Gamma, \Delta}\ \text{cut}}{\epsilon_y.(P||_x Q) \vdash \Gamma, \Delta, y : \bot}$$

and we have
$\lfloor(\epsilon_y.P)||_x Q\rfloor = \lfloor P\rfloor|\lfloor Q\rfloor = \lfloor\epsilon_y.(P||_x Q)\rfloor$.

ii: the last rule is a lambda rule

$$\cfrac{\cfrac{\cfrac{(\pi_1)}{P \vdash \Gamma, x : E, y : F, z : G}}{\lambda_x y.P \vdash \Gamma, x : E⅋F, z : G} \quad \cfrac{(\pi_2)}{Q \vdash \Delta, z : G^\perp}}{(\lambda_x y.P)||_z Q \vdash \Gamma, \Delta, x : E⅋F}\ \text{cut}$$

$$\rightsquigarrow \quad \cfrac{\cfrac{\cfrac{(\pi_1)}{P \vdash \Gamma, x : E, y : F, z : G} \quad \cfrac{(\pi_2)}{Q \vdash \Delta, z : G^\perp}}{P||_z Q \vdash \Gamma, \Delta, x : E, y : F}\ \text{cut}}{\lambda_x y.(P||_z Q) \vdash \Gamma, \Delta, x : E⅋F}$$

and we have
$\lfloor(\lambda_x y.P)||_z Q\rfloor = \lfloor P\rfloor|\lfloor Q\rfloor = \lfloor\lambda_x y.(P||_z Q)\rfloor$.

<u>If both last rules introduces the variable we cut against:</u>

i: cut rule for neutrals:

$$\cfrac{\cfrac{}{0_x \vdash x : 1} \quad \cfrac{\cfrac{(\pi_1)}{P \vdash \Gamma}}{\epsilon_x.P \vdash \Gamma, x : \bot}}{0_x||_x\epsilon_x.P \vdash \Gamma}\ \text{cut}$$

$$\rightsquigarrow \quad \cfrac{(\pi_1)}{P \vdash \Gamma}$$

and we have $\lfloor 0_x||_x\epsilon_x.P\rfloor = 0|\lfloor P\rfloor \equiv \lfloor P\rfloor$.

ii: cut rule for axiom:

$$\cfrac{\cfrac{(\pi)}{P \vdash \Gamma, x : E} \quad \cfrac{}{x \to y \vdash x : E^\perp, y : E}}{P||_x x \to y \vdash \Gamma, y : E}\ \text{cut}$$

$$\rightsquigarrow \quad \cfrac{(\pi[y/x])}{P[y/x] \vdash \Gamma, y : E}$$

and we have $\lfloor P||_x x \to y\rfloor = \lfloor P\rfloor|0 \equiv \lfloor P\rfloor = \lfloor P[y/x]\rfloor$.
This general form treats all rules that can be cut against an axiom rule.

iii: cutting construction rules:

$$\cfrac{\cfrac{\cfrac{(\pi_1)}{P \vdash \Gamma, x : E} \quad \cfrac{(\pi_2)}{Q \vdash \Delta, x : F}}{P|_x Q \vdash \Gamma, \Delta, x : E \otimes F} \quad \cfrac{\cfrac{(\pi_3)}{R \vdash \Theta, x : E^\perp, y : F^\perp}}{\lambda_x y.R \vdash \Theta, x : E^\perp ⅋ F^\perp}}{(P|_x Q)||_x\lambda_x y.R \vdash \Gamma, \Delta, \Theta}\ \text{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : E} \qquad \dfrac{\dfrac{(\pi_2[y/x])}{Q[y/x] \vdash \Delta, y : F} \qquad \dfrac{(\pi_3)}{R \vdash \Theta, x : E^\perp, y : F^\perp}}{Q[y/x]||_y R \vdash \Delta, \Theta, x : E^\perp}\text{cut}}{P||_x(Q[y/x]||_y R) \vdash \Gamma, \Delta, \Theta}\text{cut}$$

and we have
$\lfloor (P|_x Q)||_x \lambda_x y.R \rfloor = (\lfloor P \rfloor | \lfloor Q \rfloor) | \lfloor R \rfloor \equiv \lfloor P \rfloor | (\lfloor Q \rfloor | \lfloor R \rfloor) = \lfloor P||_x(Q[y/x]||_y R) \rfloor$.

With all cases treated (and with the trees marked one can easily verify that the couples of the sizes of the trees decrease each time), this concludes the proof that the cut rule is admissible under our system. $\square$

Remark: The next step is to introduce receiving and sending rules with adequate markings and translations to have them work with the system (and potentially replace the rules for atoms).

2 Sending and receiving names

We add two interaction rules, to replace the atom rule from before:

$$\frac{P \vdash \Gamma, x : A[v/t]^{\perp}}{\bar{u}_x\langle v\rangle.P \vdash \Gamma, x :\uparrow_u \exists t.A^{\perp}} \qquad \frac{P \vdash \Gamma, x : A \quad t \notin \Gamma}{u_x(t).P \vdash \Gamma, x :\downarrow_u \forall t.A}$$

with $\lfloor \bar{u}_x\langle v\rangle.P\rfloor = \bar{u}\langle v\rangle.\lfloor P\rfloor$ and $\lfloor u_x(t).P\rfloor = u(t).\lfloor P\rfloor$.

We need to keep the arrow rule (which acts as a substitution rule), and add a new rule as follows:

$$\frac{P \vdash \Gamma, x : E \otimes E^{\perp}}{\mu_x.P \vdash \Gamma} \qquad \text{with } \lfloor \mu_x.P\rfloor = \lfloor P\rfloor.$$

Proposition

This CUT rule behaves as a reduction rule, and holds up to observation in our annotated system.

$\triangleright$ Most is already treated in the first system above, here is what's new:
If one of the last rules did not introduce the cut variable:

i: the last rule is a μ rule:

$$\dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : E \otimes E^{\perp}, y : F}}{\mu_x.P \vdash \Gamma, y : F} \qquad \dfrac{(\pi_2)}{Q \vdash \Delta, y : F^{\perp}}}{\mu_x.P||_y Q \vdash \Gamma, \Delta}\ \text{cut}$$

$$\rightsquigarrow \dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : E \otimes E^{\perp}, y : F} \qquad \dfrac{(\pi_2)}{Q \vdash \Delta, y : F^{\perp}}}{P||_y Q \vdash \Gamma, \Delta, x : E \otimes E^{\perp}}\ \text{cut}}{\mu_x.(P||_y Q) \vdash \Gamma, \Delta}$$

Here, x does not appear free in Q in the transformed tree because of the cut rule operating on y. This can be achieved by applying a substitution of x for a fresh name on Q, and has no importance as substitutions only impact free occurrences of the substituted names in the scope of the term they apply to.
Because of how we project the μ rule, the congruence holds here.

ii: the last rule is a sending rule:

$$\dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : A[v/t]^{\perp}, y : E}}{\bar{u}_x\langle v\rangle.P \vdash \Gamma, x :\uparrow_u \exists t.A^{\perp}, y : E} \qquad \dfrac{(\pi_2)}{Q \vdash \Delta, y : E^{\perp}}}{\bar{u}_x\langle v\rangle.P||_y Q \vdash \Gamma, \Delta, x :\uparrow_u \exists t.A^{\perp}}\ \text{cut}$$

$$\rightsquigarrow \dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : A[v/t]^{\perp}, y : E} \qquad \dfrac{(\pi_2)}{Q \vdash \Delta, y : E^{\perp}}}{P||_y Q \vdash \Gamma, \Delta, x : A[v/t]^{\perp}}\ \text{cut}}{\bar{u}_x\langle v\rangle.(P||_y Q) \vdash \Gamma, \Delta, x :\uparrow_u \exists t.A^{\perp}}$$

Here we note that the congruence does not hold for the projection. In the decorated calculus, this transformation is valid because x cannot be free in Q, and thus Q cannot interact with the occurrence of $\bar{u}_x$ we observe, meaning the order between the parallel and the sending does not matter to the behavior of the term (this needs some semantics work to clarify).

If the last rules did both introduce the cut variable (only the interaction case remain):

$$\dfrac{\dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, x : A \quad t \notin \Gamma}}{u_x(t).P \vdash \Gamma, x :\downarrow_u \forall t.A} \qquad \dfrac{\dfrac{(\pi_2)}{Q \vdash \Delta, x : A[v/t]^\perp}}{\bar{u}_x\langle v\rangle.Q \vdash \Delta, x :\uparrow_u \exists t.A^\perp}}{u_x(t).P||_x\bar{u}_x\langle v\rangle.Q \vdash \Gamma, \Delta}\text{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{(\pi_1[v/t])}{P[v/t] \vdash \Gamma, x : A[v/t]} \qquad \dfrac{(\pi_2)}{Q \vdash \Delta, x : A[v/t]^\perp}}{P[v/t]||_xQ \vdash \Gamma, \Delta}\text{cut}$$

Here we observe a reduction behavior. This holds because, as an exterior observer, we cannot tell if an internal action on the term has happened, and this is a basic case of internal action. $\square$

$$\dfrac{\dfrac{\dfrac{x \to y \vdash x :\downarrow_v \forall t'.A, y :\uparrow_v \exists t'.A^\perp}{\bar{u}_x\langle v\rangle.(x \to y) \vdash x :\uparrow_u \exists t. \downarrow_t \forall t'.A, y :\uparrow_v \exists t'.A^\perp} \qquad \dfrac{\dfrac{(\pi_1)}{P \vdash \Gamma, y : A \quad t' \notin \Gamma}}{v_y(t').P \vdash \Gamma, y :\downarrow_v \forall t'.A}}{\bar{u}_x\langle v\rangle.(x \to y)||_y v_y(t').P \vdash \Gamma, x :\uparrow_u \exists t. \downarrow_t \forall t'.A}\text{cut}}{(\nu v)(\bar{u}_x\langle v\rangle.(x \to y)||_y v_y(t').P) \vdash \Gamma, x :\uparrow_u \exists t. \downarrow_t \forall t'.A}$$

$$\dfrac{\dfrac{\dfrac{x \to z \vdash x : A[t/t']^\perp, z : A[t/t']}{\bar{t}_x\langle a\rangle.(x \to z) \vdash x :\uparrow_t \exists t'.A^\perp, z : A[t/t']} \qquad \dfrac{(\pi_2)}{Q \vdash \Delta, z : A[t/t']^\perp}}{\bar{t}_x\langle a\rangle.(x \to z)||_z Q \vdash \Delta, x :\uparrow_t \exists t'.A^\perp \quad t \notin \Delta}\text{cut}}{u_x(t).(\bar{t}_x\langle a\rangle.(x \to z)||_z Q) \vdash \Delta, x :\downarrow_u \forall t. \uparrow_t \exists t'.A^\perp}$$

$$\dfrac{(\nu v)(\bar{u}_x\langle v\rangle.(x \to y)||_y v_y(t').P) \vdash \Gamma, x :\uparrow_u \exists t. \downarrow_t \forall t'.A \qquad u_x(t).(\bar{t}_x\langle a\rangle.(x \to z)||_z Q) \vdash \Delta, x :\downarrow_u \forall t. \uparrow_t \exists t'.A^\perp}{(\nu v)(\bar{u}_x\langle v\rangle.(x \to y)||_y v_y(t').P)||_x u_x(t).(\bar{t}_x\langle a\rangle.(x \to z)||_z Q) \vdash \Gamma, \Delta}\text{cut}$$