# Typing a $\pi$-calculus with linear logic

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# Chapter 1

# Elimination of the CUT rule in a simplified system

We study the following annotated rules for typing simplified $\pi$-calculus with MLL:

Rules for neutral elements:

$$\frac{}{0_x \vdash x : 1} \qquad \frac{P \vdash \Gamma}{\epsilon_x.P \vdash \Gamma, x : \bot}$$

Rules for atoms:

$$\frac{}{A_x \vdash x : a} \qquad \frac{}{x \to y \vdash x : E^\perp, y : E}$$

Constructive rules:

$$\frac{P \vdash \Gamma, x : E \qquad Q \vdash \Delta, x : F}{P|_x Q \vdash \Gamma, \Delta, x : E \otimes F} \qquad \frac{P \vdash \Gamma, x : E, y : F}{\lambda_x y.P \vdash \Gamma, x : E \parr F}$$

And give a translation for the left terms:

$$\begin{aligned}
\lfloor 0_x \rfloor &= 0 \\
\lfloor x \to y \rfloor &= 0 \\
\lfloor A_x \rfloor &= A \\
\lfloor \epsilon_x.P \rfloor &= P \\
\lfloor \lambda_x y.P \rfloor &= P \\
\lfloor P|_x Q \rfloor &= \lfloor P \rfloor | \lfloor Q \rfloor
\end{aligned}$$

Let's introduce a new CUT rule to cut against terms on the right side:

$$\frac{P \vdash \Gamma, x : E \qquad Q \vdash \Delta, x : E^\perp}{P||_x Q \vdash \Gamma, \Delta}$$

as well as a translation rule for it:

$$\lfloor P||_x Q \rfloor = \lfloor P \rfloor | \lfloor Q \rfloor$$

**Proposition**

*This CUT rule is admissible in our system, ie.*
*if* $P \vdash \Gamma, x : E$ *and* $Q \vdash \Delta, x : E^{\perp}$ *then there exists* $R$ *such that* $R \vdash \Gamma, \Delta$
*with* $\lfloor R \rfloor = \lfloor P||_x Q \rfloor$*, and the proof of* $R$ *does not use the CUT rule.*

$\triangleright$ By induction on the rule we cut against in proof tree:
If one the last rules is not the one that introduced the variable against which we use the CUT rule, without loss of generality, we can assume this rule was the left rule of the cut (the other case is a perfect symmetry):
Case 1: the last rule is a tensor rule:

i: the variable was introduced in the left subtree of the last rule

$$\dfrac{\dfrac{P \vdash \Gamma, x : E \qquad Q \vdash \Delta, x : F, y : G}{P|_x Q \vdash \Gamma, \Delta, x : E \otimes F, y : G} \qquad R \vdash \Theta, y : G^{\perp}}{(P|_x Q)||_y R \vdash \Gamma, \Delta, \Theta, x : E \otimes F}\ \mathrm{cut}$$

$$\rightsquigarrow \quad \dfrac{P \vdash \Gamma, x : E \qquad \dfrac{Q \vdash \Delta, x : F, y : G \qquad R \vdash \Theta, y : G^{\perp}}{Q||_y R \vdash \Delta, \Theta, x : F}\ \mathrm{cut}}{P|_x(Q||_y R) \vdash \Gamma, \Delta, \Theta, x : E \otimes F}$$

And we have the structural congruence
$\lfloor (P|_x Q)||_y R \rfloor = (P|Q)|R \equiv P|(Q|R) = \lfloor P|_x(Q||_y R) \rfloor$.

ii: the variable was introduced in the right subtree of the last rule

$$\dfrac{\dfrac{P \vdash \Gamma, x : E, y : G \qquad Q \vdash \Delta, x : F}{P|_x Q \vdash \Gamma, \Delta, x : E \otimes F, y : G} \qquad R \vdash \Theta, y : G^{\perp}}{(P|_x Q)||_y R \vdash \Gamma, \Delta, \Theta, x : E \otimes F}\ \mathrm{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{P \vdash \Gamma, x : E, y : G \qquad R \vdash \Theta, y : G^{\perp}}{P||_y R \vdash \Gamma, \Theta, x : E}\ \mathrm{cut} \qquad Q \vdash \Delta, x : F}{(P||_y R)|_x Q \vdash \Gamma, \Delta, \Theta, x : E \otimes F}$$

And we have
$\lfloor (P|_x Q)||_y R \rfloor = (P|Q)|R \equiv P|(Q|R) \equiv P|(R|Q) \equiv (P|R)|Q = \lfloor (P||_y R)|_x Q \rfloor$.

Case 2: the last rule is not a tensor rule:

i: the last rule is an epsilon rule

$$\dfrac{\dfrac{P \vdash \Gamma, x : E}{\epsilon_y.P \vdash \Gamma, x : E, y : \perp} \qquad Q \vdash \Delta, x : E^{\perp}}{(\epsilon_y.P)||_x Q \vdash \Gamma, \Delta, y : \perp}\ \mathrm{cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{P \vdash \Gamma, x : E \qquad Q : \Delta, x : E^{\perp}}{P||_x Q \vdash \Gamma, \Delta}\ \mathrm{cut}}{\epsilon_y.(P||_x Q) \vdash \Gamma, \Delta, y : \perp}$$

And we have
$\lfloor (\epsilon_y.P)||_x Q \rfloor = P|Q = \lfloor \epsilon_y.(P||_x Q) \rfloor$

ii: the last rule is a lambda rule

$$\dfrac{\dfrac{P \vdash \Gamma, x : E, y : F, z : G}{\lambda_x y.P \vdash \Gamma, x : E \parr F, z : G} \qquad Q \vdash \Delta, z : G^{\perp}}{(\lambda_x y.P)||_z Q \vdash \Gamma, \Delta, x : E \parr F} \text{ cut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{P \vdash \Gamma, x : E, y : F, z : G \qquad Q \vdash \Delta, z : G^{\perp}}{P||_z Q \vdash \Gamma, \Delta, x : E, y : F} \text{ cut}}{\lambda_x y.(P||_z Q) \vdash \Gamma, \Delta, x : E \parr F}$$

And we have
$\lfloor(\lambda_x y.P)||_z Q\rfloor = P|Q = \lfloor\lambda_x y.(P||_z Q)\rfloor$

$\square$