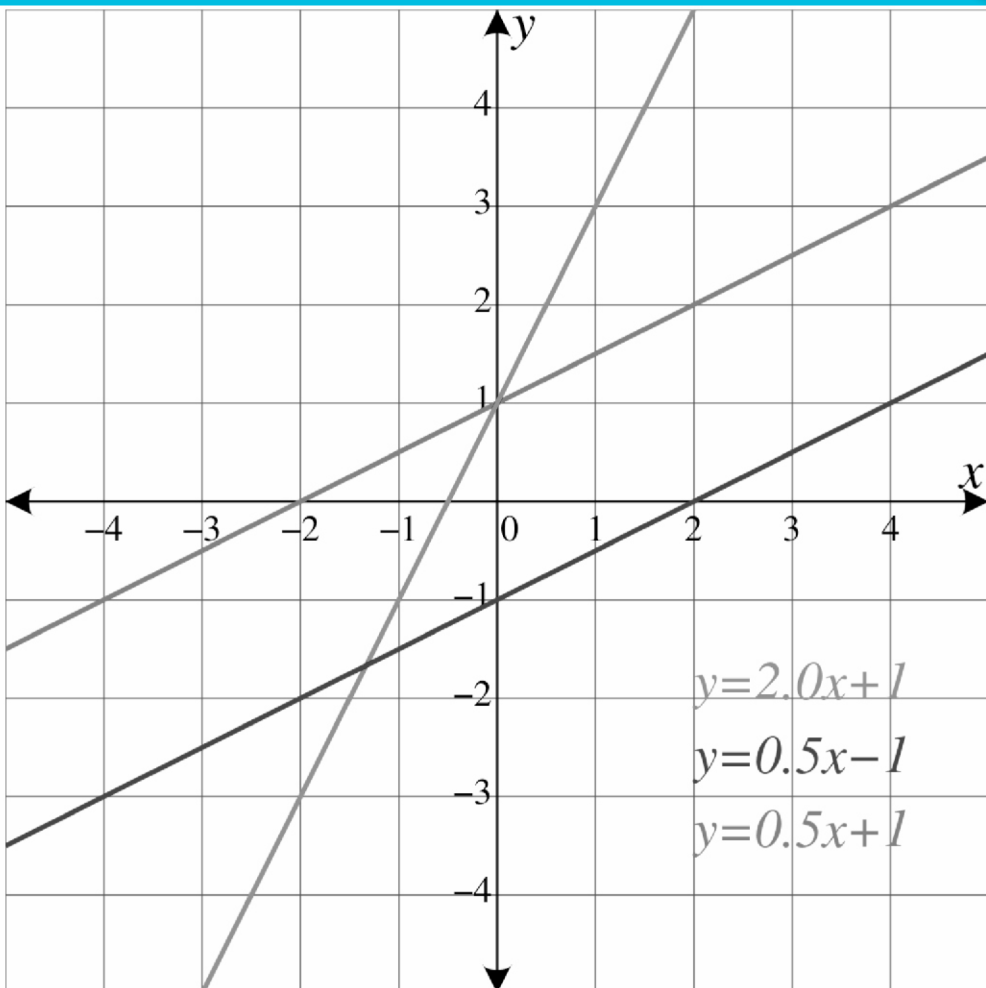
South Sudan

Secondary Additional Mathematics 3

Teacher's Guide



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South Sudan

SECONDARY
3

Additional Mathematics

Teacher's Guide 3

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Introduction

Teaching Mathematics is taking place in rapidly changing conditions. It is necessary to look for optimal didactic and educational solutions encompassing goals and contents as well as forms and teaching methods allowing for preparing students to face the challenges of the contemporary world.

The most significant role of educational system in terms of teaching Mathematics is developing and promoting subject competences as an important factor fostering student's personal development and the development of society. Well organised mathematical education facilitates logical thinking and expressing ideas, organizing own work, planning and organizing the learning process, collaboration and responsibility; it prepares for life in a modern world and enables to perform many jobs.

The teacher is required to pay more attention to students' awareness of developing learning skills and study habits, recognizing and analysing problems and predicting solutions to them. Undeniably, the implementation of modern teaching methods and techniques enhances students' curiosity about Mathematics and increases their understanding of the basis of mathematical and scientific knowledge. In accordance with the trends teaching Mathematics is supposed to help students understand and solve everyday problems.

The aim of teaching Secondary Mathematics is to encourage contemporary students to work in class, acquire knowledge and skills that are necessary in life. Moreover, research shows that teachers applying active methods assess the effectiveness of their work and how students respond to this way of teaching.

About this guide

The purpose of this guide is to offer suggestions that are helpful to Secondary 3 Mathematics teachers on planning, organizing, executing and evaluating the learning and teaching of mathematics. The suggestions will serve as useful starting points to the teachers who are expected to be dynamic innovative and creative to make the leaning process fit the learners.

The guide is to be used alongside Mathematics Students book for secondary 3. It consists of 10 units, in line with Secondary 3 Mathematics syllabus.

Each of the unit is structured to contain:

1. Introduction
2. Objectives
3. Teaching/Learning Activities
4. Answers to the exercises given in student's secondary 3 book.

In each case, the introduction highlight the relevant work than learners are expected to have covered in their previous mathematics units and what they are expected to have covered previously. It also highlights what they are expected to cover in the unit. The teacher is expected to make a quick lick up of previously learnt concepts. Learners should be able to make relevant references to their previous work. Where possible the mathematics teach ma make an entry behavior evaluation as a revision on previously learnt units related to the unit under study.

The unit objectives specify the skills (cognitive, affective, and psychomotor) that teachers will use to enable learners understand each unit. The objectives are likely to serve a useful purpose if they when stated to reflect the local conditions of the learner. For example, the type of students and the available learning resources. The teacher may break down the unit objectives to various objectives that enhance the learners understanding of the process involved and to suit different situations in the lesson, schools, society and the world at large.

Teaching/learning activities highlight the most noticeable and important. Points encountered in the learning process and suitable techniques to be used in handling each objective(s).

Answers to each exercise in the students' book are provided in these teachers guide. It is contemplated that the most conducive and favorable outcome from the guide will be realized if other sources of learning mathematics are properly organized and used.

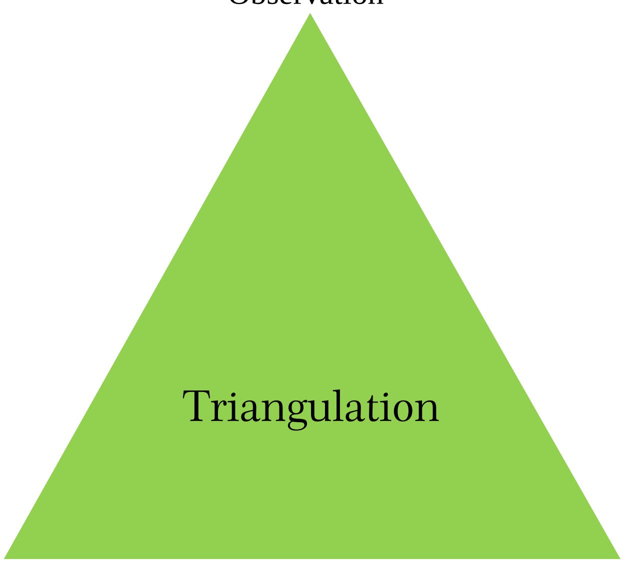
Among others, the following should be used alongside the guide:

1. The Schemes of work
2. The teacher's Lessons plans.
3. The Records of the work covered by the learners.

Making Classroom Assessment

- Observation – watching learners as they work to assess the skills learners are developing.
- Conversation – asking questions and talking to learners is good for assessing knowledge and understanding of the learner.
- Product – appraising the learner's work (writing report or finding, mathematics calculation, presentation, drawing diagram, etc.).

Observation



Product (Exercises) Conversation

To find these opportunities, look at the "Learn About' sections of the syllabus units. These describe the learning that is expected and in doing so they set out a range of opportunities for the three forms of opportunity.

UNIT 1

FUNCTIONS

<table>
<tr><th colspan="2">Additional Math   Secondary 3</th><th>Unit 1: Functions</th></tr>
<tr><td colspan="2">Learn about</td><td>Key inquiry questions</td></tr>
<tr><td colspan="2">The learners should be introduced to functions through discussion between students. They should work individually to identify domain and range and in groups describe modulus of a function and classify types of functions e.g. composite function.

Learners should evaluate the inverse or no inverse of a function and represent functions graphically. They should use the internet to supplement their learning and also investigate the application of functions in solving problems in other subjects.</td><td>- How can we describe a function?
- In what way can we justify whether a function is defined or undefined?
- How can the modulus of a function be used in our day-to- day life?
- How can we represent a function graphically?
- How can we find the inverse of a function?
- How would we use composite functions in solving problems?</td></tr>
<tr><td colspan="3">Learning outcomes</td></tr>
<tr><td>Knowledge and understanding</td><td>Skills</td><td>Attitudes</td></tr>
<tr><td>- Function, domain and range, modulus of a function, inverse (or no inverse) of a function, composite function</td><td>- Workout the value for any function
- Find inverse function from the function
- Represent functions and inverse functions graphically
- Solve problems involving functions</td><td>- Value the importance of graph use in everyday life
- Appreciate the importance of functions in solving mathematical problems and applications in business and commerce
- Develop an interest in designing graphs of</td></tr>
</table>

	Investigate the use of function graphs or charts in real life	functions using a computer

Contribution to the competencies:

Critical thinking: mapping and distinguishing different types of graphs in functions

Co-operation: team work

Communication: communicate their knowledge of functions coherently and clearly to peers teachers and others

Links to other subjects:

Science: Graphical representation in practicals.

Chemistry: Studying the rate of change in heat enthalpy, etc.

Geography: Studying the rate of growth in population

Economics: Representing the gross domestic product of a country

ICT: Internet

Learning/Teaching Materials

Graph book, a graph board, a scientific calculator for the teacher a chart on a graph of a function and its inverse, a computer, card sort etc. Learners are encouraged to use graph plotting software, for example, GeoGebra, Desmos, Microsoft Mathematics, etc.

Learning/ Teaching Materials

1) **Functions**
   a) Drawing graphs of common and familiar equations and those function.
   b) Making mapping on paired number of students by having a game of pairs and small groups.
   c) Introducing the functions by mapping activities.
   d) Defining functions and taking notes.
   e) Using function machines in defining functions by inputs outputs.
   f) Guide the learners in performing a kinesthetic activity on a functioning box machine to determine a function and equations not functions.
   g) Guide learners to draw graphs with definite and indefinite domain and ranges.
   h) Student's to use computers in confirming the graphs of various functions.

2) **Domain and Range**
   a) Guide learners in obtaining the range of a given function.
   b) Guide learners in performing an activity of matching cards on domain and range of given functions.
   c) Group learners into groups of 4-5 and guides them in think-pair-share process in their groups and the whole class.
   d) Guide learners in an exercise to complete the table of domain and range of given tables.
   e) Describing and noting the definition of domain and range.
   f) Describing function using domain and range.
   g) Using matching machine to determine a set of domain and range.

Inverse of a function

a) Teacher gives a case study of inverse of real life scenarios such as a man and his wife as having inverse and wife without a man as taking inverse and a man without a wife as being not a function.
b) Teacher assist learners discover functions with and without inverse by observing matching.
c) Teacher guide learners to deduce and get expression of inverse of given functions. Emphasis should be made on difference of inverse from that of changing the formula.
d) A discussion on similarities and differences between changing the subject of an equation and determining the inverse of a function.

Composite functions

a) Introduce composite functions by making a visual illustration (similar to that of a family).
b) A case study of a learner's family may be used in developing a composite function.
c) Guide learners in developing composite functions.
d) Guide learners in determining the domain and range of composite functions.

Modulus of functions

a) Make a visual illustration of numbers and their modulus and ask learners to continue developing the illustration hence deduce what is a modulus of a function.
b) Guide learners in defining modulus and symbolically represent them.
c) Draw graphs and modulus of functions.
d) Determine domain and range of given modulus functions.
e) Guide learners in solving real life problems involving composite functions.

Exercise 1:1

The teacher should to discuss with learners some possible solutions.

When is a graph not a graph of a function?

Exercise 1:2

Determine whether or not the graphs of the following equations are of a function.

1. $y = x^3 + x^2 + 3x + 4$
2. $y = x^3 - x$
3. $y = x^{\frac{1}{2}}$
4. $y^2 = x^2 + 3x + 4$
5. $2x^2 + 2y^2 - 4x + 2y = 0$
6. $3x^2 - 2y^2 + 3xy = 0$
7. $y = 2^x$
8. $x = (y - 3)^2 + 5$

Implicit and Explicit Functions

Exercise 1.3

Where possible, express the following implicit equations as explicit equations.

1. $y = 2x + 2 + 4y$
2. $f(x) = x^2 + 3 + 6$
3. $x^2 + y^2 = 6$
4. $x^3 + y^2 = 5$
5. $7x^6 + 4x^4 + 3y^2 = x$
6. $9xy + 3x^2y + 3x^3y + 3x^2 + 4x + 20 = 0$
7. $xy^2 - 4y = x^2 + 1$
8. $y^2x - 2y = xy$

Exercise 1.4

1. Determine whether or not the following implicit equations are a function.

a) $y^3 - 3y + 2x = 0$
b) $4x^4 + 20x^3 + 160x^2 + 80x + 20y = 0$
c) $yx^2 + 16y = x^2 - 16 - 8xy$
d) $25\,(x-3)^2 + 10(y+2)^2 = 100$

2. Determine the range and the domain for the functions above.

Modulus of a function

Exercise 1.5

1. a) y= - $\frac{2}{3}$(x+1)

   b) $x^2 + 9$

   c) y=$\sqrt{16 - x^2}$

   b) y= $\sqrt{(x - 7x^6 + 4x^4}$ x 1/3

   e) y = $\frac{\sqrt{x-7x^6+4x4+3y2}}{3}$

   f) x= - $\frac{(3x2+4x+20)}{(ax+3x2+3x3)}$

   g) y=$\frac{x2+1}{x24}$

   h) y= $\frac{x+2}{2}$

Application of Modulus

Task

In groups prepare a justification for each of the following statements:

1. $|x| \geq 0$ An absolute value is greater than or equal to zero.
2. $|0| = 0$
3. $|-x| = |x| = x$
4. $|x| + |y| \geq |x + y|$

Inverse of a function

Exercise 1.6

1. a) Note a function at $x = 0, y = +3$
   b) A function
   c) A function

d) Not a function at x=0, y = -3or y=1

2. Domain: any real number

   Range: any real number

Composite Functions

Exercise 1.7: Work in pairs.

1. Given the functions

   $f(x) = 4 - x^2, \quad g(x) = \sqrt{x+4}$ and $h(x) = \frac{1}{4x}$,

   Evaluate:

   a) $f(g(x))$
   b) $f(h(x))$
   c) $g(f(1))$
   d) $f(g(x^2))$
   e) $h(g(x))$
   f) $f(g(h(x)))$
   g) $g(f(h(x)))$
   h) $f(g(h(0)))$
   i) $f(g^{-1}(x))$
   j) $f(h^{-1}(1))$

2. Given the function $f(x) = x^2 + 3x + 2$ and $g(x) = \sqrt{x+2}$, explain why function $f(g(-3))$ does not exist.

3. Write $f(x) = \sqrt{\frac{1}{x^2+2}}$ as a composition of two functions $g(x)$ and $h(x)$.

4. Show that the following pair of functions are inverse of each other.
   a) $f(x) = x^3$ and $g(x) = \sqrt[3]{x}$.

   b) $f(x) = 3x - 3$ and $g(x) = \frac{x}{3} + 1$

c) $h(x) = \frac{1}{x^2-1}$ and $f(x) = \sqrt{\frac{1}{x}} + 1$

5. Given a function $g(x) = x^2 + 1$ has a domain $[1, \infty)$, find the domain and range of the inverse function $g^{-1}(x)$.

6. Determine the range and domain of the function $f^{-1}(x)$ given that $f(x) = x^2 - 2$.
7. Write each of the following composite functions as a composition of two functions.
   a) $f(h(x)) = \sqrt{x^3} - 1$

   b) $g(h(x)) = (3x - 4)^3$

   c) $f(g(x)) = \frac{1}{x^2-1}$

   d) $f(g(x)) = x^2$

8. For each of the following functions, specify the domain of its inverse.
   a) $f(x) = x^2 - 2x + 3$ with domain $[1, \infty)$

   b) $h(x) = |x|$ with domain $(-\infty, \infty)$

   c) $g(x) = 4 - x^2$ with domain $\mathbb{R}^+$

Operations on functions

Exercise 1.8: Work in pairs.

1. Given $g(x) = x^2 + 3x + \frac{1}{2}, find$
   a) $2g(x)$
   b) $g(x) + 2g^{-1}(x)$

2. If the function $f(x) = g(x) + 2h(x)$ and $f(x) = x^2 + 7x$ and $h(x) = x^2 + 9x + 2$, find $g(x)$.

3. If $f(x) = x^2 + 3$, $g(x) = x^2 + 2x + 3$ and $h(x) = \frac{1}{2}$, find
   a) $f(g(x))$
   b) $(f + g)(x)$
   c) $f(x) + 3g(x)$
   d) $(g + f + h)(x)$
   e) $f(g(h(x)))$
   f) $g(h(f(x)))$
   g) $(3h + 2f + 3g)(x)$
   h) $f(g(1))$
   i) $(2f + 3h)(1)$
   j) $f(0) + 2h(1) + 3h(3)$

Exercise 1.8

1. a) $f^{-1}(x) = \sqrt{x-2}$
   b) $f^{-1}(x) = \frac{x-4}{3}$
   c) $f^{-1}(x) = \sqrt{x}$
   d) $f^{+}(x) = 4x$
   e) $g^{+}(x) = \frac{3+2x}{x}$
   f) g-(x) $= \frac{2+\sqrt{4^{+}x}}{-2x}$
   g) $f^{-1}(x) = x^{1/3}$

2. i) f’(x) $= x^{1/2}$

   Domain: all positive real numbers

   Range: all real numbers

   ii) f’(x)$= x^2$

   Domain: all real numbers
   Range: all real numbers

iii) f'(x) =$\frac{1}{\sqrt{x}}$

Domain: all positive real numbers except x=1

Range: all real numbers

iv) f'(x) = $\frac{1}{\sqrt{x}}$ + 2

Domain: all positive real numbers x>0

Range: all real numbers

UNIT 2

TRIGONOMETRY

Additional Math Secondary 3	**Unit 2: Trigonometry**
Learn about	**Key inquiry questions**
Learners should be introduced to trigonometric identities and equations with more than one function through discussion. They should learn how to investigate and apply addition formulae, formulae for the tangents of compound angles, the double angle formulae and the half angle formulae to solve practical problems. They should supplement their learning by using internet and working with others.	• How can we derive trigonometric identities? • How can we evaluate equations with more than one function? • Why is the use of trigonometric identities important? • What techniques can we use to simplify trigonometric expressions? • In what way can we apply the formulae of trigonometric equations in everyday life?

Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**
• Trigonometric identities, equations with more than one function, the addition formulae, and the tangents of compound angles (A±B) • Derivation of trigonometric identities e.g. secant, cosecant	• Derive the three trigonometric identities • Solve equation related to three trigonometric identities • Derive and solve equation involving compound angles • Find expression for trigonometric function of double angles formulae	• Appreciate how trigonometry is used in maths and other fields of science such as projectiles in physics • Appreciate the importance of trigonometry in everyday life situation e.g.

and cotangent, the double angle formulae and the half angle formulae	- Derive half angle formulae and use it to solve equations - Solve real life problems involving trigonometric functions - Plan and carry out investigations of problems involving trigonometry	when applied in construction

Contribution to the competencies:

Critical thinking: Derive trigonometric formulae, investigate and develop solutions to problems using trigonometric expressions

Communication: Comprehend types of trigonometric equations

Cooperation: Team work

Links to other subjects:

Physics: vectors and wave equations

Geography: measurements of distances between land marks

ICT: Internet

Learning/Teaching materials

Graph book, Geometrical set, graph method, scientific calculator, charts, and computer.

Learning/ Teaching Activities

Introduction

a) Ask the learner to state and note trigonometric ratios previously learnt i.e. sin, cos, tan.
b) Guide students in determining the ratio for reciprocal of basic rations i.e. sec, cos e and tan.
c) Learners to state and write identities discussed in previous units i.e. cos x = sin (90-x)
d) Guide the learners in solving problems using ratios.

Trigonometric identities and equations

a) Guide learners in using geometrical and graphical properties of unit circle to get identities related to unit circle.
b) Guide learner's in **graphical constructions** to come up with identities that involve addition and subtraction of angles.
c) Learner's to work out how to derive double angle formulas. Using jig saw method the learners can derive these identities.
d) Using discussion teacher guide learners on solving trigonometric equations.
e) Using a case study, guide students in solving real life problems involving trigonometric functions.
f) Organize a field study to areas that apply trigonometry, such area may include a construction site, a garage or to a hydraulic truck, land marks.
g) Guide learners in problem solving on trigonometry and how use of internet help solve these problems.
h) Guide students in making summary of trigonometric identities using a chart and using this identities in solving equations.

Trigonometric Ratio Identities

Task 1

In groups study and discuss the figure below of a unit circle center O(0, 0) with a right triangle OQP such that, its vertex P is on the circumference and angle $POQ = \theta^0$. Answer the following questions:

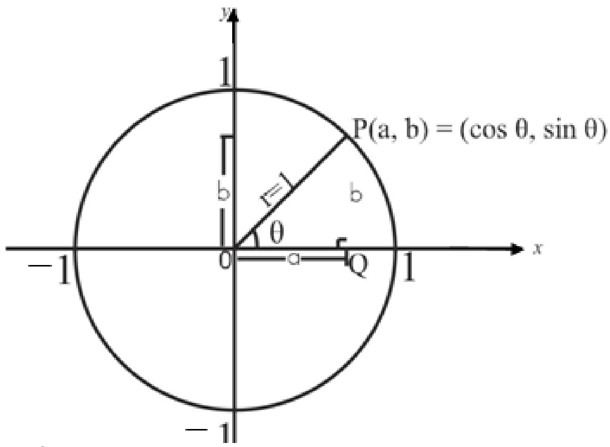


Figure 2.1

a. Write an expression for ;
   i. Sine θ
   ii. Cosine θ
   iii. Tangent θ
b. What are complementary angles?
c. Identify the pair of complementary angles in figure 2.1 above.
d. Write general expressions for;
   i. $\sin(90 - \theta)^0$
   ii. $\cos(90 - \theta)^0$
e. From the solution in a and d above what do you infer about cosine and sine of complementary angles.

i. Draw a line $x = 1$ that is parallel to *y*-axis and meets the *x*-axis at point A(1, 0). Extend the side *OP to* meet the line *x = 1* at *B*. Using similarity show $\tan\theta = \frac{\sin\theta}{\cos\theta}$

Exercise 2.1

i.

a) $\frac{4}{3}$
b) ¾
c) $\frac{4}{5}$
d) $\frac{5}{3}$
e) $\frac{5}{4}$
f) ¾

ii. a) $\frac{16}{9}$
b) $\frac{25}{16}$
c) $\frac{16}{25}$
d) $\frac{25}{9}$
e) $\frac{25}{16}$
f) $\frac{9}{25}$

iii. a) $\sqrt{\frac{y^2-x^2}{x}}$
b) $\frac{x}{(y^2-x^2)}$
c) $\frac{(y^2-x^2)}{y}$
d) $\frac{\sqrt{y^2-x^2}}{x}$
e) $\frac{y}{\left(\sqrt{y^2+x^2}\right)}$
f) $\frac{x}{\left(\sqrt{y^2+x^2}\right)}$

iv. a) $\frac{b}{\sqrt{a^2-b^2}}$
b) $\frac{\sqrt{a^2-b^2}}{b}$
c) b/a
d) $\frac{a}{\sqrt{a^2-b^2}}$
f) a/b
g) $\frac{\sqrt{a^2-b^2}}{a}$

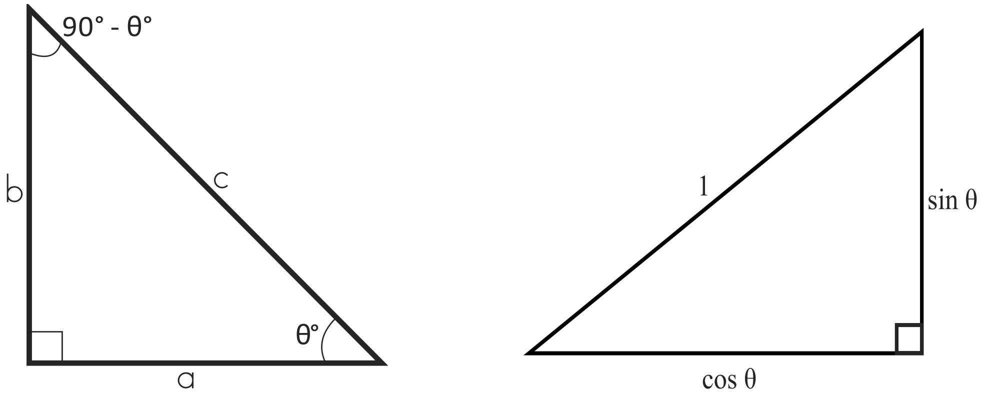
2a) $\frac{5}{13}$
b) $\frac{12}{13}$
c) $\frac{13}{5}$

3. a) $\frac{1}{2\sqrt{6}}$
b) $\frac{1}{5}$
c) $\frac{2}{5}\sqrt{16}$
d) $2\sqrt{6}$

4. a) C (12,8) or c(2,8)
b) 0 or $\frac{2}{\sqrt{5}}$
c) Undefined or ½
d) $\frac{\sqrt{5}}{2}$

Task 2

Study figure 2.2 below and compare the two triangles. Use them to derive expressions for $\cos\theta$, $\sin\theta$ and $\tan\theta$



Special Acute Angles in Trigonometry (45°, 60° and 30°)

Task 3

Determine $\cos\theta$, $\sin\theta$ and $\tan\theta$ for the isosceles right angled triangle in figure 2.3.

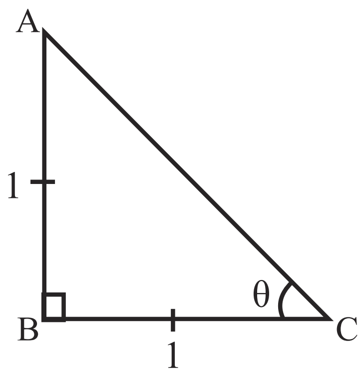
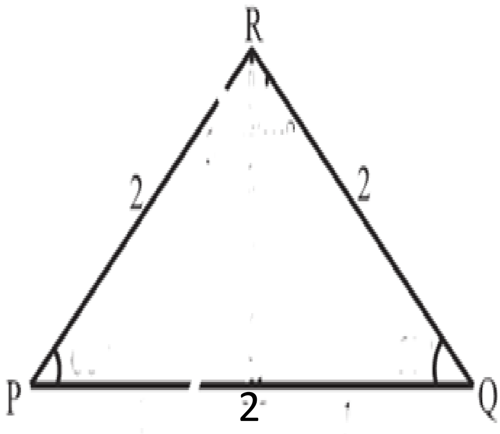


Figure 2.3

Hint: Use Pythagoras' theorem to find AC. Use angle sum of a triangle to find angle θ.

Task 4

The figure below shows an equilateral triangle $\Delta\, PQR$. $PQ = QR = RP = 2\, units$. Determine $\cos\theta$, $\sin\theta$ and $\tan\theta$ and for $90° - \theta$.



Exercise 2.2

1)

i) 60^0

ii) 30^0

iii) 43^0

iv) 45^0

v) 53^0

vi) 70^0

vii) 16^0

viii) 0^0

2

a) 45^0

b) 70^0

c) 50^0

d) 70^0

i)

e) 60^0

f) 65^0

g) 30^0

h) 30^0

3)

a) ½

b) ¼

c) $\sqrt{6}$

d) ½

e) $1 + 1/2\sqrt{2}$

f) $\frac{\sqrt{3}+1}{6}$

g) $\frac{5\sqrt{6}}{6}$

h) $\frac{4\sqrt{3}}{3}$

i) 9

4 a) $\frac{30}{\sqrt{3}}$

b) $10\sqrt{2}$ m

5 i) 3 units

ii) 3 units

6. Teacher to check the accuracy of the proof

7 i) $13\sqrt{3M}$

ii) $\sqrt{2}$ *and* 2 *units*

Finding Trigonometric values for non-acute angles

Task

Given that $\sin 30° = \frac{1}{2}$, use the graph to find:

a) $\cos 60°$
b) $\sin 150°$
c) $\sin 210°$
d) $\sin 330°$
e) $\sin(-30°)$
f) $\sin(-150°)$

Exercise 2.3

1.

2. a. $-\frac{\sqrt{3}}{2}$

b. $\frac{\sqrt{3}}{2}$

c. $\sqrt{2}$

d. 1

e. $-\frac{3}{\sqrt{3}}$

f. $\frac{2}{\sqrt{3}}$

3. a. $-\frac{\sqrt{3}}{2}$

b. -½

c. -2

d. $-\frac{2}{\sqrt{3}}$

3 a. $\sqrt{3}$

b. 1

c. $-\frac{2}{\sqrt{3}}$

4 a. $\frac{\sqrt{3}}{2}$

b. $-\frac{\sqrt{3}}{2}$

c. $-\frac{\sqrt{3}}{2}$

d. $-\frac{\sqrt{3}}{2}$

5

a) $\sqrt{3}$

b) $-\frac{\sqrt{3}}{2}$

c) $-4\sqrt{3}$

d) $\frac{-9\sqrt{2}}{4}$

e) $\frac{-3\sqrt{2}}{2}$

f) -3/2

g) $\frac{2\sqrt{3}}{11}$

Pythagoras' Theorem and Derived Trigonometric Identities

Task

The figure below shows a unit circle and a right triangle *OQB*. In groups of four students study it and answer the questions that follow

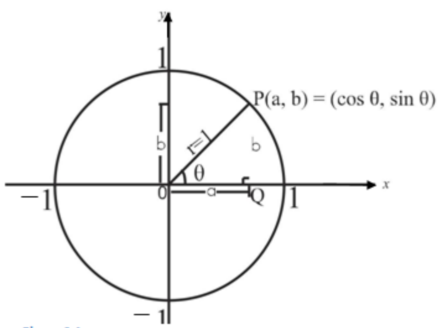


Figure 2.1

i. Use the triangle OQP to prove the identity $\cos^2\theta + \sin^2\theta = 1$
ii. Use the identity $\cos^2\theta + \sin^2\theta = 1$ to show that
   a. $1 + \tan^2\theta = \sec^2\theta$
   b. $\cot^2\theta + 1 = \csc^2\theta$

The above identities can be used to prove several other trigonometric relationships.

Exercise 2.4

The teacher to check the accuracy of the proofs made.

Compound angle formulas for Trigonometry

Task 3: Work in groups.

Use a large copy of the diagram below to derive $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.

Task 4

Replace β with $-\beta$ in your expressions for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ to derive expressions for $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$

Deriving Double angle identities

Task 5

Replace β with α in your expressions for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ to derive expressions for $\sin(2\alpha)$ and $\cos(2\alpha)$

Other exact values of trigonometric functions

Task

In groups, complete the table below with exact values

$\theta°$	$\sin\theta$	$\cos\theta$	$\tan\theta$
0			
15			
30			
45			
60			
75			
90			

Exercise 2.5

The teacher to check the accuracy of the proof made.

Relationship between Sum and Difference of Angles with the products of Trigonometric Ratios

Task

In pairs, use

i. $\sin(\theta + \beta) = \sin\theta\cos\beta + \cos\theta\sin\beta$
ii. $\sin(\theta - \beta) = \sin\theta\cos\beta - \cos\theta\sin\beta$
iii. $\cos(\theta + \beta) = \cos\theta\cos\beta - \sin\theta\sin\beta$
iv. $\cos(\theta - \beta) = \cos\theta\cos\beta + \sin\theta\cos\beta$

To derive identities for the products $\sin\theta\cos\beta$, $\cos\theta\cos\beta$ and $\sin\theta\sin\beta$

Hint: Add i and ii. Add iii and iv.

Task - Simpson's Formulas

Derive Simpson's formulas by substituting $\theta = \frac{x+y}{2}$ and $\beta = \frac{x-y}{2}$

in the identities derived in the previous task

$$\theta + \beta = \frac{x+y}{2} + \frac{x-y}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x$$

$\theta + \beta = x$

$$\theta - \beta = \frac{x+y}{2} - \frac{x-y}{2} = \frac{x+y-x+y}{2} = \frac{2y}{2} = y$$

$\theta - \beta = y$

Simpson's formulas are given below.

$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\sin x - \cos y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

$\cos x - \cos y = 2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

Exercise 2.6

1. 0
2. a) 2 tan A $\sin^2$ (x/2)
   b) Cos B (2cos B+1)
   c) Sin (a+b) sin (2-B)

3

a) $\sqrt{2 - \frac{\sqrt{2}}{4} + \sqrt{2 + \frac{23}{4}}}$

b) $2\sqrt{\frac{2+\sqrt{2}}{2}}$

c) $\frac{\sqrt{6}+\sqrt{2}}{4}$

d) $2\sqrt{2+3\sqrt{3}}$

4. a) $\frac{1-\sin A}{Cos\ A}$

b) 1-8cos^{2}A +8cos ^{4}B

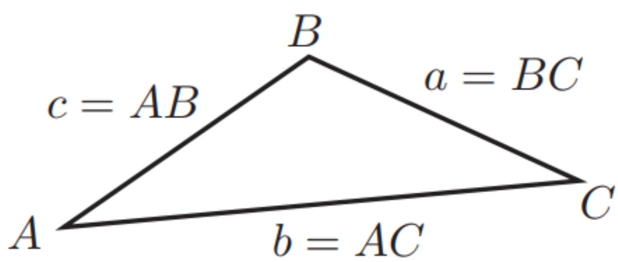
c) $\frac{Sin\ A}{Cos\ (A+1)}$

$$\frac{-3Sin\ C - 4\ sin^3x}{-3cosx + 4cos^3x}$$

The sine rule and cosine rule

Task - The sine rule

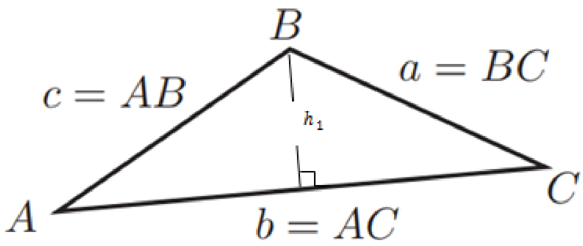
The figure below shows triangle *ABC*.



a. Drop a perpendicular line from vertex B to the opposite side of the triangle (AC) write two expressions for the length of the perpendicular.
b. Drop a perpendicular line from vertex A to the opposite side of the triangle (BC) write two expressions for the length of the perpendicular.
c. Hence, derive an expression relating the sines of the angles with the lengths.

Task – the cosine rule

In groups, derive the cosine rule $a^2 = b^2 + c^2$ +2ab Cos A for a triangle *ABC*



Hint: Use Pythagoras’ theorem in the two right-angled triangles created by the perpendicular from B to AC.

UNIT3

CALCULUS 1

Additional Math Secondary 3	Unit 3: Calculus 1
Learn about	**Key inquiry questions**
Learners should be introduced to differentiation of polynomials and composite functions by giving examples, investigate the second derivative of a function through discussion. They should use differentiation to find the tangent and normal to the tangent, maximum and minimum values of a function, velocity, acceleration and small increment: (approximate changes, connective rates of change). They should investigate and apply differentiation in solving practical problems and use the internet to supplement their learning and working with other students.	How do we differentiate polynomials and composite functions? How can we determine the second derivative of a function? What methods can we use to determine the maximum and minimum values of a function? How can we use differentiation to find the velocity and acceleration of an object? How can we use differentiation to find approximate values?

Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**

- Derivatives of polynomials, the composite functions, and the second derivative of a function - The application of differentiations to find the tangent, normal to the tangent, maximum and minimum values, velocity, acceleration, and small increments: (Approximate changes and connected rates of change)	- Apply the General rule to differentiate a polynomial and an appropriate formula to differentiate a composite function - Use graphical methods to find the maximum and minimum values of a function - Solve problems using differentiation - Carry out investigations of problems involving differentiation	- Appreciate the importance of differentiation - Develop liking for differentiation as a means of solving mathematical problems - Team work
Contribution to the competencies: Critical thinking: Analyse problems and develop solutions using differentiation Cooperation: Team work Communication: Communicate the knowledge of differentiation coherently and clearly to peers, teachers and others		
Links to other subjects: Physics: acceleration, velocity and displacement and trajectory Chemistry: heat enthalpy Geography: population studies		

Business Studies: rate of production or profit and loss; rates of change in share prices

ICT: Internet

Introduction

Learners should be introduced to differentiation of polynomials and composite functions by giving examples, and vivid illustrations, they should investigate the second derivative of a function through discussion.

They should use differentiation to find the tangent and normal to the tangent, maximum and minimum values of a function, velocity, acceleration and small increment: (approximate changes, connective rates of change). Learners should discuss the use of second derivatives in determining maximum, minimum and points of inflections.

The chain rule should be derived and used to discuss to determine the derivatives.

They should investigate and apply differentiation in solving practical problems in real life. And use the internet to supplement their learning and working with other students.

Teacher should guide learners to supplement their learning by using reference books and internet by working individually or with others.

Objectives

By the end of the lesson the learner should be able to,

1. Differentiate polynomials and composite functions.
2. Determine the second derivative of a function.
3. Determine the maximum and minimum values of a function using different techniques.
4. Use differentiation to find the tangent, normal to the tangent, maximum and minimum values, velocity, acceleration, and small increments: the velocity and acceleration of an object.
5. Use differentiation to approximate values in real life.
6. Use graph plotting software for example Geogebra or Desmos.

Using Desmos to investigate the change in gradient of a curve at different points.

Activity

In groups of four student Open Desmos in your computer carry out the activity below and answer the question that follow.

i. Define the function to investigate it change in gradient $f(x) = \sin x$ in first row

ii. Type gradient function $g(x) = \frac{d}{dx}(f(x))$, in second row hide this function

iii. Insert a general tension point $(c, f(c))$, in the third row. Animate it.

iv. Define the target line (tension line), $y = g(c)(x - c) + f(c)$ in forth row. Format it and animate by dragging $(c, f(c))$ for analysis as shown in figure 3. 3 below.

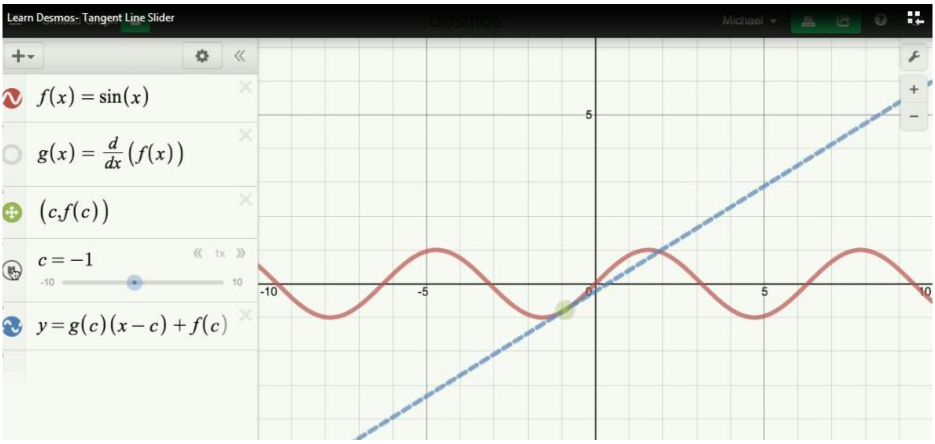


Figure 3.3

a. How many points in the interval $-10 \leq x \leq 10$ is the gradient 0?

b. How many times in the interval $-10 \leq x \leq 10$ does the wave have a positive gradient ?

c. How many times in the interval $-10 \leq x \leq 10$ does the wave have a negative gradient?

Further reading

Using the link 'https://en.savefrom.net/#url=http://youtube.com/watch?v=6K4I36_6fwM&utm_source=youtube.com&utm_medium=short_domains&utm_campaign=www.ssyoutube.com' find out technology can be use to investigate change in gradient of a function.

Exercise 3.1

1. a) 2
   b) $9x^2$
   c) $12x^2$
   d) $80x^3$
   e) $-4x^3$

2. a) 20
   b) $90x^2$
   c) 4
   d) 20
   e) $9x^2$

3. a) 96
   b) 15,309
   c) 156

Differentiation of Polynomials

Task

Find the derivative of the following functions

a) $y = 2x^3 + 3x^2 + 3x + 3$

b) $f(x) = x^4 + \frac{1}{3}x^3 + 20x^2 + 3$

Solution

a) If $y = 2x^3 + 3x^2 + 3x + 3$

$$\frac{dy}{dx} = 6x^2 + 6x + 3$$

b) $f(x) = x^4 + \frac{1}{3}x^3 + 20x^2 + 3$

$$f'(x) = 4x^3 + x^2 + 40x$$

Task

Use the function $f(x) = 3x^4 + 4x^2 + 40x$ to find the value of;-

a) $f'(x)$
b) $f'(0)$
c) $f'(20)$

Solution

a) $f(x) = 3x^4 + 4x^2 + 40x$

$$f'(x) = 12x^3 + 8x + 40$$

b) $f'(x) = 12x^3 + 8x + 40$

$$f'(0) = 0 + 0 + 40$$

$$f'(0) = 40$$

c) $f'(x) = 12x^3 + 8x + 40$

$f'(20) = 12 \times 20^3 + 8 \times 20 + 40$

$f'(x) = 96\ 200$

Exercise 3.2

1. a) $3x^2 + 8x$
   b) $3x^2 + 4x - 5 - 5x^{-6}$
   c) $2x + 1$
   d) $2x^3 + 3/2x^2 + ½$
2.) $48x^2 - 49x^2 + 6x^5$

b) $3/2x^2 + 3x$
c) $-8x^{-5} + 6x^2 + 10x$
d) $-3x + 4$
e) 2
f) $88x^{-5} - 3x^{-6/7}$
g) $x^{11} - 7x^6 - 3x^4$
h) $3 - 42^{-3}$

3. a) i) -2
   ii) -10
   iii) 12

b. i) -1

ii) -24

iii) 21

Derivative of composite functions

Exercise 3.4

1. a) $9x\,(x^2+2)^2$
   b) $24x^2$
   c) $2x$
   d) $\frac{3x^2+2}{2\sqrt{x^3+2x+1}}$
   e) $2\frac{1}{2}(2x + 3)\,(x^2 + 3x + 1)^{1/2}$
   f) $\frac{6}{7}x\,(x^2 - 4)^{-4/7}$
   g) $\frac{15}{3}(5x + 1)^{-2/3}$
   h) $\frac{5}{2}(5x + 1)^{2/5}$
   i) $42x\,(3x^2 + 3)^6$
   j) $2\,(-12x^{-5} - 4x^3 + 3x^2 + 2x)\,(3/x^4 + x - 4 + x^3 + x^2)$

2. i) 189

   ii) 300

3. a) i) 0

ii) 34,992

b. i) 0

ii) 0.898

Tangent and Normal to the Tangent of a Curve

Task

Show that at point (1, 8) on the curve $f(x) = x^3 + 2x^{2+} + 3x + 2$

The tangent has equation $y = 10x - 2$, and the normal has equation

$$y = -\frac{1}{10}x + \frac{81}{10}$$

Solution

$$f(x) = x^3 + 2x^{2+} + 3x + 2$$

$$f'(x) = 3x^2 + 4x + 3$$

$$f'(1) = 3 + 4 + 3 = 10$$

$$f'(1) = 10$$

But gradient of curve = Gradient of tangent.

The tangent pass through (1, 8), (x, y) and has gradient 10. Its equation is hence

$$\frac{(y-8)}{(x-1)} = 10$$

$$(y-8) = 10(x-1)$$

$$y = 10x - 10 + 8$$

$$y = 10x - 2$$

The gradient of normal M_2 is calculated as follows;

$M_1M_2 = -1$

$10M_2 = -1$

$M_2 = -\frac{1}{10}$

The normal line has gradient $M_2 = -\frac{1}{10}$, and passes through (1, 8) and (x, y). Its equation is hence

$$\frac{(y-8)}{(x-1)} = -\frac{1}{10}$$

$$10\,(y-8) = -1\,(x-1)$$

$$10y - 80 = -x + 1$$

$$10y = -x + 81$$

$$y = -\frac{1}{10}x + \frac{81}{10}$$

Exercise 3.5

1. i) $y = 2x + 8$

   ii) $y = ½\,x + 2\frac{1}{4}$

2. i) $y = 2x - 13$

   ii) $y = -½\,x - ½$

3. a) $y = 36x - 102$
   b) $y = -2x - 9$
   c) $y = -2$
   d) $y = 24x + 1$

4. a) $y = 2x$
   b) $y = -4$

c) $y = -\frac{1}{27} + 24/9$

d) $y = -4x + 18$

e) $y = \frac{1}{11}x + \frac{4}{3}$

f) $y = 256$

5. a) $y= \frac{2}{7}x^2 + \frac{18}{7}x + 4$

   b) $\frac{22}{7}$

   c) $y= -2/22x + \frac{123}{22}$

Second Derivative

Exercise 3.6

1. a) $528x^{10}$

   b) $600x^2 + 12x^2$

   c) $12x - 6$

   d) $24x + 48x - 1$

2. a) $6x$

   b) $40x + 18x^8$

   c) $-20/6x^{-1/6} + 360x^4$

   d) $12x + 6$

3. 328
4. 210
5. 6t

Application of differential functions

Task

You are given a function $y= f(x) = x^2 - 6x + 1$. Determine the coordinates of the stationary point.

a) Complete the table.

x	$x < c$	$x = c$	$x > c$
f'		0	
$f(x)$			

b) State the type of stationary point.

Solution

a) f has $n - 1$ stationary points, $n = 2$ hence 1 stationary point.

b) At stationary point $f'(x) = 0$

$f(x) = x^2 - 6x + 1$

$f'(x) = 2x - 6$

$f'(x) = 2x - 6 = 0$ when $x = 3$

$f(x) = x^2 - 6x + 1$

$f(3) = 3^2 - 6 \times 3 + 1 = -8$

$f(3) = -8$

The stationary point is $(3, -8)$

c)

x	$x < 3$	$x = 3$	$x > 3$
$f'(x)$	$f'(x) < 0$	$f'(x) = 0$	$f'(x) > 0$
$f(x)$	decreasing	stationary	increasing

d) The stationary point is a minimum.

Task

In pairs, show that the function $y = f(x) = x^3 - 3x^2 + 8$ has turning points at (0, 8) and (2, 4) and determine their type.

Solution

$f(x) = x^3 - 3x^2 + 8$

$f'(x) = 3x^2 - 6x = 0$ at stationary points

$3x(x - 2) = 0$

$3x = 0$ or $x - 2 = 0$

There are stationary points when $x = 0$ and $x = 2$

$f(0) = 0 - 0 + 8 = 8$

$f(2) = 2^3 + (3 \times 2^2) + 8 = 8 - 12 + 8 = 4$

The stationary points are (0, 8) and (2, 4) as required.

x	$x = -1$	$x = 0$	$x = 1$	$x = 2$	$x = 3$
$f'(x)$	$f'(-1) = 9$	$f'(0) = 0$	$f'(1) = -3$	$f'(2) = 0$	$f'(3) = 9$
$f(x)$	increasing	stationary	decreasing	stationary	increasing

Hence the maximum point is (0, 8) and the minimum point is (2, 4).

Exercise 3.7 a

1a) y- Intercept (0, 3)
x- Intercept (-3, 0), (1, 0)
Stationary point: (-1, -4) minimum points

b) x-intercept (-1.75, 0), (1.75, 0)
y-intercept (0, 3)
Stationary point (0, 3)

c) x-intercept (2, 0), (6, 0)
y-intercept (0, 12)
Stationary point (4,-4) minimum

d) x-intercept
y-intercept (0, 0)
Point of inflection (0,0) raising

2 a) x-intercept (-2, 0)

y-intercept (5, 0)

Turning point (2.5,-12.5) minimum

b) x-intercept (-3, 0) (0)
y-intercept (0,-3)
Turning point (-1, -4) minimum

c) x-intercept (-1, 0), (5, 0)
y-intercept (0, 5)
Turning point (2, 9) maximum point

d) x-intercept (0, 0)
y-intercept (0, 0)
Turning point (0, 0) falling inflection

Task

Sketch graphs of each type of stationary point (minimum, maximum and point of inflexion). Make a note on graph whether the function is increasing or decreasing (i.e. the gradient function positive or negative) on either side of the stationary point.
What can be said about the rate of change of the gradient function in each case?
Test out your answer on the functions in exercise 3.7a.

a) Maximum point

A point $(c, f\ (c))$ is a maximum point if $f'(c)\ =\ 0$ and $f''(c)\ <0$. As noted above the gradient function through a maximum is decreasing.
Figure 3.12 below illustrate these properly.

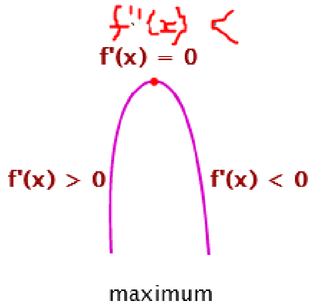


Fig 3.12

b) Minimum point

A point (c, f (c) is a minimum point if $f'(x = c) = 0$ and $f''(x = c) > 0$. At minimum point the function has first derivative equal to zero and second derivative a positive number. Figure 3.13 below illustrate the property.

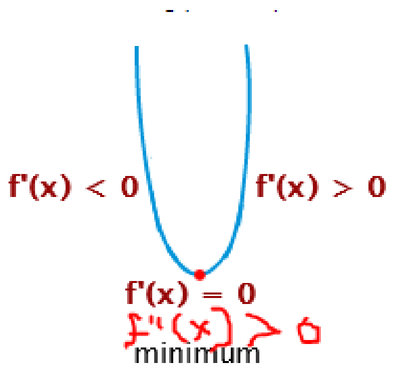


Fig 3.13

c) Point of Inflexion

If at point $(c, f(c), f'(x = c) = 0$ and $f''(x = c) = 0$ then this point is a point of inflexion.

Exercise 3.7b

1a) (0, 2) minimum

(2, 2) maximum

b) (0, 4) inflection
c) (3, 8) maximum
d) (0, 0) falling inflection
e) (0, 5) maximum, (1, 4) minimum

2. a)
 b) (0,4) falling points of inflection
 c) (0,5) maximum, (1,4) minimum
 d) (3, 27) falling point of inflection
 e) (-1, 0) minimum, (1,4) maximum
 f) (1, 5) maximum, (3,1) minimum

Displacement, Velocity and Acceleration

Task

The distance moved by a plane from point A is expressed by the function $s = 3t^4 + 6t^2 + 30$ in metres after time t in seconds. In groups of three students find

a) The distance of the plane from point A before it started moving.
b) The distance moved by the plane in the first 10 seconds.
c) Calculate the velocity of the plane at $t = 5$ seconds.
d) The acceleration of the plane at $t = 2$ seconds.

Solution

$h\ (t) = 3t^4 + 6t^2 + 30$

a) Before moving t = 0
   $h\ (t) = 3(0)^4 + 6(0)^2 + 30$
   $h\ (t)$ = 30 m

b) At t = 10

   h (10) = $3(10)^4 + 6(10)^2 + 30$

$h\ (10) = 30\ 630\text{m}$

$h\ (10) = 30.6\ \text{km}$

c) $h\ (t) = 3t^4 + 6t^2 + 30$

$v\ (t) = 12t^3 + 12t$

$v\ (t) = 12t^3 + 12t$

$v\ (5) = 12t(5)^3 + 12(5)$

$= 12(5)^3 + 12(5)$

$v\ (5) = 1560\text{m/s}$

d) $a = v'(t) = 36t^2 + 12$

$a\ (t) = 36t^2 + 12$

$a\ (2) = 36\times2^2 + 12$

$a\ (2) = 156\text{m/s}^2$

Exercise 3.8

1. t =2
2. h=1m
3. a) 70
   b) -15m/s
   c) -10
   d) t=1/2 sec
4. 34m/s
5. t=1
6. a=8
7. a) 10
   b) 44
   c) 17
   d) 40

Exercise 3.8

1. SSP 211
2. $l=w=h=1.29$
3. $r=6.2035$

   $h= 12.4070$
4. SSP 13,200
5. a) $V=108m^3$

   b) SSP 2700

UNIT 4

CALCULUS 2(INTEGRATION)

<table>
<tr><th colspan="2">Additional Math Secondary 3</th><th>Unit 4: Calculus 2</th></tr>
<tr><td colspan="2">Learn about</td><td>Key inquiry questions</td></tr>
<tr><td colspan="2">Learners should be introduced to the concept of integration to understand that integration is the opposite process of differentiation. They should investigate indefinite integration and explore different techniques of integration e.g. integration by substitution and by parts.

Learners should investigate how integration can be used in problem solving. They should supplement their learning by using the internet and working with others.</td><td>- How can we distinguish between integration and differentiation?
- How do we explain the differences between definite and indefinite integral?
- What are the concepts used in deriving formulae of integrations?</td></tr>
<tr><th colspan="3">Learning outcomes</th></tr>
<tr><td>Knowledge and understanding</td><td>Skills</td><td>Attitudes</td></tr>
<tr><td>- The definition of integration, integration as opposite of differentiation, indefinite integration, integration by substitution and by parts</td><td>- Solve problems of integration by using substitution and by parts
- Use integration to solve mathematical problems
- Carry out investigations in solving</td><td>- Value the use of integration as an approach to solving mathematical problems
- Appreciate the use of integration in real life situations
- Team work</td></tr>
</table>

	integration problems	

Contribution to the competencies:

Critical thinking: Analyse and develop solutions to problems using integration

Cooperation: Team work

Communication: Communicate the knowledge of integration coherently and clearly to peers, teachers and others

Links to other subjects:

Physics: acceleration, velocity and displacement, trajectory and moments of inertia

Chemistry: heat enthalpy and reactions

Geography: population studies

Business Studies: rate of production or profit and loss

ICT: Internet

Introduction

The term integration may be new to learners. The teacher should systematically guide earners to relate differentiation with integration. Learners should discuss how functions and derivative relate with the integrals.

Objective

By the end of the topic the learners should be able to:-

- Define integration as opposite of differentiation.
- Determine indefinite integral of basic function by substitution and by part.
- Use integration to solve mathematical problems.
- Carry out investigations in solving integration problems.

Notes and possible approaches

Objective 1

Learners should be guided to deduce how integration of a general function of x is done. Class discussion should be emphasized.

Objective 2

Class discussion should be used to deduce indefinite integral. Teacher to guide learners to relate the chain rule and integration. Class discussion and problem solving is used are used to solve examples of questions on integration by u-substitution. Only u-substitutions that apply to algebraic functions are discussed. Learners are not introduced to solve differentials in trigonometric, logarithms and exponential functions. Teacher to guide learners to algebraic functions.

Concept of definite integral should not be discussed in this area.

Task

Copy and complete table 4.1 below on derivatives of functions

Table 4.1

Function of x	Derivative $f'(x)$
$f(x) = x^3$	
$f(x) = x^3+3$	
$f(x) = x^3 + 5$	
$f(x) = x^3+8$	
$f(x) = 9$	

a. What do you notice about the $f(x)$ and $f'(x)$**?**

Solution

Function of x	Derivatives
$f(x) = x^3$	$f'(x) = 3x^2$
$f(x) = x^3+3$	$f'(x) = 3x^2$
$f(x) = x^3 + 5$	$f'(x) = 3x^2$
$f(x) = x^3+8$	$f'(x) = 3x^2$
$f(x) = 9$	$f'(x) = 0$

Exercise 4.1

a) $\frac{2}{3}x + c$

b) $x^2 + c$

c) $\frac{nx^3}{3} + c$

d) $\frac{1}{9}x^9 + c$

e) $\frac{1}{10}x^5 + c$

f) $\frac{2}{3}x + c$

g) $\frac{1}{9}x^7 + c$

2.

a) $\frac{2}{3}x^3 + c$

b) $\frac{1}{4}x^4 + c$

c) $\frac{1}{5}x^5 + c$

d) $\frac{5}{6}x^{6/5} + c$

e) $\frac{6}{7}x^{7/6} + c$

f) $\frac{4}{9}x^9 + c$

g) $\frac{1}{8}x^8 + c$

Exercise 4.2

1.

a) $\sqrt[2]{\frac{2}{3}x^{3/12} + c}$

b) $\frac{x^5}{5} + \frac{14}{3}x^3 + 49x + c$

c) $\frac{125}{x}x^7 + c$

d) $-\frac{3}{x} + \frac{9}{2x^2} + c$

e) $\frac{1}{2}x^2 + 2x + c$

f) $\frac{1}{2}x^2 + 2x + c$

2.

a) $\frac{1}{3}x^3 + \frac{11}{2}x^2 - 25x + \frac{139}{6} + c$

b) $\frac{1}{5}t^5 + \frac{1}{t}x^4 - \frac{1}{3}t^3 + \frac{1}{2}t + t + c$

c) $\frac{2}{3}x^9 - 4x^5 + \frac{1}{3}x^3 + 9x + c$

d) $\frac{1}{7}x^7 + \frac{4}{5}x^5 - \frac{1}{2}x^2 + c$

e) $10t^4 + 4t^3 - \frac{9}{2}t^2 + 14t + c$

f) $10t^4 + 4t^2 - \frac{9}{2}t^2 + c$

g) $x + c$

3. $f(x) = \frac{1}{8}x^7 + 2x^3 + c$

4. $y = \frac{3}{2}x^2 + 4x + 2$

Integration by substitution

Exercise 4.3

1.

a) $\frac{1}{4}(x+4)^4+c$

b) $\frac{x^5}{5}+\frac{14}{3}x^3+49x+c$

c) $\frac{125}{7}t^7$+c

d) $2^{-1}\theta(5x-11)^3+c$

e) $\frac{-1}{11}(3\text{-}x)^{11}+c$

f) $\frac{2}{21}(7t+9)^{3/2}+c$

2.

a) $4(3x+1)^4+c$

b) $\frac{1}{3}(x^2+3x)^3+c$

c) $\frac{2}{3}(x^2+1)^{3/2}+c$

d) $\sqrt{x^2+1}+c$

e) $\frac{1}{8}(x^2+5x)^8+c$

3.

a) $\frac{-8}{x^3+1}+c$

b) $\frac{4}{3(1-x^2)^3}+c$

c) $-\frac{3}{x}+\frac{9}{2x^2}+c$

d) $\frac{-1}{x^2+1}+c$

e) $\frac{3}{8}(1+x^4)^{2/3}+c$

1. $y=\frac{1-(1-9x^2)^2}{12}$

Application of integration

Task 1

In groups, find out what the derivatives represent given that s =displacement, v =velocity and a =acceleration.

a. $\frac{ds}{dt}$

b. $\frac{dv}{dt}$

c. $\frac{d^2s}{dt^2}$

Solution

$\frac{ds}{dt} = v$ called velocity.

$\frac{d^2s}{dt^2} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{dv}{dt} = a$ is called acceleration

Task 2

In groups, find out what the integrals represent given that
s =displacement, v =velocity and a =acceleration.

a. $\int a\, dt$
b. $\int v\, dt$

Solution

a. $\int a\, dt = v$, integral of acceleration is velocity

$\int v\, dt = s$, integral of velocity is displacement.

Exercise 4.4

1. $\frac{x^2}{4(1+x)^2} - \frac{1}{4(1-x)^2} + c$

2. $\frac{2}{5}x(1+x)^{\frac{3}{2}} - \frac{4}{5}(1+x)^{\frac{5}{2}} + c$

3. $-\frac{(3-10x^3)^5}{150}$+c

4. $\frac{-2}{5}(3-10x^3)^5 + c$

5. -1/9 $x^{3/2} + c$

6. $\frac{-3}{10(5x^2+4)} + c$

UNIT 5

MATRICES

Additional Math Secondary 3	Unit 5: Matrices
Learn about	**Key inquiry questions**
Learners should revisit their knowledge and understanding of 2x2 matrices. They should then introduced to a 3x3 matrix, workout the inverse of a 3x3 matrix, find the determinant of a 3x3 matrix and investigate the properties of its determinant through discussion. They should use co-factors and determinants to derive Crammer's rule and use it to solve simultaneous linear equations. Learners should investigate the use of matrices in problem solving and supplement their learning by using the internet and working with others.	• How can we determine 3×3 matrices? • How can we find the inverse of a 3x3 matrix? • When do we use the determinant of a 3x3 matrix? • Why is it important to know the properties of determinant? • How can we apply Crammers' rule to solve problems?

Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**
• 3x3 Matrices, determinants of matrices and Crammer's rule	• Find out the determinant and inverse of a 3x3 matrix • Use the properties of determinant in solving linear equations • Use Crammer's rule in solving simultaneous linear equations	• Show interest in the topics related to matrices • Value the use of 3x3 matrices in solving problems • Develop interest in designing and managing simple

	- Use matrices in real life situations - Investigate the use of 3x3 matrices to solve real life problems	programmes for storing information in computer - Teamwork

Contribution to the competencies: Critical thinking: Analyse and develop solutions to problems using matrices Cooperation: Team work Communication: Communicate the knowledge of matrices coherently and clearly to peers, teachers and others
Links to other subjects: Business Studies: commercial transaction Agriculture : productions Humanities: storage of information, etc. ICT: Internet

Introduction

The concept of matrixes is not new to learners. A 2x2 matrix has been discussed in details in the previous units. Teacher is to introduce 3 x3 matrices simplifications. An introduction should be done to solving systems of bigger square matrices. Emphasis should be made on determining inverse and determinant of 3 x 3 matrices using different methods. Solution of simultaneous equations of three variables should be done through discussion on real life problems and matrices.

The objectives

By the end of the lesson the learners should be able to;

1. Find out the determinant and inverse of 3 X 3 matrices and their properties.
2. Solve simultaneous linear equations with three unknowns.
3. Use calculators in solving equations.
4. Use matrices in solving real life related situation.

Methodology

Learners should investigate use of matrices and their properties through class discussion. Learners should use calculators for finding determinants and solutions to systems of simultaneous linear equations.

Using scientific calculator to operate on matrices

For example one may use Casio FX911ES PLUS CALCULATOR

Activity 1

In pairs, use calculator to simplify,

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Solution,

On your calculator, press on button, then mode button and select Matrix (using key number 6) as shown in figure 5.1 below. Select matrix A .Choose the order of matrix from the screen. and enter $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ it into the calculator then

Press the '=' and save the matrix by pressing M+ sign. Repeat the process and save matrix B as $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Open matrix memory (press shift then 4) And type the question you intent to solve, for example.

Matrix A × matrix B, press = to display the solution. The solution is displayed.

$$\begin{pmatrix} 6 & 6 \\ 3 & 3 \end{pmatrix}$$

Exercise 5.1

1a) 2×2

b) 2×1

c) 2×3

d) 4×2

2a) $\begin{pmatrix} 5 & 5 \\ 7 & 5 \end{pmatrix}$

$b)\begin{pmatrix} 8 & 4 \\ 7 & 9 \\ 13 & 8 \end{pmatrix}$

c) Not compatible

d) Not compatible

$e)\begin{pmatrix} 3 & 2 & 1 \\ -2 & -2 & -5 \\ -7 & -1 & -6 \end{pmatrix}$

3.

a) $\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 4 & 3 \\ 3 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 3 \\ 10 & 10 \end{pmatrix}$

4.

a) $\begin{pmatrix} 2 & 4 \\ 8 & 4 \end{pmatrix}$

b) $\begin{pmatrix} 9 & 6 \\ 12 & 12 \end{pmatrix}$

c) $\begin{pmatrix} 5 & 4 \\ 11 & 4 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 8 & 4 \end{pmatrix}$

5. a) -7

b) 2

c) -10

d) 4

6.

a) ½ $\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}$

b) 1/-4 $\begin{pmatrix} 2 & -4 \\ -3 & 2 \end{pmatrix}$

c) $\begin{pmatrix} 1 & -3 \\ -2 & 7 \end{pmatrix}$

d) $\frac{1}{22}\begin{pmatrix} 5 & -4 \\ -2 & 6 \end{pmatrix}$

e) $\frac{1}{3}\begin{pmatrix} 0 & -2 \\ -1 & 7 \end{pmatrix}$

f) No inverse

7. i) 1   ii) ±4   iii) 2 or -3

8.

a) $p = 1, q = 2$

b) $a = 2, b = 3$

c) $x = 10, y = 11$

d) $x = -2, y = 4$

Determinant of a 3 × 3 matrix

Using a scientific calculator

Task

Use a calculator to show the determinant of the matrix.

A = $\begin{pmatrix} 2 & 1 & 2 \\ -4 & 0 & 1 \\ 3 & 0 & -1 \end{pmatrix}$, det $A = -1$

Exercise 5.2

1. 23
2. 23
3. -2
4. 165
5. 6
6. 0

Task

In groups of four show that the determinant of matrix A = $\begin{pmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{pmatrix}$ is 165.

Solution

|A| = 0 $\begin{vmatrix} -6 & 9 \\ 6 & 1 \end{vmatrix}$ - 1 $\begin{vmatrix} 3 & 9 \\ 2 & 1 \end{vmatrix}$ + 5 $\begin{vmatrix} 3 & -6 \\ 2 & 6 \end{vmatrix}$

|A| = 0. (-60) - 1 (-15) +5(30) = 0 + 15 + 150

|A| = 165

Exercise 5.3

1. 0
2. 0
3. 6
4. 24
5. -12
6. -66
7. 39
8. -2

Inverse of a 3 × 3 and other matrices

Task 1

Show that the inverse of matrix A = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Using a calculator to determine inverse of a matrix

Task

In pairs, use a calculator to check the previous result.

Exercise 5.4

1. $\begin{pmatrix} 4 & 3 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

2. $\begin{pmatrix} 4 & 8 & 10 \\ 3 & 0 & 4 \\ 7 & 10 & 1 \end{pmatrix}$

3. $\begin{pmatrix} 12 & 3 & 0 \\ 11 & \frac{1}{2} & 4 \\ \frac{1}{2} & 1 & 7 \end{pmatrix}$

4. $\begin{pmatrix} 0 & 4 & 7 \\ 1 & 3 & 8 \\ -1 & 4 & 9 \end{pmatrix}$

5. $\frac{1}{3}\begin{pmatrix} 3 & -3 & 0 \\ -1 & -3 & -1 \\ -2 & 3 & 1 \end{pmatrix}$

6. $\frac{1}{5}\begin{pmatrix} 0 & 5 & 5 \\ 6 & -3 & 0 \\ 3 & 1 & -5 \end{pmatrix}$

7. $\frac{1}{10}\begin{pmatrix} 2 & 2 & 10 \\ -2 & 3 & 10 \\ 2 & -3 & 10 \end{pmatrix}$

8. $\frac{1}{4}\begin{pmatrix} 0 & 1 & 5 \\ 8 & -4 & 0 \\ 4 & -1 & 0-5 \end{pmatrix}$

9. $\frac{1}{4}\begin{pmatrix} -2 & -12 & -12 \\ 3 & -16 & 5 \\ -1 & 6 & -3 \end{pmatrix}$

Solving simultaneous linear equations using the inverse of a matrix

Task

Solve the simultaneous equation.

$$x + 2y - z = 7$$

$$2x - 3y - 4z = -3$$

$$x + y + z = 0$$

Solution

For the matrix equation

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & -4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 0 \end{pmatrix}$$

A B C

Where A = Coefficient Matrix

B = Variable Matrix

C = Constant matrix

$$A^{-1} = \frac{1}{16} \begin{pmatrix} -1 & 3 & 11 \\ 6 & -2 & -2 \\ -5 & -1 & 7 \end{pmatrix}$$

Multiplying both sides of AB = C by A^{-1}

$$\frac{1}{16} \begin{pmatrix} -1 & 3 & 11 \\ 6 & -2 & -2 \\ -5 & -1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & -4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{16} \begin{pmatrix} -1 & 3 & 11 \\ 6 & -2 & -2 \\ -5 & -1 & 7 \end{pmatrix} \begin{pmatrix} 7 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$ Hence $\begin{aligned} x &= -1 \\ y &= 3 \\ z &= -2 \end{aligned}$

Using calculator to solve the simultaneous equations

Task

Attempt in groups.

A quadratic equation of a curve has a general form of $y = ax^2 + bx + c$ and passes through points (-2, 4), (2, 2) and (4, 4). Determine the equation of the curve.

Hint: substitute for *x* and *y* for each of the points to derive three simultaneous equations for *a*, *b* and *c*.

Solution

The equation has three unknowns, three simultaneous equations will be enough to solve the equation. Substituting the coordinate of the point to the equation we get

4a – 2b + c = 4

4a + 2b + c = 2

16a + 4b +c = 4

To solve the four consider the coefficient and constant matrix as shown below,

$$\left\{\begin{array}{ccc|c} 4 & -2 & 1 & 4 \\ 4 & 2 & 1 & 2 \\ 16 & 4 & 1 & 4 \end{array}\right\}$$

Open the calculator, press mode and select equations (enter 5) as shown in figure 5. 2 below then equations with three unknowns and enter (press 2). Insert the system matrix and press '=' sign then press replay to list all the constants

Figure 5.2

a = $\frac{1}{4}$, b= $\frac{-1}{2}$and c = 2 and hence $y = \frac{1}{4}x^2 - \frac{1}{2}x + 2$

Exercise 5.5

1a) $x = 2, y = 1, \neq 2$

b) $x = 1, y = 3, \neq 5$

c) $x = 4, y = 1, \neq 2$

d) $x = 6, y = 2, \neq 0$

e) $x = 2, y = 0, \neq 1$

2. $p = 10, b = 20, r = 15$

3. $x = 100m, \; y = 300m, \; \neq 400m$

Exercise 5.6

1. $x = 1, y = 2, z = 1$
2. $x = -2, y = 1, z = 3$
3. $x = 2, y = 3$
4. $x = 2, y = -3, z = -1$
5. $x = 1, y = -3, z = 3$
6. $x = 2, y = 0, z = 2$
7. $x = 1, y = 2, z = 3$
8. $x = 2, y = 4, z = 0$

9. $x = 2/3, y = 7, z = -\frac{4}{3}$

10. $x = -2, y = 3$

UNIT 6

COMPLEX NUMBERS

<table>
<tr><td colspan="2">Additional Math Secondary 3</td><td>Unit 6: Complex Numbers</td></tr>
<tr><td colspan="2">Learn about</td><td>Key inquiry questions</td></tr>
<tr><td colspan="2">Learners should investigate the concepts of complex numbers and should be able to add and subtract, multiply and divide complex numbers.

They should investigate how complex numbers could be applied in other subjects and supplement their learning by using the internet and working with others.</td><td>• In what ways do complex numbers differ from real numbers?
• How can we carry out operations using complex numbers?
• How useful are complex numbers in solving problems?</td></tr>
<tr><td colspan="3">Learning outcomes</td></tr>
<tr><td>Knowledge and understanding</td><td>Skills</td><td>Attitudes</td></tr>
<tr><td>• Concepts (introduction) and definition
• Addition and subtraction, multiplication and division of complex numbers</td><td>• Carry out operation in solving problem involving complex numbers
• Investigate the use of complex numbers in everyday life situations</td><td>• Appreciate the use of complex numbers in calculating mathematical problems</td></tr>
</table>

Contribution to the competencies: Critical thinking: Analysis and ability to solve problems efficiently Co-operation: Respect others opinions Communication: Use social networks e.g. Facebook, Twitter and YouTube to create and access information
Links to other subjects: Physics: Vectors, determine amplitude in electrical circuit ICT: Internet

Introduction

The concept of complex numbers is very new to learners. Previously, learners were taught that root of numbers less than 0 do not exist. A teacher should erase this misconception by reviewing of previous solution that leads to complex numbers and by discussion orient learners on complex numbers.

In this unit only simplification and operations on complex numbers are done. Complex numbers will further be discussed in secondary 4. Teacher should encourage learners to investigate complex numbers further and supplement the class work by internet and group discussion.

The objectives

By the end of the lessons the learners should be able to:-

1. Define complex numbers, imaginary numbers and real number.
2. Distinguish complex numbers from real numbers.
3. Add, subtract, multiply and divide complex numbers.
4. Simplify complex numbers by conjugation.
5. Link complex numbers to the solution of quadratic equations and add the Argand diagram.

Methodology

Learners should use discussion and problem solving techniques to solutions of complex numbers.

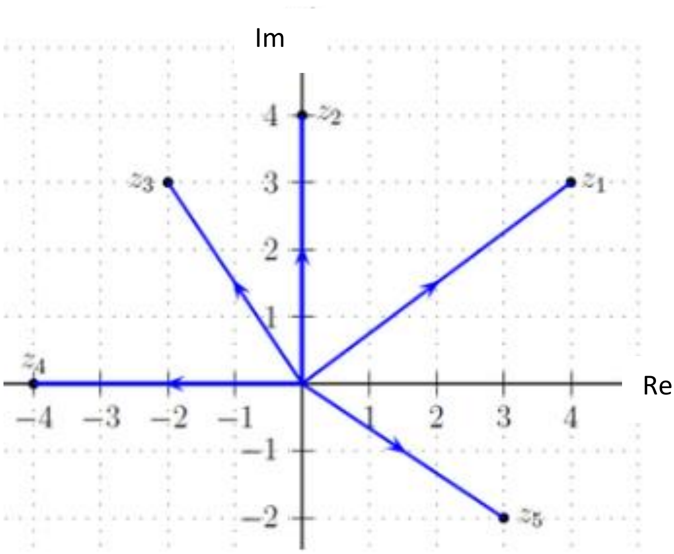
Graphical Representation of complex numbers

Task

Draw a Cartesian plane and show the position of the following complex numbers.

i) $z_1 = 4 + 3i$
ii) $z_2 = 0 + 4i$
iii) $z_3 = -2 + 3i$
iv) $z_4 = -4$
v) $z_5 = 3 - 2i$

Solution



Simplifying imaginary numbers

Exercise 6.2

1. $3i$
2. $2i$
3. $3/5\ i$
4. $5/7\ i$
5. $2i$
6. $11i$

7. 21
8. 8
9. $13i$
10. $1.4142i$
11. $1.732i$
12. $10i$
13. $9i$

Addition and subtraction of complex numbers.

Task

Given the complex numbers, $z_1 = 2 + 3i$ and $z_2 = 3 + i$. In pairs

i. Find the value of ,

a) $2z_1$
b) $3z_2$
c) $z_3 = z_1 + z_2$
d) $5z_1 + 7z_2$

i. Draw an Argand diagram of z_1, z_2, and z_3

ii. What do you notice about the addition of complex numbers?

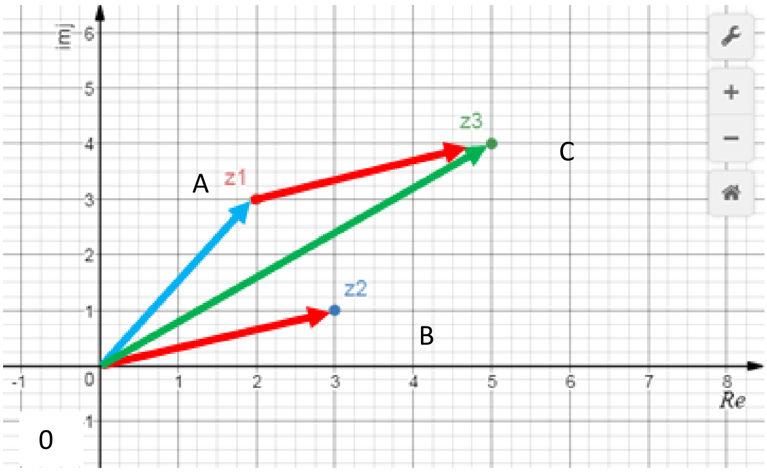
Solution

a) $2z_1 = 2(3 + i)$
$= 6 + 2i.$

b) $3z_2 = 3(3 + i)$
$= 9 + 3i.$

c) $z_3 = 2 + 3i + 3 + i$
$= 5 + 4i.$

d) $5z_1 + 7z_2 = 5(2 + 3i) + 7(3 + i)$
$= 10 + 15i + 21 + 7i.$
$= 31 + 22i.$



You notice the complex numbers are illustrated like position vector. Since vectors are free space the vector of z_2 has an equivalent vector as illustrated in the diagram above after transformation.

Exercise 6.3

1. a) 7 + 2i

   b) 7i + 7

   c) i + 4

   d) -5-8i

2. a) 6 + 3i

   b) 6 + 6i

   c) -3/2 + 2i

   d) -6 – 4i

   e) -4/3 – 2i

3. a) 9i + 12

   b) 15i -5/2

   c) 21i + 13

   d) -57i -2

   e) 33/14 I + 29/14

Multiplying complex numbers

Task

Copy and complete the table below

n	0	1	2	3	4	5	6	7	8
i^n	i^0	i^1	i^2	i^3					
simplified	1		-1						

Plot the points on an Argand diagram. What do you notice?

Task

In pairs, expand the following expressions.

1. $(a + bi)(a + bi)$
2. $(a + bi)(a - bi)$
3. $(a - bi)(a + bi)$
   What do you notice?

Solution.

By long method of expansion you notice the following relationships:

1. $(a + bi)(a + bi) = (a + bi)^2 = a^2 + 2abi - b$
2. $(a + bi)(a - bi) = (a - bi)^2 = a^2 - 2abi - b$
3. $(a - bi)(a + bi) = a^2 + b^2$

Task

You are given that $z_{1=} (3 + 2i)$ and $z_{2=} 4 + 2i$, in pairs ,

a. Find the value of $z_3 = z_1' = i\,(3 + 2i)$ and $z_4 = z_2' = i\,(4 + 2i)$
b. Illustrate $z_{1,} z_2, z_1'$ and z_2' on an Argand diagram.
c. What transformation is represented by multiplication by i?

Exercise 6.4

1. a)7- 9i
   b) +8 -14i
   c) -8 + 4i
   d) 1 – 46i
2. a) 5

b) $\sqrt{650}$
c) 5/12

3. a) -5 -6i
b) 8 -27i
c) -9 + 60i

The modulus of a complex number

Task 1

Given the complex numbers $z_1 = 5 + 2i$, $z_2 = 3 + 4i$.

In pairs, find the value of

i. $z_1 + z_2$
ii. z_1^*
iii. z_2^*
iv. $z_1 z_2^*$
v. $|z_1|$

Solution

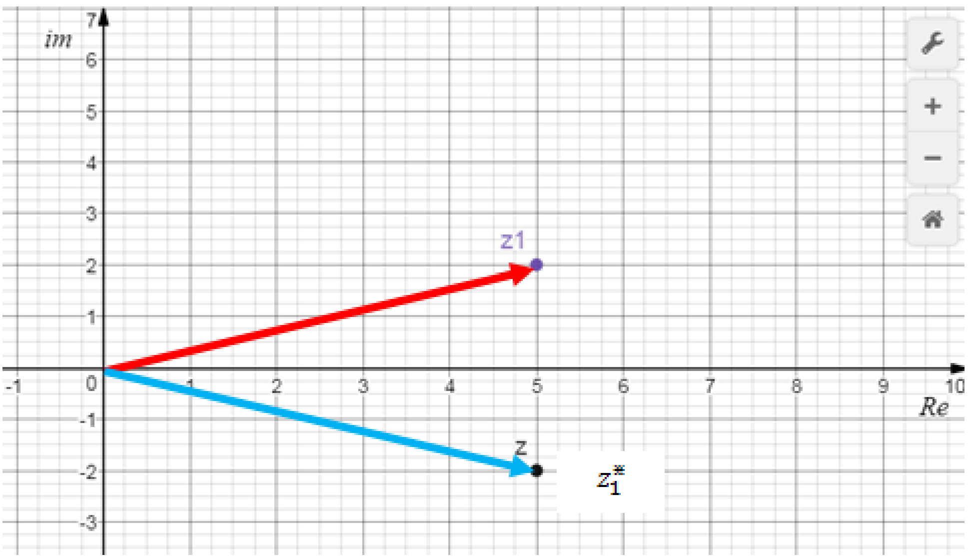
$z_1 = 5 + 2i.$

$z^* = 5 - 2i.$

$z_2 = 3 + 4i.$

a) $z_1 + z_2 = 5 + 2i + 3 + 4i$
$= 8 + 6i.$
b) $z^*{}_1 = 5 - 2i$
c) $z^*{}_2 = 3 - 4i$
d) $z_1 z^*{}_2 = (5 + 2i)(3 - 4i)$
$= 5(3 - 4i) + 2i(3 - 4i).$
$= 15 - 20i + 6i - 4(i)^2.$
$= 15 - 14i + 4.$
$= 19 + 14i.$
e) $|z_1| = \sqrt{z_1 z^*{}_1}$

$$= \sqrt{(5 + 2i)(5 - 2i)}.$$
$$= \sqrt{5(5 - 2i) + 2i(5 - 2i)}.$$
$$= \sqrt{25 - 10i + 10i - 4(i)^2}.$$
$$= \sqrt{25 - 4(-1)}.$$
$$= \sqrt{25 + 4}.$$
$$= \sqrt{29}.$$
$$= 5.385$$



vi. You notice that z_1 is reflected in the real axis to z_1^*

Exercise 6.5

1. 6
2. $a)\ \sqrt{13}$
   $b)\ \sqrt{10}$
   c) 5
   d) 1
3. a. 2 – 3i
   b. 3-2i
   c. 11+ 5i

$d. \sqrt{7}$
$e. \sqrt{26}$
f. 13
e) 95

Dividing complex numbers

Exercise 6.6

1. a) $\frac{11}{4} - 5/4$ i

   b) $1 + i$

   c) $2 + \frac{3}{2}$ i

   d) $\frac{1}{5} + \frac{2}{5}$i

2. a) $\frac{13}{5} - \frac{11}{5}i$

   b) $\frac{1}{5} - \frac{23}{10}\ i$

   c) $\frac{31}{37} - 114i$

3. a) $2 - 4i$

   b) $9 + 9i$

   c) $39 + 30i$

   d) $10 - 7i$

   e) $\frac{1}{11}(14 + 10i)$

   f) $\frac{1}{24}\ (62 - 5i)$

Solving Quadratic Equations with Complex Numbers

Exercise 6.7

1. Solve the following quadratic equations and state the types of roots.
   a) $3x^2 + 4x + 5 = 0$
   b) $14x^2 + 9x + 10 = 0$
2. A farmer intends to make a flower garden as shown in figure 6.9 of the area $75m^2$. The ratio of the sides are 3:1, what dimensions will his garden be?
   Garden will be 5m by 15m.

UNIT 7

THE CIRCLE

Additional Math Secondary 3	**Unit 7: The Circle**
Learn about	**Key inquiry questions**
Learners should investigate how to derive the equation of a circle where the centre is at the origin. They should discuss the general form of an equation of a circle; the equation of a circle that satisfies special conditions such as passing through three points, the equation when ends of diameter in it are given, and when passing through two points and its centre lies on a given straight line. Learners should determine the equation of tangent to a circle at a point on it and the length of tangent drawn to a circle from an external point. They should investigate the use of equations of circles in solving problems individually or in groups.	- How can we derive equation of a circle whose centre is the origin? - How can we demonstrate the general form of an equation of a circle? - In what way can we find the equation of a circle that satisfies 3 special conditions? - How can we find the equation of a tangent to a circle at a point on it? - How can we find the length of tangent drawn to a circle from an external point?

Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**

• The equation of the circle whose centre is at the origin • General form of equation of the circle • Equations of circles that satisfy special conditions such as passing through three points, when ends of diameter in it are given and the equation of a circle passing through two points and its centre lies on a given straight line • The equation of tangent to a circle at a point on it • Length of tangent drawn to a circle from an external point	• Derive the equation of the circle when the centre and radius are given • Use the general form of the equation of the circle in solving problems • Investigate the length of a tangent drawn from an external point to a circle • Solve mathematical problems involving equations of circles • Demonstrate how to derive equation of the circle	• Appreciate the general form of an equation for an equation • Appreciate the use of equation of a circle • Value the concept of equation of a circle • Team work

Contribution to the competencies:

Critical thinking: in the use of the application of different equations of the circle

Co-operation: team work

Communication: communicate their knowledge of circle coherently and clearly to peers teachers and others

Links to other subjects:

Fine arts: drawing circle

Geography: pie charts

TVET: Technical drawing

ICT: Internet

Introduction

The concept of circle is not new to learners, however equation of circles have not been discussed earlier. Learners should be guided in discussing circle and their equations. The geometrical properties of circle should be introduced in deriving the equation of circles, tangent and normal. Length of tangent should also be discussed.

The objectives

By the end of the lesson learners should be able to;

a) Derive and use equation of a circle given the centre and radius.
b) Determine the centre and radius of a circle given its equation.
c) Determine the equation of normal and tangent to a circle at a point.
d) Calculate the length of a tangent from an external point to a circle.
e) Determine the equation of a circle given three of its points.

Teaching method

Learner should draw circles and investigate properties of circles and their relationship to the equation, tangent and normal. Class discussion method should be used and problem solving approach should be used. Use of graphing software e.g. Geogebra or Desmos.

The equation of a circle, centre the origin

Exercise 7.1

1. a) $x^2 + y^2 = 9$

   b) $x^2 + y^2 = 25$

   c) $x^2 + y^2 = 13$

   d) $x^2 + y^2 = 1/4$

   e) $x^2 + y^2 = 9/4$

2. a) 5

   b) 10

   c) 9

   d) 1

   e) $\sqrt{29}$

   f) $\sqrt{21}$

   g) $\frac{5}{13}$

3. a) $x^2 + y^2 = 25$

   b) $x^2 + y^2 = 9$

   c) $x^2 + y^2 = 4$

   d) $x^2 + y^2 = 2$

   e) $x^2 + y^2 = 26$

   f) $x^2 + y^2 = 26$

   g) $x^2 + y^2 = 144$

   h) $x^2 + y^2 = \frac{13}{36}$

   i) $x^2 + y^2 = \frac{65}{12}$

4. a) 1544

   b) $28\frac{2}{7}$

   c) $50\frac{2}{7}$

   d) $6\frac{2}{7}$

   e) 22

   f) $56\frac{4}{7}$

The equation of a circle centre (h, k)

Exercise 7.2

1.

a) $r = 3, c\ (2, -5)$

b) $r = 2, c\ (-3, 5)$

c) $r = c, c\ (-a, -b)$

d) $r = 11, c\ (2, -4)$

e) $r = 21, c\ (-2, 7)$

f) $r = ½, c\ (½, -½)$

g) $r = \sqrt{3}, c\ (2, 3)$

h) $r = \sqrt{17}, c\ (-17, 2)$

2. i) $(x-1)^2$ -+ $(y-2)^2 = 4$

ii) $(x+1)^2 + (y-4)^2 = ¼$

iii) $(x+1)^2 + (y+3)^2 = 9$

iv) $(x-8)^2 + (y-4)^2 = 256$

v) $(x - 0.3)^2 + (y - 0.7)^2 = \frac{1}{16}$

vi) $(x + \frac{1}{3})^2 + (y + \frac{1}{3})^2 = \frac{1}{9}$

vii) $(x - 7)^2 + (y-8)^2 = 196$

3. a) $x^2 + y^2 = 10^2$

b) $(x-2)^2 + (y-3)^2 = 10$

c) $(x+4)^2 + (y+3)^2 = 25$

d) $(x + 12)^2 + (y+7)^2 = 400$

e) $(x-7)^2 + (y-2)^2 = 225$

f) $(x-7/12)^2 + (y-9/2)^2 = 25/4$

Find the equation of a circle given two points on its diameter

Exercise 7.3

1. $(x + 1)^2 + (y - 2)^2 = 10$
2. $(x-7/2)^2 + (y-5)^2 = 25$
3. $(x-1)^2 + (y-7)^2 = 100$
4. $(x - 2)^2 + (y-1)^2 = 400$
5. $(x-5.75)^2 + (y + 10)^2 = 6.25$
6. $(x + 15)^2 + (y + 10)^2 = 400$

The expanded general equations of a circle

Task

By completing the square show that the centre is (3, -1) and the radius is $\sqrt{2}$ for the circle whose equation is

$$x^2 + y^2 - 6x + 2y + 8 = 0$$

Solution

Making equation to general format

$$(x - h)^2 + (y - k)^2 = r^2$$

Rearrange x and y terms: $x^2 - 6x + y^2 + 2y = -8.$

Make a perfect square of x and y by adding $(\frac{-6}{2})^2$ and $(\frac{2}{2})^2$

On both sides

$$x^2 - 6x + y^2 + 2y = -8$$

$$x^2 - 6x + \left(\frac{-6}{2}\right)^2 + y^2 + 2y + \left(\frac{2}{2}\right)^2 = -8 + \left(\frac{-6}{3}\right)^2 + \left(\frac{2}{2}\right)^2$$

$$x^2 - 6x + (-3)^2 + y^2 + 2y + (1)^2 = -8 + 9 + 1$$

By factorization and generating perfect squares

$$(x-3)^2+(y+1)^2=2$$

Comparing to general equation

H = 3, k = 1 and $r=\sqrt{2}$

Hence center r (3, 1) and radius $\sqrt{2}$

Exercise 7.4

1. a) $r = 0.4, c\ (0,0)$

$$b)\ r = 3, c\ (2,-5)$$

$$c)\ r = 5, c\ (2,3)$$

$$d)\ r = 2, c\ (-5,-7)$$

$$e)\ r = 100, c\ (0,5)$$

$$f)\ r = 4, c\ (2,-1)$$

2. $a)\ r = 5/2, c\ (0.5,1)$

$$b)\ r = 2, c\ (2,3)$$

$$c)\ r = 2, c\ (2,3)$$

$$d)\ r = \sqrt{13}, c\ (0,4)$$

$$e)\ r = \sqrt{13}, c\ (4,0)$$

$$f)\ r = 3, c\ (2,5)$$

$$g)\ r = 2, c\ (-3,5)$$

3. $a)\ r = 20, c\ (-12,-7)$

$$b)\ r = 10, c\ (2,3)$$

$$c)\ r = 5, c\ (-4,3)$$

Determining the equation of a circle given three points on its circumference

Exercise 7.5

1. $x^2 + y^2 = 25$
2. $(x-2)^2 + (y+3)^2 = 9$
3. $(x-4)^2 + (y-2)^2 = 9$
4. $x^2 + y^2 = 1$
5. $(x + 4)^2 + (y - 4)^2 = 25$
6. $(x-4)^2 + (y-3)^2 = 25$
7. $x^2 + (y- 3)^2 = 4$
8. $x^2 + (y- 1)^2 = 4$

Given a point, the equation of the circle and the length of the tangent

Exercise 7.6

1. $y = x + 5$
2. $y = \frac{2}{3}x - 4$
3. Follow the prove
4. $\sqrt[2]{2}$
5. 9

UNIT 8

KINEMATICS

<table>
<tr><td>Additional Math Secondary 3</td><td colspan="2">Unit 8: Kinematics</td></tr>
<tr><td>Learn about</td><td colspan="2">Key inquiry questions</td></tr>
<tr><td>Learners should be introduced to the interpretation of the area under velocity-time graph through and investigation. They should derive equations of motion for a body travelling in a straight line with constant acceleration and, determine vertical motion under gravity and investigate and solve problems in Kinematics.

They should supplement their learning by using the internet, reference textbooks and working with others.</td><td colspan="2">• What factors affect motion in a straight line?
• How do we find the area under velocity-time graph?
• How can we determine the velocity from distance-time graph?
• In what way can we determine acceleration of velocity-time graph?
• How can we derive a set of equations which will be useful in solving problem connected with constant acceleration?
• How does acceleration due to gravity affect vertical motion?</td></tr>
<tr><td colspan="3">Learning outcomes</td></tr>
<tr><td>Knowledge and understanding</td><td>Skills</td><td>Attitudes</td></tr>
</table>

- The interpretation of the area under velocity–time graph, straight line motion with constant acceleration, and vertical motion under gravity	- Construct velocity-time graphs to determine the area - Demonstrate how to derive equations of motion in straight line with constant acceleration - Solve equations of motion in a straight line - Investigate the effect of acceleration due to gravity in a vertical motion	- Develop interest in graphing motion of objects or bodies - Value the importance of graphs used in everyday life - Talk about the importance of equation of motion on objects in everyday life - Team work
Contribution to the competencies: Critical thinking: deriving equations of motion in straight line and equation of vertical motion Cooperation: Carry out investigation in groups to determine acceleration due to gravity (g)		
Links to other subjects: Physics: Kinematics TVET: Kinematics ICT: Internet		

Introduction

The concept of kinematic is not new to learners. Discussion method should be used to introduce to the learners the relationship of distance, velocity, speed and acceleration. Interpretation of linear motion graph should be emphasized.

By the end of the lesson the learner should be able to;-

1. State factor that accept motion on straight line.
2. Construct velocity time and distance time graph and solve related problems.
3. Derive equations of motion with constant acceleration.
4. Explain the effect of acceleration due to gravity in vertical motion.

Methodology

Learner should use discussion method and problem solving to solve these problems. Emphasis should be made on interpretation and graphs.

Kinematics

Exercise 8.1

1. 40km/h
2. 1600km/h
3. 10km/h

Distance-time graphs

Exercise 8.2

1. a)
   A- Acceleration

   B- Constant velocity

   C- Deceleration

   b)

   A- 2m/s

   B- 0m/s

   C- $\frac{4}{3}$m/s

   c) 350M

2. Teach to check accuracy

3. a) Teacher to check accuracy of learner.

   b) 63 ½ km

4a) A- Acceleration

B- Constant velocity

b) i) 2.0m/s

ii) 4m/s

c) 240km

Linear motion

Exercise 8.3

1. 250m
2. t= 6 $\frac{2}{3}$hs
3. 500m
4. 61 1/2s
5. 1.5 m/s
6. a) -0.94m/h

   b) 21 $\frac{1}{3}$s

   c) 13 $\frac{1}{3}$s
7. 13.32s
8. 2500m
9. 2.83m/s
10. 525m
11. 1.5m/s
12. 0.3m/s
13. 615sec
14. 500m

Movement under gravity

Exercise 8.4

1. 1.25m
2. 55m/s
3. 50.176m
4. 5 sec
5. 0.816 sec
6. 125.22 m/s
7. 29.7 m/s
8. 4.06 sec
9. 217.66 sec
10. 172.96 m

11. 70.22 sec, 214.66 m

12. 32.14m, 2.56s

UNIT 9

VELOCITY

<table>
<tr><th colspan="2">Additional Math Secondary 3</th><th>Unit 9: Velocity</th></tr>
<tr><td colspan="2">Learn about</td><td>Key inquiry questions</td></tr>
<tr><td colspan="2">Learners should be introduced to the concepts of composition and resolution of velocities through discussion. They should investigate the use of resolution of velocity in solving problems. They should supplement their learning by using the internet, reference textbooks and working with others.</td><td>• How can we resolve velocity vertically and horizontally?
• How can we use resolution of velocity in solving problems in real life situations?</td></tr>
<tr><td colspan="3">Learning outcomes</td></tr>
<tr><td>Knowledge and understanding</td><td>Skills</td><td>Attitudes</td></tr>
<tr><td>• Composition of velocities and resolution of velocities</td><td>• Work out resultant velocity of objects moving down or up a slope
• Resolve velocity vertical and horizontal and calculate the resultant velocity
• Use the concept of composition of velocities to work out mathematical problems
• Carry out investigations in problems related to velocity components and resultants</td><td>• Develop interest when working with composition of velocity and resolution of velocity
• Value the importance of velocity used in everyday life
• Appreciate solving problems involving velocity components
• Teamwork</td></tr>
</table>

Contribution to the competencies: Critical thinking: Analyse and resolve velocity vertically and horizontally Communication: Discuss and suggest solutions to problems involving velocity components and resultants Co-operation: Team work
Links to other subjects: Physics: relative and resultant velocity TVET: operate specialized industrial machines ICT: Internet

Introduction

The concept of velocity is not new to learners. It has also been discussed in unit 8. However the concept of composition of velocities and resolution of velocity need to be introduced to learners. Teacher should clearly relate these aspects to real life situation to enable learners comprehend these aspects of velocity. Problem solving approach and discussion method should be emphasized. Learners should supplement their work using internet and reference books.

Objectives

By the end of the lesson the learner should be able to;-

1. Define and use composition law of velocities.
2. Describe and determine the resolution of velocities.

Components of velocity vertically and horizontally

Exercise 9.1

1	i) 21 m/s	ii) 12.2 m/s	
2	a) 17 m/s	ii) 306^0	
3	a) =	b) 16.95 m/s	c) 30.6^0

UNIT 10

FORCE AND MOMENTUM

Additional Math Secondary 3	Unit 10: Force and Momentum
Learn about	**Key inquiry questions**
Learners should be introduced to the concepts of the unit of force, classify types of forces such as weight, reaction, tension, friction and thrust. They should describe composition of forces, resolution of forces, co-planar forces acting on a point and equilibrium of a particle and verify triangle of forces, Lami's theorem and polygons of forces. Learners should define momentum and impulse through discussion between the teacher and themselves. They should investigate and verify the principles of conservation of momentum to solve problems individually and in groups. Learners should investigate the use of forces in problems solving and supplement their learning by using the internet, reference textbooks and working with others.	- How can we classify types of forces? - In what way can we combine forces? - How can we find the resultant of any number of co-planar forces acting on a particle? - How can we resolve force? - When is a particle in equilibrium? - How can we verify Lami's theorem? - How can we extend the ideas of triangle forces to more than three forces acting at a point? - How can we use the idea of resultant forces to solve real life problems? - How can we find the relationship between the momentum and impulse of a body? - In what way can we verify the conservation of linear momentum of moving bodies in straight lines? - How can we use the idea of the 'principles of conservation of momentum' to solve real life problems?

Learning outcomes		
Knowledge and understanding	**Skills**	**Attitudes**
• Types of forces- weight, reaction, tension, friction, thrust, composition of two forces resolution of forces, coplanar forces acting on a point, equilibrium of a particle • Triangle of forces, Lami's theorem, polygons of forces • The conservation of momentum	• Solve problem involving types of forces, composition of two forces, resolution of forces, co-planar forces acting on a point. Equilibrium of a particle, triangle of forces, Lami's theorem; polygon of forces • Solve problems involving polygon of forces in everyday life • Carry out investigation on Lami's theorem • Work out problems involving momentum and impulse • Use the principles of conservation of momentum in problem solving • Carry out investigation on principles of conservation of momentum	• Appreciate the knowledge of forces • Value the importance of force used in everyday life • Build interest in working with forces in various dimensions • Appreciate the use of resultant forces in real life situation • Teamwork • Appreciate the knowledge of what momentum is • Value the importance momentum used in everyday life • Enjoy the application of principles of conservation of momentum in real life situation • Teamwork

Contribution to the competencies:

Critical thinking: Analyse the components of forces; analyse and investigate principles of conservation of momentum

Cooperation: Work collaboratively with one another to verify Lami's theorem and investigate principles of conservation of momentum

<u>Communication</u>: Communicate the principles of conservation of momentum to peers, teachers and others
Links to other subjects: Physics: forces Geography: active volcanoes Chemistry: reactions, chemical bonding

Introduction

The concept of force and momentum is not new to learners, they have been deal with in other subjects. However, in mathematics class, these units are introduced for the first time. Learners should be introduced to these concepts through illustrations, experimentation and demonstration class discussion should be emphasized.

Objectives

By the end of the lesson the learners should be able to;

1. Define and classify different types of forces.
2. Describe the composition and resolution of forces in coplanar surface.
3. Describe how force can be resolved and calculate such a force.
4. Define equilibrium forces and make their calculations.
5. Solve questions using triangle and polygon of forces.
6. Describe and use the Lami's theorem in problem solving.
7. Define momentum principles of conservation of momentum, and impulse and solve problems related to momentum, impulse and law l=of conservation of momentum

Types of forces

Exercise 10.1

1.
   a) 19.6 N
   b) 0.098 N
   c) 29400N
   d) 2.45N
   e) 2940N
   f) 1960N

2. a) 29.4N
   b) 24.5N
   c) 49N
   d) 20.58N
   e) 24.5N

3. a) 0.204Kg
   b) 1.224Kg
   c) 0.04286Kg
   d)0.08571Kg
   e) 0.1Kg

4. 59400N
   c) 98N

5. a) 98N
   b) 98N
   d) 29.4N

Composition of forces

Exercise 10.2

1. 100N

2. i) 9N

ii) Yes

3. a) i)12642N ii) 735N iii) 13375N

b) 13386

c) 87. 78^0

4. a) 77.2^0 b) 2256N

Triangle and polygon of forces

Exercise 10.3

1 a) 50N b) 86.6N

2 a) i) 8.66N ii) 50N

3 i) 66 ii) 260.2N, 17.4^0

4. 400N
5. 400N
6. 22N
7. $\theta = 66.18°$, R=53.36KN
8. $R = 113.580N, \infty = 38.21°$

Equilibrium of forces and Lami's Theorem

Exercise 10.4

1. AC= 8.49

   BC = 12

2. T1= 33.54N

   T2 = 17.86N

3. R = 7.07N

   T = 21.21N

4. TBC = 150N

   TAB = 173.21N

   W2 = 259. 81N

   W2 = 86. 60N

Momentum and Impulse

Momentum

Momentum is a quantity that can be used to understand things like collisions, without specifically needing to know the forces that act on the objects involved. For example, when a tennis racket hits a ball, the strings of the racket bend, the ball compresses, and numerous other forces affect the outcome. However, by analyzing the momentum of these objects, it is possible to understand a lot about the collision.

Momentum is a vector quantity, and is represented by the symbol $\vec{p}$. It is the product of the mass of an object, m, and its velocity vector, $\vec{v}$. The definition of momentum is,

$\vec{p} = m\vec{v}$

The magnitude of the momentum is,

$|\vec{p}| = |m\vec{v}|$

$\therefore p = mv$

The unit of magnitude of the momentum is the unit of mass times the unit of speed,

$(1\ kg)(1\ m/s) = 1\ kg \cdot m/s$

Momentum is expressed in units of $kg \cdot m/s$

As a vector quantity, momentum can be expressed in terms of components in the principal three axes (x, y, and z). The subscripts for these axes indicate the direction in the formulas,

$p_x = mv_x$

$p_y = mv_y$

$p_z = mv_z$

Impulse

A quantity that is closely related to momentum is called impulse. Impulse is also a vector quantity, and is represented by the symbol $\vec{J}$. Impulse is the product of a constant net force, $\sum \vec{F}$, and a time interval Δt, which is the time interval for which the force is applied. The constant net force is the vector sum of all forces,

$$\sum \vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \cdots$$

The time interval is the time difference between a starting time t_1 and a later time t_2,

$\Delta t = t_2 - t_1$

In this formula, the Greek letter Δ ("delta") is used to mean "the change in".

The formula for impulse (assuming a constant net force) is,

$$\vec{J} = \sum \vec{F} \Delta t$$

The unit of impulse is the Newton-second, $\mathrm{N} \cdot s$.
By Newton's Second Law, the net force is equal to the mass of an object times its acceleration,

$$\sum \vec{F} = m\vec{a}$$

In a previous section, the acceleration was defined to be the change in velocity divided by the change in time,

$$\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$$

The constant net force is equal to,

$$\sum \vec{F} = m\vec{a}$$

$$\therefore \sum \vec{F} = m\left(\frac{\Delta \vec{v}}{\Delta t}\right)$$

This can be substituted in to the equation for impulse,

$$\vec{J} = \sum \vec{F} \Delta t$$

$$\therefore \vec{J} = m\left(\frac{\Delta \vec{v}}{\Delta t}\right) \Delta t$$

$$\therefore \vec{J} = m\left(\frac{\Delta \vec{v}}{\Delta t}\right) \Delta t$$

$$\therefore \vec{J} = m\Delta\vec{v}$$

The change in velocity is the difference between the velocities at the starting and ending times,

$$\Delta\vec{v} = \overrightarrow{v_2} - \overrightarrow{v_1}$$

The formula for impulse becomes,

$$\vec{J} = m(\overrightarrow{v_2} - \overrightarrow{v_1})$$

$$\therefore \vec{J} = m\overrightarrow{v_2} - m\overrightarrow{v_1}$$

The equation for momentum is $\vec{p} = m\vec{v}$, and so the two terms in the above equation for impulse are momenta at an initial and a final time,

$$\therefore \vec{J} = \overrightarrow{p_2} - \overrightarrow{p_1}$$

This equation is called the *impulse-momentum theorem*. In words, it states that the change in momentum of an object in a certain time interval is equal to the impulse of the net force that acts on the object in the time interval. Using this formula, it is possible to relate changes in momentum to the forces that were applied to cause the change. It also shows that the time over which a force is applied has an effect on the change in momentum that results.

It is also important to note that the units for momentum and impulse are effectively the same. The unit of momentum is $kg \cdot m/s$, and the unit of impulse is the Newton-second, $\mathrm{N} \cdot s$. The Newton is a compound unit, defined as,

$$1\ \mathrm{N} = 1\ kg \cdot {}^{m}/_{s^2}$$

The Newton-second is thus,

$$1\ \mathrm{N} \cdot s = \left(1\ kg \cdot {}^{m}/_{s^2}\right) \cdot s$$

$$\therefore 1\ \mathrm{N} \cdot s = 1\ kg \cdot m\left(\frac{s}{s^2}\right)$$

$$\therefore 1\ \mathrm{N} \cdot s = 1\ kg \cdot {}^{m}/_{s}$$

Exercise 10.5

1. 8N
2. a) 3500N b) 87500N
3. 33 Kgms1
4. 192 Kgm/s
5. I = 40Kgm/s
6. Dp = 40Kgm/s
   a) -21Kgm/s
   b) Heat energy, sound energy, friction force.

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