1 Ex. Moment Generating Function

1.1 Gaussian

Assume that a random variable X follows the Gaussian distribution, $N(\mu, \sigma^2)$.

(i) Derive the moment generating function.

(ii) Derive the first moment of X via the MGF.

(iii) Derive the variance of X via the MGF.

1.2 Hints

Hints:

(i) Just compute the expectation $\mathbb{E}[\exp(tX)]$.

(ii) The number of differentiations corresponds to the order of the moment.

(iii) The variance is the second central moment, i.e., $\mathbb{E}[(X - \mu)^2]$.

2 Solutions

1.1

(i)

$$\begin{aligned}
E[\exp(tX)] &= \int \exp(tx) f(x) dx \\
&= \int \exp(tx) \left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2} \frac{(x-\mu)^2}{\sigma^2} \right) \right) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left(\frac{-1}{2} \frac{(x-\mu)^2}{\sigma^2} + tx \right) dx
\end{aligned}$$

$$\begin{aligned}
&\exp\left(\frac{-1}{2} \frac{(x-\mu)^2}{\sigma^2} + tx \right) \\
&= \exp\left(-\frac{1}{2\sigma^2} (x^2 - 2\mu x + \mu^2 - 2\sigma^2 tx) \right) \\
&= \exp\left(-\frac{1}{2\sigma^2} ((x - (\mu + \sigma^2 t))^2 - 2\mu\sigma^2 - \sigma^4 t^2) \right) \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2} \right) \exp\left(-\frac{1}{2\sigma^2} ((x - (\mu + \sigma^2 t))^2) \right)
\end{aligned}$$

So we have:

$$\begin{aligned}
E[\exp(tX)] &= \int \exp(tx) f(x) dx \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int \exp\left(\mu t + \frac{\sigma^2 t^2}{2} \right) \exp\left(-\frac{1}{2\sigma^2} ((x - (\mu + \sigma^2 t))^2) \right) dx \\
&= \exp\left(\mu t + \frac{\sigma^2 t^2}{2} \right) \int \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2} ((x - (\mu + \sigma^2 t))^2) \right) dx
\end{aligned}$$

$\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}((x-(\mu+\sigma^2 t))^2)\right)$ is exactly the pdf of a normal distribution whose mean is $\mu + \sigma^2 t$ and variance is σ^2. Since $\int f(x)dx = 1$, we have:

$$M_X(t) = E[\exp(tX)] = \exp\left(\mu t + \frac{\sigma^2 t^2}{2} \right)$$

(ii)

$$\begin{aligned}
E[X] &= \partial_t M_X(t)\big|_{t=0} \\
&= \partial_t \exp\left(\mu t + \frac{\sigma^2 t^2}{2} \right)\Bigg|_{t=0} \\
&= (\mu + \sigma^2 t) \exp\left(\mu t + \frac{\sigma^2 t^2}{2} \right)\Bigg|_{t=0} \\
&= \mu.
\end{aligned}$$

(iii)

$$
\begin{aligned}
E[X^2] &= \partial_t^2 M_X(t)\big|_{t=0} \\
&= \sigma^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right) + (\mu + \sigma^2 t)^2 \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\Bigg|_{t=0} \\
&= \sigma^2 + \mu^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
&= (\sigma^2 + \mu^2) - (\mu)^2 \\
&= \sigma^2
\end{aligned}
$$