Math Revision Session

2: Matrix Algebra (2)

Jukina HATAKEYAMA

The University of Osaka, Department of Economics

February 22, 2025

1. Rank
2. Eigenvalues and Eigenvectors
3. Diagonalisation

Matrix Rank

Definition: The rank of a matrix A is the maximum number of **linearly independent** rows or columns in the matrix.

Key Properties:

- $\text{rank}(A) \leq \min(m, n)$ for $A \in \mathbb{R}^{m \times n}$.
- $\text{rank}(A) = \text{rank}(A^T)$.
- A is full rank if $\text{rank}(A) = \min(m, n)$.
- If $\text{rank}(A) < \min(m, n)$, the matrix has linearly dependent rows or columns.

Calculation Methods:

- Row echelon form (Gaussian elimination).
- Determinants (for square matrices).
- Singular value decomposition (SVD).

Linear Independence

Definition:
A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ in a vector space is said to be **linearly independent** if the only solution to the equation:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

is $c_1 = c_2 = \cdots = c_n = 0$.

Otherwise, they are linearly dependent.
Key Points:

- If at least one vector in the set can be written as a linear combination of the others, the set is **linearly dependent**.
- Linear independence is crucial for solving systems of equations and understanding vector spaces.
- The rank of a matrix can help determine if its columns are linearly independent.

Example of Linear Independence

Example 1: Linearly Independent Vectors

Consider the vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Check if they are linearly independent by solving:

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solution:

$$c_1 = 0, \quad c_2 = 0 \quad \Rightarrow \text{Linearly Independent}$$

Example 2: Linearly Dependent Vectors

Consider the vectors:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

We can see that $\mathbf{v}_2 = 2\mathbf{v}_1$, hence they are **linearly dependent**.

Exercise: Matrix Rank

Example 1: Find the rank of the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Solution: Using row operations:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of A is $\mathbf{2}$.

Example 2: Determine the rank of the matrix:

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 6 \end{pmatrix}$$

Solution: Since row 2 is all zeros and row 3 is a multiple of row 1, rank$(B) = \mathbf{1}$.

Importance of Matrix Rank

The rank of a matrix is an essential concept in linear algebra because it provides valuable information about the matrix's properties and applications. Some key reasons why rank is important include:

- **Solving Linear Systems:** The rank determines the number of independent equations in a system, helping identify whether a unique solution exists.
- **Invertibility:** **A square matrix is invertible if and only if its rank equals its dimension (i.e., full rank)**.
- **Dimension of Column Space:** The rank indicates the number of linearly independent columns, which corresponds to the dimension of the column space.
- **Consistency of Equations:** The rank helps determine whether a system of equations is consistent by comparing the rank of the coefficient matrix to the augmented matrix.
- **Data Compression and Dimensionality Reduction:** In applications like Principal Component Analysis (PCA), rank helps identify the intrinsic dimensionality of data.

Eigenvalues and Eigenvectors

Given a square matrix $A \in \mathbb{R}^{n \times n}$, a scalar λ and a nonzero vector $x \in \mathbb{R}^n$ satisfy the equation:

$$Ax = \lambda x \tag{1}$$

Definitions:

- λ is called an **eigenvalue** of A.
- x is called an **eigenvector** corresponding to λ.

Key Properties:

- Eigenvalues satisfy the characteristic equation: $\det(A - \lambda I) = 0$.
- Eigenvectors provide directions along which transformation by A only scales the vector.
- If A is symmetric, all eigenvalues are real and eigenvectors are orthogonal.

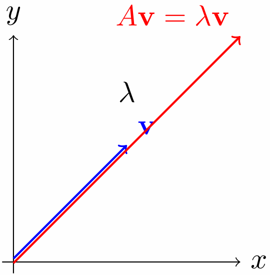
Eigenvalues and Eigenvectors

Intuition:

- Eigenvectors **preserve direction** when a matrix is applied.
- Eigenvalues **scale** the eigenvectors.

Examples:

- Imagine stretching a rubber band:
  - Some directions stretch without changing orientation (eigenvectors).
  - The stretching factor represents eigenvalues.
- Think of water flow:
  - The main flow direction is an eigenvector.
  - The speed of flow is the eigenvalue.



Mathematical Example

Consider the matrix:

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors satisfy:

$$A\mathbf{x} = \lambda\mathbf{x}$$

where λ is a scalar.

- This means multiplying by A only scales $\mathbf{x}$, without changing its direction.

Why Eigenvalues and Eigenvectors Matter?

1. **Understanding Linear Transformations:**
   - Eigenvectors **preserve direction** during linear transformations.
   - Eigenvalues **scale** the eigenvectors by a certain factor.
   - This allows us to understand how transformations affect space.
2. **Applications in Data Analysis:**
   - Eigenvectors and eigenvalues are key to Principal Component Analysis (PCA).
   - PCA reduces the dimensionality of large datasets by finding the most important directions (principal components).
3. **Applications in Physics and Engineering:**
   - Eigenvalue problems arise in the study of vibrations, stability analysis, and quantum mechanics.
   - They help identify the natural modes of a system (e.g., vibration modes of a bridge).

Summary: Eigenvalues and eigenvectors provide deep insight into the behavior of systems, from data analysis to physical systems.

Eigenvalues and Eigenvectors Calculation

1. **Calculation of Eigenvalues:**
   - Solve the characteristic equation:

     $$\det(A - \lambda I) = 0$$

     where A is the matrix and I is the identity matrix.
2. **Calculation of Eigenvectors:**
   - For each eigenvalue λ, solve:

     $$(A - \lambda I)\mathbf{v} = 0$$

     to find the corresponding eigenvector $\mathbf{v}$.

Example: Eigenvalue and Eigenvector Calculation

Given matrix:

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

Step 1: Find the eigenvalues.

- The characteristic equation is:

$$\det(A - \lambda I) = \det \begin{pmatrix} 4 - \lambda & 1 \\ 2 & 3 - \lambda \end{pmatrix} = 0$$

- Solve:

$$(4 - \lambda)(3 - \lambda) - 2 = 0$$

- This simplifies to:

$$\lambda^2 - 7\lambda + 10 = 0$$

- Solve the quadratic equation:

$$\lambda = 5,\ 2$$

Step 2: Find the eigenvectors.

- For $\lambda = 5$, solve:

$$(A - 5I)\mathbf{v} = 0 \quad \text{or} \quad \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

- This gives the eigenvector $\mathbf{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.
- For $\lambda = 2$, solve:

$$(A - 2I)\mathbf{v} = 0 \quad \text{or} \quad \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$$

- This gives the eigenvector $\mathbf{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

Diagonalisation of a Matrix

Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. If there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1},$$

then A is said to be **diagonalisable**.

Why is Diagonalisation Useful?

Diagonalisation simplifies matrix operations, such as:

- Computing matrix powers: $A^k = PD^kP^{-1}$
- Solving systems of linear differential equations
- Understanding the geometric interpretation of linear transformations

Conditions for Diagonalisation

A matrix A is diagonalisable if and only if it has n linearly independent eigenvectors. This occurs when:

- The geometric multiplicity of each eigenvalue equals its algebraic multiplicity.
- For a symmetric matrix A, diagonalisation is always possible as A has an orthonormal basis of eigenvectors.

How to Diagonalise a Matrix

1. Find the eigenvalues of A by solving $\det(A - \lambda I) = 0$.
2. Find the corresponding eigenvectors for each eigenvalue.
3. Form P using the eigenvectors as columns.
4. Construct D as the diagonal matrix with eigenvalues on the diagonal.
5. Verify that $A = PDP^{-1}$.

Example: Diagonalising a Matrix

Consider the matrix:

$$A = \begin{pmatrix} 4 & 1 \\ 6 & 3 \end{pmatrix}.$$

Step 1: Compute the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1 \\ 6 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - (6 \times 1) = 0.$$

Expanding,

$$12 - 4\lambda - 3\lambda + \lambda^2 - 6 = \lambda^2 - 7\lambda + 6 = 0.$$

Solving $\lambda^2 - 7\lambda + 6 = 0$, we get $\lambda_1 = 6, \lambda_2 = 1$.

- For $\lambda_1 = 6$: Solve $(A - 6I)x = 0$, i.e.,

$$\begin{pmatrix} -2 & 1 \\ 6 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  This gives the eigenvector $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- For $\lambda_2 = 1$: Solve $(A - I)x = 0$, i.e.,

$$\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

  This gives the eigenvector $v_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

Step 3: Construct the matrices

- Form the matrix P using the eigenvectors as columns:

$$P = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}.$$

- Construct the diagonal matrix D:

$$D = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}.$$

Step 4: Verify that $A = PDP^{-1}$

- Compute P^{-1}:

$$P^{-1} = \frac{1}{(1)(3) - (-1)(2)} \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}.$$

- Compute PDP^{-1} and confirm that it equals A.

Matrix Powers

Diagonalisation simplifies matrix exponentiation:

$$A^k = PD^kP^{-1}$$

Example: Compute A^5 for

$$A = \begin{bmatrix} 4 & 1 \\ 6 & 3 \end{bmatrix}$$

1. Find eigenvalues: $\lambda_1 = 6, \lambda_2 = 1$.
2. Find eigenvectors and form P, D.
3. Compute D^5, then find A^5 using $A^5 = PD^5P^{-1}$.

Solving Differential Equations

Diagonalisation helps solve systems of ODEs:

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x}, \quad A = PDP^{-1}$$

Steps:

1. Transform variables: $\mathbf{z} = P^{-1}\mathbf{x}$.
2. Solve the diagonal system: $\frac{dz_i}{dt} = \lambda_i z_i$.
3. Convert back to $\mathbf{x}$ using $\mathbf{x} = P\mathbf{z}$.

Understanding Linear Transformations

Diagonalisation reveals geometric properties:

- Eigenvectors define invariant directions.
- Eigenvalues determine scaling along these directions.

Example:

- Rotation matrices are not diagonalisable (except for identity).
- Symmetric matrices are always diagonalisable.