Math Revision Session

1: Matrix Algebra (1)

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April 17, 2025

Course Information

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- Day and Time of Classes: Refer the CLE.
- Number of Classes: 10 sessions (60 minutes each)

All handouts will be uploaded on my website[1] after the final lesson.

[1] Website: https://jukinahatakeyama.github.io/jhatakeyama/

Introduction

Matrices are useful because they can efficiently represent and manipulate large amounts of data, simplify calculations, and organise complex equations in a compact form.

For example, in undergraduate econometrics, we consider the following linear model:

$$
\begin{aligned}
y_i &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_K x_{i,K} + u_i && (1) \\
&= \beta_0 + \sum_{k=1}^{K} \beta_k x_{i,k} + u_i && (2)
\end{aligned}
$$

for $i = 1, \ldots, N$.

When this equation is summarised (or stacked) over i, it becomes...:

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & \dots & x_{1,K} \\ 1 & x_{2,1} & x_{2,2} & \dots & x_{2,K} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{N,1} & x_{N,2} & \dots & x_{N,K} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_K \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \tag{3}$$

This is a form of matrix representation, which can further be written as follows:

$$y = X\beta + u$$

where $y \in \mathbb{R}^N, u \in \mathbb{R}^N, X \in \mathbb{R}^{N \times K}$, and $X \in \mathbb{R}^K$.

$\mathbb{R}$ represents real numbers (set of natural Number).

Dimension

If x is ...

- scalar, $x \in \mathbb{R}$.
  This is equivalent to "x is in the set of real numbers."
  ex.) $x = 2 \in \mathbb{R}$
- a vector, $x \in \mathbb{R}^m$, where m is a natural number.
  This is equivalent to "x contains m elements."
  ex.) $x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3$
- a matrix, $x \in \mathbb{R}^{l \times m}$, where both l and m are a natural number.
  This is equivalent to "x contains $l \times m$ elements."
  ex.) $x = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \in \mathbb{R}^{3 \times 2}$

we call l as rows and m as columns.

General expression of Matrix

Let X be a matrix whose dimension is $a \times b$, then the element of X at the i-th row and j-th column is denoted as x_{ij}, where $i \in \{1, 2, \ldots, a\}$ and $j \in \{1, 2, \ldots, b\}$. Thus, we can write X as:

$$X = \begin{pmatrix} x_{11} & x_{12} & \ldots & x_{1b} \\ x_{21} & x_{22} & \ldots & x_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ x_{a1} & x_{a2} & \ldots & x_{ab} \end{pmatrix} \in \mathbb{R}^{a \times b}$$

Each element x_{ij} represents the value in the i-th row and j-th column of the matrix.

A vector can be considered as an $a \times 1$ matrix.

Square Matrix

A square matrix is a matrix that has the same number of rows and columns. In other words, a matrix of size $n \times n$, where n is a positive integer, is called a square matrix. The elements on the main diagonal (from the top-left to the bottom-right) are often important in various mathematical operations.
Example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

This is a square matrix of size 3×3, where the number of rows and columns are equal.

Symmetric Matrix

A matrix A is called a symmetric matrix if it is equal to its own transpose. In other words, a matrix A is symmetric if:

$$A = A^T$$

For a matrix $A = [a_{ij}]$, the condition of symmetry means that for all i and j:

$$a_{ij} = a_{ji}$$

This implies that the elements of the matrix are mirrored along the main diagonal.

Ex.)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

This matrix is symmetric because $A = A^T$. Specifically:

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$$

Notice that $a_{ij} = a_{ji}$ for all elements of the matrix.

Diagonal Matrix

A matrix $D \in \mathbb{R}^{n \times n}$ is called a **diagonal matrix** if all its off-diagonal entries are zero, i.e.,

$$D = \begin{pmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{pmatrix}$$

where $d_1, d_2, \dots, d_n$ are the diagonal entries of the matrix. In other words, D has non-zero entries only on its main diagonal.

Example:

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

Diagonal matrices are particularly useful in various applications such as simplifying matrix powers, eigenvalue problems, and diagonalization of matrices.

Identity Matrix

An identity matrix is a special type of square matrix where all the elements on the main diagonal are 1, and all other elements are 0. The identity matrix is denoted by I_n, where n is the size of the matrix. The identity matrix acts as the multiplicative identity in matrix multiplication, meaning that when a matrix is multiplied by the identity matrix, it remains unchanged.
Example:

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is a 3×3 identity matrix. It has 1s on the main diagonal and 0s elsewhere.

Matrix addition

Matrix addition involves adding two matrices of the same size. The sum of two matrices is a new matrix where each element is the sum of the corresponding elements from the original matrices.
If A and B are two matrices of the same dimension, say $m \times n$, the matrix sum $C = A + B$ is given by:

$$C_{ij} = A_{ij} + B_{ij}$$

where i represents the row number and j represents the column number.

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

The sum $C = A + B$ is:

$$C = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Thus, the result of the addition is the matrix:

$$C = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Matrix addition is performed pairwise on corresponding elements.

Matrix subtraction

Matrix subtraction involves subtracting two matrices of the same size. Each element of the resulting matrix is obtained by subtracting the corresponding elements of the two matrices.
If A and B are matrices of the same dimension, the matrix difference $C = A - B$ is given by:

$$C_{ij} = A_{ij} - B_{ij}$$

Example:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

The difference $C = A - B$ is:

$$C = \begin{pmatrix} 1-5 & 2-6 \\ 3-7 & 4-8 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}$$

Subtraction of matrices is also done pairwise, element by element.

Properties of Matrix Addition

Let $A, B, C \in \mathbb{R}^{m \times n}$ be matrices. The following properties hold:

1. **Commutative Property:**

   $$A + B = B + A$$

   Matrix addition is commutative.

2. **Associative Property:**

   $$(A + B) + C = A + (B + C)$$

   Matrix addition is associative.

3. **Identity Element:**

   $$A + \mathbf{0} = \mathbf{0} + A = A$$

   The zero matrix $\mathbf{0}$ is the additive identity, meaning adding it to any matrix A leaves A unchanged.

4. **Inverse Element:**

   $$A - A = \mathbf{0}$$

   Subtracting a matrix from itself results in the zero matrix.

Zero Matrix: A zero matrix is a matrix where all entries are zero, i.e.,

$$\mathbf{0} = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

Scalar Multiplication of a Matrix

Let A be an $m \times n$ matrix and c be a scalar. The scalar multiplication of A by c, denoted as cA, results in a new matrix where each element of A is multiplied by c.
If

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

then

$$cA = \begin{pmatrix} c \cdot a_{11} & c \cdot a_{12} & \dots & c \cdot a_{1n} \\ c \cdot a_{21} & c \cdot a_{22} & \dots & c \cdot a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m1} & c \cdot a_{m2} & \dots & c \cdot a_{mn} \end{pmatrix}$$

For matrices A, B of the same size and scalars c, d, the following properties hold:

1. **Distributive Property (Scalar over Matrix Addition):**

$$c(A + B) = cA + cB$$

2. **Distributive Property (Scalars over Matrix):**

$$(c + d)A = cA + dA$$

3. **Associative Property (Scalars with Matrix):**

$$c(dA) = (cd)A$$

4. **Multiplicative Identity:**

$$1A = A$$

Scalar multiplication is useful in scaling matrices and manipulating linear equations.

Exercise: Scalar Multiplication of a Matrix

Solve the following problems related to scalar multiplication of matrices.

1. Compute the result of multiplying the scalar $c = 3$ with the matrix
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$
2. Verify the distributive property by calculating both sides of the equation:
$$2\left(\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}\right) = 2\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + 2\begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix}$$
3. Show that scalar multiplication satisfies the associative property using the values $c = 2$ and $d = 3$ for the matrix
$$B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$$
4. Find the result of multiplying any matrix X by the scalar 1, and explain why this holds true for any matrix.

Matrix Operations: Addition, Subtraction, and Scalar Multiplication

Let the following matrices be given:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}, \quad c = 3$$

Now, solve the following problems using these matrices:

1. **Matrix Addition:** Compute $A + B$.
2. **Matrix Subtraction:** Compute $B - A$.
3. **Scalar Multiplication:** Compute $3A$.
4. **Combination of Operations:** Compute $2A + cB$.

Matrix Multiplication

For two matrices A and B, the matrix product AB is defined **only when the number of columns of A is equal to the number of rows of B**. In other words, **the number of columns of the first matrix must match the number of rows of the second matrix**.

If A is an $m \times n$ matrix and B is a $n \times p$ matrix, then the product AB will be an $m \times p$ matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{pmatrix}$$

The product AB is given by:

$$AB = \begin{pmatrix} \sum_{k=1}^{n} a_{1k}b_{k1} & \sum_{k=1}^{n} a_{1k}b_{k2} & \dots & \sum_{k=1}^{n} a_{1k}b_{kp} \\ \sum_{k=1}^{n} a_{2k}b_{k1} & \sum_{k=1}^{n} a_{2k}b_{k2} & \dots & \sum_{k=1}^{n} a_{2k}b_{kp} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} a_{mk}b_{k1} & \sum_{k=1}^{n} a_{mk}b_{k2} & \dots & \sum_{k=1}^{n} a_{mk}b_{kp} \end{pmatrix}$$

Important Note: - The number of columns in A must equal the number of rows in B for the multiplication to be possible. - If the dimensions do not match, matrix multiplication is not defined.

Matrix Multiplication: Example

Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

The product AB is given by:

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} (1 \times 5 + 2 \times 7) & (1 \times 6 + 2 \times 8) \\ (3 \times 5 + 4 \times 7) & (3 \times 6 + 4 \times 8) \end{pmatrix}$$

Performing the calculations:

$$AB = \begin{pmatrix} 5 + 14 & 6 + 16 \\ 15 + 28 & 18 + 32 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Thus, the product AB is:

$$AB = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$$

Properties of Matrix Multiplication

Let $\alpha \in \mathbb{R}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{h\times k}$, and $C \in \mathbb{R}^{p\times q}$, where $\mathbb{R}^{a\times b}$ denotes the space of $a \times b$ matrices with real entries. The following properties hold:

1. **Scalar Multiplication:**

$$\alpha(AB) = (\alpha A)B \quad \text{when} \quad n = h$$

2. **Associative Property of Matrix Multiplication:**

$$(AB)C = A(BC) \quad \text{when} \quad n = h \text{ and } k = p$$

3. **Distributive Property (over matrix addition):**

$$A(B + C) = AB + AC \quad \text{when} \quad n = h = p$$

4. **Distributive Property (over matrix addition on the left):**

$$(A + B)C = AC + BC \quad \text{when} \quad n = k = p$$

5. **Non-commutativity of Matrix Multiplication:**

$$AB \neq BA \quad \text{even when both products are defined.}$$

Identity Matrix Multiplication

The **identity matrix** $I_n \in \mathbb{R}^{n\times n}$ is a square matrix where all the diagonal entries are 1, and all off-diagonal entries are 0. It is denoted as:

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

The identity matrix has the following important property:

$$I_n A = A I_n = A$$

for any matrix $A \in \mathbb{R}^{n\times m}$ where the multiplication is defined. This means that multiplying a matrix by the identity matrix does not change the matrix.

Example: Let

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

and

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Multiplying A by I_2:

$$I_2 A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} = A$$

Thus, multiplying any matrix by the identity matrix results in the original matrix.

Transpose of a Matrix

The **transpose** of a matrix A, denoted by A^T, is obtained by flipping the matrix over its diagonal. In other words, the rows of A become the columns of A^T, and the columns of A become the rows of A^T.
If $A = \left(a_{ij}\right)$ is an $m \times n$ matrix, then the transpose A^T is an $n \times m$ matrix given by:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

Thus, for each element a_{ij} of A, the corresponding element in A^T is a_{ji}, i.e., the element in the i-th row and j-th column of A becomes the element in the j-th row and i-th column of A^T.

Example: Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Then the transpose of A, denoted A^T, is:

$$A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

The **transpose** of a matrix A, denoted by A^T or sometimes by A', is obtained by flipping the matrix over its diagonal. In other words, the rows of A become the columns of A^T, and the columns of A become the rows of A^T.

In some contexts, instead of A^T, we may also use A' to denote the transpose. Both notations are commonly used.

Matrix Representation of the Model

Consider the linear model for $i = 1, \ldots, N$:

$$\begin{aligned} y_i &= \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_K x_{i,K} + u_i \\ &= \beta_0 + \sum_{k=1}^{K} \beta_k x_{i,k} + u_i \end{aligned}$$

We can express the summation part using vector notation. Let $x_i = (x_{i,1}, x_{i,2}, \cdots, x_{i,K})^T \in \mathbb{R}^K$ be the vector of independent variables for observation i, and $\beta = (\beta_1, \beta_2, \cdots, \beta_K)^T \in \mathbb{R}^K$ be the vector of coefficients. Then the model can be written as:

$$y_i = \beta_0 + \mathbf{x}_i^T \beta + u_i$$

Where $x_i^T \beta = \sum_{k=1}^{K} \beta_k x_{i,k}$ represents the dot product between the independent variable vector x_i and the coefficient vector β.

Properties of Matrix Transpose

For matrices $A \in R^{n\times k}$, $B \in R^{k\times p}$, and vector $x \in \mathbb{R}^n$, the following properties of transpose hold:

1. $(A')' = A$
   The transpose of the transpose of a matrix returns the original matrix.
2. $(\alpha A)' = \alpha A'$
   The transpose of a scalar multiple of a matrix is the scalar multiplied by the transpose of the matrix.
3. $(A + B)' = A' + B'$
   The transpose of the sum of two matrices is the sum of the transposes of the matrices.

4. $(AB)' = B'A'$

   The transpose of the product of two matrices is the product of their transposes, with the order of multiplication reversed.

5. $x'x = \sum_{i=1}^{n} x_i^2$

   The transpose of a vector x multiplied by x is equal to the sum of the squares of the elements of the vector.

These properties help simplify matrix and vector computations, especially when dealing with transposition.

6. If A is an $n \times k$ matrix with rows given by the $1 \times k$ vectors $a_1, a_2, \ldots, a_n$, such that we can write:

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{pmatrix}$$

Then the transpose of A, denoted A', is the matrix whose columns are the rows of A, and is given by:

$$A' = (a_1', a_2', \ldots, a_n') = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1k} & a_{2k} & \cdots & a_{nk} \end{pmatrix}$$

$X^T X$

Let X be an $n \times k$ matrix. The matrix X^T is the transpose of X, and it is a $k \times n$ matrix. We want to show that $X^T X$ is symmetric, i.e.,

$$(X^T X)^T = X^T X$$

- Using the property of transposes, we know that:

$$(AB)^T = B^T A^T$$

- Applying this to $X^T X$:

$$(X^T X)^T = X^T (X^T)^T$$

- Since the transpose of the transpose of a matrix is the original matrix, we have:

$$(X^T)^T = X$$

- Thus, we get:

$$(X^T X)^T = X^T X$$

Conclusion: Since $(X^T X)^T = X^T X$, the matrix $X^T X$ is symmetric.

Idempotent Matrix

A matrix A is called **idempotent** if it satisfies the condition:

$$A^2 = A$$

In other words, when the matrix is multiplied by itself, it returns the same matrix. This property is important in various applications, especially in linear transformations and projection matrices.

- For a matrix to be idempotent, multiplying the matrix by itself should yield the same result as the original matrix.
- Example: Consider the matrix

  $$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

  Then:

  $$A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = A$$

- This matrix is idempotent because $A^2 = A$.

Symmetric and Idempotent Matrices

- **Symmetric Matrix:** A matrix A is symmetric if:

  $$A^T = A$$

  This means that the matrix is equal to its transpose. Example:

  $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(This is the identity matrix, which is symmetric.)}$$

- **Idempotent Matrix:** A matrix A is idempotent if:

  $$A^2 = A$$

  This means that when the matrix is multiplied by itself, it returns the same matrix. Example:

  $$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{(This is also the identity matrix, which is idempotent.)}$$

- The identity matrix I is both symmetric and idempotent:

  $$I^T = I \quad \text{and} \quad I^2 = I$$

Determinant of a Matrix

- The determinant is a scalar value that can be computed from the elements of a square matrix.
- For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is given by:

$$\det(A) = ad - bc$$

- For a 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, the determinant is:

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

- The determinant has the following important properties:
  1. If $\det(A) = 0$, the matrix A is **singular** and does not have an inverse.
  2. If $\det(A) \neq 0$, the matrix A is **non-singular** and has an inverse.
  3. The determinant of the identity matrix is 1: $\det(I) = 1$.
  4. The determinant of the product of two matrices is the product of their determinants: $\det(AB) = \det(A)\det(B)$.

Inverse Matrix

Definition: The inverse of a matrix A, denoted as A^{-1}, is the matrix such that:

$$A \cdot A^{-1} = A^{-1} \cdot A = I$$

where I is the identity matrix of appropriate size.

Existence Condition:

- A matrix has an inverse if and only if its determinant is non-zero:

$$\det(A) \neq 0$$

- If $\det(A) = 0$, the matrix is singular and does not have an inverse.

Calculation of the Inverse: If A is a square matrix with a non-zero determinant, the inverse is given by:

$$A^{-1} = \frac{1}{\det(A)} \cdot \mathsf{adj}(A)$$

where adj(A) is the adjoint (or adjugate) matrix of A, which is the transpose of the cofactor matrix of A.

Steps to Calculate the Inverse:

1. Find the determinant $\det(A)$.
2. If $\det(A) \neq 0$, compute the adjoint of A.
3. Multiply the adjoint by $\frac{1}{\det(A)}$ to obtain A^{-1}.

Adjugate Matrix and Inverse Calculation

Example Matrix A:

$$A = \begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix}$$

Step 1: Compute the Determinant

$$\det(A) = (4 \times 6) - (7 \times 2) = 24 - 14 = 10$$

Since $\det(A) \neq 0$, the inverse of A exists.

Step 2: Compute the Cofactor Matrix The cofactor matrix is calculated by finding the cofactors of each element:

$$C_{11} = \det\left((6)\right) = 6, \quad C_{12} = -\det\left((2)\right) = -2,$$

$$C_{21} = -\det\left((7)\right) = -7, \quad C_{22} = \det\left((4)\right) = 4$$

Thus, the cofactor matrix is:

$$\text{Cofactor}(A) = \begin{pmatrix} 6 & -2 \\ -7 & 4 \end{pmatrix}$$

Step 3: Compute the Adjugate Matrix (Transpose of Cofactor Matrix)

$$\mathsf{adj}(A) = \begin{pmatrix} 6 & -7 \\ -2 & 4 \end{pmatrix}$$

Step 4: Compute the Inverse Matrix The inverse of A is given by:

$$A^{-1} = \frac{1}{\det(A)} \cdot \mathsf{adj}(A) = \frac{1}{10} \cdot \begin{pmatrix} 6 & -7 \\ -2 & 4 \end{pmatrix}$$

Thus:

$$A^{-1} = \begin{pmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{pmatrix}$$

Inverse Matrix Calculation using Gaussian Elimination

$$A = \begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix}$$

Step 1: Augment the matrix with the identity matrix.

$$(A|I) = \left(\begin{pmatrix} 4 & 7 \\ 2 & 6 \end{pmatrix} \middle| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

Step 2: Perform row operations to get the left side to the identity matrix.

First, divide the first row by 4 to make the leading entry in the first column 1:

$$R_1 \to \frac{1}{4}R_1 \quad \Rightarrow \quad \left(\begin{pmatrix} 1 & \frac{7}{4} \\ 2 & 6 \end{pmatrix} \middle| \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{pmatrix} \right)$$

Next, subtract 2 times the first row from the second row to eliminate the 2 in the second row, first column:

$$R_2 \to R_2 - 2R_1 \quad \Rightarrow \quad \left(\begin{pmatrix} 1 & \frac{7}{4} \\ 0 & \frac{5}{2} \end{pmatrix} \middle| \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{2} & 1 \end{pmatrix} \right)$$

Step 3: Normalize the second row to make the leading entry in the second column 1.

$$R_2 \to \frac{2}{5}R_2 \quad \Rightarrow \quad \left(\begin{pmatrix} 1 & \frac{7}{4} \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \right)$$

Step 4: Subtract $\frac{7}{4}$ times the second row from the first row to eliminate the $\frac{7}{4}$ in the first row, second column.

$$R_1 \to R_1 - \frac{7}{4}R_2 \quad \Rightarrow \quad \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \middle| \begin{pmatrix} \frac{3}{5} & -\frac{14}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \right)$$

Step 5: The matrix on the right is now the inverse of A.

$$A^{-1} = \begin{pmatrix} \frac{3}{5} & -\frac{14}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}$$

Matrix Types: Definite and Semi-Definite

- **Positive Definite Matrix**: A matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if for any non-zero vector $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^T A \mathbf{x} > 0.$$

  This means the quadratic form is always positive.

- **Positive Semi-Definite Matrix**: A matrix $A \in \mathbb{R}^{n \times n}$ is positive semi-definite if for any vector $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^T A \mathbf{x} \geq 0.$$

  This means the quadratic form is non-negative.

- **Negative Definite**: A matrix $A \in \mathbb{R}^{n \times n}$ is indefinite if there exists at least one vector $\mathbf{x}$ such that

$$\mathbf{x}^T A\mathbf{x} > 0 \quad \text{and at least one vector} \quad \mathbf{x}^T A\mathbf{x} < 0.$$

  This means the quadratic form can take both positive and negative values.

- **Negative Semi-Definite Matrix**: A matrix $A \in \mathbb{R}^{n \times n}$ is negative semi-definite if for any vector $\mathbf{x} \in \mathbb{R}^n$,

$$\mathbf{x}^T A\mathbf{x} \leq 0.$$

  This means the quadratic form is non-positive.

Positive Definite Matrices

A matrix A is said to be **positive definite** if for all non-zero vectors $\mathbf{x}$, the quadratic form $\mathbf{x}^T A\mathbf{x} > 0$.

Positive definite matrices have the following properties:

- The determinant of A is positive, i.e., $\det(A) > 0$.
- The matrix is **invertible**, meaning that it has an inverse matrix A^{-1}.

Thus, if a matrix is positive definite, we can be certain that it has a non-zero determinant and an inverse.