Math Revision Session

Statistics (3): Continuous Random Variables and their famous distributions

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February 16, 2025

Continuous Random Variable

- A continuous random variable can take on an infinite number of possible values within a given range.
- Examples include height, weight, temperature, and time.
- Unlike discrete random variables, they are defined over an interval and cannot be counted one by one.

Ex.)

- **Height of a Person:** The height of an individual can take any value within a certain range, such as 150.3 cm or 172.8 cm.
- **Temperature:** The temperature in a city can take on any real number within a range.
- **Time Taken to Run a Race:** The time a runner takes to complete a 100m race is a continuous variable, as it can be measured to arbitrary precision.

Probability in Continuous Random Variables

- Unlike discrete random variables, assigning probabilities to individual values in a continuous random variable leads to divergence.
- Instead, probability is defined over **intervals** rather than specific values.

Key Differences:

- **Discrete case**: The probability of a single value can be nonzero, e.g.,

$$P(X = 3) = \frac{1}{6} \text{ for a fair dice.}$$

- **Continuous case**: The probability of a single value is **always zero**, i.e.,

$$P(X = 1.5) = 0 \text{ for a normal distribution.}$$

- Instead, we calculate the probability **over an interval**:

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

Inequalities in Continuous Probability

- In continuous probability distributions, the probability of a single point is always **zero**
- Therefore, including or excluding the endpoints in an interval does not change the probability.

Example: For a continuous random variable X,

$$P(0 \leq X \leq 1) = P(0 < X < 1) = P(0 \leq X < 1) = P(0 < X \leq 1).$$

Since $P(X = 0) = P(X = 1) = 0$, adding or removing these points has no effect.

Cumulative Distribution Function (CDF)

- The **Cumulative Distribution Function** (: CDF denoted by $F_X(x)$) describes the probability that a random variable X takes on a value less than or equal to x.
- Mathematically, it is defined as:

$$F_X(x) = P(X \leq x).$$

Key Properties:

- $F_X(x)$ is a monotonically **non-decreasing function**.
- $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.
- For a continuous random variable, $F_X(x)$ is differentiable and its derivative gives the **probability density function** (: PDF)

$$f_X(x) = \frac{d}{dx} F_X(x).$$

Relationship Between CDF and Interval Probability

- The probability that X falls within an interval $(a, b]$ can be computed using the CDF:

$$P(a < X \leq b) = F_X(b) - F_X(a).$$

- Similarly, for an open interval (a, b):

$$P(a < X < b) = F_X(b) - F_X(a).$$

Probability Density Function (PDF)

- The **Probability Density Function** (:PDF) denoted by $f_X(x)$, describes how probability is distributed over the values of a continuous random variable X.
- Unlike discrete probabilities, $f_X(x)$ itself does not represent probability but rather **density**.

Definition:

- The PDF is the derivative of the Cumulative Distribution Function (: CDF):

$$f_X(x) = \frac{d}{dx} F_X(x).$$

- The probability that X lies in an interval $(a, b]$ is given by:

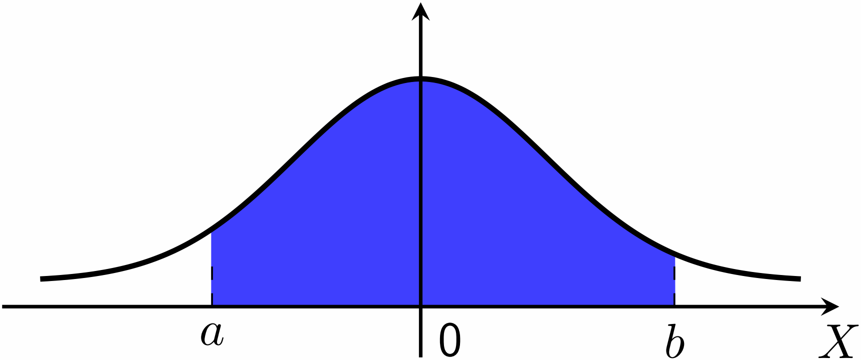
$$P(a < X \leq b) = F(b) - F(a) = \int_a^b f_X(x) dx.$$

Key Properties:

- $f_X(x) \geq 0$ for all x.
- The total area under the PDF is 1:

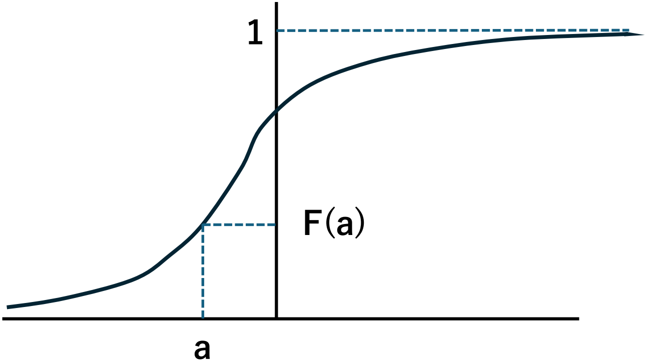
$$\int_{-\infty}^{\infty} f_X(x)dx = 1.$$

- Unlike discrete probability mass functions (PMFs), $P(X = x) = 0$ for any single point x.



a PDF

a CDF



Uniform Distribution

- A random variable X is said to follow a **uniform distribution** if every outcome in a given interval is equally likely.
- The probability density function (PDF) for a continuous uniform distribution is given by:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

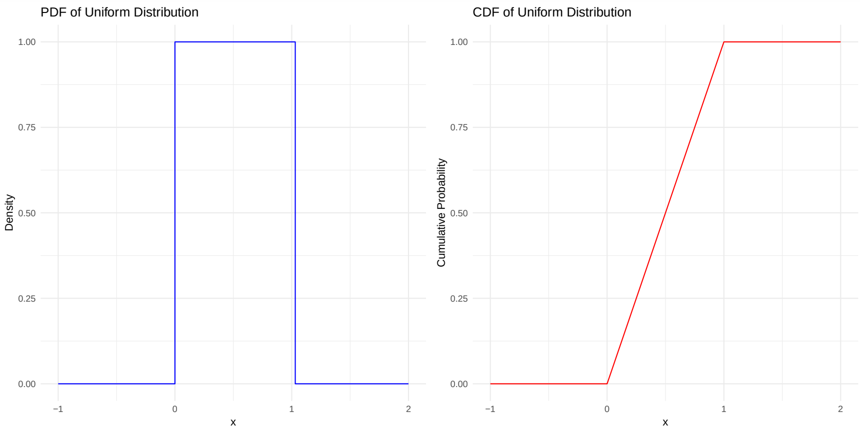
  where a and b are the lower and upper bounds of the interval.

- The cumulative distribution function (CDF) is:

$$F_X(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

- Example: If $X \sim \text{Uniform}(0, 1)$, then the PDF is:

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



The PDF is actually discontinuous at 0 and 1, with a gap at those points.

Expectation and Variance of Continuous Random Variables

- For a continuous random variable X with probability density function (PDF) $f_X(x)$, the **expectation** (or mean) of X is defined as:

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

- The **variance** of X is defined as:

$$\mathrm{Var}(X) = \mathbb{E}[X - \mathbb{E}[X]]^2 = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

where $\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$.

- **Example:** For a random variable $X \sim \mathrm{Uniform}(a, b)$, the PDF is:

$$f_X(x) = \frac{1}{b-a} \quad \text{for} \quad a \leq x \leq b$$

The expectation is:

$$\mathbb{E}[X] = \frac{a+b}{2}$$

The variance is:

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

- In general, the variance provides a measure of how much the values of a random variable deviate from the mean.
- If $\text{Var}(X) = 0$, then X is a constant (i.e., it takes a single value with probability 1).

Joint Distribution of Continuous Random Variables

- **Joint Distribution of Continuous Variables:** For two continuous random variables X and Y, the joint probability density function (PDF) is denoted as:

$$f_{X,Y}(x,y) := P(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx.$$

  The joint PDF represents the probability density over a two-dimensional region and the area under the surface corresponds to the probability.

- The differential of the joint distribution is the joint probability density function:

$$F_{X,Y}(x, y) = P(X \leq x, Y \leq y)$$

and

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y).$$

- The joint probability density function $f_{X,Y}(x, y)$ is obtained by taking the partial derivatives of the joint cumulative distribution function $F_{X,Y}(x, y)$.

- **Comparison with Discrete Case:**
  - For discrete random variables X and Y, the joint probability mass function (PMF) is given by:

$$P(X = x, Y = y) = f_{X,Y}(x, y).$$

  - The probability in the discrete case is obtained by summing over possible values of X and Y:

$$P(X = x, Y = y) = \sum_{x} \sum_{y} f_{X,Y}(x, y).$$

  - In contrast, for continuous variables, we need to integrate over ranges of X and Y, as probabilities of specific values for continuous variables are zero.

- **Marginal Distribution for Continuous Variables:** The marginal distribution of X is obtained by integrating the joint PDF over all possible values of Y:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy.$$

  Similarly, the marginal distribution of Y is:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx.$$

- **Key Difference:**
  - In the discrete case, probabilities are summed for each combination of X and Y, while in the continuous case, we integrate the joint PDF over intervals.
  - In both cases, the joint distribution describes the relationship between two random variables, but for continuous variables, the probability of exact values is zero, and we work with intervals instead.

Covariance and Correlation Coefficient

- The covariance between two continuous random variables X and Y is defined as:

$$\begin{aligned}\text{Cov}(X, Y) &= \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - \mathbb{E}[X])(y - \mathbb{E}[Y])f_{X,Y}(x, y)\,dx\,dy.\end{aligned}$$

- The covariance measures the extent to which two variables change together. If $\text{Cov}(X, Y) > 0$, they tend to increase together, while if $\text{Cov}(X, Y) < 0$, one tends to increase when the other decreases.

- The correlation coefficient is given by:

$$\rho(X, Y) = \frac{\mathsf{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\mathsf{Cov}(X, Y)}{\sqrt{Var[X]}\sqrt{Var[Y]}},$$

  where σ_X and σ_Y are the standard deviations of X and Y, respectively.
- The correlation coefficient normalizes the covariance by the standard deviations, providing a value between -1 and 1.
- A correlation of 1 indicates a perfect positive linear relationship, -1 indicates a perfect negative linear relationship, and 0 indicates no linear relationship.

Independence of Continuous Random Variables

- Two continuous random variables X and Y are independent if their joint probability density function (PDF) factorizes as:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y).$$

- This means that the occurrence of one event does not affect the probability of the other event.
- Mathematically, for independent variables:

$$\mathbb{P}(X \leq x, Y \leq y) = \mathbb{P}(X \leq x) \cdot \mathbb{P}(Y \leq y).$$

- For independent random variables, the covariance is zero:

$$\mathrm{Cov}(X, Y) = 0.$$

- Independence implies that the correlation coefficient is also zero:

$$\rho(X, Y) = 0.$$

Note: A correlation coefficient of 0 does not imply independence. It only suggests no linear relationship between the variables. Nonlinear dependencies may still exist.

Conditional Probability

- **Conditional Probability:** The conditional probability of X given $Y = y$ is defined as:

$$P(X \leq x \mid Y = y) = \frac{P(X \leq x, Y = y)}{P(Y = y)}$$

  where $P(Y = y)$ is the marginal probability of Y.

- **Conditional Probability Density:** The conditional probability density function is:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

  where $f_{X,Y}(x, y)$ is the joint probability density function and $f_Y(y)$ is the marginal density of Y.

Conditional Expectation

- **Conditional Expectation:** The conditional expectation of X given $Y = y$ is:

$$\mathbb{E}[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx$$

- **Law of Total Expectation:** The law of total expectation states:

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X \mid Y]]$$

  This states that the expectation of X is the expectation of its conditional expectation given Y.