fourier_transform_draft

April 21, 2025

1 Volume 2: The Discrete Fourier Transform

[25]:
```
import IPython
from IPython.display import Audio, display
import numpy as np
from scipy.io import wavfile
from scipy.fftpack import fft
from matplotlib import pyplot as plt
from scipy.io.wavfile import write
```

[26]:
```
plt.rcParams["figure.dpi"] = 300              # Fix plot quality.
plt.rcParams["figure.figsize"] = (12, 3)       # Change plot size / aspect (you␣
↪may adjust this).
```

[27]:
```
class SoundWave(object):
    """A class for working with digital audio signals."""

    # Problem 1
    def __init__(self, rate, samples):
        """Set the SoundWave class attributes.

        Parameters:
            rate (int): The sample rate of the sound.
            samples ((n,) ndarray): NumPy array of samples.
        """
        self.rate = rate
        self.samples = samples
        self.time = np.arange(len(samples)) / rate
        self.freq = np.fft.rfftfreq(len(samples), d=1/rate)

    # Problems 1 and 7
    def plot(self, show_dft=False):
        """Plot the graph of the sound wave (time versus amplitude)."""
        if show_dft:
            fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(12, 6))

            # Time-domain plot
```

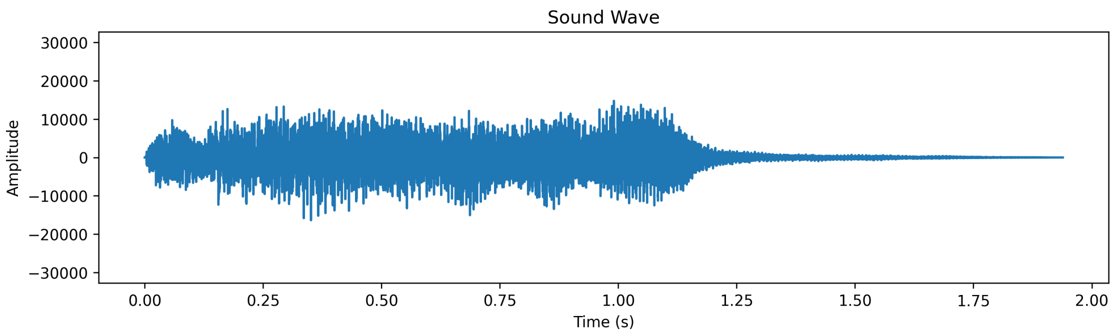
```
            ax1.plot(self.time, self.samples)
            ax1.set_xlabel("Time (s)")
            ax1.set_ylabel("Amplitude")
            ax1.set_title("Sound Wave")

            # Frequency-domain plot
            n = len(self.samples)
            freq_data = fft(self.samples)
            magnitude = np.abs(freq_data)[:n // 2]
            freqs = np.arange(n // 2) * self.rate / n

            ax2.plot(freqs, magnitude)
            ax2.set_xlabel("Frequency (Hz)")
            ax2.set_ylabel("Magnitude")
            ax2.set_title("Frequency Spectrum (DFT)")

            plt.tight_layout()

        else:
            plt.plot(self.time, self.samples)
            plt.xlabel("Time (s)")
            plt.ylabel("Amplitude")
            plt.title("Sound Wave")
            plt.ylim([-32768, 32767])

    # Problem 2
    def export(self, filename: str, force=False) -> None:
        """Generate a wav file from the sample rate and samples.
        If the array of samples is not of type np.int16, scale it before␣
↪exporting.

        Parameters:
            filename (str): The name of the wav file to export the sound to.
        """
        if self.samples.dtype == np.int16 and not force:
            write(filename, self.rate, self.samples)
            print(f"Exported without scaling: {filename}")
            return

        # Scale to int16 using formula (7.2)
        data = self.samples.astype(np.float64)
        peak = np.max(np.abs(data)) or 1.0  # Prevent divide-by-zero
        scaled = (data / peak * 32767).astype(np.int16)
        write(filename, self.rate, scaled)
        print(f"Exported with scaling: {filename}")

    # Problem 4
```

```python
    def __add__(self, other):
        """Combine the samples from two SoundWave objects.

        Parameters:
            other (SoundWave): An object containing the samples to add
                to the samples contained in this object.

        Returns:
            (SoundWave): A new SoundWave instance with the combined samples.

        Raises:
            ValueError: if the two sample arrays are not the same length.
        """
        if not isinstance(other, SoundWave):
            return NotImplemented                       # allow NumPy to try␣
↪other dtypes

        if len(self.samples) != len(other.samples):
            raise ValueError("SoundWave objects must have the same number of␣
↪samples")

        # Promote to float64 to minimise clipping/overflow when summing
        combined = self.samples.astype(np.float64) + other.samples.astype(np.
↪float64)
        return SoundWave(self.rate, combined)


    # Problem 4
    def __rshift__(self, other):
        """Concatentate the samples from two SoundWave objects.

        Parameters:
            other (SoundWave): An object containing the samples to concatenate
                to the samples contained in this object.

        Raises:
            ValueError: if the two sample rates are not equal.
        """
        if not isinstance(other, SoundWave):
            return NotImplemented                      # allow NumPy fallback

        if self.rate != other.rate:
            raise ValueError("Sample rates differ: {} Hz vs {} Hz"
                             .format(self.rate, other.rate))

        concatenated = np.concatenate([self.samples, other.samples])
        return SoundWave(self.rate, concatenated)
```

1.1 Problem 1

- Implement `SoundWave.__init__()`.
- Implement `SoundWave.plot()`.
- Use SciPy's `wavfile.read()` and the `SoundWave` class to plot `tada.wav` in the cell below.

[28]:
```python
rate, samples = wavfile.read("tada.wav")
sw = SoundWave(rate, samples)
try:
    sw.plot()
except NotImplementedError:
    pass

plt.show()
```



1.2 Problem 2

- Implement `SoundWave.export()`.
- Use the `export()` method to create two new files containing the same sound as `tada.wav`: one without scaling, and one with scaling (use `force=True`).
- Use `IPython.display.Audio()` to embed the original and two new versions of `tada.wav` in the cells below.

[29]:
```python
# (a) No scaling
sw.export("tada_noscale.wav")

# (b) With scaling - force=True
sw.export("tada_scaled.wav", force=True)
```

```
Exported without scaling: tada_noscale.wav
Exported with scaling: tada_scaled.wav
```

[30]:
```python
display(Audio("tada.wav", autoplay=False, element_id='orig'))
display(Audio("tada_noscale.wav", autoplay=False, element_id='noscale'))
display(Audio("tada_scaled.wav", autoplay=False, element_id='scaled'))
```

```
<IPython.lib.display.Audio object>
```

```
<IPython.lib.display.Audio object>
```

```
<IPython.lib.display.Audio object>
```

1.3 Problem 3

- Implement `generate_note()`.
- Use `generate_note()` to create an A tone that lasts for two seconds. Embed it in the cell below.

[31]:
```python
def generate_note(frequency, duration):
    """Generate an instance of the SoundWave class corresponding to
    the desired soundwave. Uses sample rate of 44100 Hz.

    Parameters:
        frequency (float): The frequency of the desired sound.
        duration (float): The length of the desired sound in seconds.

    Returns:
        sound (SoundWave): An instance of the SoundWave class.
    """
    sample_rate = 44100  # Standard sample rate for audio
    t = np.linspace(0, duration, int(sample_rate * duration), endpoint=False)
    samples = np.sin(2.0 * np.pi * frequency * t).astype(np.float32)
    return SoundWave(sample_rate, samples)
```

[32]:
```python
a_tone = generate_note(440.00, 2)
b_tone = generate_note(493.88, 2)
c_tone = generate_note(523.25, 2)
d_tone = generate_note(587.33, 2)
e_tone = generate_note(659.25, 2)
f_tone = generate_note(698.46, 2)
g_tone = generate_note(783.99, 2)
a_prime_tone = generate_note(880.00, 2)
```

1.4 Problem 4

- Implement `SoundWave.__add__()`.
- Generate a three-second A minor chord (A, C, and E) and embed it in the first cell below.
- Implement `SoundWave.__rshift__()`.
- Generate the arpeggio A → C → E, where each tone lasts one second, and embed it in the second cell below.

[33]:
```python
# Fundamental frequencies
# Source: https://www1.swarthmore.edu/NatSci/ceverba1/Class/e5_2006/
 ↪MusicalScales.html
a = generate_note(440, 3)    # A4
c = generate_note(261.63, 3) # C4
e = generate_note(329.63, 3) # E4
```

```
chord = a + c + e
chord.export("am_chord.wav", force=True)
display(Audio("am_chord.wav", autoplay=False, element_id="am-chord"))
```

```
Exported with scaling: am_chord.wav
```

```
<IPython.lib.display.Audio object>
```

[34]:
```
a1 = generate_note(440, 1)
c1 = generate_note(261.63, 1)
e1 = generate_note(329.63, 1)

arpeggio = a1 >> c1 >> e1
arpeggio.export("am_arpeggio.wav", force=True)
display(Audio("am_arpeggio.wav", autoplay=False, element_id="am-arpeggio"))
```

```
Exported with scaling: am_arpeggio.wav
```

```
<IPython.lib.display.Audio object>
```

1.5 Problem 5

- Implement `simple_dft()` with the formula $c = F_n f$, where F_n is the n-dimensional DFT matrix.
- In the cell below, use `np.allclose()` to check that `simple_dft()` and `scipy.fftpack.fft()` give the same result (after scaling).

[35]:
```
def simple_dft(samples):
    """Compute the DFT of an array of samples.

    Parameters:
        samples ((n,) ndarray): an array of samples.

    Returns:
        ((n,) ndarray): The DFT of the given array.
    """
    n = len(samples)
    j, k = np.meshgrid(np.arange(n), np.arange(n))  # shape (n, n)
    W = np.exp(-2j * np.pi * j * k / n)             # DFT matrix F_n
    return W @ samples
```

[36]:
```
samples = np.random.rand(64)
assert np.allclose(simple_dft(samples), fft(samples))
# Should return True if `simple_dft()` and `scipy.fftpack.fft()` give the same result (after scaling).
```

1.6 Problem 6

- Implement `simple_fft()`.

- In the cell below:
  - Generate an array of 8192 random samples and take its DFT using `simple_dft()`, `simple_fft()`, and `scipy.fftpack.fft()`.
  - Print the runtimes of each computation.
  - Use `np.allclose()` to check that `simple_fft()` and `scipy.fftpack.fft()` give the same result (after scaling).

[37]:
```python
def simple_fft(samples, threshold=1):
    """Compute the DFT using the FFT algorithm.

    Parameters:
        samples ((n,) ndarray): an array of samples.
        threshold (int): when a subarray of samples has fewer
            elements than this integer, use simple_dft() to
            compute the DFT of that subarray.

    Returns:
        ((n,) ndarray): The DFT of the given array.
    """
    n = len(samples)
    if n <= threshold:
        return simple_dft(samples)

    if n % 2 != 0:
        raise ValueError("Input size must be a power of 2")

    even = simple_fft(samples[::2], threshold)
    odd = simple_fft(samples[1::2], threshold)

    factor = np.exp(-2j * np.pi * np.arange(n) / n)
    return np.concatenate([even + factor[:n // 2] * odd,
                           even - factor[:n // 2] * odd])
```

[38]:
```python
import time

# Generate random signal of length 8192
samples = np.random.rand(8192)

# Time simple_dft
start = time.time()
dft_result = simple_dft(samples)
dft_time = time.time() - start

# Time simple_fft
start = time.time()
fft_result = simple_fft(samples, threshold=32)  # threshold chosen for speed
fft_time = time.time() - start
```

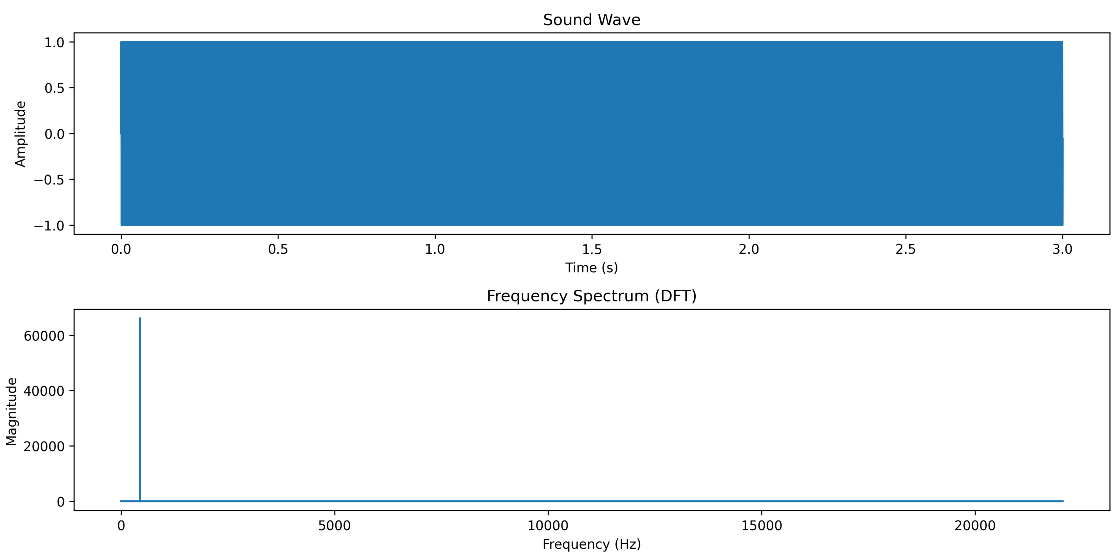
```
# Time scipy.fft
start = time.time()
scipy_result = fft(samples)
scipy_time = time.time() - start

# Output runtimes
print(f"simple_dft: {dft_time:.4f} s")
print(f"simple_fft: {fft_time:.4f} s")
print(f"scipy.fft: {scipy_time:.4f} s")

# Check correctness
print("simple_fft   scipy.fft:", np.allclose(fft_result, scipy_result))
```

```
simple_dft: 5.3490 s
simple_fft: 0.0130 s
scipy.fft: 0.0002 s
simple_fft   scipy.fft: True
```

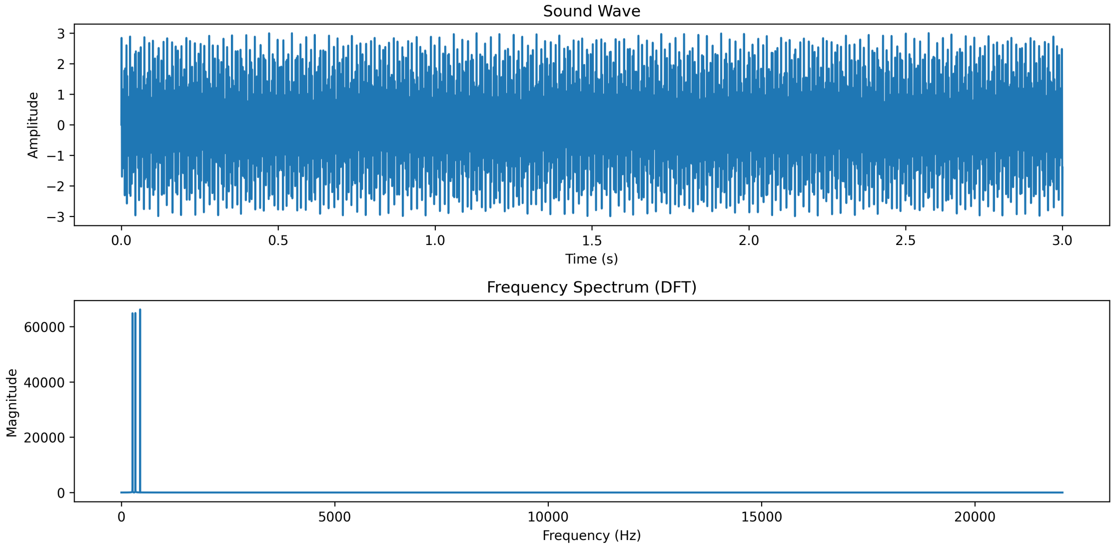
1.7 Problem 7

- Modify `SoundWave.plot()` so that it accepts a boolean. When the boolean is `True`, take the DFT of the stored samples and plot (in a new subplot) the frequencies present on the x-axis and the magnituds of those frequences on the y-axis. Only the display the first half of the plot, and adjust the x-axis so that it correctly shows the frequencies in Hertz.
- Display the plot of the DFT of the A tone from Problem 4 the first cell below.
- Display the plot of the DFT of the A minor chord from Problem 4 in the second cell below.

[39]:
```
a.plot(show_dft=True)
plt.show()
```



[40]:
```
chord.plot(show_dft=True)
plt.show()
```

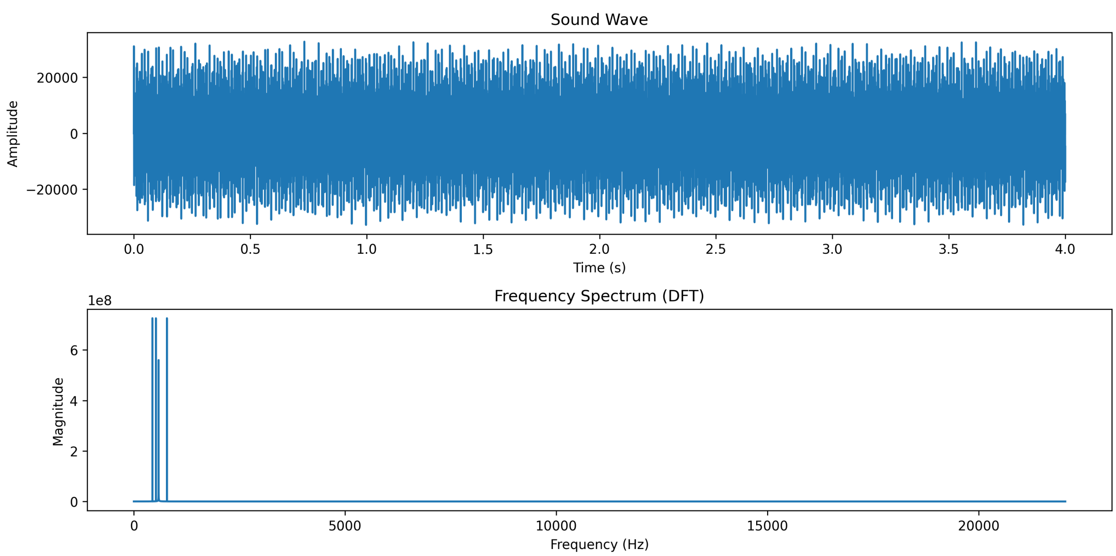


1.8 Problem 8

Use the DFT to determine the individual notes that are present in `mystery_chord.wav`.

[41]:
```
note_freqs = {
    "A": 440.00,
    "B": 493.88,
    "C": 523.25,
    "D": 587.33,
    "E": 659.25,
    "F": 698.46,
    "G": 783.99,
    "A ": 880.00
}

rate, samples = wavfile.read("mystery_chord.wav")
mystery_chord = SoundWave(rate, samples)

mystery_chord.plot(show_dft=True)
plt.show()

n = len(mystery_chord.samples)

freqs = mystery_chord.freq
```

```
magnitudes = np.abs(np.fft.rfft(mystery_chord.samples))
magnitudes[0] = 0  # Ignore DC component
freqs = np.fft.rfftfreq(n, d=1/rate)
top_indices = np.argsort(magnitudes)[-10:][::-1]
top_freqs = freqs[top_indices]
top_freqs = np.unique(np.round(top_freqs, 1))[:5]  # Round and deduplicate

detected_notes = []
for f in top_freqs:
    closest_note = min(note_freqs, key=lambda note: abs(note_freqs[note] - f))
    detected_notes.append((closest_note, f))
print("Detected notes and frequencies:")
for note, freq in detected_notes:
    print(f"{note}: {freq:.2f} Hz")

top_indices = np.argsort(magnitudes)[-10:][::-1]
top_freqs = freqs[top_indices]
```




```
Detected notes and frequencies:
A: 440.00 Hz
C: 523.20 Hz
D: 586.80 Hz
D: 587.00 Hz
D: 587.20 Hz
```