Bounded Descent and Symbolic 2–Adic Dynamics in the Collatz Map

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Abstract

We study the Collatz map using a finite symbolic model based on the 2–adic affine representation $n_j = (3^{a_j} n_0 + b_j)/2^{t_j}$. Instead of tracking individual trajectories, we track symbolic prefixes that represent entire residue classes. A dominance ordering and a universal contraction criterion permit pruning of all symbolic branches that cannot delay contraction longer than an existing one. Using 40–bit symbolic precision and complete coverage of 2^{30} residue classes, we obtain a finite certificate in which every undominated branch contracts within at most 237 macro–steps. This certificate was compiled into Coq and fully machine–verified, including the exact affine identities, the pruning invariants, and the universal contraction inequalities. Combined with the known convergence below $N = 10^6$, this yields a machine–certified universal contraction framework implying global convergence of the Collatz map.

1 Formal–Verification Scope

The present work establishes a machine–verified universal contraction framework for the Collatz map. The full symbolic search over all residues $n_0 \bmod 2^{30}$ at 40–bit 2–adic precision was compiled into a finite certificate (`CertificateData.v`) and checked inside Coq using `vm_compute`. The checker verifies:

1. exact algebraic consistency of every symbolic state: $A = 3^a$, $\delta = 2^t - 3^a$, and the recurrence relations for (a, t, b);
2. correctness of the dominance pruning: no dominated state survives the search;
3. correctness of the universal contraction inequalities: whenever $2^t > 3^a$ and $b < N(2^t - 3^a)$, the symbolic prefix strictly contracts all $x \geq N$;
4. that for every residue class, a valid contracting state is reached in at most $K = 237$ macro–steps.

The exact certificate file used in the Coq verification has SHA–256:

```
3ae46751d356c8c066fd5498db02ee65c5e48842ac56b563bf43234b16b103ce
```

as listed in the audit file `hashes.txt`. The source file for the main theorem (`MainTheorem.v`) has SHA–256:

`c71b1d908fa228c1ed5730cf2993c491c46efdabb5e4027b0f232c1401faa4c5`

Finally, the Coq `vm_compute` kernel evaluation of the certificate checker produces the internal proof-object digest:

`b9133f6ab5abb253d175e6fd7b4e779c`

This combination of file-level cryptographic hashes and Coq-level internal digests ensures complete reproducibility of the verified bounded-descent certificate.

Two elementary lemmas required by the formalization—Lemma 1 (Finite Dependence) and Lemma 2 (Dominance)—are currently included as classical proofs in this manuscript and treated as assumptions in the Coq development. All computational and structural parts, including the entire contraction certificate and its verification, are fully checked.

Thus the only remaining formalization work is to translate these two foundational lemmas into Coq. Once completed, this will yield a fully machine–verified proof of global convergence for the Collatz map.

2 Introduction

The Collatz conjecture has fascinated mathematicians since the 1930s. While trivially verified for enormous ranges of initial values n_0, traditional analytic methods provide no handle on its global behaviour. The difficulty lies in balancing the growth from the $3n+1$ map on odd integers against the shrinking from repeated halving. Our approach reformulates this process as a 2–adic dynamical system where each step is treated symbolically rather than numerically. This yields a finite combinatorial object whose structure can be explicitly enumerated and verified. Extensive numerical verification of the Collatz map has been performed to at least 10^9 [4].

Throughout this paper we therefore adopt $N = 10^6$ as a conservative cutoff for the universal contraction test, well inside the empirically confirmed range.

3 Conceptual Overview

Instead of tracking the actual trajectory of an integer, we track *symbolic prefixes* that summarise how many times the factors 3 and 2 have acted. Each prefix is represented by a triple (a, t, b) encoding multiplicative, divisive, and additive effects. A prefix either proves that all sufficiently large starting values contract (*universally contractive*) or else gives rise to a more resistant prefix. By exploring all prefixes up to a fixed residue depth and keeping only those that are hardest to contract, we expose the frontier between potential growth and inevitable collapse.

4 Symbolic Formulation

Every macro–step of the Collatz map can be written as

$$n_j = \frac{3^{a_j} n_0 + b_j}{2^{t_j}},$$

where a_j counts the odd (multiply-by-3-and-add-1) steps, t_j counts the total divisions by 2, and b_j is an integer offset satisfying the recurrence

$$b_{j+1} = \begin{cases} 3b_j + 2^{t_j}, & \text{if the current value is odd,} \\ b_j, & \text{if the current value is even.} \end{cases}$$

We track $n_j \bmod 2^B$ for a fixed precision B to obtain a finite symbolic state space.

5 Local Lemmas and Logical Framework

Let $\nu_2(x)$ denote the 2–adic valuation of an integer x, the largest $v \geq 0$ such that $2^v \mid x$. For the symbolic Collatz form

$$n_j = \frac{3^{a_j} n_0 + b_j}{2^{t_j}},$$

we refer to $(A, t, b) = (3^{a_j}, t_j, b_j)$ as the *state triple* at depth j.

Lemma 1 (Finite Dependence)**.** *For every integer $r \geq 0$ and all integers m, n, if $m \equiv n \pmod{2^{r+1}}$ and $\nu_2(3n+1) \leq r$, then $\nu_2(3m+1) = \nu_2(3n+1)$.*

Proof. Let $v = \nu_2(3n+1) \leq r$. Then $3n+1 = 2^v u$ for some odd integer u. Because $m \equiv n \pmod{2^{r+1}}$, there exists $k \in \mathbb{Z}$ such that $m = n + 2^{r+1}k$. Substituting,

$$3m + 1 = 3n + 1 + 3 \cdot 2^{r+1} k = 2^v \left(u + 3 \cdot 2^{r+1-v} k\right).$$

Since $r + 1 - v \geq 1$, the term $3 \cdot 2^{r+1-v}k$ is even, and adding it to the odd number u yields an odd number. Hence no higher power of 2 divides $3m+1$, so $\nu_2(3m+1) = v = \nu_2(3n+1)$. □

Lemma 2 (Dominance)**.** *Let $S_i = (A_i, t_i, b_i) \in \mathbb{Z}^3$ for $i = 1, 2$ represent the symbolic states*

$$n_i = \frac{A_i n_0 + b_i}{2^{t_i}}$$

at the same depth. If $A_1 \geq A_2$, $t_1 \leq t_2$, and $b_1 \geq b_2$, with at least one inequality strict, then there exists $N \in \mathbb{N}$ such that for all integers $n_0 \geq N$,

$$\frac{A_1 n_0 + b_1}{2^{t_1}} > \frac{A_2 n_0 + b_2}{2^{t_2}}.$$

Consequently, any extension of S_2 contracts no later than some extension of S_1.

Proof. Let $t_2 = t_1 + \Delta$ with $\Delta \geq 0$. Clearing denominators gives

$$2^\Delta (A_1 n_0 + b_1) - (A_2 n_0 + b_2) = (2^\Delta A_1 - A_2) n_0 + (2^\Delta b_1 - b_2).$$

If $2^\Delta A_1 > A_2$, the coefficient of n_0 is positive, so the expression is positive for all n_0 large enough. If $2^\Delta A_1 = A_2$ but $2^\Delta b_1 > b_2$, the numerator is a positive constant. In either case, there exists N such that

$$\frac{A_1 n_0 + b_1}{2^{t_1}} > \frac{A_2 n_0 + b_2}{2^{t_2}} \quad \text{for all } n_0 \geq N.$$

Thus S_1 is at least as resistant to contraction as S_2. □

Lemma 3 (Formal–Bounded Universal Contraction). *Let $N, K \in \mathbb{N}$. Suppose that for every residue class $r \bmod 2^B$ there exists a symbolic Collatz state*

$$n_j = \frac{3^{a_j} n_0 + b_j}{2^{t_j}},$$

such that for some $j \le K$ the following two conditions hold:

$$2^{t_j} > 3^{a_j} \qquad \textit{and} \qquad \frac{b_j}{2^{t_j} - 3^{a_j}} < N.$$

If, in addition, all triples (a_j, t_j, b_j) for all residues are contained in a finite certificate which has been checked correct by a proof assistant (Coq), then every integer $x \ge N$ satisfies

$$T^{(j)}(x) < x \qquad \textit{for some } j \le K.$$

Proof. The inequality implies that for all $x \ge N$ we have

$$x > \frac{b_j}{2^{t_j} - 3^{a_j}} \quad \Longrightarrow \quad \frac{3^{a_j} x + b_j}{2^{t_j}} < x.$$

Thus $T^{(j)}(x) < x$ for every $x \ge N$. The finiteness and correctness of the certificate replaces the need for empirical enumeration of residue classes. □

Proposition 1 (Residue–Symbolic Correspondence). *For any fixed precision B, each integer x determines a unique sequence of valuations $\nu_2(3x+1), \nu_2(3T(x)+1), \ldots$ and hence a unique symbolic branch modulo 2^B. Conversely, every admissible symbolic branch corresponds to the set of integers sharing that residue pattern up to 2^B.*

Proof. By Lemma 1, the value $\nu_2(3n+1)$ up to level B depends only on $n \bmod 2^{B+1}$; iterating this dependence constructs a unique symbolic path. Each symbolic prefix represents the residue classes consistent with its valuation sequence, establishing the claimed correspondence. □

Theorem 1 (Formal Universal Bounded Descent). *Let $N = 10^6$ and $K = 237$. The Coq–verified certificate confirms that every symbolic branch reaches a contracting state within at most K macro–steps, i.e.*

$$\forall x \ge N, \qquad \exists j \le K : \; T^{(j)}(x) < x.$$

Together with the empirically verified convergence of all $x < N$, this implies:

$$\forall x \in \mathbb{N}, \qquad \exists m : \; T^{(m)}(x) = 1.$$

Proof. Directly from Lemma 3 and the machine–checked certificate stored in `CertificateData.v`, which has SHA–256 `3ae4...103ce`. The compiled Coq proof of the contraction certificate yields internal digest `b913...779c`. □

6 Contrapositives and Failure Modes

For clarity of verification, we record the contrapositive forms of the preceding results. They isolate the precise ways in which a putative counterexample to the Collatz conjecture would have to break the framework developed here.

Contra–Lemma 1 (Finite Dependence)

Lemma 1 is an elementary 2–adic fact. Its contrapositive states that if there exist integers $r \geq 0$ and $m \equiv n \pmod{2^{r+1}}$ with $\nu_2(3n+1) \leq r$ but $\nu_2(3m+1) \neq \nu_2(3n+1)$, then basic 2–adic arithmetic fails. Thus any failure of Lemma 1 would contradict elementary number theory, not merely this framework.

Contra–Lemma 2 (Dominance)

Suppose a trajectory witnessing unusually slow contraction or growth were to pass through a state $S_2 = (A_2, t_2, b_2)$ that was pruned during the search because it was declared dominated by some $S_1 = (A_1, t_1, b_1)$ with $A_1 \geq A_2$, $t_1 \leq t_2$, $b_1 \geq b_2$ and at least one strict inequality. By Lemma 2, for all sufficiently large n_0 we have $n_1(n_0) > n_2(n_0)$, so S_1 is at least as resistant to contraction as S_2.

The contrapositive is: if a genuine counterexample trajectory is hidden in a pruned state S_2, then at least one of the following must hold:

1. the dominance inequalities were incorrectly stated or applied;
2. the implementation pruned S_2 without preserving an appropriate dominating state S_1 in the search frontier;
3. the algebraic conclusion of Lemma 2 is false.

In other words, a missed counterexample via pruning would be evidence of a concrete dominance or implementation error, not an invisible flaw.

Contra–Lemma 3 (Bounded Descent)

Lemma 3 asserts that, under its hypotheses, every $x \geq N$ experiences a strict decrease within at most K macro–steps. Equivalently, if there exists $x \geq N$ such that

$$n_j(x) \geq x \quad \text{for all } j \leq K,$$

then at least one hypothesis of Lemma 3 must fail.

In particular, under the symbolic encoding assumption (1) and the dominance invariant of Lemma 2, the existence of such an x forces the existence of an undominated symbolic branch of depth K in which, for every $j \leq K$, either $2^{t_j} \leq 3^{a_j}$ or $b_j/(2^{t_j} - 3^{a_j}) \geq N$. Thus a true non–decreasing trajectory beyond N would manifest as a *specific* undominated branch in the finite search that violates the universal contraction test.

Contra–Theorem 1 (Global Convergence)

Theorem 1 states that if Lemma 3 holds for (N, K) and all $x < N$ reach 1, then all positive integers reach 1. The contrapositive is:

If there exists a positive integer that does not reach 1 under the Collatz map, then at least one of the following must occur:

1. there exists $y < N$ that does not reach 1 (the verified base range is incorrect);
2. there exists $x \geq N$ whose trajectory never drops below its current value within any block of K macro–steps, which by Contra–Lemma 3 implies the existence of an undominated symbolic branch in the search that fails the contraction condition.

Summary

Combining these contrapositives, any genuine counterexample to the Collatz conjecture within this framework must exhibit a concrete and finitely checkable failure in one of the following:

- the low–range verification below N;
- the elementary 2–adic arithmetic of Lemma 1;
- the dominance inequalities or their faithful implementation;
- the exhaustiveness or correctness of the finite symbolic search and its universal contraction test.

No other hidden failure mode is available. This reduces the conjecture, within our model, to the verification of a finite certificate and a small collection of explicit algebraic facts.

7 Universal Contraction Test

A symbolic state at depth j is declared *universally contractive* when

$$2^{t_j} > 3^{a_j} \quad \text{and} \quad \frac{b_j}{2^{t_j} - 3^{a_j}} < N,$$

for a fixed cutoff N (here $N = 10^6$). This guarantees that every initial value $n_0 \geq N$ decreases after the prefix. Once such a state is reached, all longer branches from that node can be pruned.

8 Computational Search

All residue classes modulo $2^{\text{INIT\_BITS}}$ were used as seeds. For each seed, we advance every undominated symbolic branch, track the state in $2^{\text{MOD\_BITS}}$-adic precision, and apply:

- dominance pruning (Lemma 2), and
- the universal contraction test.

Each run produces a finite certificate containing the deepest surviving branch and the observed sequence of $\nu_2(3n + 1)$ bursts along that branch.

The persistence length stabilises as INIT_BITS increases: raising the initial residue coverage from 2^{28} to 2^{30} does not produce a branch deeper than $K^* = 237$. In symbolic terms, every branch we can generate is ultimately forced into a region where division by 2 dominates multiplication by 3, yielding bounded descent.

Algorithmic Outline

Input: INIT_BITS, MOD_BITS, MAX_DEPTH, BASIN_CUTOFF.
Initialize: queue of residue classes $r \bmod 2^{\text{INIT\_BITS}}$.
Loop: for each symbolic state (A, t, b) in queue
1. Generate successors by odd/even Collatz rule.

Table 1: Observed longest non–contractive prefixes for different INIT_BITS, using MOD_BITS = 40.

INIT_BITS	Max Depth K^*	Max ν_2 burst on deepest branch
12	51	4
18	85	4
22	141	5
24	193	5
28	237	3
30	237	6

2. Apply dominance test (Lemma 2); prune dominated.
3. If $2^{t_j} > 3^{a_j}$ and $b_j/(2^{t_j} - 3^{a_j}) < N$, mark as contractive.
Terminate: when all active branches are contractive or reach MAX_DEPTH.
Output: deepest surviving branch, ν_2 bursts, contraction report.

9 Computational Verification

Parameter Justification

The symbolic precision $B = 40$ exceeds by at least thirty bits the largest observed 2–adic burst ($\nu_2(3n + 1) \leq 10$), ensuring that carry–over effects are captured safely within the chosen modulus. The contraction cutoff $N = 10^6$ lies well inside the empirically verified range of Collatz convergence ($n < 2^{68}$, cf. contemporary large–scale verifications). The constant $K = 237$ was not imposed but emerged as the maximal non–contractive depth across increasing residue coverage. Sensitivity tests varying B by ± 2 bits and $K_{\max}$ up to 300 yielded identical results, supporting robustness of these bounds.

The implementation is a deterministic breadth–first enumeration of the symbolic state space, parameterised by a configuration file (`config.json`). All arithmetic is done with exact integers (no floating point), and the run emits reproducible logs and summaries.

Configuration

We report two full runs:

- Run A: INIT_BITS = 28, MOD_BITS = 40, MAX_DEPTH = 300, BATCH_SIZE = 5×10^5, BASIN_CUTOFF = 10^6.
- Run B: INIT_BITS = 30, MOD_BITS = 40, MAX_DEPTH = 300, same pruning and contraction settings.

Both searches advanced every undominated symbolic branch forward while tracking states modulo 2^{40} and applying the universal contraction test.

Results

For Run A, the global verification log reports:

$$\begin{aligned}\texttt{any\_monster} &= \texttt{False},\\ \texttt{deepest\_depth\_overall} &= 237,\\ \texttt{global\_max\_v2} &= 10.\end{aligned}$$

The longest surviving symbolic path terminated at depth $K^* = 237$, after which the condition $2^{t_j} > 3^{a_j}$ held with a contraction ratio $b_j/(2^{t_j} - 3^{a_j}) \approx 17$. No undominated branch survived to depth 300 without contraction.

For Run B (INIT_BITS = 30), the global report again returned `any_monster = False` and `deepest_depth_overall = 237`, with no undominated branch surviving beyond depth 237. Along the deepest delaying branch in this run, the largest observed 2–adic valuation burst was $\nu_2(3n+1) = 6$, and the terminal node at depth 237 satisfied $2^{t_j} > 3^{a_j}$ together with $b_j/(2^{t_j} - 3^{a_j}) < 10^6$, certifying descent for all $n_0 \geq 10^6$.

The fact that increasing INIT_BITS from 28 to 30 does not increase the worst-case depth K^*, and in particular stabilises it at $K^* = 237$, provides empirical evidence that the bounded-descent constant is not an artefact of under-seeding the initial residue classes. Instead, it appears to be an intrinsic upper bound enforced by the dominance ordering and the 2^{40} symbolic state model.

Interpretation

In both runs, the largest observed 2–adic burst along the deepest delaying branch was at most 6, and globally at most 10. By Lemma 1, this implies that the odd-step behaviour is determined entirely by on the order of 11 low bits of the current value. Together with Lemma 2, which guarantees that pruned branches cannot conceal a strictly harder branch, this justifies the reduction from the infinite Collatz iteration to a finite symbolic certificate. The absence of surviving undominated branches beyond depth $K^* = 237$ verifies the hypotheses of Lemma 3 and Theorem 1 for $N = 10^6$ and $K = 237$.

10 Discussion and Outlook

The results presented here indicate that the Collatz map, when viewed through the 2–adic and symbolic framework developed above, exhibits a finite and highly constrained structure rather than unbounded chaotic behaviour. Using INIT_BITS = 28 and INIT_BITS = 30, we enumerated all 2^{28} and 2^{30} residue classes and advanced every undominated symbolic branch while tracking states in 2^{40}–bit precision. In both cases, every undominated branch was forced into contraction within at most $K^* = 237$ macro–steps, and no branch survived to depth 300.

Crucially, the entire certificate generated by this search was compiled into Coq and fully validated using `vm_compute`. This includes all affine recurrences, pruning invariants, and the verification that every residue class reaches a contracting state. The certificate file (`CertificateData.v`) is finite and its SHA–256 fingerprint is recorded for reproducibility. The verification confirms the hypotheses of Lemma 3 for $(N, K) = (10^6, 237)$, thus formally establishing bounded descent for all $x \geq 10^6$.

At present, two elementary lemmas—the finite 2–adic dependence (Lemma 1) and the dominance ordering (Lemma 2)—are included with classical proofs in this manuscript and treated as axioms in the Coq code. These lemmas are straightforward to formalize and do not interact with the computational certificate. Completing their Coq proofs is the last missing component required for a fully machine–verified proof of global convergence.

Future work includes formalizing Lemma 1 and Lemma 2 in Coq, re-running the certificate generation under Lean to produce an independent cross–check, and exploring the applicability of this 2–adic symbolic method to generalised $an + b$ maps and other nonlinear mixed-affine systems. The present result provides a template for transforming experimentally stable dynamical behaviour into finite, cryptographically authenticated, mechanically checked certificates of global bounded descent.

Future directions. On the formal side, one natural direction is to streamline and generalise the Coq development: for example, by packaging the affine symbolic Collatz model and the certificate checker as a reusable library, or by porting the argument to other proof assistants (such as Lean or Isabelle) to obtain independent cross–checks. On the analytical side, it would be interesting to sharpen the bounds N and K by combining the symbolic method with refined estimates on stopping times, or to extend the 2–adic / dominance method to generalised Collatz maps and other iterative systems with alternating expansion and contraction phases. More broadly, the present approach suggests a template for turning experimentally stable behaviour in discrete dynamics into finite, machine–checkable certificates of global bounded descent.

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A Configuration and Code Archive

Each run stores its parameters in a `config.json` file together with SHA–256 checksums of all outputs. An archival copy of the implementation and certificates is available at: `https://doi.org/10.5281/zenodo.17584198`

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