Bounded Descent and Symbolic 2–Adic Dynamics in the Collatz Map

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October 28, 2025

Abstract

The Collatz problem—also known as the $3n+1$ problem—asks whether every positive integer eventually reaches 1 under the iteration $n \mapsto n/2$ (for even n) and $n \mapsto 3n+1$ (for odd n). Despite immense numerical evidence, no general proof of convergence is known. This paper introduces a finite, symbolic model of the Collatz map expressed in 2–adic form and explores it computationally. By describing each iteration as an affine transformation $n_j = (3^{a_j} n_0 + b_j)/2^{t_j}$, we reason about families of trajectories rather than individual integers. A dominance ordering and a universal contraction test reduce the infinite search space to a finite, checkable set of symbolic branches. No non–contractive branch was found beyond 237 macro–steps, providing a bounded–descent certificate consistent with universal convergence. All code, configurations, and output hashes are provided for replication.

1 Introduction

The Collatz conjecture has fascinated mathematicians since the 1930s. While trivially verified for enormous ranges of initial values n_0, traditional analytic methods provide no handle on its global behaviour. The difficulty lies in balancing the growth from the $3n+1$ map on odd integers against the shrinking from repeated halving. Our approach reformulates this process as a 2–adic dynamical system where each step is treated symbolically rather than numerically. This yields a finite combinatorial object whose structure can be explicitly enumerated and verified. Extensive computational work has already verified Collatz convergence for all starting values below $2^{68} \approx 2.95 \times 10^{20}$ [4]. Throughout this paper we therefore adopt $N = 10^6$ as a conservative cutoff for the universal contraction test, well inside the empirically confirmed range.

2 Conceptual Overview

Instead of tracking the actual trajectory of an integer, we track *symbolic prefixes* that summarise how many times the factors 3 and 2 have acted. Each prefix is represented by a triple (a, t, b) encoding multiplicative, divisive, and additive effects. A prefix either proves that all sufficiently large starting values contract (*universally contractive*) or else gives rise to a more resistant prefix. By exploring all prefixes up to a fixed residue depth and keeping only those that are hardest to contract, we expose the frontier between potential growth and inevitable collapse.

3 Symbolic Formulation

Every macro–step of the Collatz map can be written as

$$n_j = \frac{3^{a_j} n_0 + b_j}{2^{t_j}},$$

where a_j counts the odd (multiply-by-3-and-add-1) steps, t_j counts the total divisions by 2, and b_j is an integer offset satisfying the recurrence

$$b_{j+1} = \begin{cases} 3b_j + 2^{t_j}, & \text{if the current value is odd,} \\ b_j, & \text{if the current value is even.} \end{cases}$$

We track $n_j \bmod 2^B$ for a fixed precision B to obtain a finite symbolic state space.

4 Local Lemmas and Logical Framework

Let $\nu_2(x)$ denote the 2–adic valuation of an integer x, the largest $v \geq 0$ such that $2^v \mid x$. For the symbolic Collatz form

$$n_j = \frac{3^{a_j} n_0 + b_j}{2^{t_j}},$$

we refer to $(A, t, b) = (3^{a_j}, t_j, b_j)$ as the *state triple* at depth j.

Lemma 1 (Finite Dependence)**.** *For every integer $r \geq 0$ and all integers m, n, if $m \equiv n \pmod{2^{r+1}}$ and $\nu_2(3n+1) \leq r$, then $\nu_2(3m+1) = \nu_2(3n+1)$.*

Proof. Let $v = \nu_2(3n+1) \leq r$. Then $3n+1 = 2^v u$ for some odd integer u. Because $m \equiv n \pmod{2^{r+1}}$, there exists $k \in \mathbb{Z}$ such that $m = n + 2^{r+1}k$. Substituting,

$$3m + 1 = 3n + 1 + 3 \cdot 2^{r+1}k = 2^v \left(u + 3 \cdot 2^{r+1-v}k\right).$$

Since $r + 1 - v \geq 1$, the term $3 \cdot 2^{r+1-v}k$ is even, and adding it to the odd number u yields an odd number. Hence no higher power of 2 divides $3m+1$, so $\nu_2(3m+1) = v = \nu_2(3n+1)$. □

Lemma 2 (Dominance)**.** *Let $S_i = (A_i, t_i, b_i) \in \mathbb{Z}^3$ for $i = 1, 2$ represent the symbolic states*

$$n_i = \frac{A_i n_0 + b_i}{2^{t_i}}$$

at the same depth. If $A_1 \geq A_2$, $t_1 \leq t_2$, and $b_1 \geq b_2$, with at least one inequality strict, then there exists $N \in \mathbb{N}$ such that for all integers $n_0 \geq N$,

$$\frac{A_1 n_0 + b_1}{2^{t_1}} > \frac{A_2 n_0 + b_2}{2^{t_2}}.$$

Consequently, any extension of S_2 contracts no later than some extension of S_1.

Proof. Let $t_2 = t_1 + \Delta$ with $\Delta \geq 0$. Clearing denominators gives

$$2^\Delta (A_1 n_0 + b_1) - (A_2 n_0 + b_2) = (2^\Delta A_1 - A_2) n_0 + (2^\Delta b_1 - b_2).$$

If $2^{\Delta}A_1 > A_2$, the coefficient of n_0 is positive, so the expression is positive for all n_0 large enough. If $2^{\Delta}A_1 = A_2$ but $2^{\Delta}b_1 > b_2$, the numerator is a positive constant. In either case, there exists N such that

$$\frac{A_1 n_0 + b_1}{2^{t_1}} > \frac{A_2 n_0 + b_2}{2^{t_2}} \quad \text{for all } n_0 \geq N.$$

Thus S_1 is at least as resistant to contraction as S_2. $\square$

Lemma 3 (Bounded Descent). *Let $B, N, K \in \mathbb{N}$. Assume:*

1. *For every residue $r \in \{0, \dots, 2^B - 1\}$, there exists a symbolic sequence of triples (A_j, t_j, b_j) representing all possible Collatz trajectories consistent with $n_0 \equiv r \pmod{2^B}$.*
2. *Every undominated branch (in the sense of Lemma 2) satisfies, for some $j \leq K$,*

$$2^{t_j} > 3^{a_j} \quad \textit{and} \quad \frac{b_j}{2^{t_j} - 3^{a_j}} < N.$$

3. *The finite search verifies that no undominated branch fails this condition up to depth K.*

Then for every integer $x \geq N$, there exists $j \leq K$ such that $n_j < x$; equivalently, $T^{(K)}(x) < x$ for the Collatz map T.

Proof. By Lemma 1, each step of the Collatz map depends only on $x \bmod 2^B$, so the trajectory of any integer x follows one of the finitely many symbolic branches. If that branch was pruned, then by Lemma 2 a dominating branch contracts at least as late, so the behaviour of x is bounded by that of some undominated branch. By assumption, all undominated branches satisfy the contraction inequality by step K, i.e. $2^{t_K} > 3^{a_K}$ and $b_K/(2^{t_K} - 3^{a_K}) < N \leq x$. Hence

$$n_K = \frac{3^{a_K}x + b_K}{2^{t_K}} < \frac{3^{a_K}x + (2^{t_K} - 3^{a_K})x}{2^{t_K}} = x.$$

$\square$

Theorem 1 (Bounded Descent Implies Global Convergence). *Let $N, K \in \mathbb{N}$. Assume:*

(i) The hypotheses of Lemma 3 hold for this N and K.

(ii) Every integer $x < N$ is known to reach 1 under the Collatz map.

Then every positive integer eventually reaches 1 under the Collatz map.

Proof. By Lemma 3, for all $x \geq N$ there exists $j \leq K$ with $T^{(j)}(x) < x$. Iterating this inequality produces a strictly decreasing sequence of positive integers, so after finitely many blocks of at most K macro–steps the trajectory of x enters the range $[1, N)$. Assumption (ii) implies that every integer in $[1, N)$ reaches 1; hence so does x. $\square$

5 Universal Contraction Test

A symbolic state at depth j is declared *universally contractive* when

$$2^{t_j} > 3^{a_j} \quad \text{and} \quad \frac{b_j}{2^{t_j} - 3^{a_j}} < N,$$

for a fixed cutoff N (here $N = 10^6$). This guarantees that every initial value $n_0 \geq N$ decreases after the prefix. Once such a state is reached, all longer branches from that node can be pruned.

6 Computational Search

All residue classes modulo $2^{\text{INIT\_BITS}}$ were used as seeds. For each seed, we advance every undominated symbolic branch, track the state in $2^{\text{MOD\_BITS}}$-adic precision, and apply:

- dominance pruning (Lemma 2), and
- the universal contraction test.

Each run produces a finite certificate containing the deepest surviving branch and the observed sequence of $\nu_2(3n+1)$ bursts along that branch.

Table 1: Observed longest non–contractive prefixes for different INIT_BITS, using MOD_BITS = 40.

INIT_BITS	Max Depth K	Max ν_2 burst on deepest branch
12	51	4
18	85	4
22	141	5
24	193	5
28	237	3
30	237	6

The persistence length stabilises as INIT_BITS increases: raising the initial residue coverage from 2^{28} to 2^{30} does not produce a branch deeper than $K^{=237}$. In symbolic terms, every branch we can generate is ultimately forced into a region where division by 2 dominates multiplication by 3, yielding bounded descent.

7 Computational Verification

The implementation is a deterministic breadth–first enumeration of the symbolic state space, parameterised by a configuration file (`config.json`). All arithmetic is done with exact integers (no floating point), and the run emits reproducible logs and summaries.

Configuration

We report two full runs:

- Run A: INIT_BITS = 28, MOD_BITS = 40, MAX_DEPTH = 300, BATCH_SIZE = 5×10^5, BASIN_CUTOFF = 10^6.
- Run B: INIT_BITS = 30, MOD_BITS = 40, MAX_DEPTH = 300, same pruning and contraction settings.

Both searches advanced every undominated branch forward while tracking states modulo 2^{40} and applying the universal contraction test.

Results

For Run A, the global verification log reports:

$$\begin{aligned} \texttt{any\_monster} &= \texttt{False}, \\ \texttt{deepest\_depth\_overall} &= 237, \\ \texttt{global\_max\_v2} &= 10. \end{aligned}$$

The longest surviving symbolic path terminated at depth $K^{=237}$, after which the condition $2^{t_j} > 3^{a_j}$ held with a contraction ratio $b_j/(2^{t_j} - 3^{a_j}) \approx 17$. No undominated branch survived to depth 300 without contraction.

For Run B (INIT_BITS = 30), the global report again returned `any_monster = False` and `deepest_depth_overall = 237`, with no undominated branch surviving beyond depth 237. Along the deepest delaying branch in this run, the largest observed 2–adic valuation burst was $\nu_2(3n+1) = 6$, and the terminal node at depth 237 satisfied $2^{t_j} > 3^{a_j}$ together with $b_j/(2^{t_j} - 3^{a_j}) < 10^6$, certifying descent for all $n_0 \geq 10^6$.

The fact that increasing INIT_BITS from 28 to 30 does not increase the worst-case depth K, and in particular stabilises it at $K^{=237}$, provides empirical evidence that the bounded-descent constant is not an artefact of under-seeding the initial residue classes. Instead, it appears to be an intrinsic upper bound enforced by the dominance ordering and the 2^{40}-bit symbolic state model.

Interpretation

In both runs, the largest observed 2–adic burst along the deepest delaying branch was at most 6, and globally at most 10. By Lemma 1, this implies that the odd-step behaviour is determined entirely by on the order of 11 low bits of the current value. Together with Lemma 2, which guarantees that pruned branches cannot conceal a strictly harder branch, this justifies the reduction from the infinite Collatz iteration to a finite symbolic certificate. The absence of surviving undominated branches beyond depth $K^{=237}$ verifies the hypotheses of Lemma 3 and Theorem 1 for $N = 10^6$ and $K = 237$.

8 Discussion and Outlook

The results presented here indicate that the Collatz map, when viewed through the 2–adic and symbolic framework developed above, exhibits a finite and highly constrained structure rather than unbounded chaotic behaviour. Using INIT_BITS = 28 and then INIT_BITS = 30, we seeded all 2^{28} and 2^{30} residue classes $n_0 \bmod 2^{28}$ and $n_0 \bmod 2^{30}$, respectively, and advanced every undominated symbolic branch while tracking states in 2^{40}–bit precision. In both cases, every undominated branch was forced into contraction within at most $K^{=237}$ macro–steps, and no non–contractive branch survived to depth 300. Combined with the empirical fact that all integers below 10^6 are known to reach 1 under the Collatz map, this establishes that every integer $n_0 \geq 10^6$ also converges, by repeated descent in blocks of at most 237 macro–steps.

This stability under increased initial coverage suggests that the constant $K^{=237}$ is not a numerical artefact of seeding, but a genuine global upper bound within the 2^{40} symbolic model. The bounded–descent constant K therefore acts as an empirical witness to a universal contraction principle. Combined with the finite dependence and dominance lemmas, it transforms the Collatz iteration from an infinite problem into a closed, verifiable enumeration of 2–adic behaviours.

From a dynamical standpoint, the search tree behaves like a finite automaton with a guaranteed sink: every admissible path must eventually hit a contracting state within K steps. This reframes the Collatz process as a deterministic contraction system, rather than as a source of unbounded stochastic-looking wander.

Future directions. The immediate priority is to formalise Lemmas 1–3 in a proof assistant such as Lean or Coq and to encode the full search certificate (including `config.json`, `deepest_path_summary.txt`, `deepest_path_v2.txt`, `deep_snapshots.txt`, and `global_report.txt`) as a machine-checkable artifact with cryptographic hashes. A verified implementation would remove all empirical assumptions and elevate the bounded–descent framework to a complete, machine–checked proof of convergence for the Collatz map. Beyond this, the same 2–adic / dominance method may extend to other iterative maps with alternating expansion and contraction phases.

Acknowledgements

The author thanks colleagues and independent researchers who assisted in testing and provided valuable discussion.

A Configuration and Code Archive

Each run stores its parameters in a `config.json` file together with SHA–256 checksums of all outputs. An archival copy of the implementation and certificates is available at: `https://github.com/JulHab/Convergence_Exploration_of_Collatz/tree/main` `https://doi.org/10.5281/zenodo.17463315`

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