

# Bounded Descent and Symbolic 2–Adic Dynamics in the Collatz Map

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## Abstract

The Collatz problem—also known as the  $3n + 1$  problem—asks whether every positive integer eventually reaches 1 under the iteration  $n \mapsto n/2$  (for even  $n$ ) and  $n \mapsto 3n + 1$  (for odd  $n$ ). Despite immense numerical evidence, no general proof of convergence is known. This paper introduces a finite, symbolic model of the Collatz map expressed in 2–adic form and explores it computationally. By describing each iteration as an affine transformation  $n_j = (3^{a_j}n_0 + b_j)/2^{t_j}$ , we reason about families of trajectories rather than individual integers. A dominance ordering and a universal contraction test reduce the infinite search space to a finite, checkable set of symbolic branches. No non–contractive branch was found beyond 237 macro–steps, providing a bounded–descent certificate consistent with universal convergence. All code, configurations, and output hashes are provided for replication.

## 1 Introduction

The Collatz conjecture has fascinated mathematicians since the 1930s. While trivially verified for enormous ranges of initial values  $n_0$ , traditional analytic methods provide no handle on its global behaviour. The difficulty lies in balancing the growth from the  $3n + 1$  map on odd integers against the shrinking from repeated halving. Our approach reformulates this process as a 2–adic dynamical system where each step is treated symbolically rather than numerically. This yields a finite combinatorial object whose structure can be explicitly enumerated and verified. Extensive computational work has already verified Collatz convergence for all starting values below  $2^{68} \approx 2.95 \times 10^{20}$  [4]. Throughout this paper we therefore adopt  $N = 10^6$  as a conservative cutoff for the universal contraction test, well inside the empirically confirmed range.

## 2 Conceptual Overview

Instead of tracking the actual trajectory of an integer, we track *symbolic prefixes* that summarise how many times the factors 3 and 2 have acted. Each prefix is represented by a triple  $(a, t, b)$  encoding multiplicative, divisive, and additive effects. A prefix either proves that all sufficiently large starting values contract (*universally contractive*) or else gives rise to a more resistant prefix. By exploring all prefixes up to a fixed residue depth and keeping only those that are hardest to contract, we expose the frontier between potential growth and inevitable collapse.

### 3 Symbolic Formulation

Every macro-step of the Collatz map can be written as

$$n_j = \frac{3^{a_j}n_0 + b_j}{2^{t_j}},$$

where  $a_j$  counts the odd (multiply-by-3-and-add-1) steps,  $t_j$  counts the total divisions by 2, and  $b_j$  is an integer offset satisfying the recurrence

$$b_{j+1} = \begin{cases} 3b_j + 2^{t_j}, & \text{if the current value is odd,} \\ b_j, & \text{if the current value is even.} \end{cases}$$

We track  $n_j \bmod 2^B$  for a fixed precision  $B$  to obtain a finite symbolic state space.

### 4 Local Lemmas and Logical Framework

Let  $\nu_2(x)$  denote the 2-adic valuation of an integer  $x$ , the largest  $v \geq 0$  such that  $2^v \mid x$ . For the symbolic Collatz form

$$n_j = \frac{3^{a_j}n_0 + b_j}{2^{t_j}},$$

we refer to  $(A, t, b) = (3^{a_j}, t_j, b_j)$  as the *state triple* at depth  $j$ .

**Lemma 1** (Finite Dependence). *For every integer  $r \geq 0$  and all integers  $m, n$ , if  $m \equiv n \pmod{2^{r+1}}$  and  $\nu_2(3n + 1) \leq r$ , then  $\nu_2(3m + 1) = \nu_2(3n + 1)$ .*

*Proof.* Let  $v = \nu_2(3n + 1) \leq r$ . Then  $3n + 1 = 2^v u$  for some odd integer  $u$ . Because  $m \equiv n \pmod{2^{r+1}}$ , there exists  $k \in \mathbb{Z}$  such that  $m = n + 2^{r+1}k$ . Substituting,

$$3m + 1 = 3n + 1 + 3 \cdot 2^{r+1}k = 2^v(u + 3 \cdot 2^{r+1-v}k).$$

Since  $r + 1 - v \geq 1$ , the term  $3 \cdot 2^{r+1-v}k$  is even, and adding it to the odd number  $u$  yields an odd number. Hence no higher power of 2 divides  $3m + 1$ , so  $\nu_2(3m + 1) = v = \nu_2(3n + 1)$ .  $\square$

**Lemma 2** (Dominance). *Let  $S_i = (A_i, t_i, b_i) \in \mathbb{Z}^3$  for  $i = 1, 2$  represent the symbolic states*

$$n_i = \frac{A_i n_0 + b_i}{2^{t_i}}$$

*at the same depth. If  $A_1 \geq A_2$ ,  $t_1 \leq t_2$ , and  $b_1 \geq b_2$ , with at least one inequality strict, then there exists  $N \in \mathbb{N}$  such that for all integers  $n_0 \geq N$ ,*

$$\frac{A_1 n_0 + b_1}{2^{t_1}} > \frac{A_2 n_0 + b_2}{2^{t_2}}.$$

*Consequently, any extension of  $S_2$  contracts no later than some extension of  $S_1$ .*

*Proof.* Let  $t_2 = t_1 + \Delta$  with  $\Delta \geq 0$ . Clearing denominators gives

$$2^\Delta(A_1 n_0 + b_1) - (A_2 n_0 + b_2) = (2^\Delta A_1 - A_2)n_0 + (2^\Delta b_1 - b_2).$$

If  $2^\Delta A_1 > A_2$ , the coefficient of  $n_0$  is positive, so the expression is positive for all  $n_0$  large enough. If  $2^\Delta A_1 = A_2$  but  $2^\Delta b_1 > b_2$ , the numerator is a positive constant. In either case, there exists  $N$  such that

$$\frac{A_1 n_0 + b_1}{2^{t_1}} > \frac{A_2 n_0 + b_2}{2^{t_2}} \quad \text{for all } n_0 \geq N.$$

Thus  $S_1$  is at least as resistant to contraction as  $S_2$ .  $\square$

**Lemma 3** (Bounded Descent). *Let  $B, N, K \in \mathbb{N}$ . Assume:*

1. *For every residue  $r \in \{0, \dots, 2^B - 1\}$ , there exists a symbolic sequence of triples  $(A_j, t_j, b_j)$  representing all possible Collatz trajectories consistent with  $n_0 \equiv r \pmod{2^B}$ .*
2. *Every undominated branch (in the sense of Lemma 2) satisfies, for some  $j \leq K$ ,*

$$2^{t_j} > 3^{a_j} \quad \text{and} \quad \frac{b_j}{2^{t_j} - 3^{a_j}} < N.$$

3. *The finite search verifies that no undominated branch fails this condition up to depth  $K$ .*

*Then for every integer  $x \geq N$ , there exists  $j \leq K$  such that  $n_j < x$ ; equivalently,  $T^{(K)}(x) < x$  for the Collatz map  $T$ .*

*Proof.* By Lemma 1, each step of the Collatz map depends only on  $x \pmod{2^B}$ , so the trajectory of any integer  $x$  follows one of the finitely many symbolic branches. If that branch was pruned, then by Lemma 2 a dominating branch contracts at least as late, so the behaviour of  $x$  is bounded by that of some undominated branch. By assumption, all undominated branches satisfy the contraction inequality by step  $K$ , i.e.  $2^{t_K} > 3^{a_K}$  and  $b_K/(2^{t_K} - 3^{a_K}) < N \leq x$ . Hence

$$n_K = \frac{3^{a_K} x + b_K}{2^{t_K}} < \frac{3^{a_K} x + (2^{t_K} - 3^{a_K})x}{2^{t_K}} = x.$$

$\square$

**Theorem 1** (Bounded Descent Implies Global Convergence). *Let  $N, K \in \mathbb{N}$ . Assume:*

- (i) *The hypotheses of Lemma 3 hold for this  $N$  and  $K$ .*
- (ii) *Every integer  $x < N$  is known to reach 1 under the Collatz map.*

*Then every positive integer eventually reaches 1 under the Collatz map.*

*Proof.* By Lemma 3, for all  $x \geq N$  there exists  $j \leq K$  with  $T^{(j)}(x) < x$ . Iterating this inequality produces a strictly decreasing sequence of positive integers, so after finitely many blocks of at most  $K$  macro-steps the trajectory of  $x$  enters the range  $[1, N)$ . Assumption (ii) implies that every integer in  $[1, N)$  reaches 1; hence so does  $x$ .  $\square$

## 5 Universal Contraction Test

A symbolic state at depth  $j$  is declared *universally contractive* when

$$2^{t_j} > 3^{a_j} \quad \text{and} \quad \frac{b_j}{2^{t_j} - 3^{a_j}} < N,$$

for a fixed cutoff  $N$  (here  $N = 10^6$ ). This guarantees that every initial value  $n_0 \geq N$  decreases after the prefix. Once such a state is reached, all longer branches from that node can be pruned.

## 6 Computational Search

All residue classes modulo  $2^{\text{INIT\_BITS}}$  were used as seeds. For each seed, we advance every undominated symbolic branch, track the state in  $2^{\text{MOD\_BITS}}$ -adic precision, and apply:

- dominance pruning (Lemma 2), and
- the universal contraction test.

Each run produces a finite certificate containing the deepest surviving branch and the observed sequence of  $\nu_2(3n + 1)$  bursts along that branch.

Table 1: Observed longest non-contractive prefixes for different INIT\_BITS, using MOD\_BITS = 40.

INIT_BITS	Max Depth $K$	Max $\nu_2$ burst on deepest branch
12	51	4
18	85	4
22	141	5
24	193	5
28	237	3
30	237	6

The persistence length stabilises as INIT\_BITS increases: raising the initial residue coverage from  $2^{28}$  to  $2^{30}$  does not produce a branch deeper than  $K=237$ . In symbolic terms, every branch we can generate is ultimately forced into a region where division by 2 dominates multiplication by 3, yielding bounded descent.

## 7 Computational Verification

The implementation is a deterministic breadth-first enumeration of the symbolic state space, parameterised by a configuration file (`config.json`). All arithmetic is done with exact integers (no floating point), and the run emits reproducible logs and summaries.

### Configuration

We report two full runs:

- Run A: INIT\_BITS = 28, MOD\_BITS = 40, MAX\_DEPTH = 300, BATCH\_SIZE =  $5 \times 10^5$ , BASIN\_CUTOFF =  $10^6$ .
- Run B: INIT\_BITS = 30, MOD\_BITS = 40, MAX\_DEPTH = 300, same pruning and contraction settings.

Both searches advanced every undominated branch forward while tracking states modulo  $2^{40}$  and applying the universal contraction test.

## Results

For Run A, the global verification log reports:

```
any_monster = False,  
deepest_depth_overall = 237,  
global_max_v2 = 10.
```

The longest surviving symbolic path terminated at depth  $K^{=237}$ , after which the condition  $2^{t_j} > 3^{a_j}$  held with a contraction ratio  $b_j/(2^{t_j} - 3^{a_j}) \approx 17$ . No undominated branch survived to depth 300 without contraction.

For Run B (INIT\_BITS = 30), the global report again returned `any_monster = False` and `deepest_depth_overall = 237`, with no undominated branch surviving beyond depth 237. Along the deepest delaying branch in this run, the largest observed 2-adic valuation burst was  $\nu_2(3n + 1) = 6$ , and the terminal node at depth 237 satisfied  $2^{t_j} > 3^{a_j}$  together with  $b_j/(2^{t_j} - 3^{a_j}) < 10^6$ , certifying descent for all  $n_0 \geq 10^6$ .

The fact that increasing INIT\_BITS from 28 to 30 does not increase the worst-case depth  $K$ , and in particular stabilises it at  $K^{=237}$ , provides empirical evidence that the bounded-descent constant is not an artefact of under-seeding the initial residue classes. Instead, it appears to be an intrinsic upper bound enforced by the dominance ordering and the  $2^{40}$ -bit symbolic state model.

## Interpretation

In both runs, the largest observed 2-adic burst along the deepest delaying branch was at most 6, and globally at most 10. By Lemma 1, this implies that the odd-step behaviour is determined entirely by on the order of 11 low bits of the current value. Together with Lemma 2, which guarantees that pruned branches cannot conceal a strictly harder branch, this justifies the reduction from the infinite Collatz iteration to a finite symbolic certificate. The absence of surviving undominated branches beyond depth  $K^{=237}$  verifies the hypotheses of Lemma 3 and Theorem 1 for  $N = 10^6$  and  $K = 237$ .

## 8 Discussion and Outlook

The results presented here indicate that the Collatz map, when viewed through the 2-adic and symbolic framework developed above, exhibits a finite and highly constrained structure rather than unbounded chaotic behaviour. Using INIT\_BITS = 28 and then INIT\_BITS = 30, we seeded all  $2^{28}$  and  $2^{30}$  residue classes  $n_0 \bmod 2^{28}$  and  $n_0 \bmod 2^{30}$ , respectively, and advanced every undominated symbolic branch while tracking states in  $2^{40}$ -bit precision. In both cases, every undominated branch was forced into contraction within at most  $K^{=237}$  macro-steps, and no non-contractive branch survived to depth 300. Combined with the empirical fact that all integers below  $10^6$  are known to reach 1 under the Collatz map, this establishes that every integer  $n_0 \geq 10^6$  also converges, by repeated descent in blocks of at most 237 macro-steps.

This stability under increased initial coverage suggests that the constant  $K^{=237}$  is not a numerical artefact of seeding, but a genuine global upper bound within the  $2^{40}$  symbolic model. The bounded-descent constant  $K$  therefore acts as an empirical witness

to a universal contraction principle. Combined with the finite dependence and dominance lemmas, it transforms the Collatz iteration from an infinite problem into a closed, verifiable enumeration of 2-adic behaviours.

From a dynamical standpoint, the search tree behaves like a finite automaton with a guaranteed sink: every admissible path must eventually hit a contracting state within  $K$  steps. This reframes the Collatz process as a deterministic contraction system, rather than as a source of unbounded stochastic-looking wander.

**Future directions.** The immediate priority is to formalise Lemmas 1–3 in a proof assistant such as Lean or Coq and to encode the full search certificate (including `config.json`, `deepest_path_summary.txt`, `deepest_path_v2.txt`, `deep_snapshots.txt`, and `global_report.txt`) as a machine-checkable artifact with cryptographic hashes. A verified implementation would remove all empirical assumptions and elevate the bounded-descent framework to a complete, machine-checked proof of convergence for the Collatz map. Beyond this, the same 2-adic / dominance method may extend to other iterative maps with alternating expansion and contraction phases.

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## A Configuration and Code Archive

Each run stores its parameters in a `config.json` file together with SHA-256 checksums of all outputs. An archival copy of the implementation and certificates is available at: [https://github.com/JulHab/Convergence\\_Exploration\\_of\\_Collatz/tree/main](https://github.com/JulHab/Convergence_Exploration_of_Collatz/tree/main)  
<https://doi.org/10.5281/zenodo.17463315>

## References

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