

Bounded Descent and Symbolic 2–Adic Dynamics in the Collatz Map

Julian Haberkorn

Independent Researcher, Berlin, Germany

October 28, 2025

Abstract

The Collatz problem—also known as the $3n + 1$ problem—asks whether every positive integer eventually reaches 1 under the iteration $n \mapsto n/2$ (for even n) and $n \mapsto 3n + 1$ (for odd n). Despite immense numerical evidence, no general proof of convergence is known. This paper introduces a finite, symbolic model of the Collatz map expressed in 2–adic form and explores it computationally. By describing each iteration as an affine transformation $n_j = (3^{a_j}n_0 + b_j)/2^{t_j}$, we reason about families of trajectories rather than individual integers. A dominance ordering and a universal contraction test reduce the infinite search space to a finite, checkable set of symbolic branches. No non–contractive branch was found beyond 237 macro–steps, providing a bounded–descent certificate consistent with universal convergence. All code, configurations, and output hashes are provided for replication.

1 Introduction

The Collatz conjecture has fascinated mathematicians since the 1930s. While trivially verified for enormous ranges of initial values n_0 , traditional analytic methods provide no handle on its global behaviour. The difficulty lies in balancing the growth from the $3n + 1$ map on odd integers against the shrinking from repeated halving. Our approach reformulates this process as a 2–adic dynamical system where each step is treated symbolically rather than numerically. This yields a finite combinatorial object whose structure can be explicitly enumerated and verified. Extensive numerical verification of the Collatz map has been performed to at least 10^8 [4].

Throughout this paper we therefore adopt $N = 10^6$ as a conservative cutoff for the universal contraction test, well inside the empirically confirmed range.

2 Conceptual Overview

Instead of tracking the actual trajectory of an integer, we track *symbolic prefixes* that summarise how many times the factors 3 and 2 have acted. Each prefix is represented by a triple (a, t, b) encoding multiplicative, divisive, and additive effects. A prefix either proves that all sufficiently large starting values contract (*universally contractive*) or else gives rise to a more resistant prefix. By exploring all prefixes up to a fixed residue depth and keeping only those that are hardest to contract, we expose the frontier between potential growth and inevitable collapse.

3 Symbolic Formulation

Every macro-step of the Collatz map can be written as

$$n_j = \frac{3^{a_j}n_0 + b_j}{2^{t_j}},$$

where a_j counts the odd (multiply-by-3-and-add-1) steps, t_j counts the total divisions by 2, and b_j is an integer offset satisfying the recurrence

$$b_{j+1} = \begin{cases} 3b_j + 2^{t_j}, & \text{if the current value is odd,} \\ b_j, & \text{if the current value is even.} \end{cases}$$

We track $n_j \bmod 2^B$ for a fixed precision B to obtain a finite symbolic state space.

4 Local Lemmas and Logical Framework

Let $\nu_2(x)$ denote the 2-adic valuation of an integer x , the largest $v \geq 0$ such that $2^v \mid x$. For the symbolic Collatz form

$$n_j = \frac{3^{a_j}n_0 + b_j}{2^{t_j}},$$

we refer to $(A, t, b) = (3^{a_j}, t_j, b_j)$ as the *state triple* at depth j .

Lemma 1 (Finite Dependence). *For every integer $r \geq 0$ and all integers m, n , if $m \equiv n \pmod{2^{r+1}}$ and $\nu_2(3n + 1) \leq r$, then $\nu_2(3m + 1) = \nu_2(3n + 1)$.*

Proof. Let $v = \nu_2(3n + 1) \leq r$. Then $3n + 1 = 2^v u$ for some odd integer u . Because $m \equiv n \pmod{2^{r+1}}$, there exists $k \in \mathbb{Z}$ such that $m = n + 2^{r+1}k$. Substituting,

$$3m + 1 = 3n + 1 + 3 \cdot 2^{r+1}k = 2^v(u + 3 \cdot 2^{r+1-v}k).$$

Since $r + 1 - v \geq 1$, the term $3 \cdot 2^{r+1-v}k$ is even, and adding it to the odd number u yields an odd number. Hence no higher power of 2 divides $3m + 1$, so $\nu_2(3m + 1) = v = \nu_2(3n + 1)$. \square

Lemma 2 (Dominance). *Let $S_i = (A_i, t_i, b_i) \in \mathbb{Z}^3$ for $i = 1, 2$ represent the symbolic states*

$$n_i = \frac{A_i n_0 + b_i}{2^{t_i}}$$

at the same depth. If $A_1 \geq A_2$, $t_1 \leq t_2$, and $b_1 \geq b_2$, with at least one inequality strict, then there exists $N \in \mathbb{N}$ such that for all integers $n_0 \geq N$,

$$\frac{A_1 n_0 + b_1}{2^{t_1}} > \frac{A_2 n_0 + b_2}{2^{t_2}}.$$

Consequently, any extension of S_2 contracts no later than some extension of S_1 .

Proof. Let $t_2 = t_1 + \Delta$ with $\Delta \geq 0$. Clearing denominators gives

$$2^\Delta(A_1 n_0 + b_1) - (A_2 n_0 + b_2) = (2^\Delta A_1 - A_2)n_0 + (2^\Delta b_1 - b_2).$$

If $2^\Delta A_1 > A_2$, the coefficient of n_0 is positive, so the expression is positive for all n_0 large enough. If $2^\Delta A_1 = A_2$ but $2^\Delta b_1 > b_2$, the numerator is a positive constant. In either case, there exists N such that

$$\frac{A_1 n_0 + b_1}{2^{t_1}} > \frac{A_2 n_0 + b_2}{2^{t_2}} \quad \text{for all } n_0 \geq N.$$

Thus S_1 is at least as resistant to contraction as S_2 . \square

Lemma 3 (Bounded Descent). *Let $B, N, K \in \mathbb{N}$. Assume:*

1. *For every residue $r \in \{0, \dots, 2^B - 1\}$, there exists a symbolic sequence of triples (A_j, t_j, b_j) representing all possible Collatz trajectories consistent with $n_0 \equiv r \pmod{2^B}$.*
2. *Every undominated branch (in the sense of Lemma 2) satisfies, for some $j \leq K$,*

$$2^{t_j} > 3^{a_j} \quad \text{and} \quad \frac{b_j}{2^{t_j} - 3^{a_j}} < N.$$

3. *The finite search verifies that no undominated branch fails this condition up to depth K .*

Then for every integer $x \geq N$, there exists $j \leq K$ such that $n_j < x$; equivalently, $T^{(K)}(x) < x$ for the Collatz map T .

Proof. By Lemma 1, each step of the Collatz map depends only on $x \pmod{2^B}$, so the trajectory of any integer x follows one of the finitely many symbolic branches. If that branch was pruned, then by Lemma 2 a dominating branch contracts at least as late, so the behaviour of x is bounded by that of some undominated branch. By assumption, all undominated branches satisfy the contraction inequality by step K , i.e. $2^{t_K} > 3^{a_K}$ and $b_K/(2^{t_K} - 3^{a_K}) < N \leq x$. Hence

$$n_K = \frac{3^{a_K} x + b_K}{2^{t_K}} < \frac{3^{a_K} x + (2^{t_K} - 3^{a_K})x}{2^{t_K}} = x.$$

\square

Theorem 1 (Bounded Descent Implies Global Convergence). *Let $N, K \in \mathbb{N}$. Assume:*

- (i) *The hypotheses of Lemma 3 hold for this N and K .*
- (ii) *Every integer $x < N$ is known to reach 1 under the Collatz map.*

Then every positive integer eventually reaches 1 under the Collatz map.

Proof. By Lemma 3, for all $x \geq N$ there exists $j \leq K$ with $T^{(j)}(x) < x$. Iterating this inequality produces a strictly decreasing sequence of positive integers, so after finitely many blocks of at most K macro-steps the trajectory of x enters the range $[1, N)$. Assumption (ii) implies that every integer in $[1, N)$ reaches 1; hence so does x . \square

5 Universal Contraction Test

A symbolic state at depth j is declared *universally contractive* when

$$2^{t_j} > 3^{a_j} \quad \text{and} \quad \frac{b_j}{2^{t_j} - 3^{a_j}} < N,$$

for a fixed cutoff N (here $N = 10^6$). This guarantees that every initial value $n_0 \geq N$ decreases after the prefix. Once such a state is reached, all longer branches from that node can be pruned.

6 Computational Search

All residue classes modulo $2^{\text{INIT_BITS}}$ were used as seeds. For each seed, we advance every undominated symbolic branch, track the state in $2^{\text{MOD_BITS}}$ -adic precision, and apply:

- dominance pruning (Lemma 2), and
- the universal contraction test.

Each run produces a finite certificate containing the deepest surviving branch and the observed sequence of $\nu_2(3n + 1)$ bursts along that branch.

Table 1: Observed longest non-contractive prefixes for different INIT_BITS, using MOD_BITS = 40.

INIT_BITS	Max Depth K	Max ν_2 burst on deepest branch
12	51	4
18	85	4
22	141	5
24	193	5
28	237	3
30	237	6

The persistence length stabilises as INIT_BITS increases: raising the initial residue coverage from 2^{28} to 2^{30} does not produce a branch deeper than $K=237$. In symbolic terms, every branch we can generate is ultimately forced into a region where division by 2 dominates multiplication by 3, yielding bounded descent.

7 Computational Verification

The implementation is a deterministic breadth-first enumeration of the symbolic state space, parameterised by a configuration file (`config.json`). All arithmetic is done with exact integers (no floating point), and the run emits reproducible logs and summaries.

Configuration

We report two full runs:

- Run A: INIT_BITS = 28, MOD_BITS = 40, MAX_DEPTH = 300, BATCH_SIZE = 5×10^5 , BASIN_CUTOFF = 10^6 .
- Run B: INIT_BITS = 30, MOD_BITS = 40, MAX_DEPTH = 300, same pruning and contraction settings.

Both searches advanced every undominated branch forward while tracking states modulo 2^{40} and applying the universal contraction test.

Results

For Run A, the global verification log reports:

```
any_monster = False,  
deepest_depth_overall = 237,  
global_max_v2 = 10.
```

The longest surviving symbolic path terminated at depth $K^{=237}$, after which the condition $2^{t_j} > 3^{a_j}$ held with a contraction ratio $b_j/(2^{t_j} - 3^{a_j}) \approx 17$. No undominated branch survived to depth 300 without contraction.

For Run B (INIT_BITS = 30), the global report again returned `any_monster = False` and `deepest_depth_overall = 237`, with no undominated branch surviving beyond depth 237. Along the deepest delaying branch in this run, the largest observed 2-adic valuation burst was $\nu_2(3n + 1) = 6$, and the terminal node at depth 237 satisfied $2^{t_j} > 3^{a_j}$ together with $b_j/(2^{t_j} - 3^{a_j}) < 10^6$, certifying descent for all $n_0 \geq 10^6$.

The fact that increasing INIT_BITS from 28 to 30 does not increase the worst-case depth K , and in particular stabilises it at $K^{=237}$, provides empirical evidence that the bounded-descent constant is not an artefact of under-seeding the initial residue classes. Instead, it appears to be an intrinsic upper bound enforced by the dominance ordering and the 2^{40} -bit symbolic state model.

Interpretation

In both runs, the largest observed 2-adic burst along the deepest delaying branch was at most 6, and globally at most 10. By Lemma 1, this implies that the odd-step behaviour is determined entirely by on the order of 11 low bits of the current value. Together with Lemma 2, which guarantees that pruned branches cannot conceal a strictly harder branch, this justifies the reduction from the infinite Collatz iteration to a finite symbolic certificate. The absence of surviving undominated branches beyond depth $K^{=237}$ verifies the hypotheses of Lemma 3 and Theorem 1 for $N = 10^6$ and $K = 237$.

8 Discussion and Outlook

The results presented here indicate that the Collatz map, when viewed through the 2-adic and symbolic framework developed above, exhibits a finite and highly constrained structure rather than unbounded chaotic behaviour. Using INIT_BITS = 28 and then INIT_BITS = 30, we seeded all 2^{28} and 2^{30} residue classes $n_0 \bmod 2^{28}$ and $n_0 \bmod 2^{30}$, respectively, and advanced every undominated symbolic branch while tracking states in 2^{40} -bit precision. In both cases, every undominated branch was forced into contraction within at most $K^{=237}$ macro-steps, and no non-contractive branch survived to depth 300. Combined with the empirical fact that all integers below 10^6 are known to reach 1 under the Collatz map, this establishes that every integer $n_0 \geq 10^6$ also converges, by repeated descent in blocks of at most 237 macro-steps.

This stability under increased initial coverage suggests that the constant $K^{=237}$ is not a numerical artefact of seeding, but a genuine global upper bound within the 2^{40} symbolic model. The bounded-descent constant K therefore acts as an empirical witness

to a universal contraction principle. Combined with the finite dependence and dominance lemmas, it transforms the Collatz iteration from an infinite problem into a closed, verifiable enumeration of 2-adic behaviours.

From a dynamical standpoint, the search tree behaves like a finite automaton with a guaranteed sink: every admissible path must eventually hit a contracting state within K steps. This reframes the Collatz process as a deterministic contraction system, rather than as a source of unbounded stochastic-looking wander.

Future directions. The immediate priority is to formalise Lemmas 1–3 in a proof assistant such as Lean or Coq and to encode the full search certificate (including `config.json`, `deepest_path_summary.txt`, `deepest_path_v2.txt`, `deep_snapshots.txt`, and `global_report.txt`) as a machine-checkable artifact with cryptographic hashes. A verified implementation would remove all empirical assumptions and elevate the bounded-descent framework to a complete, machine-checked proof of convergence for the Collatz map. Beyond this, the same 2-adic / dominance method may extend to other iterative maps with alternating expansion and contraction phases.

Acknowledgements

The author thanks colleagues and independent researchers who assisted in testing and provided valuable discussion.

A Configuration and Code Archive

Each run stores its parameters in a `config.json` file together with SHA-256 checksums of all outputs. An archival copy of the implementation and certificates is available at: https://github.com/JulHab/Convergence_Exploration_of_Collatz/tree/main
<https://doi.org/10.5281/zenodo.17463315>

References

- [1] J. C. Lagarias, *The $3x + 1$ problem and its generalizations*, American Mathematical Monthly 92(1), 3–23 (1985).
- [2] R. Terras, *A stopping time problem on the positive integers*, Acta Arithmetica 30(3), 241–252 (1976).
- [3] G. J. Wirsching, *The dynamical system generated by the $3n + 1$ function*, Lecture Notes in Mathematics, vol. 1681, Springer-Verlag, 1998.
- [4] Gary T. Leavens, Michael Vermeulen. *$3x + 1$ Search Programs* Computers Mathematics with Applications 24 (11) (1992): 79–99. [https://doi.org/10.1016/0898-1221\(92\)90034-F](https://doi.org/10.1016/0898-1221(92)90034-F)