Companion Models for Basic Non-Linear and Transient Devices

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1 PN Diode

As a two-terminal device with only one distinct operating region, the PN diode was a natural starting point for non-linear simulation. The current model does not include parasitic capacitances, but this is planned for the future.

The i-v relation that describes a diode is the following:

$$i(v) = I_S(e^{v/V_t} - 1) \tag{1}$$

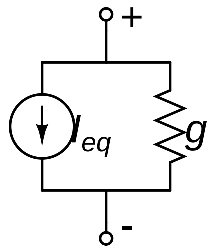
where I_S is the reverse saturation current and V_t is the thermal voltage (kT/q). From this, we derive the small-signal conductance:

$$g = \frac{di}{dv} = (I_S/V_t)e^{v/V_T} \approx i(v)/V_t \tag{2}$$

A subtlety of non-linear simulation is that devices cannot simply be replaced by their small-signal linear equivalent models. We must apply a bias current such that branch current evolves as governed by Newton-Raphson iteration:

$$i_{n+1} = i_n + (di/dv)(v_{n+1} - v_n) \tag{3}$$

where v_n is the port voltage computed in the previous iteration. By rearranging this equation, we find that $i_{n+1} = (i(v_n) - gv_n) + gv_{n+1}$. Hence the appropriate companion model for a diode is a conductance g in parallel with a current source $i(v_n) - gv_n$, not simply $i(v_n)$, as one might suspect. The result is shown in Figure 1.



$$
\begin{aligned}
g &= i(v_n)/V_t \\
I_{eq} &= i(v_n) - g_{eq} v_n
\end{aligned}
$$

Figure 1: Linear companion model for diode

2 Metal-Oxide-Silicon Field-Effect Transistor

The next simplest non-linear devices are n- and p-channel MOSFETs, as no current flows into the gate, and the non-linear source-to-drain port depends on only two parameters, v_{GS} and v_{DS}. The model is complicated slightly, however, by the fact that MOSFETs have three distinct operating regions. For brevity, we derive only the results for n-channel MOSFETs. Similar results for p-channel MOSFETs are presented at the end of this section.

2.1 Cutoff: $v_{GS} - V_T < 0$

Cutoff is a trivial operating region, as all of the companion model parameters (depicted in Fig. 2) zero. No current flows into any of the nodes.

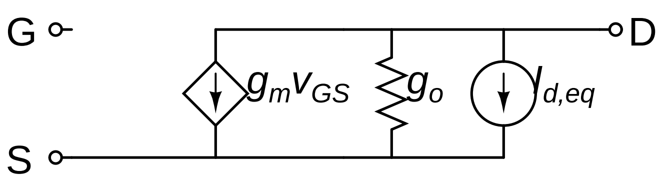


Figure 2: Linear companion model for n-channel MOSFET

2.2 Saturation: $0 \leq v_{GS} - V_T < v_{DS}$

In saturation, the following large-signal model holds:

$$I_D = \frac{K}{2}(v_{GS} - V_T)^2(1 + (v_{DS} - v_{DS,sat})/V_a) \tag{4}$$

where V_a is the Early voltage and $v_{DS,sat} = v_{GS} - V_T$. Since I_D is a function of two variables, we now use partial derivatives to find the small-signal conductances across the DS port.

$$
\begin{aligned}
g_m &= \frac{\partial I_D}{\partial v_{GS}} = K(v_{GS} - V_T)(1 + (v_{DS} - v_{DS,sat})/V_a) = \sqrt{2KI_D} && (5) \\
g_o &= \frac{\partial I_D}{\partial v_{DS}} = \frac{K}{2}(v_{GS} - V_T)^2/V_a = I_S/V_a && (6)
\end{aligned}
$$

We seek to define a bias current from drain to source that will cause the node voltages to evolve by Newton-Raphson iteration, as we did for the diode in the previous section. We now use the multivariable form of the iteration equation:

$$
I_{D,n+1} = I_D + g_m(v_{GS,n+1} - v_{GS,n}) + g_o(v_{DS,n+1} - v_{DS,n}) \tag{7}
$$

Rearranging this equation, we find that $I_{D,n+1} = (I_D - g_m v_{GS,n} - g_o v_{DS,n}) + g_m v_{GS,n+1} + g_o v_{DS,n+1}$. Hence the bias current is $I_{D,eq} = I_D - g_m v_{GS,n} - g_o v_{DS,n}$. Although we will not derive it here, it should now be apparent that the following general result holds when dermining bias currents for Newton-Raphson iteration:

$$
I_{eq} = I(\mathcal{V}_i) - \sum_{v \in \mathcal{V}} \frac{\partial I}{\partial v} v_i \tag{8}
$$

where $\mathcal{V}$ is the set of device port voltages and $\mathcal{V}_i$ is the set of calculated node voltages from the previous iteration.

2.3 Triode: $0 \leq v_{DS} \leq v_{GS} - V_T$

The drain current in this operating region is related to the v_{GS} and v_{DS} by Equation 9. Note that this model does not include the Early effect.

$$
I_S = K((v_{GS} - V_T) - v_{DS}/2)v_{DS} \tag{9}
$$

Applying partial derivatives, we find the small-signal conductances:

$$
\begin{aligned}
g_m &= \frac{\partial I_D}{\partial v_{GS}} = K v_{DS} && (10) \\
g_o &= \frac{\partial I_D}{\partial v_{DS}} = K((v_{GS} - V_T) - v_{DS}) && (11)
\end{aligned}
$$

Again, $I_{D,eq} = I_D - g_m v_{GS} - g_o v_{DS}$.

2.4 p-channel Results

The linear companion model for a p-channel MOSFET is shown in Figure 3, and the values of g_m, g_o, and $I_{S,eq}$ for the various operating regions are summarized in Table 1. For all operating regions, $I_{S,eq}$ is related to I_S, g_m, and g_o by the following equation:

$$I_{S,eq} = I_S - g_m v_{SG} - g_o v_{SD} \tag{12}$$

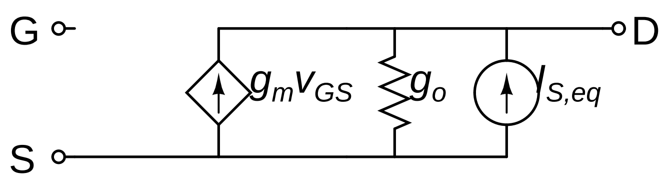


Figure 3: Linear companion model for p-channel MOSFET

Table 1: PMOS companion model parameters

Operating Region	Parameters
Cutoff: $v_{SG} - V_T < 0$	$I_S = 0$ $g_m = 0$ $g_o = 0$
Saturation: $0 \le v_{GS} - V_T < v_{DS}$	$I_S = \frac{K}{2}(v_{SG} - V_T)^2(1 + (v_{SD} - v_{SD,sat})/V_a)$ $g_m = \sqrt{2KI_S}$ $g_o = I_S/V_a$
Triode: $0 \le v_{DS} \le v_{GS} - V_T$	$I_S = K((v_{SG} - V_T) - v_{SD}/2)v_{SD}$ $g_m = Kv_{SD}$ $g_o = K((v_{SG} - V_T) - v_{SD})$

3 Bipolar Junction Transistor

Companion models for BJTs are the most complex circuits presented here because current flows between all nodes. The model is not quite as unwieldy as it looks, however, because there is only one operating region.

We start with a simplified, mid-band Gummel-Poon NPN model, as shown in Figure 4. The base currents are defined as follows:

$$i_{bf} = (I_{fs}/\beta_f)(e^{v_{BE}/V_t} - 1) \quad (13)$$

$$i_{br} = (I_{rs}/\beta_r)(e^{v_{BC}/V_t} - 1) \quad (14)$$

Then, defining the small-signal conductances as in Figure 5, we have

$$g_{\pi,f} = \frac{\partial i_{bf}}{\partial v_{BE}} = (I_{fs}/\beta_f)e^{v_{BE}/V_t}/V_t \approx I_{bf}/V_t \quad (15)$$

$$g_{\pi,r} = \frac{\partial i_{br}}{\partial v_{BC}} = (I_{rs}/\beta_r)e^{v_{BC}/V_t}/V_t \approx I_{br}/V_t \quad (16)$$

$$g_{m,f} = \frac{\partial i_C}{\partial v_{BE}} = I_{fs}e^{v_{BE}/V_t}/V_t = \beta_f g_{\pi,f} \quad (17)$$

$$g_{m,r} = \frac{\partial i_C}{\partial v_{BC}} = I_{rs}e^{v_{BC}/V_t}/V_t = \beta_r g_{\pi,r} \tag{18}$$

We add the Early effect by placing an output resistance across the CE port, with conductance I_C/V_a.

Using the general bias current formula (Eqn. 8), we find

$$I_{bf,eq} = I_b f(v_{BE}) - g_{\pi,f} v_{BE} \tag{19}$$

$$I_{br,eq} = I_b r(v_{BC}) - g_{\pi,r} v_{BC} \tag{20}$$

$$I_{C,eq} = I_C(v_{BE}, v_{BC}) - g_{m,f} v_{BE} + g_{m,r} v_{BC} - g_o v_{CE} \tag{21}$$

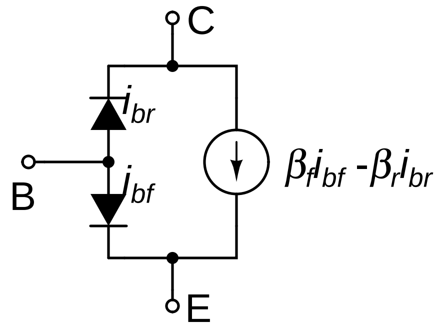


Figure 4: Mid-band Gummel-Poon BJT Model

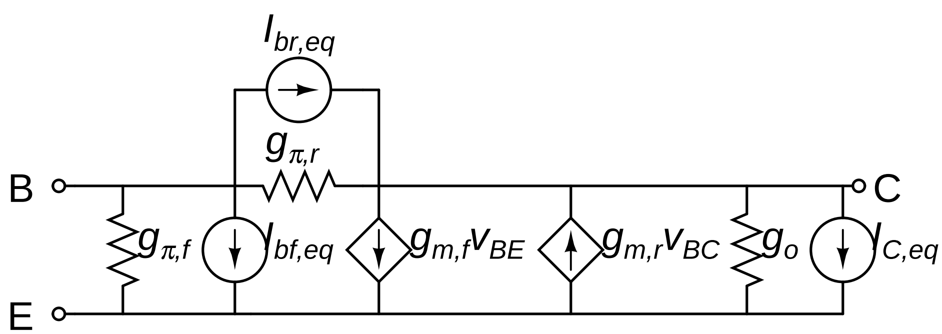


Figure 5: Linear companion model for NPN BJT

The analogous model for a PNP BJT is shown in Figure 6, and the corresponding parameters are listed below:

$$g_{\pi,f} = \frac{\partial i_{bf}}{\partial v_{EB}} = (I_{fs}/\beta_f)e^{v_{EB}/V_t}/V_t \approx I_{bf}/V_t \tag{22}$$

$$
\begin{aligned}
g_{\pi,r} &= \frac{\partial i_{br}}{\partial v_{CB}} = (I_{rs}/\beta_r)e^{v_{CB}/V_t}/V_t \approx I_{br}/V_t && (23)\\
g_{m,f} &= \frac{\partial i_E}{\partial v_{EB}} = I_{fs}e^{v_{EB}/V_t}/V_t = \beta_f g_{\pi,f} && (24)\\
g_{m,r} &= \frac{\partial i_E}{\partial v_{CB}} = I_{rs}e^{v_{CB}/V_t}/V_t = \beta_r g_{\pi,r} && (25)\\
g_o &= I_E/V_a && (26)
\end{aligned}
$$

$$
\begin{aligned}
I_{bf,eq} &= I_b f(v_{EB}) - g_{\pi,f}v_{EB} && (27)\\
I_{br,eq} &= I_b r(v_{CB}) - g_{\pi,r}v_{CB} && (28)\\
I_{C,eq} &= I_C(v_{EB}, v_{CB}) - g_{m,f}v_{EB} + g_{m,r}v_{CB} - g_o v_{EC} && (29)
\end{aligned}
$$

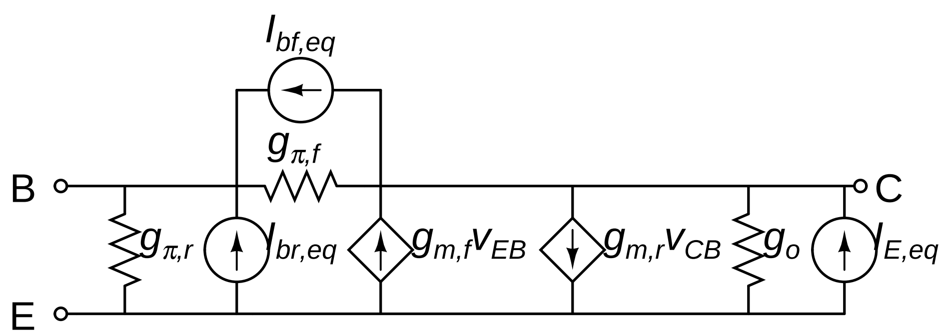


Figure 6: Linear companion model for PNP BJT

4 Capacitor

The companion models presented for capacitors and inductors in the next two sections are somewhat different from the companion models presented so far. Both devices are linear, but require special models to keep track of state variables from timestep to timestep. The state variable for a capacitor is voltage, so a Thevenin equivalent circuit, shown in Figure 7, is the natural choice for a capacitor companion model.

In deriving the model parameters, we start with the differential equation relating current to voltage in a capacitor:

$$I = C(dv/dt) \tag{30}$$

Discretizing this equation, we find

$$v_{n+1} \approx v_n + (\Delta t/C)I \tag{31}$$

How we define I determines the integration method implemented by the companion model. Table 2 summarizes three common integration methods and the corresponding model parameters.

Table 2: Capacitor companion model parameters for integration methods

Method	Parameters
Forward Euler: $I = I_{n+1}$	$V_{eq} = v_n + (\Delta t/C)i_n$, $R = 0$
Backward Euler: $I = I_n$	$V_{eq} = v_n$, $R = \Delta t/C$
Trapezoidal: $I = (I_n + I_{n+1})/2$	$V_{eq} = v_n + (\Delta t/2C)i_n$, $R = \Delta t/2C$

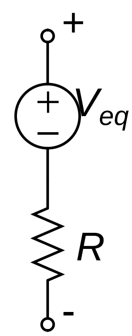


Figure 7: Linear companion model for capacitor

5 Inductor

The state variable for an inductor is current, so a Norton equivalent model is better suited to the inductor (see Fig. 8). The model parameters can be found in the same way as for the capacitor: discretize the i-v differential equation and select an integration method.

We start with the relation V=L(di/dt), and make the following approximation for small timesteps

$$i_{n+1} \approx i_n + (\Delta t/L)V \tag{32}$$

Again, our choice of V will determine the integration method used. The results are displayed in Table 3.

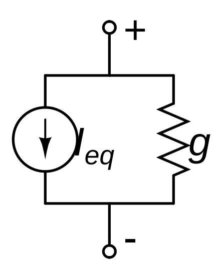


Figure 8: Linear companion model for inductor

Table 3: Induction companion model parameters for integration methods

Method	Parameters
Forward Euler: $V = V_{n+1}$	$I_{eq} = i_n + (\Delta t/L)v_n$ $g = 0$
Backward Euler: $V = V_n$	$I_{eq} = i_n$ $R = \Delta t/L$
Trapezoidal: $V = (V_n + V_{n+1})/2$	$I_{eq} = i_n + (\Delta t/2L)v_n$ $R = \Delta t/2L$

References

[1] T. L. Pillage, R. A. Rohrer, and C. Visweswariah. *Electronic Circuit & System Simulation Methods*. McGraw-Hill, Inc., 1994.