Companion Models for Basic Non-Linear and Transient Devices

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1 Introduction

1.1 Linear DC analysis

1.1.1 Nodal analysis

The most simple kind of circuit simulation deals with constant sources and resistors. There are a number of different techniques to formulate the network equations in a general way, the most convenient for computer simulation being nodal analysis (NA). This method formulates equation in the matrix form $GV = I$, which is straightforward for current sources and resistors by application of the Kirchoff current law (KCL) at each node.
An interesting side effect is that we can formulate the equations in a per-component basis : each component "stamps" itself in the G matrix and I right hand side. For instance, a resistor between node i and node j contributes to the current flowing to node j by $\frac{1}{R}(v_j - v_i)$ and to the current flowing to node i by the negative of the same amount. So its "stamp" is four entries in the G matrix : R in (i,i) and (j,j), and $-R$ in (i,j) and (j,i). The stamp for the current source is derived in the same way : a current source of known current J between node i and j adds J to entry number i of I, and $-J$ to entry number j of I.

1.1.2 Modified nodal analysis

Using NA, we can treat circuits comprised of resistors and current sources. For the other basic building block, voltage source, a trick is required. Since we can not incorporate the missing equations directly, we add another unknown for every voltage source : this is known as modified nodal analysis (MNA) and is at the core of agar-eda. The idea is to split the unknown vector (which is called x in the simulation) and the right hand side (called z) in two parts : the first corresponds to classical nodal analysis, the second is used to treat voltage sources.
The second part of the unknowns are the current flowing through the voltage

source. A voltage source of known voltage V between nodes i and j adds the unknown $I_{i,j}$ to the unknowns vector, and contributes to the equations with by adding $I_{i,j}$ to i_i and $-I_{i,j}$ to i_j, and the new equation $V_j - V_i = V$. These equations, like the previous ones, can be incorporated by stamping.
The new matrix relation becomes $Ax = z$:

- z is (z_{NA}, z_{MNA}), z_{NA} being the stamps of current sources, and z_{MNA} the stamps of voltage sources.
- x is (x_{NA}, x_{MNA}), x_{NA} being the voltages at the nodes, and x_{MNA} the currents flowing through the voltage sources.
- A is the block matrix $(G, C; B, D)$, G being the admittance matrix of NA, and B, C, D representing various constitutive equations concerning the voltage sources.

1.2 Non-linear DC analysis

TODO

1.3 Transient analysis

TODO

2 Non-linear components

2.1 PN Diode

As a two-terminal device with only one distinct operating region, the PN diode was a natural starting point for non-linear simulation. The current model does not include parasitic capacitances, but this is planned for the future.

The i-v relation that describes a diode is the following:

$$i(v) = I_S(e^{v/V_t} - 1) \tag{1}$$

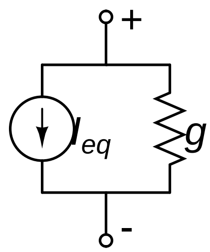
where I_S is the reverse saturation current and V_t is the thermal voltage (kT/q). From this, we derive the small-signal conductance:

$$g = \frac{di}{dv} = (I_S/V_t)e^{v/V_T} \approx i(v)/V_t \tag{2}$$

The error committed in the equation is I_S/V_t, which for common diodes is around $10^{-12}S$, which is negligibly for common applications and allow us to avoid computing the exp function twice. In simulation we use the Newton-Raphson method to solve circuits with non-linear elements. The method works by linearizing the constitutive equations, solve for a new operating point, and iterate until convergence. The linearised curve around operating point v_n is :

$$i_{lin}(v) = i(v_n) + (di/dv)(v_n)(v - v_n) \tag{3}$$

By rearranging this equation, we find the i-v relation $i_{lin} = (i(v_n) - gv_n) + gv$. Hence the appropriate companion model for a diode is a conductance g in parallel with a current source $i(v_n) - gv_n$, not simply $i(v_n)$, as one might suspect. The result is shown in Figure **??**.



$$\begin{aligned} g_{eq} &= i(v_n)/V_t \\ I_{eq} &= i(v_n) - g_{eq}v_n \end{aligned}$$

Figure 1: Linear companion model for diode

2.2 Metal-Oxide-Silicon Field-Effect Transistor

The next simplest non-linear devices are n- and p-channel MOSFETs, as no current flows into the gate, and the non-linear source-to-drain port depends on only two parameters, v_{GS} and v_{DS}. The model is complicated slightly, however, by the fact that MOSFETs have three distinct operating regions. For brevity, we derive only the results for n-channel MOSFETs. Similar results for p-channel MOSFETs are presented at the end of this subsection.

2.2.1 Cutoff: $v_{GS} - V_T < 0$

Cutoff is a trivial operating region, as all of the companion model parameters (depicted in Fig. **??**) zero. No current flows into any of the nodes.

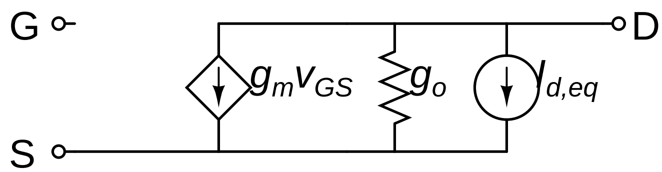


Figure 2: Linear companion model for n-channel MOSFET

2.2.2 Saturation: $0 \leq v_{GS} - V_T < v_{DS}$

In saturation, the following large-signal model holds:

$$I_D = \frac{K}{2}(v_{GS} - V_T)^2(1 + (v_{DS} - v_{DS,sat})/V_a) \tag{4}$$

where V_a is the Early voltage and $v_{DS,sat} = v_{GS} - V_T$. Since I_D is a function of two variables, we now use partial derivatives to find the small-signal conductances across the DS port.

$$g_m = \frac{\partial I_D}{\partial v_{GS}} = K(v_{GS} - V_T)(1 + (v_{DS} - v_{DS,sat})/V_a) = \sqrt{2KI_D} \tag{5}$$

$$g_o = \frac{\partial I_D}{\partial v_{DS}} = \frac{K}{2}(v_{GS} - V_T)^2/V_a = I_S/V_a \tag{6}$$

We seek to define a bias current from drain to source that will cause the node voltages to evolve by Newton-Raphson iteration, as we did for the diode in the previous subsection. We now use the multivariable form of the iteration equation:

$$I_{D,n+1} = I_D + g_m(v_{GS,n+1} - v_{GS,n}) + g_o(v_{DS,n+1} - v_{DS,n}) \tag{7}$$

Rearranging this equation, we find that $I_{D,n+1} = (I_D - g_m v_{GS,n} - g_o v_{DS,n}) + g_m v_{GS,n+1} + g_o v_{DS,n+1}$. Hence the bias current is $I_{D,eq} = I_D - g_m v_{GS,n} - g_o v_{DS,n}$. Although we will not derive it here, it should now be apparent that the following general result holds when dermining bias currents for Newton-Raphson iteration:

$$I_{eq} = I(\mathcal{V}_i) - \sum_{v \in \mathcal{V}} \frac{\partial I}{\partial v} v_i \tag{8}$$

where $\mathcal{V}$ is the set of device port voltages and $\mathcal{V}_i$ is the set of calculated node voltages from the previous iteration.

2.2.3 Triode: $0 \leq v_{DS} \leq v_{GS} - V_T$

The drain current in this operating region is related to the v_{GS} and v_{DS} by Equation **??**. Note that this model does not include the Early effect.

$$I_S = K((v_{GS} - V_T) - v_{DS}/2)v_{DS} \tag{9}$$

Applying partial derivatives, we find the small-signal conductances:

$$g_m = \frac{\partial I_D}{\partial v_{GS}} = K v_{DS} \tag{10}$$

$$g_o = \frac{\partial I_D}{\partial v_{DS}} = K((v_{GS} - V_T) - v_{DS}) \tag{11}$$

Again, $I_{D,eq} = I_D - g_m v_{GS} - g_o v_{DS}$.

2.2.4 p-channel Results

The linear companion model for a p-channel MOSFET is shown in Figure **??**, and the values of g_m, g_o, and $I_{S,eq}$ for the various operating regions are summarized in Table **??**. For all operating regions, $I_{S,eq}$ is related to I_S, g_m, and g_o by the following equation:

$$I_{S,eq} = I_S - g_m v_{SG} - g_o v_{SD} \tag{12}$$

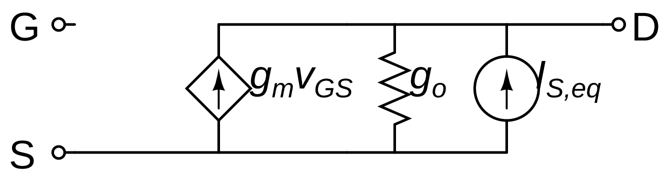


Figure 3: Linear companion model for p-channel MOSFET

Table 1: PMOS companion model parameters

Operating Region	Parameters
Cutoff: $v_{SG} - V_T < 0$	$I_S = 0$, $g_m = 0$, $g_o = 0$
Saturation: $0 \le v_{GS} - V_T < v_{DS}$	$I_S = \frac{K}{2}(v_{SG} - V_T)^2(1 + (v_{SD} - v_{SD,sat})/V_a)$, $g_m = \sqrt{2KI_S}$, $g_o = I_S/V_a$
Triode: $0 \le v_{DS} \le v_{GS} - V_T$	$I_S = K((v_{SG} - V_T) - v_{SD}/2)v_{SD}$, $g_m = Kv_{SD}$, $g_o = K((v_{SG} - V_T) - v_{SD})$

2.3 Bipolar Junction Transistor

Companion models for BJTs are the most complex circuits presented here because current flows between all nodes. The model is not quite as unwieldy as it looks, however, because there is only one operating region.

We start with a simplified, mid-band Gummel-Poon NPN model, as shown in Figure **??**. The base currents are defined as follows:

$$i_{bf} = (I_{fs}/\beta_f)(e^{v_{BE}/V_t} - 1) \tag{13}$$
$$i_{br} = (I_{rs}/\beta_r)(e^{v_{BC}/V_t} - 1) \tag{14}$$

Then, defining the small-signal conductances as in Figure **??**, we have

$$g_{\pi,f} = \frac{\partial i_{bf}}{\partial v_{BE}} = (I_{fs}/\beta_f)e^{v_{BE}/V_t}/V_t \approx I_{bf}/V_t \tag{15}$$
$$g_{\pi,r} = \frac{\partial i_{br}}{\partial v_{BC}} = (I_{rs}/\beta_r)e^{v_{BC}/V_t}/V_t \approx I_{br}/V_t \tag{16}$$
$$g_{m,f} = \frac{\partial i_C}{\partial v_{BE}} = I_{fs}e^{v_{BE}/V_t}/V_t = \beta_f g_{\pi,f} \tag{17}$$