A Notations

- x: Denotation for a classified Quanitity of presence.
- H_x: Hits, which is the number of observations that are reference presence and correctly classified at Quantity x
- F_x: False Alarms, which is the number of observations that are reference absence and are incorrectly classified at Quantity x
- M_x: Misses, which is the number of observations that are reference presence and are incorrectly classified at Quantity x
- P: Number of observations that are reference presence
- Q: Number of observations that are reference absence
- $E = P + Q$: Extent
- $\mathrm{Pr} = \frac{P}{P+Q}$ = Prevalence, proportion of Presence in the Extent
- $Q = \text{Absence} = \frac{P(1-\mathrm{Pr})}{\mathrm{Pr}} = (1 - \mathrm{Pr})E$

B Reference Areas

B.1 Area of the Upper Bound CSI

The Upper Bound of CSI follows the equation:

$$\begin{Bmatrix} \text{if } x < P, \text{ then } F_x = 0; H_x = x \\ \text{if } x \geq P, \text{ then } H_x = P; F_x = x - P \end{Bmatrix}$$

B.1.1 Area in the range $[0, P]$

$$ACSI(x) = \min\left(\int_0^x \frac{x}{P}dx, \frac{P}{2}\right) + \max\left(0, \int_P^x \frac{P}{x}dx\right)$$

$$ACSI(x) = \min\left(\frac{x^2}{2P}, \frac{P}{2}\right) + \max\left(0, P\ln\left(\frac{x}{P}\right)\right)$$

B.1.2 Area in the range $(P, E]$

The area is equal to the sum of the area minus the lower right triangle.

$$ACSI(x) = \int_{x=P}^{E} \left(\frac{P}{F_x + P}\right) dx - \frac{P}{P+Q}$$

$$= \int_{x=P}^{E} \left(\frac{P}{c}\right) dx - \frac{P}{P+Q} = P\left[\ln(P+Q) - \ln(P)\right] - \frac{P}{P+Q}$$

$$ACSI(x) = P\ln\left(\frac{P+Q}{P}\right) - \frac{P}{P+Q} = P\ln\left(\frac{1}{\mathrm{Pr}}\right) - \mathrm{Pr}$$

B.1.3 Total Area Covered by the Upper Bound

$$ACSI_E = ACSI_P + ACSI_{E-P}$$

$$= \frac{P}{2} + P\ln\left(\frac{P+Q}{P}\right) - \frac{P}{P+Q}$$

$$ACSI_E = P\left[\frac{1}{2} - \ln(\mathrm{Pr}) - \frac{1}{P+Q}\right]$$

B.2 Area of the Baseline CSI

B.2.1 Area as a Function of Hits + False Alarms

$$BACSI(x) = \int_0^x \frac{H_x}{H_x + F_x + P}\, dx = \int_0^x \frac{\mathrm{Pr}x}{(1-\mathrm{Pr})x + P}\, dx$$

Let $u = (1-\mathrm{Pr})x + P$ and $du = (1-\mathrm{Pr})\, dx$.

$$\int \frac{\mathrm{Pr}x}{(1-\mathrm{Pr})x + P}\, dx = \frac{\mathrm{Pr}}{1-\mathrm{Pr}}\left[(1-\mathrm{Pr})x + P - P\ln((1-\mathrm{Pr})x + P)\right]$$

$$\int_0^x \frac{\mathrm{Pr}x}{(1-\mathrm{Pr})x + P}\, dx =$$

$$= \frac{\mathrm{Pr}}{1-\mathrm{Pr}}\left[(1-\mathrm{Pr})x + P - P\ln((1-\mathrm{Pr})x + P)\right] - \frac{\mathrm{Pr}}{1-\mathrm{Pr}}\left[P - P\ln P\right]$$

$$= \frac{\mathrm{Pr}}{1-\mathrm{Pr}}\left[(1-\mathrm{Pr})x + P\ln P - P\ln((1-\mathrm{Pr})x + P)\right]$$

$$BACSI(x) = \frac{\mathrm{Pr}}{1-\mathrm{Pr}}\left[(1-\mathrm{Pr})x + P\ln\left(\frac{P}{(1-\mathrm{Pr})x + P}\right)\right]$$

B.2.2 Total Area Covered by the Baseline

$$BACSI_E = \int_{x=0}^{E} \frac{\mathrm{Pr}x}{(1-\mathrm{Pr})x+P}\,dx$$

$$= \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left[(1-\mathrm{Pr})E - P\ln(\frac{P}{(1-\mathrm{Pr})E+P}\right]$$

$$= \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left[(1-\mathrm{Pr})E - P\ln\mathrm{Pr}\right]$$

$$BACSI_E = \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left[Q - P\ln\mathrm{Pr}\right]$$

B.3 ACSI(x) as a Proportion of the Area Between the Upper Bound and the Baseline

$$ReferenceArea(x) = \frac{\text{Area between UpperCSI and uniform line (x)}}{\text{Total Area between UpperCSI and uniform line}}$$

$$= \frac{ACSI(x) - BACSI(x)}{ACSI_E - BACSI_E}$$

$$= \frac{\min\left(\frac{x^2}{2P}, \frac{P}{2}\right) + \max\left(0, P\ln\left(\frac{x}{P}\right)\right) - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left(P\ln\left(\frac{P}{(1-\mathrm{Pr})x+P}\right) + (1-\mathrm{Pr})x\right)}{\frac{P}{2} - P\ln(\mathrm{Pr}) - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}(P\ln(\mathrm{Pr}) + Q)}$$

$$= \frac{\min\left(\frac{1}{2\mathrm{Pr}^2}(\frac{x}{E})^2, \frac{1}{2}\right) + \max\left(0, \ln\left(\frac{x}{\mathrm{Pr}E}\right)\right) + \frac{1}{(1-\mathrm{Pr})^2}\left(\mathrm{Pr}\ln\left(\frac{1-\mathrm{Pr}}{\mathrm{Pr}}\frac{x}{E} + 1\right) + (1-\mathrm{Pr})\frac{x}{E}\right)}{\frac{1}{2} - \ln(\mathrm{Pr}) - \frac{1}{(1-\mathrm{Pr})^2}(\mathrm{Pr}\ln(\mathrm{Pr}) + (1-\mathrm{Pr}))}$$

B.4 Area of the Upper Bound TOC

B.4.1 Area in the range $[0, P]$

$$AUC_P = \frac{P^2}{2}$$

B.4.2 Area in the range $(P, E]$

The area is equal to the sum of the area minus the lower right triangle:

$$AUC_{(E-P)} = P \cdot Q - \frac{P^2}{2}$$

B.4.3 Total Area Covered by the Upper Bound

$$AUC_E = AUC_P + AUC_{(E-P)} = P \cdot Q$$

B.5 Area of the Baseline as a Function of Hits + False Alarms

$$BAUC(x) = \min\left(\frac{x^2}{2}, \frac{P^2}{2}\right) + \max(0, P(x-P)) - \max(0, x-Q)$$

B.6 AUC(x) as a Proportion of the Area Between the Upper Bound and the Baseline

The proportion of the area between the Upper TOC and the uniform line is given by:

$$\frac{\text{Area between Upper TOC and uniform line (x)}}{\text{Total Area between Upper TOC and uniform line}}$$

$$= \frac{AUC(x) - BAUC(x)}{AUC_E - AUC_E}$$

$$= \frac{\min\left(\frac{x^2}{2}, \frac{P^2}{2}\right) + \max\left(0, P(x-P)\right) - \frac{\mathrm{Pr}x^2}{2}}{\frac{P \cdot Q}{2}}$$

$$= \frac{\min\left((\frac{x}{E})^2, \mathrm{Pr}^2\right) + \max\left(0, \mathrm{Pr}(\frac{x}{E} - \mathrm{Pr})\right) - \frac{\mathrm{Pr}}{2}(\frac{x}{E})^2}{\mathrm{Pr}(1-\mathrm{Pr})}$$

C Questions

C.1 Question (1): Does CSI Concentrate More Area Around P Than TOC?

To answer this, we analyze the contribution to Reference area at P, i.e., the derivatives at P of the equations (7) and (8). We calculate the elasticity to compare the values for any extent.

C.1.1 TOC

For $x < P$:

$$AUC(x) = \frac{x^2 - \mathrm{Pr}x^2}{P \cdot Q}$$

$$AUC'(x) = \frac{x - \mathrm{Pr}x}{\frac{PQ}{2}} = \frac{2x(1-\mathrm{Pr})}{P \cdot Q} = \frac{2x}{P \cdot E}$$

$$\varepsilon(x) = \frac{AUC'(x)x}{AUC(x)} = \frac{\frac{2x^2}{P \cdot E}}{\frac{x^2 - \mathrm{Pr}x^2}{P \cdot Q}} = \frac{2x^2 \cdot Q}{Ex^2 \cdot (1-\mathrm{Pr})} = 2$$

At $x = P$:

$$\varepsilon(P) = 2$$

For $x \geq P$:

$$AUC(x) = \frac{\frac{P^2}{2} + P(x-P) - \frac{\mathrm{Pr}x^2}{2}}{\frac{PQ}{2}}$$

$$AUC'(x) = \frac{P - \mathrm{Pr}x}{\frac{PQ}{2}} = \frac{2(P \cdot E - \mathrm{Pr}x)}{P \cdot (1-\mathrm{Pr}) \cdot E^2} = \frac{2(E-x)}{(1-\mathrm{Pr}) \cdot E^2}$$

$$\varepsilon(x) = \frac{AUC'(x)x}{AUC(x)} = \frac{(P - \mathrm{Pr}x)x}{\frac{P^2}{2} + P(x-P) - \frac{\mathrm{Pr}x^2}{2}}$$

At $x = P$:

$$\varepsilon(P) = \frac{AUC'(x)x}{AUC(x)} = \frac{P - \mathrm{Pr}P}{\frac{P}{2} - \frac{\mathrm{Pr}P}{2}} = 2$$

C.1.2 CSI

For $x < P$:

$$ACSI'(x) = \frac{\frac{x}{P} - \mathrm{Pr} \cdot \frac{x}{(1-\mathrm{Pr})x + P}}{ACSI_E}$$

$$\varepsilon(x) = \frac{ACSI'(x)}{ACSI(x)} \cdot x = \frac{\frac{\frac{x}{P} - \mathrm{Pr} \cdot \frac{x}{(1-\mathrm{Pr})x+P}}{ACSI_E} x}{\frac{\frac{x^2}{2P} - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left(P \ln\left(\frac{P}{(1-\mathrm{Pr})x+P}\right) + (1-\mathrm{Pr})x\right)}{ACSI_E}}$$

$$= \frac{\frac{x^2}{P} - \mathrm{Pr} \cdot \frac{x^2}{(1-\mathrm{Pr})x+P}}{\frac{x^2}{2P} - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left(P \ln\left(\frac{P}{(1-\mathrm{Pr})x+P}\right) + (1-\mathrm{Pr})x\right)}$$

At $x = P$:

$$\varepsilon(P) = \frac{\frac{P^2}{P} - \mathrm{Pr}\cdot\frac{P^2}{(1-\mathrm{Pr})P+P}}{\frac{P^2}{2P} - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left(P\ln\left(\frac{P}{(1-\mathrm{Pr})P+P}\right) + (1-\mathrm{Pr})P\right)}$$

$$= \frac{P(1-\frac{\mathrm{Pr}}{2-\mathrm{Pr}})}{\frac{P}{2} - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left(P\ln(\frac{1}{2-\mathrm{Pr}}) + (1-\mathrm{Pr})P\right)}$$

$$= \frac{1-\frac{\mathrm{Pr}}{2-\mathrm{Pr}}}{\frac{1}{2} - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left(\ln(\frac{1}{2-\mathrm{Pr}}) + (1-\mathrm{Pr})\right)}$$

$$= \frac{1-\frac{\mathrm{Pr}}{2-\mathrm{Pr}}}{\frac{1}{2} + \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\ln(2-\mathrm{Pr}) - \frac{\mathrm{Pr}}{1-\mathrm{Pr}}}$$

For $x \geq P$:

$$ACSI'(x) = \frac{\frac{P}{x} - Pr\left(\frac{x}{(1-Pr)x+P}\right)}{ACSI_E}$$

$$\varepsilon(x) = \frac{ACSI'(x)}{ACSI(x)}\cdot x = \frac{\frac{\frac{P}{x}-Pr\frac{x}{(1-Pr)x+P}}{ACSI_E}\cdot x}{\frac{\frac{P}{2}+Pln(\frac{x}{P})-\frac{Pr}{(1-Pr)^2}\left(Pln\left(\frac{P}{(1-Pr)x+P}\right)+(1-Pr)x\right)}{ACSI_E}}$$

$$= \frac{P - Pr\frac{x^2}{(1-Pr)x+P}}{\frac{P}{2} + Pln(\frac{x}{P}) - \frac{Pr}{(1-Pr)^2}\left(Pln\left(\frac{P}{(1-Pr)x+P}\right) + (1-Pr)x\right)}$$

At $x = P$:

$$\varepsilon(P) = \frac{P - Pr\frac{P^2}{(1-Pr)P+P}}{\frac{P}{2} + Pln(1) - \frac{Pr}{(1-Pr)^2}\left(Pln\left(\frac{P}{(1-Pr)P+P}\right) + (1-Pr)P\right)}$$

$$= \frac{1-\frac{\mathrm{Pr}}{2-\mathrm{Pr}}}{\frac{1}{2} + \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\ln(2-\mathrm{Pr}) - \frac{\mathrm{Pr}}{1-\mathrm{Pr}}}$$

C.2 Question (2): Asymmetry Between the Right and Left Neighborhoods of P

To analyze asymmetry, we calculate the areas to the right and left of P for both metrics and determine their ratio.

C.2.1 TOC

Let $h = \min(P, Q)$.

Right Area:

$$\text{Right Area} = \int_{P}^{P+h} (P - \text{Pr}x)\, dx$$

$$= \left[Px - \frac{\text{Pr}x^2}{2} \right]_{P}^{P+h}$$

$$= \left(P(P+h) - \frac{\text{Pr} \cdot (P+h)^2}{2} \right) - \left(P^2 - \frac{\text{Pr} \cdot P^2}{2} \right)$$

$$= Ph - \frac{\text{Pr} \cdot (P+h)^2 - \text{Pr} \cdot P^2}{2}$$

$$= Ph - \frac{2\text{Pr} \cdot Ph + \text{Pr} \cdot h^2}{2} = (1 - \text{Pr}) \cdot Ph - \frac{\text{Pr} \cdot h^2}{2}$$

$$RightArea = h \left((1 - \text{Pr}) \cdot P - \frac{\text{Pr} \cdot h}{2} \right)$$

Left Area:

$$\text{Left Area} = \int_{P-h}^{P} (x - \text{Pr}x)\ dx$$

$$= \left[\frac{x^2}{2} - \frac{\text{Pr}x^2}{2} \right]_{P-h}^{P}$$

$$= \frac{P^2(1 - \text{Pr})}{2} - \frac{(P-h)^2(1 - \text{Pr})}{2}$$

$$= \frac{(1 - \text{Pr})}{2} \left(2Ph - h^2 \right)$$

$$\text{Left Area} = h \left((1 - \text{Pr}) \cdot P - \frac{(1 - \text{Pr}) \cdot h}{2} \right)$$

Asymmetry: The ratio of Right Area to Left Area is:

$$\frac{\text{Right Area}}{\text{Left Area}} = \frac{\frac{2P(1-\text{Pr}) - \text{Pr} \cdot h}{2}}{\frac{2P(1-\text{Pr}) - (1-\text{Pr}) \cdot h}{2}}$$

$$= \frac{2P(1 - \text{Pr}) - \text{Pr} \cdot h}{2P(1 - \text{Pr}) - (1 - \text{Pr}) \cdot h}$$

$$= \frac{2P - \frac{\text{Pr} \cdot h}{(1-\text{Pr})}}{2P - \frac{h}{(1-\text{Pr})}}$$

If $h = P$:

$$\frac{\text{Right Area}}{\text{Left Area}} = 2 - \frac{\text{Pr}}{1 - \text{Pr}}$$

For $\Pr = 0.5$:

$$\frac{\text{Right Area}}{\text{Left Area}} = 1$$

C.2.2 CSI

Let $h = \min(P, Q)$.

Right Area:

$$\text{Right Area} = \int_P^{P+h} \left(\frac{P}{x} - \frac{\Pr x}{(1-\Pr)x + P} \right) dx$$

$$= \left[P \ln x - \frac{\Pr}{(1-\Pr)^2} \left(\ln\left((1-\Pr)x + P\right) - (1-\Pr)x + \ln P \right) \right]_P^{P+h}$$

$$= P \ln\left(1 + \frac{h}{P}\right) - \frac{\Pr \cdot h}{1-\Pr} + \frac{\Pr \cdot P}{(1-\Pr)^2} \ln\left(1 + \frac{h \cdot (1-\Pr)}{P \cdot (2-\Pr)}\right)$$

Left Area:

$$\text{Left Area} = \int_{P-h}^{P} \left(\frac{x}{P} - \frac{\Pr x}{(1-\Pr)x + P} \right) dx$$

$$= \left[\frac{x^2}{2P} - \frac{\Pr}{(1-\Pr)^2} \left(\ln\left((1-\Pr)x + P\right) - (1-\Pr)x + \ln P \right) \right]_{P-h}^{P}$$

$$= h - \frac{h^2}{2P} - \frac{\Pr \cdot h}{1-\Pr} + \frac{\Pr \cdot P}{(1-\Pr)^2} \ln\left(1 + \frac{h \cdot (1-\Pr)}{P \cdot (2-\Pr)}\right)$$

Assymetry: If $h = P$:

$$\frac{\text{Right Area}}{\text{Left Area}} = \frac{\ln 2 - \frac{\Pr}{1-\Pr} + \frac{\Pr}{(1-\Pr)^2} \ln\left(1 + \frac{1-\Pr}{2-\Pr}\right)}{\frac{1}{2} - \frac{\Pr}{1-\Pr} - \frac{\Pr}{(1-\Pr)^2} \ln\left(1 + \frac{1-\Pr}{2-\Pr}\right)}$$

C.3 Question (3): Does ACSI Put Less Weight than AUC to the Last Thresholds of the Graph?

To determine if ACSI puts less weight on the last thresholds compared to TOC, we compare the proportion of the area covered around P by each metric. A higher proportion of area covered around P implies less area is covered by the metric elsewhere.

C.3.1 TOC

Proportion at $x = P$:

$$\mathrm{AUC}(P) = \frac{\frac{P^2}{2} - \frac{\mathrm{Pr}\cdot P^2}{2}}{\frac{PQ}{2}}$$

$$= \frac{P - \mathrm{Pr}\cdot P}{Q} = \frac{P}{Q}\cdot(1-\mathrm{Pr})$$

$$\mathrm{AUC}(P) = \frac{\mathrm{Pr}}{1+\mathrm{Pr}}\cdot(1+\mathrm{Pr}) = \mathrm{Pr}$$

C.3.2 CSI

Proportion at $x = P$:

$$\mathrm{ACSI}(P) = \frac{\frac{1}{2} - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left(\ln(P(2-\mathrm{Pr})) - (1-\mathrm{Pr}) - \ln P\right)}{\frac{1}{2} - \ln\mathrm{Pr} - \frac{\mathrm{Pr}\ln\mathrm{Pr} + (1-\mathrm{Pr})}{(1-\mathrm{Pr})^2}}$$

$$= \frac{\frac{1}{2} - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})^2}\left(\ln(2-\mathrm{Pr}) - (1-\mathrm{Pr})\right)}{\frac{1}{2} - \ln\mathrm{Pr} - \frac{\mathrm{Pr}\ln\mathrm{Pr} + (1-\mathrm{Pr})}{(1-\mathrm{Pr})^2}}$$

$$= \frac{\frac{1-\mathrm{Pr}\ln(2-\mathrm{Pr})}{(1-\mathrm{Pr})^2} + \mathrm{Pr}}{\frac{1}{2} - \ln\mathrm{Pr} - \frac{(1-\mathrm{Pr})^2+\mathrm{Pr}}{(1-\mathrm{Pr})^2}\ln\mathrm{Pr}}$$

$$= \frac{\frac{1-3\mathrm{Pr}}{2} - \frac{\mathrm{Pr}}{(1-\mathrm{Pr})}\ln(2-\mathrm{Pr})}{-\frac{1+\mathrm{Pr}}{2} - \frac{(1-\mathrm{Pr})^2+\mathrm{Pr}}{(1-\mathrm{Pr})}\ln\mathrm{Pr}}$$